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Numerical simulation of an electro-thermal model for superconducting nanowire single-photon detectors ¹

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Abstract

The electro-thermal model for Superconducting Nanowire Single-Photon Detectors is a nonlinear free boundary problem involving the temperature and the current, which are coupled together by a nonlinear parabolic interface equation and a second order ordinary differential equation. In this paper, we propose a novel method to numerically solve the preceding electro-thermal model. A series of numerical experiments are provided to demonstrate the effectiveness of the method proposed.

Keywords. Electro-thermal model, free boundary problem, finite difference method, shooting method

1 Introduction

In recent years, superconducting nanowires single photon detection (SNSPD) has emerged as a new and promising single photon detection technology and has received wide attention in the field of applied superconductivity (cf. [1, 8]). The corresponding device structures nanometer zigzag line on the ultra-thin superconducting material, and uses the highly sensitive response of superconducting nanowire to realize single-photon detection. As shown in Figure 1 (see [1]), the key step of SNSPD is to discover the variation of the photon-induced hotspot.

In 2007, some researchers in MIT (cf. [15]) proposed a relevant electro-thermal mechanism to account for the variation of the photon-induced hotspot in SNSPD, after a small resistive hotspot forms along the nanowire. In this model, the SNSPD is approximated as a one-dimensional structure, the thermal response is modeled by a one-dimensional nonlinear parabolic interface equation involving the current flowing through the nanowire, and the electrical response is modeled by a second order ordinary differential equation. The two equations are coupled together to form a free boundary problem (cf. [3]).

To be more precise, let L denote the length of the superconducting nanowire under discussion, d the wire thickness, and W the width of nanowire. The domain occupying the nanowire is simply written as $\tilde{\Omega} = (-L/2, L/2)$. Due to the symmetry of the physical process, it suffices for us to discuss the variation of physical quantities in the half part $\Omega = (0, L/2)$. This domain is further split into two regions, $\Omega_{norm}(t) = (0, l(t))$ and $\Omega_{super}(t) = (l(t), L/2)$, corresponding to the normal/resistive and superconducting states, respectively. The interface $x = l(t)$ is used to separate the two states at time t . Let $T(x, t)$ represent the temperature of the material in the point x at time t . Then, as given in [15], $T(x, t)$ is determined by the parabolic interface equation

$$J^2 \rho + \kappa_n \frac{\partial^2 T}{\partial x^2} - \frac{\alpha}{d}(T - T_{sub}) = \frac{\partial C_n T}{\partial t}, \quad 0 < x < l(t), \quad t > 0, \quad (1.1)$$

$$\kappa_s \frac{\partial^2 T}{\partial x^2} - \frac{\alpha}{d}(T - T_{sub}) = \frac{\partial C_s T}{\partial t}, \quad l(t) < x < L/2, \quad t > 0, \quad (1.2)$$

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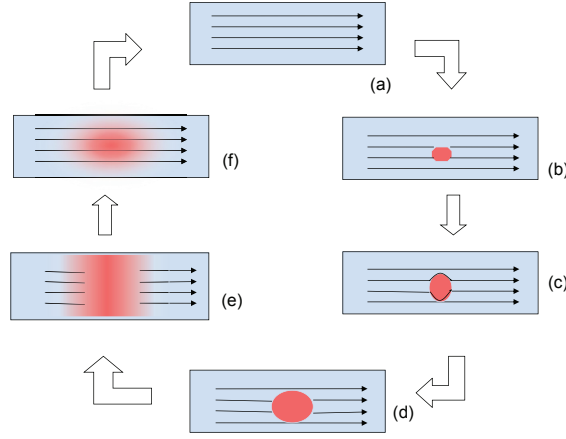


Figure 1: The variation of the photon-induced hotspot in SNSPD. (a) Bias direct current close to (but less than) its critical current, and set the nanowire temperature well below its superconducting critical temperature. (b) Form a small resistive hotspot. (c) The hotspot region forces the supercurrent to flow around the periphery of the hotspot, since the hotspot itself is not large enough to span the width of the nanowire. (d) Form a resistive barrier across the width of the nanowire, results in an easily measurable voltage pulse. (e) Resistive region is increased, the bias current is shunted by the external circuit. (f) The NbN nanowire becomes fully superconducting again.

with the initial condition

$$T(x, 0) = T_0, \quad 0 \leq x \leq L/2.$$

Observe that $T(x, t)$ is symmetric about $x = 0$ with respect to x , and L is taken large enough such that the temperature at $x = L/2$ almost coincides with the substrate temperature. Then we impose the following boundary conditions:

$$\left. \frac{\partial T(x, t)}{\partial x} \right|_{x=0} = 0, \quad T(L/2, t) = T_{sub}, \quad t > 0. \quad (1.3)$$

Moreover, at the interface point $x = l(t)$ we impose the standard interface conditions:

$$T(x, t)|_{l^-} = T(x, t)|_{l^+}, \quad \kappa_n \left. \frac{\partial T(x, t)}{\partial x} \right|_{l^-} = \kappa_s \left. \frac{\partial T(x, t)}{\partial x} \right|_{l^+}, \quad t > 0, \quad (1.4)$$

as well as a phase transition condition

$$I_c(T) = I_c(0) \times (1 - (T/T_c)^2)^2. \quad (1.5)$$

Here, $J = \frac{I(t)}{Wd}$ is the current density through the nanowire, ρ is the electrical resistivity, κ_n and κ_s are the thermal conductivity coefficients, α is the thermal boundary conductance between the film and the substrate, T_{sub} is the substrate temperature (since the nanowire is thin enough), C_n and C_s are the heat capacity (per unit volume) of the superconducting film, $I_c(0)$ is the initial critical current, and T_c is the critical temperature. We mention that the transition condition is an empirical relation (see [15, p. 582]), which was obtained from an excellent fit with experimental measurements. Using this expression, one can determine

a segment to be resistive when $I > I_c(T)$, where $T = T(x)$ is the temperature of a nanowire at the position x . The remaining part of the nanowire then belongs to the superconducting state.

On the other hand, using Kirchhoff's first law, we can find as in [3] that the current $I(t)$ through the nanowire satisfies an ordinary differential equation

$$I(t)R(t) + L_k \frac{dI(t)}{dt} = \frac{1}{C_{bt}} \int_0^t (I_{bias} - I(s))ds + (I_{bias} - I(t))Z_0, \quad t > 0, \quad (1.6)$$

with the initial condition

$$I(0) = I_0.$$

Differentiating (1.6) once with respect to t gives

$$C_{bt} \left(\frac{d^2 L_k I}{dt^2} + \frac{d(I(t)R(t))}{dt} + Z_0 \frac{dI(t)}{dt} \right) = I_{bias} - I(t), \quad t > 0, \quad (1.7)$$

where C_{bt} is the capacitor, an inductor L_k and a resistor $R(t)$ represent respectively the kinetic inductance of the superconducting nanowire and the time-dependent hotspot resistance, the time-dependent hotspot resistance respectively is given as $R(t) = 2\rho \frac{l(t)}{S} = 2\rho \frac{l(t)}{Wd}$, Z_0 is the impedance of the transmission line connecting the probe to RF amplifiers (cf. [9]), I_{bias} is the bias current of the SNSPD.

It is easy to see that the above electro-thermal model is a nonlinear free boundary problem with the interface $x = l(t)$ to be determined. Observe that the quantity J appearing in (1.1) satisfies that $J = \frac{I(t)}{Wd}$, and the quantity $R(t)$ appearing in (1.6) satisfies that $R(t) = 2\rho \frac{l(t)}{Wd}$. Hence, the temperature $T(x, t)$ and the current $I(t)$ are coupled together by the equations (1.1)-(1.2) and (1.7). Therefore, it is very challenging to devise an efficient method for numerically approximating the solution of this model. As far as we know, there is no work discussing numerical solution for the previous model systematically in the literature. The goal of this paper is intended to design some efficient algorithms for such a problem.

Before designing our algorithm, let us review some typical methods for numerically solving free boundary problems. First of all, front-tracking methods which use an explicit representation of the interface has always been a common way of solving moving boundary problems. Juric and Tryggvason presented in [5] a front-tracking method which use a fixed grid in space and explicit tracking of the liquid-solid interface, the method performs well in approximating the exact solution. The moving grid method can also be used to solve free boundary problems, which focuses on increasing the order of accuracy in discretization. For example, Javiera (cf. [4]) located the interface in the r th node and the grid should be adapted at each time step. Compared to the level set method, the accuracy of first-order convergence in the interface position was slightly higher. The level set method (cf. [2, 6]) is also a widely used method for moving boundary problems. The main idea behind the method is that the interface position is represented by the zero level set, and it captures the interface position implicitly. Compared to the moving grid method, the level set has a main advantage that a fixed grid can be used, which avoids the mesh generation at every time step. Phase-field methods (cf. [4, 7]) have become increasingly popular for phase transition models over the past decade. These methods are based on phase field models, a free boundary arising from a phase field transition is assumed to have finite thickness, which differ from the classical model of a sharp interface. Phase-field methods present an advantage over front-tracking methods, because Phase-field methods only have an approximate representation of the front location. The main difference between the level set and phase-field methods is that

the level set method can capture the front on a fixed grid, in order to apply discretizations that depend on the exact interface location. In contrast, in the phase-field model, the front is not being explicitly tracked, and thus near the front the discretization of the diffusion field is less accurate.

However, although there have developed many numerical methods for free boundary problems, it seems very difficult to simulate the above electro-thermal model effectively with these methods. Concretely speaking, since the moving grid method requires to introduce a transformation mapping to map a fictitious domain into the physical domain to form space grid points, it will lead to essential difficulty in discretization of the thermal equation, which is a nonlinear parabolic interface equation; for the level set method, it is inconvenient to establish a level set equation coupled with the original equations governing the variation of the temperature and current; for the phase-field method, it is very difficult to construct a relevant phase-field functional which involves very deep physical interpretation of the model. Hence, we develop a new approach to solve the electro-thermal model under discussion.

The main novelty of our method proposed here is that we determine the interface $x = l(t)$ at time t by means of the idea of the shooting method (cf. [11]) combined with the phase transition condition (1.5). We notice that the shooting method is often used in solving nonlinear two-point boundary value problems (cf. [11]). Our algorithm can be briefly described as follows. We use the finite difference method with fixed mesh to discretize the thermal equations (1.1)-(1.2) and the current equation (1.7). Assume the temperature T and the current I are available at time $t = t_n$. We then select a grid position \tilde{l} as the guess of the interface position $x = l(t_{n+1})$ at $t = t_{n+1}$. Next, we compute the critical current $I_c(T)$ at $t = t_{n+1}$ in view of (1.5) at all grid points. If there exists a grid point $x = \tilde{l}_1$ such that the numerical current I_{n+1} is greater than the critical current at the left point of $x = \tilde{l}_1$, and less at the right side point, then we update the guess interface position \tilde{l} as \tilde{l}_1 . Repeat the above computation process until it converges. We present some numerical examples to show the computational performance of our method.

The rest of this paper is organized as follows. In section 2, we describe the Crank-Nicholson finite difference method and implicit-explicit scheme for the discretization of the thermal equation, and the trapezoidal rule for the discretization of the current equation. The algorithm for determination of the interface positions is given in section 3. A series of numerical results are given in section 4. In the final section, we present a short conclusion about our investigation in this paper.

2 Discretization of the governing equations

In order to numerically solve the electro-thermal model, we first partition the space region $[0, L/2]$ into N intervals with equal width Δx , to get the spatial nodes $0 = x_0 < x_1 < \cdots < x_N = L/2$ with $x_i = i\Delta x$, and then construct the time nodes $t_n = n\tau$ with $\tau > 0$ as the time stepsize, $n = 0, 1, \cdots$. We denote by T_i^n the approximate solution of the temperature T at a grid point (x_i, t_n) and denote by I_n the approximate solution of the current I at a grid point t_n . In this section, we will design effective finite difference methods for solving T_i^n and I_n , respectively.

2.1 Discretization of the thermal equation

2.1.1 The Crank-Nicholson method

Because the physical parameters rely on the temperature T itself, the thermal equations (1.1)-(1.2) are highly nonlinear. Hence, we use linearized schemes to carry out discretization, in order to avoid heavy cost in solving a nonlinear system of algebraic equations.

Let $x = l = l_0^{n+1} = x_j$ be the approximate interface position at the time $t = t_{n+1} = (n+1)\tau$. For a spatial point $x = x_i = i\Delta x$ in $(0, l)$, we view the physical parameters to be constant in the time interval $[t_n, t_{n+1}]$, equal to the ones corresponding to the temperature at $t = t_n$. Then we use the standard Crank-Nicholson finite difference method to discretize the equation (1.1) (cf. [12]), to get the following difference equation:

$$\left(\frac{1}{Wd} \times \frac{I_{n+1} + I_n}{2}\right)^2 \rho + \kappa_n(T_i^n) \times \frac{1}{2} \left(\frac{T_{i-1}^n - 2T_i^n + T_{i+1}^n}{\Delta x^2} + \frac{T_{i-1}^{n+1} - 2T_i^{n+1} + T_{i+1}^{n+1}}{\Delta x^2} \right) - \frac{1}{d} \alpha(T_i^n) \times \left(\frac{T_i^{n+1} + T_i^n}{2} - T_{sub} \right) = M(T_i^n) \times \frac{T_i^{n+1} - T_i^n}{\tau}, \quad (2.1)$$

where $M(T) := C_n(T) + TC'_n(T)$ so that $\frac{\partial C_n T}{\partial t} = M(T) \frac{\partial T}{\partial t}$.

Similarly, for $x = x_i = i\Delta x$ in $(l, L/2)$ we can derive the following difference equation from (1.2):

$$\kappa_s(T_i^n) \times \frac{1}{2} \left(\frac{T_{i-1}^n - 2T_i^n + T_{i+1}^n}{\Delta x^2} + \frac{T_{i-1}^{n+1} - 2T_i^{n+1} + T_{i+1}^{n+1}}{\Delta x^2} \right) - \frac{1}{d} \alpha(T_i^n) \times \left(\frac{T_i^{n+1} + T_i^n}{2} - T_{sub} \right) = H(T_i^n) \times \frac{T_i^{n+1} - T_i^n}{\tau}, \quad (2.2)$$

where $H(T) := C_s(T) + TC'_s(T)$ so that $\frac{\partial C_s T}{\partial t} = H(T) \frac{\partial T}{\partial t}$.

Next, let us deal with discretization of the boundary conditions. The homogeneous Neumann boundary condition is imposed at the left boundary point $x = x_0$. To ensure second order accuracy of approximation, we use the ghost point method (cf. [12]). We introduce a ghost point $x_{-1} = -\Delta x$ outside the solution region $[0, L/2]$ and let T_{-1}^m denote the approximate solution of T at the grid point (x_{-1}, t_m) fictitiously. Then using the central difference scheme we have from (1.3) that

$$\frac{T_1^m - T_{-1}^m}{2\Delta x} = 0. \quad (2.3)$$

On the other hand, we assume the difference scheme (2.1) holds at $x = x_0$ to get

$$\left(\frac{1}{Wd} \times \frac{I_{n+1} + I_n}{2}\right)^2 \rho + \kappa_n(T_0^n) \times \frac{1}{2} \left(\frac{T_{-1}^n - 2T_0^n + T_1^n}{\Delta x^2} + \frac{T_{-1}^{n+1} - 2T_0^{n+1} + T_1^{n+1}}{\Delta x^2} \right) - \frac{1}{d} \alpha(T_0^n) \times \left(\frac{T_0^{n+1} + T_0^n}{2} - T_{sub} \right) = M(T_0^n) \times \frac{T_0^{n+1} - T_0^n}{\tau}. \quad (2.4)$$

From (2.3) we know $T_{-1}^n = T_1^n$ and $T_{-1}^{n+1} = T_1^{n+1}$, and plugging them into (2.4) we obtain

$$\left(\frac{1}{Wd} \times \frac{I_{n+1} + I_n}{2}\right)^2 \rho + \kappa_n(T_0^n) \times \frac{1}{2} \left(\frac{-2T_0^n + 2T_1^n}{\Delta x^2} + \frac{-2T_0^{n+1} + 2T_1^{n+1}}{\Delta x^2} \right) - \frac{1}{d} \alpha(T_0^n) \times \left(\frac{T_0^{n+1} + T_0^n}{2} - T_{sub} \right) = H(T_0^n) \times \frac{T_0^{n+1} - T_0^n}{\tau}. \quad (2.5)$$

The Dirichlet condition is imposed at the right boundary point $x = L/2$, so we directly have

$$T_N = T_{sub}. \quad (2.6)$$

To discretize the interface condition (1.4) at the interface point $x = x_j$, we use a backward (resp. forward) scheme to approximate $\frac{\partial T(x,t)}{\partial x}$ from the left (resp. right) at $x = x_j$. So we have from (1.4) that

$$\kappa_n \frac{T_j - T_{j-1}}{\Delta x} = \kappa_s \frac{T_{j+1} - T_j}{\Delta x}. \quad (2.7)$$

The combination of the difference equations (2.1), (2.2), (2.5)-(2.7) can uniquely determine the grid function $\{T_i^{n+1}\}_{i=0}^{N-1}$. Obviously the scheme is implicit, and can be expressed in matrix notation as a linear system with a tridiagonal coefficient matrix. So we can obtain $\{T_i^{n+1}\}_{i=0}^{N-1}$ in an efficient way.

2.1.2 The Implicit-Explicit (IMEX) method

In order to derive an efficient implicit-explicit scheme for solving the thermal model given before, we first make a reformulation for the equations (1.1) and (1.2). As a matter of fact, from some direct and routine manipulation, the two equations can be rewritten as follows.

$$\frac{J^2 \rho}{M(T)} + G_n(T) \frac{\partial^2 T}{\partial x^2} + \frac{F(T)}{d} T - \frac{F(T)}{d} T_{sub} = \frac{\partial T}{\partial t}, \quad (2.8)$$

$$G_s(T) \frac{\partial^2 T}{\partial x^2} + \frac{E(T)}{d} T - \frac{E(T)}{d} T_{sub} = \frac{\partial T}{\partial t}. \quad (2.9)$$

where $G_n(T) = \frac{\kappa_n(T)}{M(T)}$, $G_s(T) = \frac{\kappa_s(T)}{H(T)}$, $F(T) = -\frac{\alpha(T)}{M(T)}$, $E(T) = -\frac{\alpha(T)}{H(T)}$.

Next, we choose a positive constant G_0 , large enough, such that G_0 is no less than $G_n(T)$ and $G_s(T)$ at least. The constant can be obtained by some additional calculation in terms of the explicit form of the underlying function. In our numerical experiments developed in section 4, G_0 is taken such that

$$G_0 = \max_{T_{sub} \leq T \leq T_C} \{G_n(T), G_s(T)\}. \quad (2.10)$$

Therefore, the above equations can be reformulated further as

$$G_0 \frac{\partial^2 T}{\partial x^2} + \frac{J^2 \rho}{M(T)} + [G_n(T) - G_0] \frac{\partial^2 T}{\partial x^2} + \frac{F(T)}{d} T - \frac{F(T)}{d} T_{sub} = \frac{\partial T}{\partial t}, \quad (2.11)$$

$$G_0 \frac{\partial^2 T}{\partial x^2} + [G_s(T) - G_0] \frac{\partial^2 T}{\partial x^2} + \frac{E(T)}{d} T - \frac{E(T)}{d} T_{sub} = \frac{\partial T}{\partial t}. \quad (2.12)$$

Hence, borrowing the same ideas to treat the variable coefficients as for the Crank-Nicholson method and using the technique that we discretize the partial derives of T with constant coefficients via implicit schemes and the other terms via explicit schemes (cf. [10]), we obtain from (2.11) that, at $x = x_i = i\Delta x$, $x \in (0, l)$ and $t = t_{n+1} = (n+1)\tau$, the difference equation for (1.1) reads

$$\begin{aligned} G_0 \times \frac{T_{i-1}^{n+1} - 2T_i^{n+1} + T_{i+1}^{n+1}}{\Delta x^2} + \frac{1}{M(T_i^n)} \times \left(\frac{1}{Wd} \times \frac{I^{n+1} + I^n}{2} \right)^2 \rho \\ + [G_n(T_i^n) - G_0] \times \frac{T_{i-1}^n - 2T_i^n + T_{i+1}^n}{\Delta x^2} + \frac{F(T_i^n)}{d} \times (T_i^n - T_{sub}) = \frac{T_i^{n+1} - T_i^n}{\tau}. \end{aligned} \quad (2.13)$$

Similarly, we have for $x = x_i = i\Delta x$, $x \in (l, L/2)$, the difference equation for (1.2) reads

$$G_0 \times \frac{T_{i-1}^{n+1} - 2T_i^{n+1} + T_{i+1}^{n+1}}{\Delta x^2} + [G_s(T_i^n) - G_0] \times \frac{T_{i-1}^n - 2T_i^n + T_{i+1}^n}{\Delta x^2} + \frac{E(T_i^n)}{d} \times (T_i^n - T_{sub}) = \frac{T_i^{n+1} - T_i^n}{\tau}. \quad (2.14)$$

Following the same ideas for construction of the Crank-Nicholson method mentioned above, we can derive the difference equations corresponding to the boundary conditions and the interface condition.

Compared to the Crank-Nicholson method, the present implicit-explicit (IMEX) scheme has an advantage. That is, if the generic constant G_0 is chosen feasibly, we only require to solve a linear system with the same coefficient matrix at different time nodes $t = t_n$. This will reduce the computational cost greatly, in particular, in high-dimensional case.

2.2 Discretization of the current equation

The current equation (1.7) is a second order ordinary differential equation, we rewrite it as a system of first-order equations and then carry out discretization. To this end, let $K(t) = \frac{1}{C_{bt}} \int_0^t (I_{bias} - I(s))ds + I_{bias}Z_0$. Hence, by some direct manipulation, (1.7) is equivalent to

$$K'(t) = \frac{I_{bias} - I(t)}{C_{bt}}, \quad (2.15)$$

$$L_k I'(t) = K(t) - (R(t) + Z_0)I(t), \quad (2.16)$$

where

$$R(t) = 2\rho \frac{l(t)}{S} = 2\rho \frac{l(t)}{Wd}.$$

The corresponding initial conditions are given by

$$I(0) = I_0, \quad K(0) = I_{bias}Z_0.$$

Integrating both sides of the equation (2.15) in the domain $[t_n, t_{n+1}]$ implies

$$\int_{t_n}^{t_{n+1}} K'(t)dt = \int_{t_n}^{t_{n+1}} \frac{I_{bias} - I(t)}{C_{bt}} dt,$$

and using the trapezoid method for numerical integration to the right side term we further have

$$K_{n+1} = K_n + \frac{\tau}{2C_{bt}}(2I_{bias} - I_n - I_{n+1}), \quad (2.17)$$

Similarly, integrating both sides of the equation (2.16) in the domain $[t_n, t_{n+1}]$, we have

$$\int_{t_n}^{t_{n+1}} L_k I'(t)dt = \int_{t_n}^{t_{n+1}} (K(t) - (R(t) + Z_0)I(t))dt,$$

which, in conjunction with the trapezoid method, implies

$$\left(L_k + \frac{\tau}{2}R_{n+1} + \frac{\tau}{2}Z_0\right)I_{n+1} = \left(L_k - \frac{\tau}{2}R_n - \frac{\tau}{2}Z_0\right)I_n + \frac{\tau}{2}K_n + \frac{\tau}{2}K_{n+1}. \quad (2.18)$$

where $R_n = R_n(t)$.

Making use of (2.17) and (2.18) immediately gives

$$aI_{n+1} = bI_n + \tau K_n + \frac{\tau^2}{2C_{bt}} I_{bias}, \quad (2.19)$$

where

$$a = L_k + \frac{\tau^2}{4C_{bt}} + \frac{\tau}{2} R_{n+1} + \frac{\tau}{2} Z_0, \quad b = L_k - \frac{\tau^2}{4C_{bt}} - \frac{\tau}{2} R_n - \frac{\tau}{2} Z_0,$$

$$R_n = 2\rho \frac{l_n}{Wd}, \quad R_{n+1} = 2\rho \frac{l_{n+1}}{Wd}.$$

We remark that in the real applications, the interface positions l_n and l_{n+1} at $t = t_n$ and $t = t_{n+1}$ should be replaced by their approximate values l_0^n and l_0^{n+1} , respectively. Therefore, it is clear that we can get the current I_{n+1} whenever the unknowns at the time $t = t_n$ and the interface position $x = l_0^{n+1}$ are available.

Observing that for the superconducting nanowire single-photon detector described in Figure 1, while the current drops below critical current and the resistive region subsides, the wire becomes fully superconducting again, the bias current through the wire returns to the original value. Thus, the time-dependent hotspot resistance $R_n(t) = 0$, and the current equation (1.7) becomes

$$\frac{d^2 I(t)}{dt^2} + a_1 \frac{dI(t)}{dt} + a_2 I(t) = a_3, \quad (2.20)$$

where $a_1 = \frac{Z_0}{L_k}$, $a_2 = \frac{1}{C_{bt}L_k}$, $a_3 = \frac{I_{bias}}{C_{bt}L_k}$.

Since the equation (2.20) is a constant second order inhomogeneous linear equation, we can easily derive its closed form of the solution:

$$I(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \frac{a_3}{a_2}. \quad (2.21)$$

If the initial conditions are given by $I(\hat{t}) = a_4$ and $I'(\hat{t}) = a_5$, then we know by a direct manipulation that the undetermined coefficients in (2.21) are

$$\lambda_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2}, \quad \lambda_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2},$$

$$c_1 = \frac{a_4 \lambda_2 - a_5 - \frac{a_3}{a_2} \lambda_2}{(\lambda_2 - \lambda_1) e^{\lambda_1 \hat{t}}}, \quad c_2 = \frac{a_4 \lambda_1 - a_5 - \frac{a_3}{a_2} \lambda_1}{(\lambda_1 - \lambda_2) e^{\lambda_2 \hat{t}}}.$$

Therefore, if the nanowire returns to superconducting state again, we are able to get the current from the expression (2.21) explicitly, instead of the numerical solution. This will increase the computational efficiency greatly.

3 Determination of the interface position

Similar to the standard numerical method for solving evolutionary equations, we will conduct numerical simulation for the electro-thermal model along the time direction. That means, once the numerical results at $t = t_n$ are obtained, we will try to get the numerical results at $t = t_{n+1}$. From our discussion given in the above section, we easily know the key difficulty is to derive the interface position at this instant.

Our key points to overcome the above obstacle are as follows. First of all, we make the partition of the region $[0, L/2]$ fine enough, i.e. Δx is taken small enough, so that we

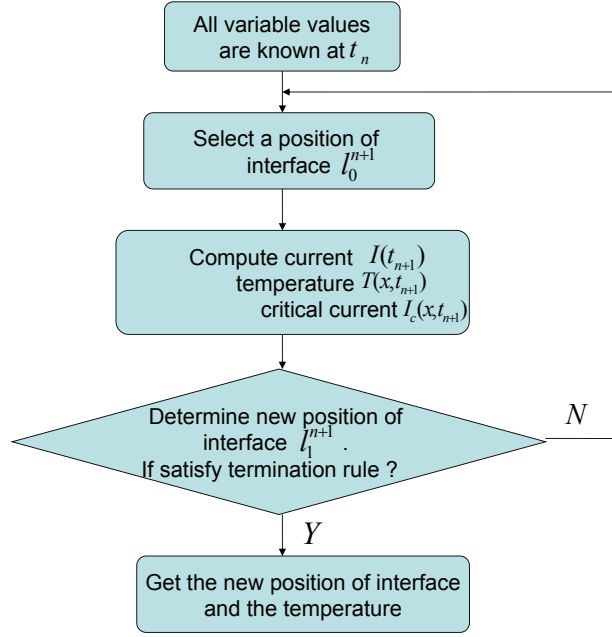


Figure 2: The flow chart of the iterative algorithm to form the interface position at different time nodes.

can assume that the interface positions always lie in the spatial grid points, approximately with desired accuracy. Next, we will use the shooting method to determine the interface position at $t = t_{n+1}$, in view of the idea of the shooting method (cf. [11]) combined with the phase transition condition (1.5). To be more precise, we choose $x = \tilde{l} = l^n$ as the initial guess of the interface position at $t = t_{n+1}$. Then, by means of the finite difference methods in section 2, we can derive the approximate temperature values T_i^{n+1} at all grid points as well as the approximate current I_{n+1} , and compute the critical current $I_c(T)$ at $t = t_{n+1}$ in view of (1.5) at all grid points. If the initial guess $x = \tilde{l}$ satisfies that the numerical current I_{n+1} is greater than the critical current at the left point of $x = \tilde{l}$, and less at the right side point, then we take $l_0^{n+1} = \tilde{l}$. Otherwise, we try to find a grid $x = \tilde{l}_1$ such that the numerical current I_{n+1} is greater than the critical current at the left point of $x = \tilde{l}$, and less at the right side point. And then replace the guess interface position \tilde{l} by \tilde{l}_1 . Repeat the above computation process until it converges, and choose the final result \tilde{l} as l_0^{n+1} . In all the calculations presented in the following section, the termination rule is taken as $|\tilde{l}_1 - \tilde{l}| < tol$, with $tol = 1 \times 10^{-6}$.

For preciseness, the above algorithm is shown in a flow chart, described in Figure 2.

4 Numerical results

In this section, we give some numerical experiments to illustrate the performance and accuracy of our method introduced in sections 2 and 3, from which we can observe the evolution of interface positions, namely the growth of the normal region along the wire, the change in current through the wire, the change of resistance along the wire after a

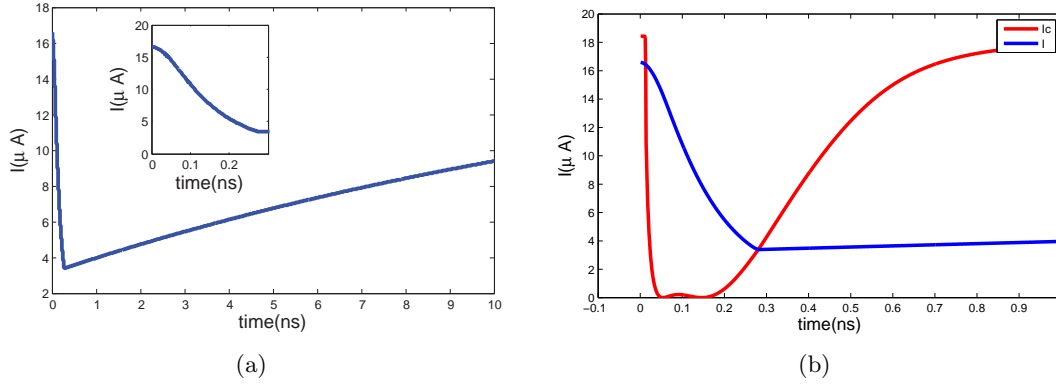


Figure 3: The current variation in the electro-thermal model.(a) The calculated current through the wire vs. time. (b) The calculated current and critical current at $x = 100nm$ vs time.

small resistive hotspot is formed. Here the photon-induced resistive barrier forms at $t = 0$. Most of the physical parameters are taken from the monograph [13] about the theory of superconductivity, and the other ones are taken from the related literature. In particular, since the hotspot only forms and exists for several nanoseconds (cf. [1]), we choose in our numerical simulation the terminal time to be $t_{end} = 10ns$. Then we choose the stepsize in t to be $\tau = 1ps$, where $1ps = 10^{-3}ns$. If τ is taken a little larger, say $\tau = 5ps$, our algorithms will not converge.

In the normal/resistive state, the electrical resistivity $\rho = 2.4 \times 10^{-6}\Omega m$. According to the Wiedemann-Franz law, the ratio of the electronic contribution of the thermal conductivity κ_n to the electrical conductivity ρ of a metal, is proportional to the temperature T ($\kappa_n = \mathcal{L} \frac{T}{\rho}$, where $\mathcal{L} = 2.45 \times 10^{-8}W\Omega/K^2$ is the Lorenz number). The heat capacity (per unit volume) of the superconducting film C_n includes electron specific C_{en} and phonon specific heat C_{pn} , where C_{en} is proportional to the temperature T ($C_{en} = \gamma T$, where $\gamma = 240$), and C_{pn} is proportional to T^3 such that $C_{pn} = 9.8T^3$ (cf. [8]). The thermal boundary conductivity α between NbN and sapphire we used is obtained from [15], and we only considered its cubic dependence on temperature ($\alpha = BT^3$, where $B = 800$).

In the region of superconducting state, ρ is taken to be zero naturally. We express the thermal conductivity as $\kappa_s = \mathcal{L} \frac{T^2}{\rho T_c}$ (cf. [14]), where $T_c = 10K$ is the critical temperature. The heat capacity C_s also include two parts, the electron specific was calculated such that $C_{es} = Ae^{-\frac{3.5T_c}{T}}$ with $A = 1.93 \times 10^5$ (cf. [13]), and the phonon specific heat is state independent such that $C_{ps} = 9.8T^3$. The thermal boundary conductivity is given as $\alpha = BT^3$.

Some more data used in all the computations are given as follows. The length of superconducting nanowire $L = 2000nm$, the wire thickness $d = 4nm$, the width of nanowire $W = 100nm$, the substrate temperature $T_{sub} = 2K$, the initial critical current $I_c(0) = 20\mu A$, the capacitor $C_{bt} = 20 \times 10^{-9}F$, the kinetic inductance of the superconducting nanowire $L_k = 807.7nH$, the impedance of the transmission line connecting the probe to RF amplifiers $Z_0 = 50\Omega$, the current of the SNSPD $I_{bias} = 16.589\mu A$, the initial interface position $l_0 = 15nm$, the initial temperature $T_0 = 5K$ where the wire is normal and $T_0 = T_{sub} = 2K$ where the wire is superconducting.

For the Crank-Nicholson method and the IMEX method, we take $N = 1000$, so the space step is $\Delta x = L/2N = 1nm$. The time step is $\tau = 1ps$, as introduced at the beginning

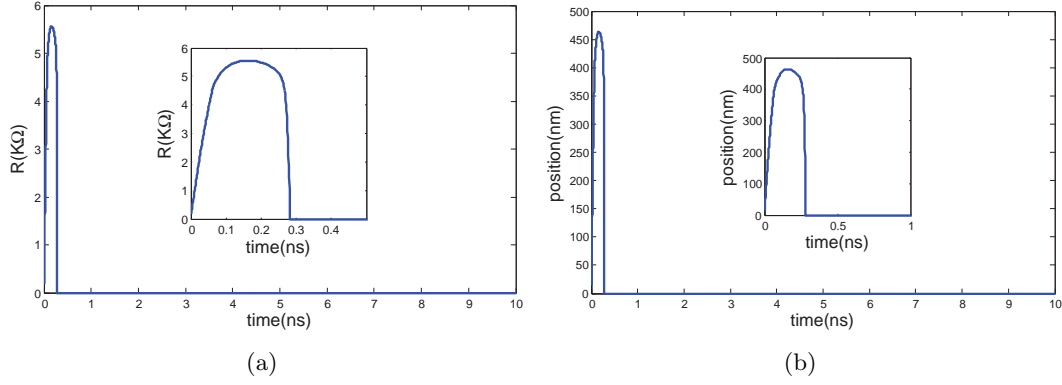


Figure 4: The resistance variation in the electro-thermal model. (a) The calculated total normal state resistance vs. time, and the inset shows in greater detail the change of the resistance. (b) The interface position vs. time, and the inset shows in greater detail the change of the position.

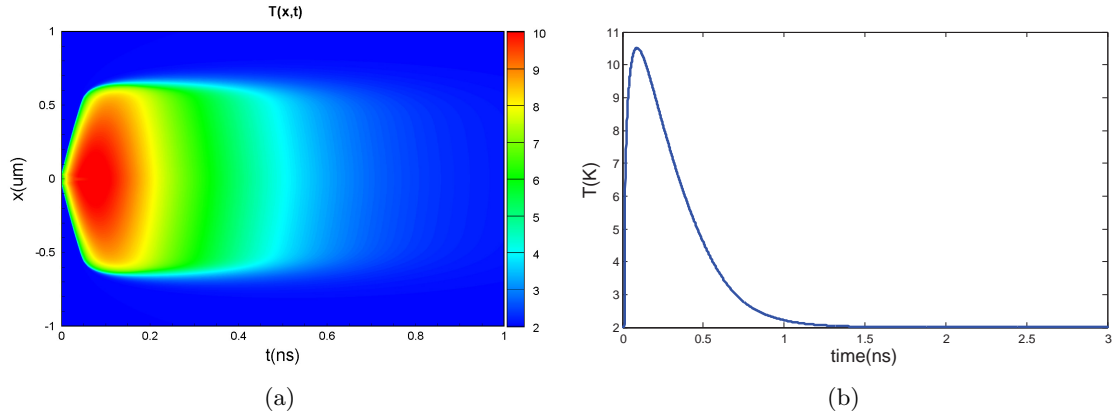


Figure 5: The temperature variation in the electro-thermal model. (a) The calculated temperature (shown using colors) at different positions along the wire and in time. (b) The calculated temperature history at $x = 100nm$.

of this section. Furthermore, for the IMEX method, we choose the parameter G_0 in view of the formulation (2.10) to get $G_0 = 2 * 10^{-5}$.

We first use the Crank-Nicholson method for solving the thermal equations, combined with the numerical method for solving the current equation and the algorithm in section 3 to search for interface positions, to implement numerical simulation.

It is shown in Figure 3(a) the calculated current through the SNSPD. We find the curve first forms a sharp decline within a short time period. Afterwards, the nanowire under consideration switches to superconducting state, and the calculated current increases at an exponential rate. It is shown in Figure 3(b) the calculated current and the critical current at $x = 100nm$ vs. time.

It is shown in Figure 4(a) the calculated total normal state resistance along the wire. It appears that the resistance increases gradually and then decreases to 0 sharply, and the inset shows in greater detail the change of the resistance. It is shown in Figure 4(b) the evolution of the interface position in the electro-thermal model. The initial interface position is taken as $l_0 = 15nm$. The interface point moves forward to about the position $x = 463nm$ at $t = 170ps$, where the resistance increases to a maximum value. Then the

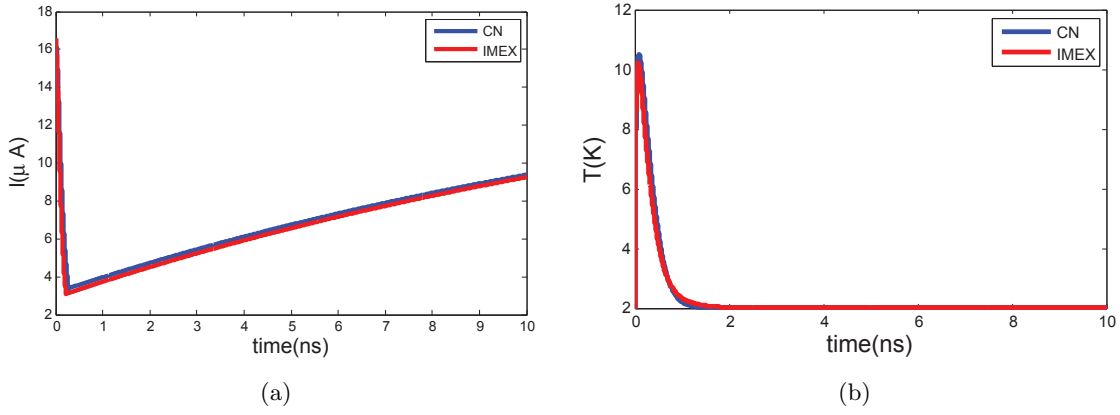


Figure 6: Some numerical comparison between the Crank-Nicholson finite difference method and the IMEX method. (a) The calculated current through the wire vs. time. (b) The calculated temperature history at $x = 100nm$.

interface point returns gradually to the central position with $x = 0$, and the nanowire becomes superconducting state.

It is shown in Figure 5(a) the calculated temperature (shown using colors) at different positions along the wire and in time, the temperature at each segment show the segment under consideration switches into the normal state or remains superconducting. And it is shown in Figure 5(b) the calculated temperature history at $x = 100nm$. At that position, the initial temperature is $T = 5K$ where the wire is normal, then it increases to a maximum value of about $10.7K$, after that the temperature gradually returns to $2K$ and the position lies in superconducting state.

All the numerical results given above coincide with the physical phenomenon observed by experiments (cf. [15]).

We also compare the numerical results with the thermal equations numerically solved by the Crank-Nicholson method and the IMEX method, respectively. We observe from the numerical data in Figure 6 that the two methods which perform in the similar manners, can produce very similar numerical results.

5 Conclusions

In this paper, we propose two algorithms for numerically solving the electro-thermal model for Superconducting Nanowire Single-Photon Detectors. Such a model is governed by a nonlinear free boundary problem involving the temperature and the current, which are coupled together by a nonlinear parabolic interface equation and a second order ordinary differential equation (see the equations (1.1)-(1.2) and (1.7) for details). In our numerical experiments, for a fixed spatial size Δx , only if the stepsize in time τ is taken small enough, our numerical methods are convergent. Therefore, we only develop in this paper some initial but interesting results for the coupled system of equations (1.1)-(1.2) and (1.7). Due to the complexity of the model, it is very challenging to establish mathematical theory for this model and discuss convergence analysis of the methods proposed in this paper.

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On the Existence of Meromorphic Solutions of Some Nonlinear Differential-Difference Equations

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Abstract: In this paper, we investigate the conditions concerning the existence or non-existence of transcendental meromorphic or entire solutions of some kinds of differential-difference equations. We also give examples to illustrate the sharpness of our results.

Key words: Differential-difference equation, meromorphic solution, entire solution.

AMS Classification No. 39B32, 34M05, 30D35

1 Introduction and main results

Throughout this paper, we assume that $f(z)$ is a meromorphic function in the whole complex plane, and use standard notations, such as $m(r, f)$, $T(r, f)$, $N(r, f)$, in the Nevanlinna theory (see e.g. [3, 7, 8, 17]). And we also use $\sigma(f)$ and $\sigma_2(f)$ to denote respectively the order and the hyper order of $f(z)$. Moreover, we say that a meromorphic function $g(z)$ is small with respect to $f(z)$, if $T(r, g) = S(r, f)$, where $S(r, f)$ means any real quantity satisfying $S(r, f) = o(T(r, f))$ as $r \rightarrow \infty$ outside of a possible exceptional set of finite logarithmic measure.

Recently, with some establishments of difference analogues of the classic Nevanlinna theory (two typical and most important ones can be seen in [2, 4–6]), there has been a renewed interest in the properties of complex difference expressions and meromorphic solutions of complex difference equations (see e.g. [10–12, 18]). Further, Yang-Laine gave analogies between nonlinear difference and differential equations in [15]. From then on, some results concerning nonlinear differential-difference equations were found (see e.g. [13]).

In what follows, we use the definition of the differential-difference polynomial in [15, 19]. A differential-difference polynomial is a polynomial in $f(z)$, its shifts, its derivatives and derivatives of its shifts, that is, an expression of the form

$$\begin{aligned} P(z, f) = & \sum_{\lambda \in I} a_{\lambda}(z) f(z)^{\lambda_{0,0}} f'(z)^{\lambda_{0,1}} \cdots f^{(k)}(z)^{\lambda_{0,k}} \\ & \cdot f(z + c_1)^{\lambda_{1,0}} f'(z + c_1)^{\lambda_{1,1}} \cdots f^{(k)}(z + c_1)^{\lambda_{1,k}} \cdots \\ & \cdot f(z + c_l)^{\lambda_{l,0}} f'(z + c_l)^{\lambda_{l,1}} \cdots f^{(k)}(z + c_l)^{\lambda_{l,k}} \end{aligned}$$

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$$= \sum_{\lambda \in I} a_{\lambda}(z) \prod_{i=0}^l \prod_{j=0}^k f^{(j)}(z + c_i)^{\lambda_{i,j}}, \quad (1.1)$$

where I is a finite set of multi-indices $\lambda = (\lambda_{0,0}, \dots, \lambda_{0,k}, \lambda_{1,0}, \dots, \lambda_{1,k}, \dots, \lambda_{l,0}, \dots, \lambda_{l,k})$, and $c_0 (= 0)$, c_1, \dots, c_l are distinct complex constants. And we assume that the meromorphic coefficients $a_{\lambda}(z)$, $\lambda \in I$ of $P(z, f)$ are of growth $S(r, f)$. We denote the degree and the weight of the monomial $\prod_{i=0}^l \prod_{j=0}^k f^{(j)}(z + c_i)^{\lambda_{i,j}}$ of $P(z, f)$ respectively by

$$d(\lambda) = \sum_{i=0}^l \sum_{j=0}^k \lambda_{i,j} \quad \text{and} \quad w(\lambda) = \sum_{i=0}^l \sum_{j=0}^k (j+1) \lambda_{i,j}.$$

Then we denote the degree and the weight of $P(z, f)$ respectively by

$$d(P) = \max_{\lambda \in I} \{d(\lambda)\} \quad \text{and} \quad w(P) = \max_{\lambda \in I} \{w(\lambda)\}.$$

In the following, we assume $d(P) \geq 1$.

We recall the following result due to Wang-Li [13] by rewriting the original differential-difference polynomial in [13] as the one of the form (1.1).

Theorem A. Suppose that a nonlinear differential-difference equation is

$$f^n(z) + P(z, f) = p(z), \quad (1.2)$$

where $n \in \mathbb{N}$, $p(z)$ is a polynomial, and $P(z, f)$ is a differential-difference polynomial of the form (1.1) with polynomial coefficients. If

$$n > (s+1)d(P) - \sum_{\lambda \in I} d(\lambda), \quad (1.3)$$

where s is the number of components of I , then the equation (1.2) has no transcendental entire solutions of finite order.

Remark 1.1 Obviously, (1.3) results in

$$n > (s+1)d(P) - \sum_{\lambda \in I} d(\lambda) \geq (s+1)d(P) - sd(P) = d(P) \geq 1.$$

Then, our first main purpose is to improve Theorem A. On the one hand, we improve the restrict on n by introducing an important lemma of our own. On the other hand, we also consider the non-existence of meromorphic solutions of the equation (1.2). Our result is as follows.

Theorem 1.1 Consider the nonlinear differential-difference equation

$$f^n(z) + P(z, f) = c(z), \quad n \in \mathbb{N}, \quad (1.4)$$

where $P(z, f)$ is a differential-difference polynomial of the form (1.1) with meromorphic coefficients $a_{\lambda}(z)$, $\lambda \in I$, and $c(z)$ is a meromorphic functions.

(i) If $n > d(P)$, then the equation (1.4) has no admissible transcendental entire solutions with hyper order less than 1.

(ii) If $n > w(P)$, then the equation (1.4) has no admissible transcendental meromorphic solutions with hyper order less than 1.

Remark 1.2 Here, a meromorphic or entire solution $f(z)$ of the equation (1.4) is called admissible, if $a_\lambda(z), \lambda \in I$ and $c(z)$ are small with respect to $f(z)$, that is, $T(r, a_\lambda) = S(r, f), \lambda \in I$ and $T(r, c) = S(r, f)$.

Wang-Li also investigated another kind of nonlinear differential-difference equation in [13] as follows.

Theorem B For two integers $n \geq 3, k > 0$ and a nonlinear differential-difference equation

$$f^n(z) + q(z)f^{(k)}(z+t) = ae^{ibz} + de^{-ibz}, \quad (1.5)$$

where $q(z)$ is a polynomial and t, a, b, d are complex numbers such that $|a| + |d| \neq 0, bt \neq 0$.

(i) Let $n = 3$. If $q(z)$ is nonconstant, then the equation (1.5) does not admit entire solutions of finite order. If $q = q(z)$ is constant, then the equation (1.5) admits three distinct transcendental entire solutions of finite order, provided that

$$bt = 3m\pi \ (m \neq 0, \text{ if } q \neq 0), \quad q^3 = (-1)^{m+1} \left(\frac{3i}{b}\right)^{3k} 27ad,$$

when k is even, or

$$bt = \frac{3\pi}{2} + 3m\pi \ (\text{if } q \neq 0), \quad q^3 = i(-1)^m \left(\frac{3i}{b}\right)^{3k} 27ad,$$

when k is odd, for an integer m .

(ii) Let $n > 3$. If $ad \neq 0$, then the equation (1.5) does not admit entire solutions of finite order. If $ad = 0$, then the equation (1.5) admits n distinct transcendental entire solutions of finite order, provided that $q = q(z) \equiv 0$.

Moreover, they proposed a question in [13]: for the differential-difference equation of the form

$$f^n(z) + L(z, f) = ae^{ibz} + de^{-ibz}, \quad n \geq 3,$$

where $L(z, f)$ is some linear differential-difference polynomial of $f(z)$ with polynomial coefficients, what can we say considering Theorem B.

Then, our second main purpose is to give the following results, which answer the above question to some extent.

Theorem 1.2 Consider the nonlinear differential-difference equation

$$f^n(z) + \sum_{s=0}^l \sum_{t=0}^k A_{s,t}(z) f^{(t)}(z+c_s) = ae^{ibz} + de^{-ibz}, \quad n \in \mathbb{N}, n \geq 3, \quad (1.6)$$

where $c_0 (= 0), c_1, \dots, c_l$ are distinct complex constants, $A_{s,t}(z), s = 0, 1, \dots, l, t = 0, 1, \dots, k$ are polynomials, and $a, b, d \in \mathbb{C}$ such that $b \neq 0$ and $|a| + |d| \neq 0$.

(i) Let $n = 3$. If

$$ad \neq 0 \quad \text{and} \quad \sum_{s=0}^l \sum_{t=0}^k A_{s,t}(z) \left(e^{\frac{ibc_s}{3}} \left(\frac{ib}{3}\right)^t - e^{\frac{-ibc_s}{3}} \left(\frac{-ib}{3}\right)^t \right) \neq 0,$$

then the equation (1.6) has no transcendental entire solutions of finite order. If

$$\sum_{s=0}^l \sum_{t=0}^k A_{s,t}(z) e^{\frac{ibc_s}{3}} \left(\frac{ib}{3}\right)^t \equiv 0 = d, \quad \text{or} \quad \sum_{s=0}^l \sum_{t=0}^k A_{s,t}(z) e^{\frac{-ibc_s}{3}} \left(\frac{-ib}{3}\right)^t \equiv 0 = a,$$

or

$$\sum_{s=0}^l \sum_{t=0}^k A_{s,t}(z) e^{\frac{ibcs}{3}} \left(\frac{ib}{3}\right)^t \equiv \sum_{s=0}^l \sum_{t=0}^k A_{s,t}(z) e^{\frac{-ibcs}{3}} \left(\frac{-ib}{3}\right)^t \equiv -3d_1d_2, \quad d_1^3 = a, d_2^3 = d,$$

then the equation (1.6) has three transcendental entire solutions of finite order.

(ii) Let $n > 3$. If $ad \neq 0$, or

$$ad = 0 \quad \text{and} \quad \sum_{s=0}^l \sum_{t=0}^k A_{s,t}(z) e^{\frac{ibcs}{n}} \left(\frac{ib}{n}\right)^t \not\equiv 0,$$

then the equation (1.6) has no transcendental entire solutions of finite order. If

$$ad = 0 \quad \text{and} \quad \sum_{s=0}^l \sum_{t=0}^k A_{s,t}(z) e^{\frac{ibcs}{n}} \left(\frac{ib}{n}\right)^t \equiv 0,$$

then the equation (1.6) has n transcendental entire solutions of finite order.

In particular, we obtain more concrete results for a special linear difference polynomial $L(z, f)$ as follows.

Theorem 1.3 Consider the nonlinear difference equation

$$f^n(z) + q(z)\Delta^m f(z) = ae^{ibz} + de^{-ibz}, \quad n, m \in \mathbb{N}, n \geq 3, \quad (1.7)$$

where $q(z)$ is a polynomial, $a, b, d \in \mathbb{C}$ such that $b \neq 0$ and $|a| + |d| \neq 0$.

(i) Let $n = 3$. If $ad \neq 0$ and $q(z)$ is a nonconstant, then the equation (1.7) has no transcendental entire solutions of finite order. If $ad \neq 0$ and $q(z)$ is a constant q , then the equation (1.7) has three transcendental entire solutions of the form $f(z) = d_1 e^{\frac{ibz}{3}} + d_2 e^{\frac{-ibz}{3}}$, $d_1^3 = a, d_2^3 = d$, provided that

$$bc = 6k\pi + 3\pi + \frac{6s\pi}{m} \quad (k \in \mathbb{Z}, s \in \{0, 1, \dots, m-1\}) \quad \text{and} \quad q^3 = (-1)^{m+1} \frac{27ad}{(e^{\frac{2s\pi i}{m}} + 1)^{3m}}.$$

If $ad = 0$, then the equation (1.7) has three transcendental entire solutions of the form $f(z) = d_1 e^{\frac{ibz}{3}} + d_2 e^{\frac{-ibz}{3}}$, $d_1^3 = a, d_2^3 = d$, provided that $q(z) \equiv 0$ or $bc = 6k\pi, k \in \mathbb{Z}$.

(ii) Let $n > 3$. If $ad \neq 0$, then the equation (1.7) has no transcendental entire solutions of finite order. If $ad = 0$, then the equation (1.7) has n transcendental entire solutions of the form $f(z) = d_1 e^{\frac{ibz}{n}} + d_2 e^{\frac{-ibz}{n}}$, $d_1^n = a, d_2^n = d$, provided that $q(z) \equiv 0$ or $bc = 2kn\pi, k \in \mathbb{Z}$.

Remark 1.3 Here, the forward difference $\Delta^m f(z)$ for $m \in \mathbb{N}$ and $c \in \mathbb{C} \setminus \{0\}$ is defined in the standard way [14, p. 52] by

$$\Delta f(z) = f(z+c) - f(z), \quad \Delta^m f(z) = \Delta(\Delta^{m-1} f(z)) = \Delta^{m-1} f(z+c) - \Delta^{m-1} f(z), \quad m \geq 2.$$

And it is shown as in [2] that

$$\Delta^m f(z) = \sum_{j=0}^m C_m^j (-1)^{m-j} f(z+jc), \quad f(z+mc) = \sum_{j=0}^m C_m^j \Delta^j f(z),$$

where $C_m^j, j = 0, 1, \dots, m$ are the binomial coefficients.

2 Lemmas

Lemma 2.1. ([19]) Let $f(z)$ be a transcendental meromorphic function of $\sigma_2(f) < 1$, and $P(z, f)$ be a differential-difference polynomial of the form (1.1), then we have

$$m(r, P(z, f)) \leq d(P)m(r, f) + S(r, f).$$

Furthermore, if $f(z)$ also satisfies $N(r, f) = S(r, f)$, then we have

$$T(r, P(z, f)) \leq d(P)T(r, f) + S(r, f).$$

Lemma 2.2. ([6]) Let $T : [0, +\infty) \rightarrow [0, +\infty)$ be a non-decreasing continuous function and let $s \in (0, +\infty)$. If the hyper order of T is strictly less than one, i.e. $\overline{\lim}_{r \rightarrow \infty} \frac{\log_2 T(r)}{\log r} = \zeta < 1$, and $\delta \in (0, 1 - \zeta)$, then

$$T(r + s) = T(r) + o\left(\frac{T(r)}{r^\delta}\right),$$

where r runs to infinity outside of a set of finite logarithmic measure.

It is shown in [3, p.66] and [1, Lemma 1] that the inequality

$$(1 + o(1))T(r - |c|, f) \leq T(r, f(z + c)) \leq (1 + o(1))T(r + |c|, f)$$

holds for $c \neq 0$ and $r \rightarrow \infty$. And from its proof, the above relation is also true for counting function. By combining Lemma 2.2 and these inequalities, we immediately deduce the following lemma.

Lemma 2.3. Let $f(z)$ be a nonconstant meromorphic function of $\sigma_2(f) < 1$, and c be a nonzero complex constant. Then we have

$$T(r, f(z + c)) = T(r, f) + S(r, f),$$

$$N(r, f(z + c)) = N(r, f) + S(r, f), \quad N(r, \frac{1}{f(z + c)}) = N(r, \frac{1}{f}) + S(r, f).$$

Laine-Yang [9] gave a difference analogue of Clunie lemma as follows.

Lemma 2.4. ([9]) Let $f(z)$ be a transcendental finite order meromorphic solution of

$$U(z, f)P(z, f) = Q(z, f),$$

where $U(z, f), P(z, f), Q(z, f)$ are difference polynomials in $f(z)$ with small meromorphic coefficients, $\deg_f U = n$ and $\deg_f Q \leq n$. Moreover, we assume that $U(z, f)$ contains just one term of maximal total degree. Then

$$m(r, P(z, f)) = S(r, f).$$

Remark 2.1. Yang-Laine [15] also pointed out that Lemma 2.4 is also true if $P(z, f), Q(z, f)$ are differential-difference polynomials in $f(z)$. Further, by a careful inspection of the proof of Lemma 2.4, we see that the same conclusion holds for the differential-difference case, if the coefficients $b_\mu(z)$ of $P(z, f), Q(z, f)$ satisfy $m(r, b_\mu) = S(r, f)$ instead of $T(r, b_\mu) = S(r, f)$.

Lemma 2.5. ([16]) Suppose that c is a nonzero complex constant, $\alpha(z)$ is a nonconstant meromorphic function. Then the differential equation

$$f^2(z) + (cf^{(n)}(z))^2 = \alpha(z)$$

has no transcendental meromorphic solutions satisfying $T(r, \alpha) = S(r, f)$.

3 Proofs of Theorems 1.1-1.3

Proof of Theorem 1.1. (i) Let $f(z)$ be an admissible transcendental entire solution of (1.4) with $\sigma_2(f) < 1$. By Lemma 2.1, we see that

$$m(r, P(z, f)) \leq d(P)m(r, f) + S(r, f). \quad (3.1)$$

By (1.4) and (3.1), we obtain that

$$nT(r, f) = T(r, P(z, f)) + S(r, f) = m(r, P(z, f)) + S(r, f) \leq d(P)T(r, f) + S(r, f). \quad (3.2)$$

Since $n > d(P)$, (3.2) is a contradiction. Thus, (1.4) has no admissible transcendental entire solutions with hyper order less than 1.

(ii) Let $f(z)$ be an admissible transcendental meromorphic solution of (1.4) with $\sigma_2(f) < 1$. We consider each pole of $P(z, f)$. Since each pole of $P(z, f)$ in $|z| < r$ comes from the poles of $f(z + c_i)$, $i = 0, \dots, l$ and $a_\lambda(z)$, $\lambda \in I$ in $|z| < r$, and each pole of $f(z + c_i)$ with multiplicity p_i is a pole of $P(z, f)$ with multiplicity at most

$$p_i \lambda_{i,0} + (p_i + 1) \lambda_{i,1} + \dots + (p_i + k) \lambda_{i,k} \leq p_i (\lambda_{i,0} + 2\lambda_{i,1} + \dots + (k+1)\lambda_{i,k}) = p_i \sum_{j=0}^k (j+1) \lambda_{i,j},$$

we have by Lemma 2.3 that

$$\begin{aligned} N(r, P(z, f)) &\leq \max_{\lambda \in I} \left\{ \sum_{i=0}^l \sum_{j=0}^k (j+1) \lambda_{i,j} N(r, f(z + c_i)) \right\} + S(r, f) \\ &= \max_{\lambda \in I} \left\{ \sum_{i=0}^l \sum_{j=0}^k (j+1) \lambda_{i,j} N(r, f) \right\} + S(r, f) \\ &= \max_{\lambda \in I} w(\lambda) N(r, f) + S(r, f) = w(P) N(r, f) + S(r, f). \end{aligned} \quad (3.3)$$

Clearly, (3.1) holds by Lemma 2.1 again. By (1.4), (3.1) and (3.3), we obtain that

$$\begin{aligned} nT(r, f) &= T(r, P(z, f)) + S(r, f) = m(r, P(z, f)) + N(r, P(z, f)) + S(r, f) \\ &\leq d(P)m(r, f) + w(P)N(r, f) + S(r, f) \leq w(P)T(r, f) + S(r, f). \end{aligned} \quad (3.4)$$

Since $n > w(P)$, (3.4) is a contradiction. Thus, (1.4) has no admissible transcendental meromorphic solutions with hyper order less than 1. \square

Proof of Theorem 1.3. Suppose that $f(z)$ is a transcendental entire solution of finite order of (1.7). Differentiating (1.7), we have

$$\begin{aligned} n f^{n-1}(z) f'(z) + q'(z) \sum_{j=0}^m C_m^j (-1)^{m-j} f(z + jc) + q(z) \sum_{j=0}^m C_m^j (-1)^{m-j} f'(z + jc) \\ = ib(ae^{ibz} - de^{-ibz}). \end{aligned} \quad (3.5)$$

By combining (1.7) and (3.5), we have

$$\left(n f^{n-1}(z) f'(z) + q'(z) \sum_{j=0}^m C_m^j (-1)^{m-j} f(z + jc) + q(z) \sum_{j=0}^m C_m^j (-1)^{m-j} f'(z + jc) \right)^2$$

$$+b^2\left(f^n(z) + q(z) \sum_{j=0}^m C_m^j (-1)^{m-j} f(z+jc)\right)^2 = 4adb^2,$$

consequently,

$$f^{2n-2}(z)(b^2 f^2(z) + n^2 f'^2(z)) = Q(z, f), \quad (3.6)$$

where $Q(z, f)$ is a differential-difference polynomial of $f(z)$ with the total degree at most $n+1$.

If $b^2 f^2(z) + n^2 f'^2(z) \equiv 0$, we differentiate it and obtain that

$$n^2 f''(z) + b^2 f(z) = 0, \quad (3.7)$$

which implies the solution must be

$$f(z) = d_1 e^{\frac{ibz}{n}} + d_2 e^{\frac{-ibz}{n}}, \quad (3.8)$$

where d_1 and d_2 are arbitrary complex constants. If $b^2 f^2(z) + n^2 f'^2(z) \not\equiv 0$, we may apply Lemma 2.4 and Remark 2.1 to (3.6) and obtain that

$$T(r, b^2 f^2 + n^2 f'^2) = m(r, b^2 f^2 + n^2 f'^2) = S(r, f).$$

Thus, by Lemma 2.5, we see that $b^2 f^2(z) + n^2 f'^2(z)$ must be a constant M . Differentiating $b^2 f^2(z) + n^2 f'^2(z) = M$, we obtain (3.7) and (3.8) again.

Substituting (3.8) into (1.7) and denoting $w = w(z) = e^{\frac{ibz}{n}}$, we obtain that

$$\begin{aligned} & d_1^n w^{2n} + C_n^1 d_1^{n-1} d_2 w^{2n-2} + C_n^2 d_1^{n-2} d_2^2 w^{2n-4} + \cdots + C_n^{n-2} d_1^2 d_2^{n-2} w^4 + C_n^{n-1} d_1 d_2^{n-1} w^2 + d_2^n \\ & + d_1 q(z) \sum_{j=0}^m C_m^j (-1)^{m-j} e^{\frac{ijbc}{n}} w^{n+1} + d_2 q(z) \sum_{j=0}^m C_m^j (-1)^{m-j} e^{\frac{-ijbc}{n}} w^{n-1} = aw^{2n} + d. \end{aligned} \quad (3.9)$$

(i) Let $n = 3$, then (3.9) reduces into

$$a_6 w^6 + a_4 w^4 + a_2 w^2 + a_0 = 0,$$

where

$$\begin{cases} a_6 = d_1^3 - a, \\ a_4 = 3d_1^2 d_2 + d_1 q(z) \sum_{j=0}^m C_m^j (-1)^{m-j} e^{\frac{ijbc}{3}} = 3d_1^2 d_2 + d_1 q(z) (e^{\frac{ibc}{3}} - 1)^m, \\ a_2 = 3d_1 d_2^2 + d_2 q(z) \sum_{j=0}^m C_m^j (-1)^{m-j} e^{\frac{-ijbc}{3}} = 3d_1 d_2^2 + d_2 q(z) (e^{\frac{-ibc}{3}} - 1)^m, \\ a_0 = d_2^3 - d. \end{cases}$$

Since $w(z)$ is transcendental, we have

$$a_6 = a_4 = a_2 = a_0 = 0.$$

If $ad \neq 0$, then $d_1^3 = a \neq 0$, $d_2^3 = d \neq 0$. It follows from $a_4 = a_2 = 0$ that

$$3d_1 d_2 + q(z) (e^{\frac{ibc}{3}} - 1)^m = 3d_1 d_2 + q(z) (e^{\frac{-ibc}{3}} - 1)^m = 0. \quad (3.10)$$

If $q(z)$ is a nonconstant, then (3.10) results in $e^{\frac{ibc}{3}} - 1 = e^{\frac{-ibc}{3}} - 1 = 0$, which implies a contradiction that $d_1 = d_2 = 0$. Thus, (1.7) has no transcendental entire solutions of finite order for this case. If $q(z)$ is a constant q , then (3.10) results in $(e^{\frac{ibc}{3}} - 1)^m = (e^{\frac{-ibc}{3}} - 1)^m$. Denoting $v = e^{\frac{ibc}{3}}$, we have $(v - 1)^m = (\frac{1}{v} - 1)^m$, consequently,

$$v - 1 = u_s \left(\frac{1}{v} - 1 \right), \quad s = 0, \dots, m-1,$$

where $u_s = e^{\frac{2s\pi i}{m}} = \varepsilon^s$, $\varepsilon = e^{\frac{2\pi i}{m}}$, $s = 0, \dots, m-1$. If $s = 0$ (that is, $u_0 = 1$), then $v^2 = e^{\frac{2ibc}{3}} = 1$, that is, $bc = 3k\pi$, where $k \in \mathbb{Z}$. Substituting it into (3.10), we deduce that

$$3d_1d_2 + q((-1)^k - 1)^m = 0.$$

Then k is odd, and $q^3 = (-1)^{m+1} \frac{27ad}{8^m}$. Thus, (1.7) has three distinct transcendental entire solutions of finite order for this case. If $s \in \{1, \dots, m-1\}$ (that is, $u_s = \varepsilon^s$), then $v = 1$ or $-\varepsilon^s$, that is, $bc = 6k\pi$ or $bc = 6k\pi + 3\pi + \frac{6s\pi}{m}$, where $k \in \mathbb{Z}$. Substituting $bc = 6k\pi$ into (3.10), we deduce that $d_1d_2 = 0$, which is a contradiction. Substituting $bc = 6k\pi + 3\pi + \frac{6s\pi}{m}$ into (3.10), we deduce that

$$3d_1d_2 + q(-\varepsilon^s - 1)^m = 0.$$

Then $q^3 = (-1)^{m+1} \frac{27ad}{(e^{\frac{2s\pi i}{m}} + 1)^{3m}}$. Thus, (1.7) has three distinct transcendental entire solutions of finite order for this case.

If $a \neq 0$ and $d = 0$, then $d_1^3 = a \neq 0$ and $d_2 = 0$. If $q(z) \equiv 0$, then (1.7) has three distinct transcendental entire solutions of finite order for this case. If $q(z) \not\equiv 0$, it follows from $a_4 = a_2 = 0$ that $e^{\frac{ibc}{3}} - 1 = e^{\frac{-ibc}{3}} - 1 = 0$, that is, $bc = 6k\pi$, where $k \in \mathbb{Z}$. Substituting $bc = 6k\pi$ into (1.7), we see that (1.7) has three distinct transcendental entire solutions of finite order for this case.

If $a = 0$ and $d \neq 0$, then $d_1 = 0$ and $d_2^3 = d \neq 0$. We can deduce similar results as the above.

(ii) Let $n > 3$ (which implies $2n - 2 > n + 1$ and $2 < n - 1$), then we deduce from (3.9) that

$$a_{2n}w^{2n} + a_{2n-2}w^{2n-2} + \dots + a_2w^2 + a_0 = 0, \quad (3.11)$$

where

$$\begin{cases} a_{2n} = d_1^n - a, \\ a_{2n-2} = nd_1^{n-1}d_2, \\ \dots \dots \\ a_2 = nd_1d_2^{n-1}, \\ a_0 = d_2^n - d. \end{cases}$$

Since $w(z)$ is transcendental, we have

$$a_{2n} = a_{2n-2} = \dots = a_2 = a_0 = 0.$$

If $ad \neq 0$, then $d_1^n = a \neq 0$, $d_2^n = d \neq 0$. It follows from $a_{2n-2} = a_2 = 0$ that $d_1d_2 = 0$, which is a contradiction. Thus, (1.7) has no transcendental entire solutions of finite order for this case.

If $a \neq 0$ and $d = 0$, then $d_1^n = a \neq 0$ and $d_2 = 0$. If n is even, then $n + 1$ is odd. Hence, the coefficient of w^{n+1} in (3.11) is

$$a_{n+1} = d_1 q(z) \sum_{j=0}^m C_m^j (-1)^{m-j} e^{\frac{ijbc}{n}} = d_1 q(z) (e^{\frac{ibc}{n}} - 1)^m.$$

Since $a_{n+1} = 0$, we have $q(z) \equiv 0$ or $e^{\frac{ibc}{n}} - 1 = 0$ (that is, $bc = 2kn\pi$, where $k \in \mathbb{Z}$). If n is odd, then $n + 1$ is even. Hence, the coefficient of w^{n+1} in (3.11) is

$$a_{n+1} = C_n^{\frac{n-1}{2}} d_1^{\frac{n+1}{2}} d_2^{\frac{n-1}{2}} + d_1 q(z) \sum_{j=0}^m C_m^j (-1)^{m-j} e^{\frac{ijbc}{n}} = d_1 q(z) (e^{\frac{ibc}{n}} - 1)^m.$$

Hence, we deduce the same result as the above, that is, $q(z) \equiv 0$ or $bc = 2kn\pi$, where $k \in \mathbb{Z}$. Thus, (1.7) has n distinct transcendental entire solutions of finite order for this case.

If $a = 0$ and $d \neq 0$, then $d_1 = 0$ and $d_2^n = d \neq 0$. We can deduce similar results as the above. \square

Proof of Theorem 1.2. The proof of Theorem 1.2 is similar as the one of Theorem 1.3. \square

4 Examples

Example 4.1. In the following, we give examples to show the sharpness of Theorem 1.1.

Consider the nonlinear differential-difference equation

$$f^2(z) + P_1(z, f) = 1 + 4(z - \pi)^2, \quad (4.1)$$

where

$$P_1(z, f) = \frac{1}{4z^2} f'^2(z) + f'^2(z - \pi) + 4(z - \pi)^2 f^2(z - \pi) + f(z + \sqrt{\pi}) + f(z - \sqrt{\pi}) + 2 \cos(2\sqrt{\pi}z) f(z).$$

Clearly, $n = 2 = d(P)$, and $f_1(z) = \sin z^2$ is an admissible transcendental entire solution of (4.1). This shows our assumption “ $n > d(P)$ ” in Theorem 1.1(i) is sharp.

Consider the nonlinear differential-difference equation

$$f^4(z) + P_2(z, f) = 1 + z, \quad (4.2)$$

where

$$P_2(z, f) = 2f'(z) - f'^2(z + \pi) - zf(z)f(z + \frac{\pi}{2}).$$

Clearly, $n = 4 = w(P)$, and $f_2(z) = \tan z$ is an admissible transcendental meromorphic solution of (4.2). This shows our assumption “ $n > w(P)$ ” in Theorem 1.1(ii) is sharp.

Example 4.2. In the following, we give examples to illustrate the existence of entire solutions of finite order of (1.7) under the assumptions in Theorem 1.3.

Denote $\varepsilon = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$, which is a cubit root of unity, and consider the nonlinear difference equation

$$f^3(z) + q\Delta^m f(z) = \pi^3 e^{ibz} - e^{-ibz}. \quad (4.3)$$

If $m = 2$, $q = \frac{3}{4}\pi$, $b = \frac{3}{2}\pi$, $c = 2$, then (4.3) has three solutions as follows.

$$\begin{aligned}f_1(z) &= \pi e^{\frac{i\pi z}{2}} - e^{-\frac{i\pi z}{2}}, \\f_2(z) &= \pi \varepsilon e^{\frac{i\pi z}{2}} - \varepsilon^2 e^{-\frac{i\pi z}{2}}, \\f_3(z) &= \pi \varepsilon^2 e^{\frac{i\pi z}{2}} - \varepsilon e^{-\frac{i\pi z}{2}}.\end{aligned}$$

If $m = 3$, $q = 3\pi$, $b = \pi$, $c = 1$, then (4.3) has three solutions as follows.

$$\begin{aligned}f_1(z) &= \pi e^{\frac{i\pi z}{3}} - e^{-\frac{i\pi z}{3}}, \\f_2(z) &= \pi \varepsilon e^{\frac{i\pi z}{3}} - \varepsilon^2 e^{-\frac{i\pi z}{3}}, \\f_3(z) &= \pi \varepsilon^2 e^{\frac{i\pi z}{3}} - \varepsilon e^{-\frac{i\pi z}{3}}.\end{aligned}$$

If $m = 4$, $q = -\frac{3}{4}\pi$, $b = -3\pi$, $c = \frac{1}{2}$, then (4.3) has three solutions as follows.

$$\begin{aligned}f_1(z) &= \pi e^{-i\pi z} - e^{i\pi z}, \\f_2(z) &= \pi \varepsilon e^{-i\pi z} - \varepsilon^2 e^{i\pi z}, \\f_3(z) &= \pi \varepsilon^2 e^{-i\pi z} - \varepsilon e^{i\pi z}.\end{aligned}$$

Consider the nonlinear difference equation

$$f^n(z) + q(z)\Delta^m f(z) = ie^{-3z}, \quad n, m \in \mathbb{N}, n \geq 3, \quad (4.4)$$

where $p(z)$ is a polynomial. If $q(z) \equiv 0$ or $c = -\frac{2kn\pi i}{3}$, $k \in \mathbb{Z}$, then (4.4) has n solutions as follows.

$$f_j(z) = d_j e^{-\frac{3z}{n}}, \quad j = 0, 1, \dots, n-1,$$

where $d_j^n = i$, $j = 0, 1, \dots, n-1$.

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Asymptotic approximations of a stable and unstable manifolds of a two-dimensional quadratic map

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Abstract. We find the asymptotic approximations of the stable and unstable manifolds of the saddle equilibrium solutions and the saddle period-two solutions of the following difference equation $x_{n+1} = cx_{n-1}^2 + dx_n + 1$, where the parameters c and d are positive numbers and initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers. These manifolds determine completely global dynamics of this equation.

Keywords. Basin of attraction, cooperative, difference equation, local stable manifold, local unstable manifold, monotonicity, period-two solutions;

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1 Introduction

In this paper we consider the difference equation

$$x_{n+1} = cx_{n-1}^2 + dx_n + 1, \quad (1)$$

where the parameters c and d are positive numbers and initial conditions x_{-1} and x_0 are arbitrary nonnegative numbers. Set

$$u_n = x_{n-1} \text{ and } v_n = x_n \text{ for } n = 0, 1, \dots \quad (2)$$

and write Eq.(1) in the equivalent form:

$$\begin{aligned} u_{n+1} &= v_n \\ v_{n+1} &= cu_n^2 + dv_n + 1. \end{aligned} \quad (3)$$

Let T be the corresponding map defined by:

$$T \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ cu^2 + dv + 1 \end{pmatrix}. \quad (4)$$

It is easy to see that

$$T^2 \begin{pmatrix} u \\ v \end{pmatrix} = T \left(T \begin{pmatrix} u \\ v \end{pmatrix} \right) = \begin{pmatrix} cu^2 + dv + 1 \\ d(cu^2 + dv + 1) + cv^2 + 1 \end{pmatrix}. \quad (5)$$

The local dynamics of the map T was derived in [1] where it was shown that the following holds:

Theorem 1 *If*

$$d < 1 \text{ and } (d-1)^2 - 4c \geq 0$$

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then Eq.(1) has the equilibrium points \bar{x}_1 and \bar{x}_2 where

$$\bar{x}_1 = \frac{1-d-\sqrt{(d-1)^2-4c}}{2c}, \quad \bar{x}_2 = \frac{1-d+\sqrt{(d-1)^2-4c}}{2c}$$

and the following holds:

i) \bar{x}_1 is locally asymptotically stable if

$$c < \frac{(d-1)^2}{4}.$$

ii) \bar{x}_1 a non-hyperbolic point if

$$c = \frac{(d-1)^2}{4}.$$

iii) \bar{x}_2 is a repeller if

$$c < \frac{(1-3d)(d+1)}{4}.$$

iv) \bar{x}_2 is a saddle point if

$$\frac{(1-3d)(d+1)}{4} < c < \frac{(d-1)^2}{4}$$

v) \bar{x}_2 a non-hyperbolic point if

$$c = \frac{(1-3d)(d+1)}{4} \text{ or } c = \frac{(d-1)^2}{4}.$$

Theorem 2 If

$$c < \frac{(1-3d)(d+1)}{4}$$

then Eq.(1) has the minimal period-two solution

$$P = \left\{ \frac{d+1-\sqrt{1-4c-d(3d+2)}}{2c}, \frac{d+1+\sqrt{1-4c-d(3d+2)}}{2c} \right\}$$

which is a saddle point.

The global dynamics of Eq.(1) is delicate and is described by the following theorem [1].

Theorem 3 Consider Eq.(1). Then the following holds:

- (i) If $c < \frac{(1+d)(1-3d)}{4}$ then Eq.(1) has two equilibrium solutions $0 < \bar{x}_- < \bar{x}_+$, where \bar{x}_- is locally asymptotically stable, \bar{x}_+ is a repeller and the minimal period-two solution \dots, Φ, Ψ, \dots , $\Phi < \Psi$ is a saddle point. All non-equilibrium solutions $\{x_n\}$ converge to \bar{x}_- , or to the period-two solution or are asymptotic to ∞ . More precisely, there exist four continuous curves $W^s(P_1), W^s(P_2)$ (stable manifolds of $P_1(\Phi, \Psi)$ and $P_2(\Psi, \Phi)$), $W^u(P_1), W^u(P_2)$, (unstable manifolds of P_1 and P_2) where $W^s(P_1), W^s(P_2)$ are passing through the point $E_+(\bar{x}_+, \bar{x}_+)$, and are graphs of decreasing functions. The curves $W^u(P_1), W^u(P_2)$ are the graphs of increasing functions and are starting at $E_-(\bar{x}_-, \bar{x}_-)$. Every solution $\{x_n\}$ which starts below $W^s(P_1) \cup W^s(P_2)$ in North-east ordering converges to $E_-(\bar{x}_-, \bar{x}_-)$ and every solution $\{x_n\}$ which starts above $W^s(P_1) \cup W^s(P_2)$ in North-east ordering satisfies $\lim x_n = \infty$.
- (ii) If $c = \frac{(1+d)(1-3d)}{4}$ then Eq.(1) has two equilibrium solutions $0 < \bar{x}_- < \bar{x}_+$, where \bar{x}_- is locally asymptotically stable and \bar{x}_+ is the non-hyperbolic equilibrium solution. There exist the continuous decreasing curve $W^s(E_+)$ passing through the point $E_+ = (\bar{x}_+, \bar{x}_+)$, such that every solution $\{x_n\}$ which starts below $W^s(E_+)$ in North-east ordering converges to $E_-(\bar{x}_-, \bar{x}_-)$ and every solution $\{x_n\}$ which starts above $W^s(E_+)$ in North-east ordering satisfies $\lim x_n = \infty$.
- (iii) If $\frac{(1+d)(1-3d)}{4} < c < \frac{(1-d)^2}{4}$ then Eq.(1) has two equilibrium solutions $0 < \bar{x}_- < \bar{x}_+$ and no minimal period-two solutions. If \bar{x}_+ is a saddle equilibrium solution, then there exist two continuous curves $W^s(E_+)$ and $W^u(E_+)$, both passing through the point $E_+ = (\bar{x}_+, \bar{x}_+)$, such that $W^s(E_+)$ is a graph of decreasing function and $W^u(E_+)$ is a graph of an increasing

function. The first quadrant of initial condition $Q_1 = \{(x_{-1}, x_0) : x_{-1} \geq 0, x_0 \geq 0\}$ is the union of three disjoint basins of attraction, namely

$$Q_1 = \mathcal{B}(E_-) \cup \mathcal{B}(E_+) \cup \mathcal{B}(E_\infty),$$

where E_- and E_∞ denote the points (x_-, x_-) and (∞, ∞) respectively, and $\mathcal{B}(E_+) = W^s(E_+)$,

$$\mathcal{B}(E_-) = \{(x, y) | (x, y) \preceq_{ne} (x_{E_+}, y_{E_+}) \text{ for some } (x_{E_+}, y_{E_+}) \in W^s(E_+)\},$$

$$\mathcal{B}(E_\infty) = \{(x, y) | (x_{E_+}, y_{E_+}) \preceq_{ne} (x, y) \text{ for some } (x_{E_+}, y_{E_+}) \in W^s(E_+)\}.$$

In addition, for every $(x_{-1}, x_0) \in Q_1 \setminus W^s(E_+)$ every solution is asymptotic to $W^u(E_+)$.

- (iv) If $c = \frac{(1-d)^2}{4}$ then Eq.(1) has one non-hyperbolic equilibrium solution \bar{x} and there exists an invariant continuous curve $W^s(E)$, where $E(\bar{x}, \bar{x})$, which is the graph of a decreasing function, such that every solution $\{x_n\}$ of Eq.(1) for which $(x_{-1}, x_0) \in W^s(E)$ is attracted to E as well as every solution $\{x_n\}$ of Eq.(1) for which $(x_{-1}, x_0) \preceq_{ne} W^s(E)$.

Every solution $\{x_n\}$ of Eq.(1) for which there exists $(x_W, y_W) \in W^s(E)$ such that $(x_W, y_W) \preceq_{ne} (x_{-1}, x_0)$, $(x_{-1}, x_0) \notin W^s(E)$ satisfies $\lim x_n = \infty$.

- (v) If $c > \frac{(1-d)^2}{4}$ then Eq.(1) neither has an equilibrium solution nor the minimal period-two solution and every solution $\{x_n\}$ of Eq.(1) satisfies $\lim_{n \rightarrow \infty} x_n = \infty$.

As one may see from Theorem 3 the boundaries of the basins of attraction of all attractors of Eq.(1) are the stable manifolds of either equilibrium points or of the period-two solution. In addition, by using the results from [9] one can see that the solutions which are asymptotic to the locally asymptotically stable equilibrium solutions are approaching the unstable manifolds of the neighboring saddle equilibrium points or period-two point. The monotonicity and smoothness of stable and unstable manifolds for the map T given with (4) is guaranteed by Theorems 4, 5, 6 of [9]. See [4, 7, 9, 12, 13] for related results about the stable manifolds for competitive maps. Our main goal here is to get the local asymptotic estimates for these manifolds for both equilibrium solutions and the period-two solutions. We will bring the considered map to the normal form around the equilibrium solutions and the period-two solutions and then use the method of undetermined coefficients to find the local approximations of the considered manifolds. Since the map T is cooperative, it is guaranteed that both stable and unstable manifolds are as smooth as the functions of the considered map and that are monotonic such that the stable manifold is decreasing and unstable manifold is increasing, see [2, 9]. See [4, 10, 14] for similar local approximations of stable and unstable manifolds. See [3, 5, 6, 11, 14] for basic results on stable and unstable manifolds for general maps.

2 Preliminaries

In this section we present some basic results for the cooperative maps which describe the existence and the properties of their invariant manifolds.

A first order system of difference equations

$$\begin{cases} x_{n+1} &= f(x_n, y_n) \\ y_{n+1} &= g(x_n, y_n) \end{cases}, \quad n = 0, 1, 2, \dots, \quad (x_0, y_0) \in \mathcal{S}, \quad (6)$$

where $\mathcal{S} \subset \mathbb{R}^2$ is nonempty, $(f, g) : \mathcal{S} \rightarrow \mathcal{S}$, f, g are continuous functions is *cooperative* if $f(x, y)$ and $g(x, y)$ are non-decreasing in x and y . *Strongly cooperative* systems of difference equations or strongly cooperative maps are those for which the functions f and g are coordinate-wise strictly monotone.

If $\mathbf{v} = (u, v) \in \mathbb{R}^2$, we denote with $\mathcal{Q}_\ell(\mathbf{v})$, $\ell \in \{1, 2, 3, 4\}$, the four quadrants in \mathbb{R}^2 relative to \mathbf{v} , i.e., $\mathcal{Q}_1(\mathbf{v}) = \{(x, y) \in \mathbb{R}^2 : x \geq u, y \geq v\}$, $\mathcal{Q}_2(\mathbf{v}) = \{(x, y) \in \mathbb{R}^2 : x \leq u, y \geq v\}$, and so on. Define the *South-East* partial order \preceq_{se} on \mathbb{R}^2 by $(x, y) \preceq_{se} (s, t)$ if and only if $x \leq s$ and $y \geq t$. Similarly, we define the *North-East* partial order \preceq_{ne} on \mathbb{R}^2 by $(x, y) \preceq_{ne} (s, t)$ if and only if $x \leq s$ and $y \leq t$. For $\mathcal{A} \subset \mathbb{R}^2$ and $x \in \mathbb{R}^2$, define the *distance from x to A* as $\text{dist}(x, \mathcal{A}) := \inf \{\|x - y\| : y \in \mathcal{A}\}$. By $\text{int } \mathcal{A}$ we denote the interior of a set \mathcal{A} .

It is easy to show that a map F is cooperative if it is non-decreasing with respect to the North-East partial order, that is if the following holds:

$$\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \preceq_{ne} \begin{pmatrix} x^2 \\ y^2 \end{pmatrix} \Rightarrow F \begin{pmatrix} x^1 \\ y^1 \end{pmatrix} \preceq_{ne} F \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}. \quad (7)$$

The following five results were proved by Kulenović and Merino [8, 9] for competitive systems in the plane, when one of the eigenvalues of the linearized system at an equilibrium (hyperbolic or non-hyperbolic) is by absolute value smaller than 1 while the other has an arbitrary value. We give the analogue versions for cooperative maps.

A region $\mathcal{R} \subset \mathbb{R}^2$ is *rectangular* if it is the cartesian product of two intervals in \mathbb{R} .

Theorem 4 *Let T be a cooperative map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\bar{x} \in \mathcal{R}$ be a fixed point of T such that $\Delta := \mathcal{R} \cap \text{int}(\mathcal{Q}_2(\bar{x}) \cup \mathcal{Q}_4(\bar{x}))$ is nonempty (i.e., \bar{x} is not the NE or SW vertex of \mathcal{R}), and T is strongly cooperative on Δ . Suppose that the following statements are true.*

- a. *The map T has a C^1 extension to a neighborhood of \bar{x} .*
- b. *The Jacobian matrix of T at \bar{x} has real eigenvalues λ, μ such that $0 < |\lambda| < \mu$, where $|\lambda| < 1$, and the eigenspace E^λ associated with λ is not a coordinate axis.*

Then there exists a curve $\mathcal{C} \subset \mathcal{R}$ through \bar{x} that is invariant and a subset of the basin of attraction of \bar{x} , such that \mathcal{C} is tangential to the eigenspace E^λ at \bar{x} , and \mathcal{C} is the graph of a strictly decreasing continuous function of the first coordinate on an interval. Any endpoints of \mathcal{C} in the interior of \mathcal{R} are either fixed points or minimal period-two points. In the latter case, the set of endpoints of \mathcal{C} is a minimal period-two orbit of T .

Corollary 1 *If T has no fixed point nor periodic points of minimal period two in Δ , then the endpoints of \mathcal{C} belong to $\partial\mathcal{R}$.*

For maps that are strongly cooperative near the fixed point, hypothesis (b). of Theorem 4 reduces just to $|\lambda| < 1$. This follows from a change of variables [13] that allows the Perron-Frobenius Theorem to be applied to give that at any point, the Jacobian matrix of a strongly cooperative map has two real and distinct eigenvalues, the larger one in absolute value being positive, and that corresponding eigenvectors may be chosen to point in the direction of the second and first quadrant, respectively. Also, one can show that in such a case no associated eigenvector is aligned with a coordinate axis.

Theorem 5 *Under the hypotheses of Theorem 4, suppose there exists a neighborhood \mathcal{U} of \bar{x} in \mathbb{R}^2 such that T is of class C^k on $\mathcal{U} \cup \Delta$ for some $k \geq 1$, and that the Jacobian of T at each $x \in \Delta$ is invertible. Then the curve \mathcal{C} in the conclusion of Theorem 4 is of class C^k .*

The following result gives a description of the global stable and unstable manifolds of a saddle point of a cooperative map. The result is the modification of Theorem 5 from [7]. See also [8].

Theorem 6 *In addition to the hypotheses of Theorem 4, suppose that $\mu > 1$ and that the eigenspace E^μ associated with μ is not a coordinate axis. If the curve \mathcal{C} of Theorem 4 has endpoints in $\partial\mathcal{R}$, then \mathcal{C} is the global stable manifold $\mathcal{W}^s(\bar{x})$ of \bar{x} , and the global unstable manifold $\mathcal{W}^u(\bar{x})$ is a curve in \mathcal{R} that is tangential to E^μ at \bar{x} and such that it is the graph of a strictly increasing function of the first coordinate on an interval. Any endpoints of $\mathcal{W}^u(\bar{x})$ in \mathcal{R} are fixed points of T .*

Theorem 7 *Assume the hypotheses of Theorem 4, and let \mathcal{C} be the curve whose existence is guaranteed by Theorem 4. If the endpoints of \mathcal{C} belong to $\partial\mathcal{R}$, then \mathcal{C} separates \mathcal{R} into two connected components, namely*

$$\mathcal{W}_- := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } x \preceq_{ne} y\} \quad \text{and} \quad \mathcal{W}_+ := \{x \in \mathcal{R} \setminus \mathcal{C} : \exists y \in \mathcal{C} \text{ with } y \preceq_{ne} x\}, \quad (8)$$

such that the following statements are true.

- (i) \mathcal{W}_- is invariant, and $\text{dist}(T^n(x), \mathcal{Q}_1(\bar{x})) \rightarrow 0$ as $n \rightarrow \infty$ for every $x \in \mathcal{W}_-$.
- (ii) \mathcal{W}_+ is invariant, and $\text{dist}(T^n(x), \mathcal{Q}_3(\bar{x})) \rightarrow 0$ as $n \rightarrow \infty$ for every $x \in \mathcal{W}_+$.

If, in addition, \bar{x} is an interior point of \mathcal{R} and T is C^2 and strongly cooperative in a neighborhood of \bar{x} , then T has no periodic points in the boundary of $\mathcal{Q}_2(\bar{x}) \cup \mathcal{Q}_4(\bar{x})$ except for \bar{x} , and the following statements are true.

- (iii) For every $x \in \mathcal{W}_-$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \text{int } \mathcal{Q}_1(\bar{x})$ for $n \geq n_0$.
- (iv) For every $x \in \mathcal{W}_+$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \text{int } \mathcal{Q}_3(\bar{x})$ for $n \geq n_0$.

Remark 1 The map T defined with (4) is strongly cooperative in the first quadrant of initial conditions. Theorems 4, 5 and 6 show that the stable and unstable manifolds of cooperative maps, which satisfies certain conditions, are simple monotonic curves which are as smooth as the functions of the map. Thus the assumed forms of these manifolds are justified. As is well-known the stable and unstable manifolds of general maps can have complicated structure consisting of many branches or being strange attractors, see [3, 5, 10, 14] for some examples of polynomial maps such as Henon with unstable manifold being a strange attractor. Finally, see [13] for examples of competitive and so cooperative maps in the plane with chaotic attractors.

3 Invariant manifolds and Normal Forms

Let

$$\begin{pmatrix} \xi_{n+1} \\ \eta_{n+1} \end{pmatrix} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} + \begin{pmatrix} g_1(\xi_n, \eta_n) \\ g_2(\xi_n, \eta_n) \end{pmatrix}, \quad (9)$$

where

$$g_1(0, 0) = 0, \quad g_2(0, 0) = 0, \quad Dg_1(0, 0) = 0 \text{ and } Dg_2(0, 0) = 0.$$

Suppose that $|\mu_1| < 1$ and $|\mu_2| > 1$. Then, there are two unique invariant manifolds \mathcal{W}^s and \mathcal{W}^u tangents to $(1, 0)$ and $(0, 1)$ at $(0, 0)$, which are graphs of the maps

$$\varphi : E_1 \rightarrow E_2 \text{ and } \psi : E_1 \rightarrow E_2,$$

such that

$$\varphi(0) = \psi(0) = 0 \text{ and } \varphi'(0) = \psi'(0) = 0.$$

See [4, 5, 10, 14]. Letting $\eta_n = \varphi(\xi_n)$ yields

$$\eta_{n+1} = \varphi(\xi_{n+1}) = \varphi(\mu_1 \xi_n + g_1(\xi_n, \varphi(\xi_n))). \quad (10)$$

On the other hand by (9)

$$\eta_{n+1} = \mu_2 \varphi(\xi_n) + g_2(\xi_n, \varphi(\xi_n)). \quad (11)$$

Equating equations (10) and (11) yields

$$\varphi(\mu_1 \xi_n + g_1(\xi_n, \varphi(\xi_n))) = \mu_2 \varphi(\xi_n) + g_2(\xi_n, \varphi(\xi_n)). \quad (12)$$

Similarly, letting $\xi_n = \psi(\eta_n)$ yields

$$\xi_{n+1} = \psi(\eta_{n+1}) = \psi(\mu_2 \eta_n + g_2(\psi(\eta_n), \eta_n)). \quad (13)$$

By using (9) we obtain

$$\xi_{n+1} = \mu_1 \psi(\eta_n) + g_1(\psi(\eta_n), \eta_n). \quad (14)$$

Equating equations (13) and (14) yields

$$\psi(\mu_2 \eta_n + g_2(\psi(\eta_n), \eta_n)) = \mu_1 \psi(\eta_n) + g_1(\psi(\eta_n), \eta_n). \quad (15)$$

Thus the functional equations (12) and (15), define the local stable manifold

$$\mathcal{W}^s = \{(\xi, \eta) \in \mathbb{R}^2 : \eta = \varphi(\xi)\},$$

and the local unstable manifold

$$\mathcal{W}^u = \{(\xi, \eta) \in \mathbb{R}^2 : \xi = \psi(\eta)\}.$$

Without loss generality, we can assume that solutions of the functional equations (12) and (15) take the forms

$$\psi(\eta) = \alpha_2 \eta^2 + \beta_2 \eta^3 + O(|\eta|^4)$$

and

$$\varphi(\xi) = \alpha_1 \xi^2 + \beta_1 \xi^3 + O(|\xi|^4),$$

where $\alpha_i, \beta_i, i = 1, 2$ are undetermined coefficients.

3.1 Normal form of the map T at \bar{x}_2

Put $y_n = x_n - \bar{x}_2$. Then Eq(1) becomes

$$y_{n+1} = c(\bar{x}_2 + y_{n-1})^2 + d(\bar{x}_2 + y_n) - \bar{x}_2 + 1. \quad (16)$$

Set

$$u_n = y_{n-1} \text{ and } v_n = y_n \text{ for } n = 0, 1, \dots \quad (17)$$

and write Eq(16) in the equivalent form:

$$\begin{aligned} u_{n+1} &= v_n \\ v_{n+1} &= c(\bar{x}_2 + u_n)^2 + d(\bar{x}_2 + v_n) - \bar{x}_2 + 1. \end{aligned} \quad (18)$$

Let F be the function defined by:

$$F \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ c(\bar{x}_2 + u)^2 + d(\bar{x}_2 + v) - \bar{x}_2 + 1 \end{pmatrix}. \quad (19)$$

Then F has the fixed point $(0, 0)$ and maps $(-\bar{x}_2, \infty)^2$ into $(-\bar{x}_2, \infty)^2$. The Jacobian matrix of F is given by

$$Jac_F(u, v) = \begin{pmatrix} 0 & 1 \\ 2c(u + \bar{x}_2) & d \end{pmatrix}.$$

At $(0, 0)$, $Jac_F(u, v)$ has the form

$$J_0 = Jac_F(0, 0) = \begin{pmatrix} 0 & 1 \\ 2c\bar{x}_2 & d \end{pmatrix}. \quad (20)$$

The eigenvalues of (20) are $\mu_{1,2}$ where

$$\mu_1 = \frac{1}{2} \left(d - \sqrt{8c\bar{x}_2 + d^2} \right) \text{ and } \mu_2 = \frac{1}{2} \left(d + \sqrt{8c\bar{x}_2 + d^2} \right),$$

and the corresponding eigenvectors are given by

$$v_1 = \left(-\frac{d + \sqrt{8c\bar{x}_2 + d^2}}{4c\bar{x}_2}, 1 \right)^T \text{ and } v_2 = \left(-\frac{d - \sqrt{8c\bar{x}_2 + d^2}}{4c\bar{x}_2}, 1 \right)^T,$$

respectively.

Then we have that

$$F \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2c\bar{x}_2 & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_1(u, v) \\ g_1(u, v) \end{pmatrix}, \quad (21)$$

where

$$\begin{aligned} f_1(u, v) &= 0 \\ g_1(u, v) &= \bar{x}_2(c\bar{x}_2 + d - 1) + cu^2 + 1. \end{aligned}$$

Then, the system (16) is equivalent to

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2c\bar{x}_2 & d \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} f_1(u_n, v_n) \\ g_1(u_n, v_n) \end{pmatrix}. \quad (22)$$

Let

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = P \cdot \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix}$$

where

$$P = \begin{pmatrix} -\frac{d + \sqrt{d^2 + 8c\bar{x}_2}}{4c\bar{x}_2} & -\frac{d - \sqrt{d^2 + 8c\bar{x}_2}}{4c\bar{x}_2} \\ 1 & 1 \end{pmatrix} \text{ and } P^{-1} = \begin{pmatrix} -\frac{2c\bar{x}_2}{\sqrt{d^2 + 8c\bar{x}_2}} & \frac{\sqrt{d^2 + 8c\bar{x}_2} - d}{2\sqrt{d^2 + 8c\bar{x}_2}} \\ \frac{2c\bar{x}_2}{\sqrt{d^2 + 8c\bar{x}_2}} & \frac{d + \sqrt{d^2 + 8c\bar{x}_2}}{2\sqrt{d^2 + 8c\bar{x}_2}} \end{pmatrix}.$$

Then system (22) is equivalent to

$$\begin{pmatrix} \xi_{n+1} \\ \eta_{n+1} \end{pmatrix} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} + P^{-1} \cdot H_1 \left(P \cdot \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} \right), \quad (23)$$

where

$$H_1 \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} f_1(u, v) \\ g_1(u, v) \end{pmatrix}.$$

Let

$$G_1 \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} \tilde{f}_1(u, v) \\ \tilde{g}_1(u, v) \end{pmatrix} = P^{-1} \cdot H_1 \left(P \cdot \begin{pmatrix} u \\ v \end{pmatrix} \right).$$

By straightforward calculation we obtain that

$$\begin{aligned} \tilde{f}_1(u, v) &= \frac{\Upsilon_1(u, v) (\sqrt{8c\bar{x}_2 + d^2} - d)}{16c\bar{x}_2^2 \sqrt{8c\bar{x}_2 + d^2}}, \\ \tilde{g}_1(u, v) &= \frac{\Upsilon_1(u, v) (\sqrt{8c\bar{x}_2 + d^2} + d)}{16c\bar{x}_2^2 \sqrt{8c\bar{x}_2 + d^2}}, \end{aligned}$$

where

$$\Upsilon_1(u, v) = 8c^2 \bar{x}_2^4 + d(u^2 - v^2) \sqrt{8c\bar{x}_2 + d^2} + 4c\bar{x}_2 (2(d-1)\bar{x}_2^2 + 2\bar{x}_2 + (u-v)^2) + d^2(u^2 + v^2).$$

3.2 Stable and unstable manifolds corresponding to \bar{x}_2

Assume that $d < 1$ and $(d-1)^2 - 4c \geq 0$. Then Eq.(1) has the equilibrium point \bar{x}_2 where

$$\bar{x}_2 = \frac{1-d + \sqrt{(d-1)^2 - 4c}}{2c}$$

which is a saddle point if

$$\frac{(1-3d)(d+1)}{4} < c < \frac{(d-1)^2}{4}.$$

Let us assume that the local stable manifold is the graph of the map φ_1 of the form

$$\varphi_1(\xi) = \alpha_1 \xi^2 + \beta_1 \xi^3 + O(|\xi|^4), \quad \alpha_1, \beta_1 \in \mathbb{R}.$$

Now we compute the constants α_1 and β_1 . The function φ_1 must satisfy the stable manifold equation

$$\varphi_1 \left(\mu_1 \xi + \tilde{f}_1(\xi, \varphi_1(\xi)) \right) = \mu_2 \varphi_1(\xi) + \tilde{g}_1(\xi, \varphi_1(\xi)),$$

This leads to the following polynomial equation

$$p_1 \xi^2 + p_2 \xi^3 + \dots + p_{18} \xi^{18} = 0$$

where the coefficients p_1 and p_2 , obtain by using *Mathematica* are in appendix A. Substituting \bar{x}_2 into (42) and (43) and solving system $p_1 = 0$ and $p_2 = 0$, we obtain the values

$$\alpha_1 = \frac{-8c^2}{\Upsilon_1(c, d) + \Upsilon_2(c, d) \sqrt{4\sqrt{(d-1)^2 - 4c} + d^2 - 4d + 4}}$$

and

$$\beta_1 = \frac{4\alpha_1 c(4c + (d+1)(3d-1))}{\Upsilon_3(c, d) + \Upsilon_4(c, d) \sqrt{4\sqrt{(d-1)^2 - 4c} + d^2 - 4d + 4}}$$

where

$$\begin{aligned} \Upsilon_1(c, d) &= \sqrt{(d-1)^2 - 4c} (d^6 - 4c(13d^2 - 4d + 8)) \\ &\quad - 4c(7d^4 - 12d^3 + 17d^2 - 12d + 8) + (d-1)d^6 + 64c^2, \\ \Upsilon_2(c, d) &= \sqrt{(d-1)^2 - 4c} (4c(5d+2) - d^5) + 4c(5d^3 - 4d^2 + 3d + 2) + (1-d)d^5, \\ \Upsilon_3(c, d) &= \sqrt{(d-1)^2 - 4c} (-4c(5d^2 - 12d + 16) - 15d^4 + 27d^3 - 13d^2 - 32d + 24) \\ &\quad + 64c^2 - 4c(d^4 - 6d^3 + 7d^2 - 32d + 28) - 3d^6 + 16d^5 - 42d^4 + 40d^3 + 19d^2 - 56d + 24, \\ \Upsilon_4(c, d) &= \sqrt{(d-1)^2 - 4c} (4c(3d-2) + 9d^3 + d^2 - 3d + 6) + 3d^5 - 10d^4 + 8d^3 + 4d^2 - 9d + 6 \\ &\quad + 4c(d^3 - 4d^2 + 5d - 6). \end{aligned} \tag{24}$$

Since

$$\begin{aligned}\eta_n &= \alpha_1 \xi_n^2 + \beta_1 \xi_n^3, \\ \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} &= P^{-1} \cdot \begin{pmatrix} u_n \\ v_n \end{pmatrix},\end{aligned}$$

and

$$u_n = x_{n-1} - \bar{x}_2 \text{ and } v_n = x_n - \bar{x}_2$$

we can approximate locally the local stable manifold $\mathcal{W}_{loc}^s(\bar{x}_2, \bar{x}_2)$ as the graph of the map $\tilde{\varphi}_1(u)$ such that $S(u, \tilde{\varphi}_1(u)) = 0$ where

$$\begin{aligned}S(u, v) := & \alpha_1 \left(\frac{(v - \bar{x}_2)(\sqrt{8c\bar{x}_2 + d^2} - d)}{2\sqrt{8c\bar{x}_2 + d^2}} - \frac{2c\bar{x}_2(u - \bar{x}_2)}{\sqrt{8c\bar{x}_2 + d^2}} \right)^2 \\ & + \beta_1 \left(\frac{(v - \bar{x}_2)(\sqrt{8c\bar{x}_2 + d^2} - d)}{2\sqrt{8c\bar{x}_2 + d^2}} - \frac{2c\bar{x}_2(u - \bar{x}_2)}{\sqrt{8c\bar{x}_2 + d^2}} \right)^3 \\ & - \frac{2c\bar{x}_2(u - \bar{x}_2)}{\sqrt{8c\bar{x}_2 + d^2}} - \frac{(v - \bar{x}_2)(\sqrt{8c\bar{x}_2 + d^2} + d)}{2\sqrt{8c\bar{x}_2 + d^2}}\end{aligned} \quad (25)$$

and which satisfies

$$\tilde{\varphi}_1(\bar{x}_2) = \bar{x}_2 \text{ and } \tilde{\varphi}_1'(\bar{x}_2) = -\frac{4c\bar{x}_2}{\sqrt{8c\bar{x}_2 + d^2} + d}.$$

Let us assume that the local unstable manifold is the graph of the map ψ that has the form

$$\psi_1(\eta) = \alpha_2 \eta^2 + \beta_2 \eta^3 + O(|\eta|^4), \quad \alpha_1, \beta_1 \in \mathbb{R}.$$

Now we compute the constants α_2 and β_2 . The function ψ_1 must satisfy the unstable manifold equation

$$\psi_1(\mu_2 \eta + \tilde{g}_1(\psi_1(\eta), \eta)) = \mu_1 \psi_1(\eta) + \tilde{f}_1(\psi_1(\eta), \eta),$$

This leads to the following polynomial equation

$$q_1 \eta^2 + q_2 \eta^3 + \dots + q_{18} \eta^{18} = 0$$

where the coefficients q_1 and q_2 are in appendix A.

Substituting \bar{x}_2 into (44) and (45) and solving system $q_1 = 0$ and $q_2 = 0$, we obtain the values

$$\alpha_2 = \frac{-8c^2}{\Gamma_1(c, d) + \Gamma_2(c, d) \sqrt{4\sqrt{(d-1)^2 - 4c} + d^2 - 4d + 4}}$$

and

$$\beta_2 = \frac{\alpha_2 \Gamma_5(c, d)}{\Gamma_3(c, d) + \Gamma_4(c, d) \sqrt{4\sqrt{(d-1)^2 - 4c} + d^2 - 4d + 4}}$$

where

$$\begin{aligned}\Gamma_1(c, d) &= 64c^2 + \sqrt{(d-1)^2 - 4c} (d^6 - 4c(13d^2 - 4d + 8)) \\ &\quad - 4c(7d^4 - 12d^3 + 17d^2 - 12d + 8) + (d-1)d^6, \\ \Gamma_2(c, d) &= \sqrt{(d-1)^2 - 4c} (d^5 - 4c(5d + 2)) - 4c(5d^3 - 4d^2 + 3d + 2) + (d-1)d^5, \\ \Gamma_3(c, d) &= 256c^2 + \sqrt{(d-1)^2 - 4c} \left((d^2 - 8d + 8)^2 (d^2 - 2d + 3) - 32c(3d^2 - 8d + 10) \right) \\ &\quad - 4c(9d^4 - 72d^3 + 208d^2 - 304d + 176) - (d^2 - 8d + 8)^2 (d^3 - 3d^2 + 5d - 3), \\ \Gamma_4(c, d) &= d\sqrt{(d-1)^2 - 4c} (-48c + d^4 - 14d^3 + 61d^2 - 88d + 40) \\ &\quad - d(4c(7d^2 - 36d + 32) + (d^3 - 13d^2 + 48d - 40)(d-1)^2), \\ \Gamma_5(c, d) &= \sqrt{4\sqrt{(d-1)^2 - 4c} + d^2 - 4d + 4} \left(-2c(d-2)^2 - 8c\sqrt{(d-1)^2 - 4c} \right) \\ &\quad - 4c(d^2 - 10d + 8) \sqrt{(d-1)^2 - 4c} + 2c(32c + 3d^3 - 22d^2 + 36d - 16).\end{aligned} \quad (26)$$

Since

$$\begin{aligned}\xi_n &= \alpha_2 \eta_n^2 + \beta_2 \eta_n^3, \\ \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} &= P^{-1} \cdot \begin{pmatrix} u_n \\ v_n \end{pmatrix},\end{aligned}$$

and

$$u_n = x_{n-1} - \bar{x}_2 \text{ and } v_n = x_n - \bar{x}_2$$

we can approximate locally the local unstable manifold $\mathcal{W}_{loc}^u(\bar{x}_2, \bar{x}_2)$ as the graph of the map $\tilde{\psi}_1(u)$ such that $U(\tilde{\psi}_1(v), v) = 0$ where

$$\begin{aligned}U(u, v) := & \alpha_2 \left(\frac{2c\bar{x}_2(u - \bar{x}_2)}{\sqrt{8c\bar{x}_2 + d^2}} + \frac{(v - \bar{x}_2)(\sqrt{8c\bar{x}_2 + d^2} + d)}{2\sqrt{8c\bar{x}_2 + d^2}} \right)^2 \\ & + \beta_2 \left(\frac{2c\bar{x}_2(u - \bar{x}_2)}{\sqrt{8c\bar{x}_2 + d^2}} + \frac{(v - \bar{x}_2)(\sqrt{8c\bar{x}_2 + d^2} + d)}{2\sqrt{8c\bar{x}_2 + d^2}} \right)^3 \\ & + \frac{2c\bar{x}_2(u - \bar{x}_2)}{\sqrt{8c\bar{x}_2 + d^2}} - \frac{(v - \bar{x}_2)(\sqrt{8c\bar{x}_2 + d^2} - d)}{2\sqrt{8c\bar{x}_2 + d^2}}\end{aligned}\quad (27)$$

and which satisfies

$$\tilde{\psi}_1(\bar{x}_2) = \bar{x}_2 \text{ and } \tilde{\psi}'_1(\bar{x}_2) = \frac{4c\bar{x}_2}{\sqrt{8c\bar{x}_2 + d^2} - d}.$$

Thus we proved the following result

Theorem 8 Consider Eq.(1) subject to the condition $\frac{(1-3d)(d+1)}{4} < c < \frac{(d-1)^2}{4}$. Then the local stable and unstable manifolds are given with the asymptotic expansions (25) and (27) respectively.

3.3 Some numerical examples

In this section we provide some numerical examples and we compare visually the asymptotic approximations of stable and unstable manifolds, obtained by using *Mathematica*, with the boundaries of the basins of attraction obtained by using the software package *Dynamica 3* [6].

For $c = 0.06$ and $d = 0.3$ we have that

$$\begin{aligned}S_1(u, v) = & -0.0000205931x^3 + x^2(0.0000492004y + 0.0322121) \\ & + x(-0.0000391827y^2 - 0.0513067y + 0.411741) \\ & + 0.0000104016y^3 + 0.0204302y^2 + 0.672096y - 10.9718\end{aligned}$$

and for $c = 0.075$ and $d = 0.42$

$$\begin{aligned}S_2(u, v) = & y^2(0.0227887 - 0.000731949x) + 0.000814449(x - 62.2685)xy \\ & - 0.000302082(x - 105.844)x(x + 12.4415) \\ & + 0.000219269y^3 + 0.642493y - 6.35325.\end{aligned}$$

Figures 1 and 2 show the graph of the functions $S_1(u, v) = 0$ and $S_2(u, v) = 0$ with the basins of attraction created with *Dynamica 3*. Figure 3 shows the graph of the functions $S_1(u, v) = 0$ and $S_2(u, v) = 0$ for different values of the parameters c and d .

For $c = 0.06$ and $d = 0.3$ we have that

$$\begin{aligned}U_1(u, v) = & (0.000113205x - 0.00441057)y^2 + 0.000108186(x - 77.922)xy \\ & + 0.0000344633(x - 183.477)x(x + 66.5943) \\ & + 0.0000394854y^3 + 0.559375y + 0.00870931\end{aligned}$$

and for $c = 0.075$ and $d = 0.42$

$$\begin{aligned}U_2(u, v) = & (0.000446197x - 0.010175)y^2 + 0.000309085(x - 45.6077)xy \\ & + 0.000071369(x - 111.063)x(x + 42.6512) \\ & + 0.00021471y^3 + 0.511958y - 0.265003\end{aligned}$$

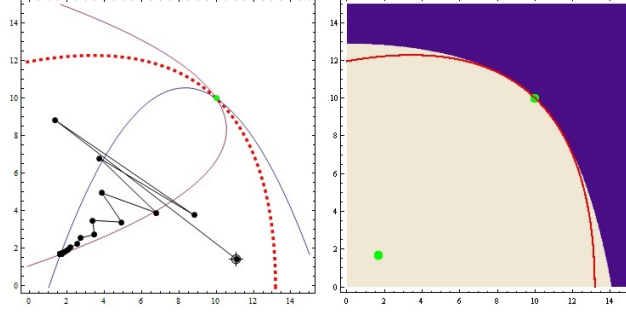


Figure 1: The graph of the function $S_1(u, v) = 0$ (red curve) for $c = 0.06$ and $d = 0.3$ with the basins of attraction generated by *Dynamica 3*.

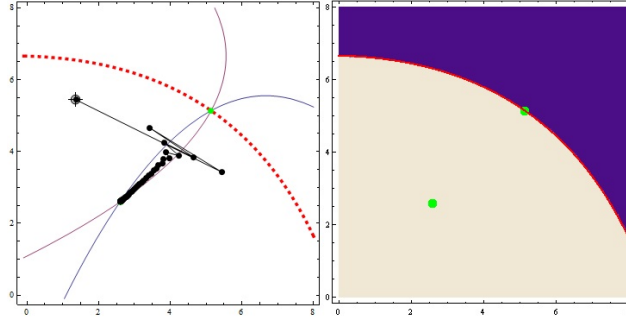


Figure 2: The graph of the function $S_2(u, v) = 0$ (red curve) for $c = 0.075$ and $d = 0.42$ with the basins of attraction generated by *Dynamica 3*.

3.4 Normal form of the map T^2 at the period-two solution

The period-two solution of (1) is given as

$$\bar{u}_1 = \frac{d+1-D}{2c} \text{ and } \bar{v}_1 = \frac{d+1+D}{2c},$$

where

$$D = \sqrt{1 - 4c - 2d - 3d^2}.$$

First we transform the period two solution (\bar{u}_1, \bar{v}_1) of (1) to the origin by the translation

$$\tilde{u} = u - \bar{u}_1 \text{ and } \tilde{v} = v - \bar{v}_1$$

under which the corresponding map (5) becomes

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \rightarrow \tilde{F} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = T^2 \begin{pmatrix} \tilde{u} + \bar{u}_1 \\ \tilde{v} + \bar{v}_1 \end{pmatrix} - \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} c\tilde{u}^2 + \tilde{u}(d-D+1) + d\tilde{v} \\ (c\tilde{u}^2 - D\tilde{u} + \tilde{u} + \tilde{v}) + \tilde{v}(c\tilde{v} + D+1) + d^2(\tilde{u} + \tilde{v}) \end{pmatrix}. \quad (28)$$

Then \tilde{F} has the fixed point at $(0, 0)$. The Jacobian matrix of \tilde{F} is given by

$$Jac_{\tilde{F}}(\tilde{u}, \tilde{v}) = \begin{pmatrix} d-D+2c\tilde{u}+1 & d \\ d^2 + (-D+2c\tilde{u}+1)d & d^2 + d + D + 2c\tilde{v} + 1 \end{pmatrix}.$$

At $(0, 0)$, $Jac_{\tilde{F}}(\tilde{u}, \tilde{v})$ has the form

$$J_0 = Jac_{\tilde{F}}(0, 0) = \begin{pmatrix} d-D+1 & d \\ d^2 + (1-D)d & d^2 + d + D + 1 \end{pmatrix}. \quad (29)$$

The eigenvalues of (29) are

$$\nu_1 = \frac{1}{2}(d(d+2) + 2 - C) \text{ and } \nu_2 = \frac{1}{2}(d(d+2) + 2 + C),$$

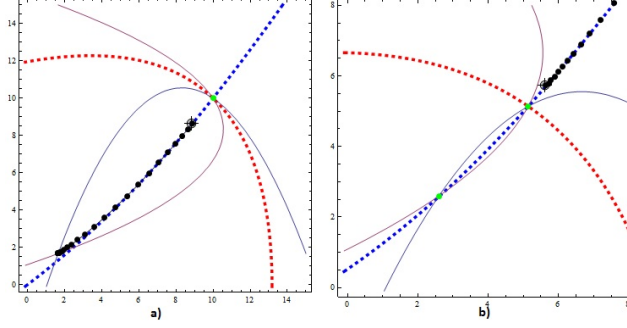


Figure 3: a) The graph of the functions $S_1(u, v) = 0$ (red curve) and $U_1(u, v) = 0$ (blue curve) for $c = 0.06$ and $d = 0.3$. b) The graph of the functions $S_2(u, v) = 0$ (red curve) and $U_2(u, v) = 0$ (blue curve) for $c = 0.075$ and $d = 0.42$.

where

$$C = \sqrt{-16c + (d-2)d(d+6) + 4} + 4.$$

The eigenvectors corresponding to the eigenvalues $\nu_{1,2}$ are given by

$$\mathbf{v}_1 = \left(\frac{2d}{-C + d^2 + 2D}, 1 \right)^T \quad \text{and} \quad \mathbf{v}_2 = \left(\frac{2d}{C + d^2 + 2D}, 1 \right)^T,$$

respectively.

Then we have that

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \rightarrow \tilde{F} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} d-D+1 & d \\ d^2 + (1-D)d & d^2 + d + D + 1 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} + \begin{pmatrix} f_2(\tilde{u}, \tilde{v}) \\ g_2(\tilde{u}, \tilde{v}) \end{pmatrix}, \quad (30)$$

where

$$f_2(\tilde{u}, \tilde{v}) = c\tilde{u}^2, \quad g_2(\tilde{u}, \tilde{v}) = c(d\tilde{u}^2 + \tilde{v}^2).$$

Let

$$\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = P \cdot \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

where

$$P = \begin{pmatrix} \frac{2d}{d^2 - C + 2D} & \frac{2d}{d^2 + C + 2D} \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad P^{-1} = \begin{pmatrix} \frac{(d^2 + 2D)^2 - C^2}{4Cd} & -\frac{d^2 - C + 2D}{2C} \\ \frac{C^2 - (d^2 + 2D)^2}{4Cd} & \frac{d^2 + C + 2D}{2C} \end{pmatrix}.$$

Then (30) leads to the corresponding normal form

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \rightarrow \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} + P^{-1} \cdot H_2 \left(P \cdot \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right), \quad (31)$$

where

$$H_2 \begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} f_2(u, v) \\ g_2(u, v) \end{pmatrix}.$$

Let

$$G_2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \tilde{f}_2(u, v) \\ \tilde{g}_2(u, v) \end{pmatrix} = P^{-1} \cdot H_2 \left(P \cdot \begin{pmatrix} u \\ v \end{pmatrix} \right).$$

By straightforward calculation we obtain that

$$\tilde{f}_2(u, v) = c\Upsilon_2(u, v) \left((d^2 + 2D)^2 - C^2 \right),$$

$$\tilde{g}_2(u, v) = -c\Upsilon_2(u, v) \left((d^2 + 2D)^2 - C^2 \right),$$

where

$$\Upsilon_2(u, v) = \frac{1}{2C} \left(2d \left(\frac{u}{-C+d^2+2D} + \frac{v}{C+d^2+2D} \right)^2 - \frac{4d^3 \left(\frac{u}{-C+d^2+2D} + \frac{v}{C+d^2+2D} \right)^2 + (u+v)^2}{C+d^2+2D} \right).$$

3.5 Stable and unstable manifolds corresponding to the saddle period-two solution

If $c < \frac{(1-3d)(d+1)}{4}$ then Eq.(1) has minimal period-two solution $\{P_1(\bar{u}_1, \bar{v}_1), P_2(\bar{v}_1, \bar{u}_1)\}$ which is a saddle point.

In view of the fact that the T^2 is cooperative map if T is cooperative map, we can assume that the stable manifold W_{loc}^s at the period-two solution $(0, 0)$, which corresponding to (\bar{u}_1, \bar{v}_1) , is the graph of the map

$$\varphi_2(\xi) = \alpha_3 \xi^2 + \beta_3 \xi^3, \quad \alpha_3, \beta_3 \in \mathbb{R}.$$

Now we compute the constants α_3 and β_3 . The function φ_2 must satisfy the stable manifold equation

$$\varphi_2(\mu_1 \xi + \tilde{g}_1(\xi, \varphi_2(\xi))) = \mu_2 \varphi_2(\xi) + \tilde{g}_2(\xi, \varphi_2(\xi)),$$

This leads to the following polynomial equation

$$\tilde{p}_1 \xi^2 + \tilde{p}_2 \xi^3 + \cdots + \tilde{p}_{18} \xi^{18} = 0$$

where

$$\begin{aligned} \tilde{p}_1 = & \frac{1}{4} \alpha_3 (-C + d(d+2) + 2)^2 + \frac{1}{2} \alpha_3 (-C - d(d+2) - 2) \\ & + \frac{cd \left((d^2 + 2D)^2 - C^2 \right)}{C(-C + d^2 + 2D)^2} - \frac{c \left((d^2 + 2D)^2 - C^2 \right)}{2C(-C + d^2 + 2D)} - \frac{2cd^3 \left((d^2 + 2D)^2 - C^2 \right)}{C(-C + d^2 + 2D)^3} \end{aligned} \quad (32)$$

and

$$\begin{aligned} \tilde{p}_2 = & \frac{1}{8} \beta_3 (-C^3 + 3C^2(d(d+2) + 2) - C(3d(d+2)(d(d+2) + 4) + 16) \\ & + d(d+2)(d(d+2) + 2)(d(d+2) + 4)) + \frac{4\alpha_3 c(-d(d+2) - 4)((d+2)d^3 + 4(d-1)dD + 4D^2)}{2C(-C + d^2 + 2D)} \\ & + \frac{4\alpha_3 c(C^3 - C^2(d(3d+4) + 4D) + C(3d^4 + 8d^3 + 8d^2(D+1) + 4dD + 4D(D+2)))}{2C(-C + d^2 + 2D)}. \end{aligned} \quad (33)$$

By solving system $\tilde{p}_1 = 0$ and $\tilde{p}_2 = 0$, we obtain the values

$$\alpha_3 = \frac{2c(C + d^2 + 2D)(d^2(4D - 2C) + 2d(C - 2D) + (C - 2D)^2 + d^4 + 2d^3)}{C(C^2 - 2C(d(d+2) + 3) + d(d+2)(d(d+2) + 2))(-C + d^2 + 2D)^2}$$

and

$$\beta_3 = \frac{\alpha_3 \Phi_1(c, d)}{\Phi_2(c, d)},$$

where

$$\begin{aligned} \Phi_1(c, d) = & 4c(-C^3 + C^2(d(3d+4) + 4D) - C(3d^4 + 8d^3 + 8d^2(D+1) + 4dD + 4D(D+2)) \\ & + (d(d+2) + 4)((d+2)d^3 + 4(d-1)dD + 4D^2)), \end{aligned} \quad (34)$$

$$\begin{aligned} \Phi_2(c, d) = & 4C(-C^3 + 3C^2(d(d+2) + 2) - C(3d(d+2)(d(d+2) + 4) + 16) \\ & + d(d+2)(d(d+2) + 2)(d(d+2) + 4))(-C + d^2 + 2D). \end{aligned} \quad (35)$$

Let us assume that the unstable manifold at the period two solution $(0, 0)$, which corresponding to (\bar{u}_1, \bar{v}_1) , is the graph of the map

$$\psi_2(\eta) = \alpha_4 \eta^2 + \beta_4 \eta^3, \quad \alpha_4, \beta_4 \in \mathbb{R}.$$

Now we compute the constants α_4 and β_4 . The function ψ_2 must satisfy the stable manifold equation

$$\psi_2(\mu_2\eta + \tilde{g}_2(\psi_2(\eta), \eta)) = \mu_1\psi_2(\eta) + \tilde{f}_2(\psi_2(\eta), \eta),$$

This leads to the following polynomial equation

$$\tilde{q}_1\eta^2 + \tilde{q}_2\eta^3 + \cdots + \tilde{q}_{18}\eta^{18} = 0$$

where

$$\begin{aligned} \tilde{q}_1 = & \frac{1}{4}A(C + d(d+2) + 2)^2 + \frac{1}{2}A(C - d(d+2) - 2) \\ & - \frac{cd((d^2 + 2D)^2 - C^2)}{C(C + d^2 + 2D)^2} + \frac{c((d^2 + 2D)^2 - C^2)}{2C(C + d^2 + 2D)} + \frac{2cd^3((d^2 + 2D)^2 - C^2)}{C(C + d^2 + 2D)^3} \end{aligned} \quad (36)$$

and

$$\begin{aligned} \tilde{q}_2 = & \frac{1}{8}B(C^3 + 3C^2(d(d+2) + 2) + C(3d(d+2)(d(d+2) + 4) + 16) \\ & + d(d+2)(d(d+2) + 2)(d(d+2) + 4)) + \frac{\alpha_3 c(d(d+2) + 4)((d+2)d^3 + 4(d-1)dD + 4D^2)}{2C(C + d^2 + 2D)} \\ & + \frac{Ac(C^3 + C^2(d(3d+4) + 4D) + C(3d^4 + 8d^3 + 8d^2(D+1) + 4dD + 4D(D+2)))}{2C(C + d^2 + 2D)}. \end{aligned} \quad (37)$$

By solving system $\tilde{q}_1 = 0$ and $\tilde{q}_2 = 0$, we obtain the values

$$\alpha_4 = -\frac{2c(-C + d^2 + 2D)(2d^2(C + 2D) - 2d(C + 2D) + (C + 2D)^2 + d^4 + 2d^3)}{C(C^2 + 2C(d(d+2) + 3) + d(d+2)(d(d+2) + 2))(C + d^2 + 2D)^2}$$

and

$$\beta_4 = \frac{\alpha_4 \tilde{\Phi}_1(c, d)}{\tilde{\Phi}_2(c, d)},$$

where

$$\begin{aligned} \tilde{\Phi}_1(c, d) = & 4c(-C^3 - C^2(d(3d+4) + 4D) \\ & - C(3d^4 + 8d^3 + 8d^2(D+1) + 4dD + 4D(D+2)) + (-d(d+2) - 4) \\ & ((d+2)d^3 + 4(d-1)dD + 4D^2)), \end{aligned} \quad (38)$$

$$\begin{aligned} \tilde{\Phi}_2(c, d) = & C(C^3 + 3C^2(d(d+2) + 2) \\ & + C(3d(d+2)(d(d+2) + 4) + 16) + d(d+2)(d(d+2) + 2)(d(d+2) + 4))(C + d^2 + 2D). \end{aligned} \quad (39)$$

As in the case of the saddle point equilibrium, one can show that we can approximate locally local stable manifold $\mathcal{W}_{loc}^s(\bar{u}_1, \bar{v}_1)$ and local unstable manifold $\mathcal{W}_{loc}^u(\bar{u}_1, \bar{v}_1)$ as the graph of the maps $\tilde{\varphi}_2(u)$ and $\tilde{\psi}_2(v)$ such that $\tilde{S}(u, \tilde{\varphi}_2(u)) = 0$ and $\tilde{U}(\tilde{\psi}_2(v), v) = 0$ hold, where

$$\begin{aligned} \tilde{S}(u, v) := & \alpha_3 \left(\frac{(u - \bar{u}_1)((d^2 + 2D)^2 - C^2)}{4Cd} - \frac{(v - \bar{v}_1)(-C + d^2 + 2D)}{2C} \right)^2 \\ & + \beta_3 \left(\frac{(u - \bar{u}_1)((d^2 + 2D)^2 - C^2)}{4Cd} - \frac{(v - \bar{v}_1)(-C + d^2 + 2D)}{2C} \right)^3 \\ & - \frac{(u - \bar{u}_1)(C^2 - (d^2 + 2D)^2)}{4Cd} - \frac{(v - \bar{v}_1)(C + d^2 + 2D)}{2C}, \end{aligned} \quad (40)$$

$$\begin{aligned}\tilde{U}(u, v) := & \alpha_4 \left(\frac{(u - \bar{u}_1)(C^2 - (d^2 + 2D)^2)}{4Cd} + \frac{(v - \bar{v}_1)(C + d^2 + 2D)}{2C} \right)^2 \\ & + \beta_4 \left(\frac{(u - \bar{u}_1)(C^2 - (d^2 + 2D)^2)}{4Cd} + \frac{(v - \bar{v}_1)(C + d^2 + 2D)}{2C} \right)^3 \\ & - \frac{(u - \bar{u}_1)((d^2 + 2D)^2 - C^2)}{4Cd} + \frac{(v - \bar{v}_1)(-C + d^2 + 2D)}{2C}\end{aligned}\quad (41)$$

Thus we proved the following result

Theorem 9 Consider Eq.(1) subject to the condition $c < \frac{(1-3d)(d+1)}{4}$. Then the local stable and local unstable manifolds of the unique period-two solution are given with the asymptotic expansions (40) and (41) respectively.

3.6 Some numerical examples

For $c = 0.09$ and $d = 0.23$ we have that

$$\begin{aligned}\tilde{S}_1(u, v) = & 0.147835(0.207875(v - 7.64414) - 0.422497(u - 6.02253))^3 \\ & - 0.418625(0.207875(v - 7.64414) - 0.422497(u - 6.02253))^2 \\ & - 0.422497(u - 6.02253) - 0.792125(v - 7.64414)\end{aligned}$$

and

$$\begin{aligned}\tilde{U}_1(u, v) = & -0.00042726(0.422497(u - 6.02253) + 0.792125(v - 7.64414))^3 \\ & + 0.00729364(0.422497(u - 6.02253) + 0.792125(v - 7.64414))^2 \\ & + 0.422497(u - 6.02253) - 0.207875(v - 7.64414).\end{aligned}$$

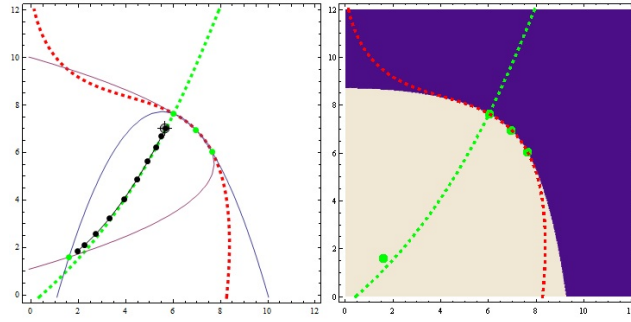


Figure 4: The graphs of the functions $\tilde{S}_1(u, v) = 0$ (red curve) and $\tilde{U}_1(u, v) = 0$ (green curve) for $c = 0.09$ and $d = 0.23$ with the basins of attraction generated by *Dynamica 3*.

For $c = 0.03$ and $d = 0.22$ we have that

$$\begin{aligned}\tilde{S}_2(u, v) = & 0.0912789(0.0236741(v - 29.3826) - 0.125096(u - 11.2841))^3 \\ & - 0.458495(0.0236741(v - 29.3826) - 0.125096(u - 11.2841))^2 \\ & - 0.125096(u - 11.2841) - 0.976326(v - 29.3826)\end{aligned}$$

and

$$\begin{aligned}\tilde{U}_2(u, v) = & 3.7 \times 10^{-6}(0.125096(u - 11.2841) + 0.976326(v - 29.3826))^3 \\ & + 0.000212577(0.125096(u - 11.2841) + 0.976326(v - 29.3826))^2 \\ & + 0.125096(u - 11.2841) - 0.0236741(v - 29.3826)\end{aligned}$$

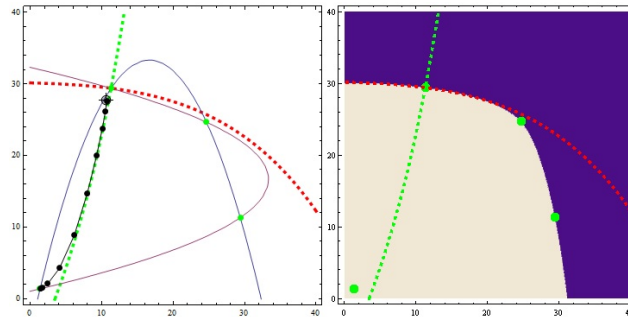


Figure 5: The graphs of the functions $\tilde{S}_2(u, v) = 0$ (red curve) and $\tilde{U}_2(u, v) = 0$ (green curve) for $c = 0.03$ and $d = 0.22$ with the basins of attraction generated by *Dynamica 3*.

Figures 4 and 5 show the graph of the functions $\tilde{S}_2(u, v) = 0$ and $\tilde{U}_2(u, v) = 0$ with the basins of attraction created with *Dynamica 3*. for different values of the parameters c and d .

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A Values of coefficients p_1, p_2, q_1 and q_2

$$\begin{aligned}
p_1 = & -12c\beta_1\bar{x}_2^3d^6 - 12c^2\beta_1\bar{x}_2^4d^5 + 12c\beta_1\bar{x}_2^3d^5 - 4c\alpha_1\bar{x}_2^2d^5 - 12c\beta_1\bar{x}_2^2d^5 - d^5 - 168c^2\beta_1\bar{x}_2^4d^4 \\
& - 4c\alpha_1\bar{x}_2^2d^4 - 168c^3\beta_1\bar{x}_2^5d^3 + 156c^2\beta_1\bar{x}_2^4d^3 - 64c^2\alpha_1\bar{x}_2^3d^3 - 168c^2\beta_1\bar{x}_2^3d^3 - 14c\bar{x}_2d^3 \\
& - 600c^3\beta_1\bar{x}_2^5d^2 + 24c^2\beta_1\bar{x}_2^4d^2 - 64c^2\alpha_1\bar{x}_2^3d^2 - 24c^2\beta_1\bar{x}_2^3d^2 - 588c^4\beta_1\bar{x}_2^6d + 600c^3\beta_1\bar{x}_2^5d \\
& - 256c^3\alpha_1\bar{x}_2^4d - 600c^3\beta_1\bar{x}_2^4d - 12c^2\beta_1\bar{x}_2^4d + 24c^2\beta_1\bar{x}_2^3d - 48c^2\bar{x}_2^2d - 12c^2\beta_1\bar{x}_2^2d - 256c^3\alpha_1\bar{x}_2^4 \\
& + \sqrt{d^2 + 8c\bar{x}_2} (204c^4\beta_1\bar{x}_2^6 - 216c^3\beta_1\bar{x}_2^5 + 120c^3d^2\beta_1\bar{x}_2^5 + 216c^3d\beta_1\bar{x}_2^5 + 144c^3\alpha_1\bar{x}_2^4 + 12c^2d^4\beta_1\bar{x}_2^4 \\
& + 216c^3\beta_1\bar{x}_2^4 + 120c^2d^3\beta_1\bar{x}_2^4 + 12c^2\beta_1\bar{x}_2^4 - 108c^2d^2\beta_1\bar{x}_2^4 - 24c^2d\beta_1\bar{x}_2^4 - 16c^2\alpha_1\bar{x}_2^3 \\
& + 48c^2d^2\alpha_1\bar{x}_2^3 - 16c^2d\alpha_1\bar{x}_2^3 + 12cd^5\beta_1\bar{x}_2^3 - 12cd^4\beta_1\bar{x}_2^3 - 24c^2\beta_1\bar{x}_2^3 + 120c^2d^2\beta_1\bar{x}_2^3 + 24c^2d\beta_1\bar{x}_2^3 \\
& - 16c^2\bar{x}_2^2 + 4cd^4\alpha_1\bar{x}_2^2 - 4cd^3\alpha_1\bar{x}_2^2 + 16c^2\alpha_1\bar{x}_2^2 + 12cd^4\beta_1\bar{x}_2^2 + 12c^2\beta_1\bar{x}_2^2 - 10cd^2\bar{x}_2 - d^4), \quad (42)
\end{aligned}$$

$$\begin{aligned}
p_2 = & -\beta_1\bar{x}_2d^6 + 3\alpha_1\beta_1\bar{x}_2^2d^5 + 6c\alpha_1\beta_1\bar{x}_2^3d^4 - 18c\beta_1\bar{x}_2^2d^4 - 6\alpha_1\beta_1\bar{x}_2^2d^4 + 2\alpha_1^2\bar{x}_2d^4 + 6\alpha_1\beta_1\bar{x}_2d^4 \\
& - \beta_1\bar{x}_2d^4 + 3c^2\alpha_1\beta_1\bar{x}_2^4d^3 + 12c\alpha_1\beta_1\bar{x}_2^3d^3 + 2\alpha_1^2d^3 + 2c\alpha_1^2\bar{x}_2^2d^3 - 6c\beta_1\bar{x}_2^2d^3 + 6c\alpha_1\beta_1\bar{x}_2^2d^3 \\
& + 3\alpha_1\beta_1\bar{x}_2^2d^3 + \alpha_1d^3 + 3\alpha_1\beta_1d^3 - 2\alpha_1^2\bar{x}_2d^3 - 6\alpha_1\beta_1\bar{x}_2d^3 + 36c^2\alpha_1\beta_1\bar{x}_2^4d^2 - 102c^2\beta_1\bar{x}_2^3d^2 \\
& - 36c\alpha_1\beta_1\bar{x}_2^3d^2 + 16c\alpha_1^2\bar{x}_2^2d^2 - 10c\beta_1\bar{x}_2^2d^2 + 36c\alpha_1\beta_1\bar{x}_2^2d^2 - 4c\alpha_1\bar{x}_2d^2 - 6c\beta_1\bar{x}_2d^2 \\
& + 18c^3\alpha_1\beta_1\bar{x}_2^5d - 36c^2\alpha_1\beta_1\bar{x}_2^4d + 16c^2\alpha_1^2\bar{x}_2^3d - 48c^2\beta_1\bar{x}_2^3d + 36c^2\alpha_1\beta_1\bar{x}_2^3d + 18c\alpha_1\beta_1\bar{x}_2^3d \\
& - 16c\alpha_1^2\bar{x}_2^2d - 36c\alpha_1\beta_1\bar{x}_2^2d + 16c\alpha_1^2\bar{x}_2d + 8c\alpha_1\bar{x}_2d + 18c\alpha_1\beta_1\bar{x}_2d - 176c^3\beta_1\bar{x}_2^4 - 16c^2\beta_1\bar{x}_2^3 \\
& - 32c^2\alpha_1\bar{x}_2^2 - 48c^2\beta_1\bar{x}_2^2 + \sqrt{d^2 + 8c\bar{x}_2} (\beta_1\bar{x}_2d^5 - 3\alpha_1\beta_1\bar{x}_2^2d^4 - 6c\alpha_1\beta_1\bar{x}_2^3d^3 + 14c\beta_1\bar{x}_2^2d^3 \\
& + 6\alpha_1\beta_1\bar{x}_2^2d^3 - 2\alpha_1^2\bar{x}_2d^3 - 6\alpha_1\beta_1\bar{x}_2d^3 - \beta_1\bar{x}_2d^3 - 3c^2\alpha_1\beta_1\bar{x}_2^4d^2 - 2\alpha_1^2d^2 - 2c\alpha_1^2\bar{x}_2^2d^2 \\
& + 6c\beta_1\bar{x}_2^2d^2 - 6c\alpha_1\beta_1\bar{x}_2^2d^2 - 3\alpha_1\beta_1\bar{x}_2^2d^2 + \alpha_1d^2 - 3\alpha_1\beta_1d^2 + 2\alpha_1^2\bar{x}_2d^2 + 6\alpha_1\beta_1\bar{x}_2d^2 \\
& - 12c^2\alpha_1\beta_1\bar{x}_2^4d + 54c^2\beta_1\bar{x}_2^3d + 12c\alpha_1\beta_1\bar{x}_2^3d - 8c\alpha_1^2\bar{x}_2^2d - 14c\beta_1\bar{x}_2^2d - 12c\alpha_1\beta_1\bar{x}_2^2d \\
& + 6c\beta_1\bar{x}_2d - 6c^3\alpha_1\beta_1\bar{x}_2^5 + 12c^2\alpha_1\beta_1\bar{x}_2^4 - 8c^2\alpha_1^2\bar{x}_2^3 - 12c^2\alpha_1\beta_1\bar{x}_2^3 - 6c\alpha_1\beta_1\bar{x}_2^3 + 8c\alpha_1^2\bar{x}_2^2 \\
& + 12c\alpha_1\beta_1\bar{x}_2^2 - 8c\alpha_1^2\bar{x}_2 + 8c\alpha_1\bar{x}_2 - 6c\alpha_1\beta_1\bar{x}_2) \quad (43)
\end{aligned}$$

$$\begin{aligned}
q_1 = & 12\beta_2c\bar{x}_2^3d^7 + 12\beta_2c^2\bar{x}_2^4d^6 - 12\beta_2c\bar{x}_2^3d^6 + 4\alpha_2c\bar{x}_2^2d^6 + 12\beta_2c\bar{x}_2^2d^6 - d^6 + 216\beta_2c^2\bar{x}_2^4d^5 - 4\alpha_2c\bar{x}_2^2d^5 \\
& + 216\beta_2c^3\bar{x}_2^5d^4 - 204\beta_2c^2\bar{x}_2^4d^4 + 80\alpha_2c^2\bar{x}_2^3d^4 + 216\beta_2c^2\bar{x}_2^3d^4 - 18c\bar{x}_2d^4 + 1176\beta_2c^3\bar{x}_2^5d^3 \\
& - 24\beta_2c^2\bar{x}_2^4d^3 - 48\alpha_2c^2\bar{x}_2^3d^3 + 24\beta_2c^2\bar{x}_2^3d^3 + 1164\beta_2c^4\bar{x}_2^6d^2 - 1080\beta_2c^3\bar{x}_2^5d^2 + 528\alpha_2c^3\bar{x}_2^4d^2 \\
& + 1176\beta_2c^3\bar{x}_2^4d^2 + 12\beta_2c^2\bar{x}_2^4d^2 - 16\alpha_2c^2\bar{x}_2^3d^2 - 24\beta_2c^2\bar{x}_2^3d^2 + 16\alpha_2c^2\bar{x}_2^2d^2 + 12\beta_2c^2\bar{x}_2^2d^2 \\
& - 96c^2\bar{x}_2^2d^2 + 1728\beta_2c^4\bar{x}_2^6d - 192\beta_2c^3\bar{x}_2^5d - 128\alpha_2c^3\bar{x}_2^4d + 192\beta_2c^3\bar{x}_2^4d + 1632\beta_2c^5\bar{x}_2^7 \\
& - 1728\beta_2c^4\bar{x}_2^6 + 1152\alpha_2c^4\bar{x}_2^5 + 1728\beta_2c^4\bar{x}_2^5 + 96\beta_2c^3\bar{x}_2^5 - 128\alpha_2c^3\bar{x}_2^4 - 192\beta_2c^3\bar{x}_2^4 + 128\alpha_2c^3\bar{x}_2^3 \\
& + 96\beta_2c^3\bar{x}_2^3 - 128c^3\bar{x}_2^3 + \sqrt{d^2 + 8c\bar{x}_2} (12\beta_2c\bar{x}_2^3d^6 + 12\beta_2c^2\bar{x}_2^4d^5 - 12\beta_2c\bar{x}_2^3d^5 + 4\alpha_2c\bar{x}_2^2d^5 \\
& + 12\beta_2c\bar{x}_2^2d^5 + d^5 + 168\beta_2c^2\bar{x}_2^4d^4 + 4\alpha_2c\bar{x}_2^2d^4 + 168\beta_2c^3\bar{x}_2^5d^3 - 156\beta_2c^2\bar{x}_2^4d^3 + 64\alpha_2c^2\bar{x}_2^3d^3 \\
& + 168\beta_2c^2\bar{x}_2^3d^3 + 14c\bar{x}_2d^3 + 600\beta_2c^3\bar{x}_2^5d^2 - 24\beta_2c^2\bar{x}_2^4d^2 + 64\alpha_2c^2\bar{x}_2^3d^2 + 24\beta_2c^2\bar{x}_2^3d^2 \\
& + 588\beta_2c^4\bar{x}_2^6d - 600\beta_2c^3\bar{x}_2^5d + 256\alpha_2c^3\bar{x}_2^4d + 600\beta_2c^3\bar{x}_2^4d + 12\beta_2c^2\bar{x}_2^4d - 24\beta_2c^2\bar{x}_2^3d \\
& + 12\beta_2c^2\bar{x}_2^2d + 48c^2\bar{x}_2^2d + 256\alpha_2c^3\bar{x}_2^4), \quad (44)
\end{aligned}$$

$$\begin{aligned}
q_2 = & \beta_2 \bar{x}_2 d^6 - 3\alpha_2 \beta_2 \bar{x}_2^2 d^5 - 6c\alpha_2 \beta_2 \bar{x}_2^3 d^4 + 18c\beta_2 \bar{x}_2^2 d^4 + 6\alpha_2 \beta_2 \bar{x}_2^2 d^4 - 2\alpha_2^2 \bar{x}_2 d^4 - 6\alpha_2 \beta_2 \bar{x}_2 d^4 \\
& + \beta_2 \bar{x}_2 d^4 - 3c^2 \alpha_2 \beta_2 \bar{x}_2^4 d^3 - 12c\alpha_2 \beta_2 \bar{x}_2^3 d^3 - 2\alpha_2^2 d^3 - 2c\alpha_2^2 \bar{x}_2^2 d^3 + 6c\beta_2 \bar{x}_2^2 d^3 - 6c\alpha_2 \beta_2 \bar{x}_2^2 d^3 \\
& - 3\alpha_2 \beta_2 \bar{x}_2^2 d^3 - \alpha_2 d^3 - 3\alpha_2 \beta_2 d^3 + 2\alpha_2^2 \bar{x}_2 d^3 + 6\alpha_2 \beta_2 \bar{x}_2 d^3 - 36c^2 \alpha_2 \beta_2 \bar{x}_2^4 d^2 + 102c^2 \beta_2 \bar{x}_2^3 d^2 \\
& + 36c\alpha_2 \beta_2 \bar{x}_2^3 d^2 - 16c\alpha_2^2 \bar{x}_2^2 d^2 + 10c\beta_2 \bar{x}_2^2 d^2 - 36c\alpha_2 \beta_2 \bar{x}_2^2 d^2 + 4c\alpha_2 \bar{x}_2 d^2 + 6c\beta_2 \bar{x}_2 d^2 \\
& - 18c^3 \alpha_2 \beta_2 \bar{x}_2^5 d + 36c^2 \alpha_2 \beta_2 \bar{x}_2^4 d - 16c^2 \alpha_2^2 \bar{x}_2^3 d + 48c^2 \beta_2 \bar{x}_2^3 d - 36c^2 \alpha_2 \beta_2 \bar{x}_2^3 d - 18c\alpha_2 \beta_2 \bar{x}_2^3 d \\
& + 16c\alpha_2^2 \bar{x}_2^2 d + 36c\alpha_2 \beta_2 \bar{x}_2^2 d - 16c\alpha_2^2 \bar{x}_2 d - 8c\alpha_2 \bar{x}_2 d - 18c\alpha_2 \beta_2 \bar{x}_2 d + 176c^3 \beta_2 \bar{x}_2^4 + 16c^2 \beta_2 \bar{x}_2^3 \\
& + 32c^2 \alpha_2 \bar{x}_2^2 + 48c^2 \beta_2 \bar{x}_2^2 + \sqrt{d^2 + 8c\bar{x}_2} (\beta_2 \bar{x}_2 d^5 - 3\alpha_2 \beta_2 \bar{x}_2^2 d^4 - 6c\alpha_2 \beta_2 \bar{x}_2^3 d^3 + 14c\beta_2 \bar{x}_2^2 d^3 \\
& + 6\alpha_2 \beta_2 \bar{x}_2^2 d^3 - 2\alpha_2^2 \bar{x}_2 d^3 - 6\alpha_2 \beta_2 \bar{x}_2 d^3 - \beta_2 \bar{x}_2 d^3 - 3c^2 \alpha_2 \beta_2 \bar{x}_2^4 d^2 - 2\alpha_2^2 d^2 - 2c\alpha_2^2 \bar{x}_2^2 d^2 \\
& + 6c\beta_2 \bar{x}_2^2 d^2 - 6c\alpha_2 \beta_2 \bar{x}_2^2 d^2 - 3\alpha_2 \beta_2 \bar{x}_2^2 d^2 + \alpha_2 d^2 - 3\alpha_2 \beta_2 d^2 + 2\alpha_2^2 \bar{x}_2 d^2 + 6\alpha_2 \beta_2 \bar{x}_2 d^2 \\
& - 12c^2 \alpha_2 \beta_2 \bar{x}_2^4 d + 54c^2 \beta_2 \bar{x}_2^3 d + 12c\alpha_2 \beta_2 \bar{x}_2^3 d - 8c\alpha_2^2 \bar{x}_2^2 d - 14c\beta_2 \bar{x}_2^2 d - 12c\alpha_2 \beta_2 \bar{x}_2^2 d \\
& + 6c\beta_2 \bar{x}_2 d - 6c^3 \alpha_2 \beta_2 \bar{x}_2^5 + 12c^2 \alpha_2 \beta_2 \bar{x}_2^4 - 8c^2 \alpha_2^2 \bar{x}_2^3 - 12c^2 \alpha_2 \beta_2 \bar{x}_2^3 - 6c\alpha_2 \beta_2 \bar{x}_2^3 + 8c\alpha_2^2 \bar{x}_2^2 \\
& + 12c\alpha_2 \beta_2 \bar{x}_2^2 - 8c\alpha_2^2 \bar{x}_2 + 8c\alpha_2 \bar{x}_2 - 6c\alpha_2 \beta_2 \bar{x}_2)
\end{aligned} \tag{45}$$

Fractional differential equations with integral and ordinary-fractional flux boundary conditions

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Abstract

We investigate the existence of solutions for a coupled system of fractional differential equations with integral and ordinary-fractional flux boundary conditions. The existence results are derived via Schauder's fixed point theorem and Leray-Schauder's alternative, while the uniqueness of solutions is established by applying Banach's contraction principle. Several new results appear as a special case of the present work with appropriate choice of the parameters involved in the problem at hand.

Key words and phrases: Fractional differential systems; nonlocal boundary conditions; integral boundary conditions; fixed point theorem.

AMS (MOS) Subject Classifications: 34A08, 34B15.

1 Introduction

In this paper, we study a coupled system of Caputo type fractional differential equations:

$$\begin{cases} {}^c D^\alpha x(t) = f(t, x(t), y(t)), & t \in [0, 1], \quad 1 < \alpha \leq 2 \\ {}^c D^\beta y(t) = h(t, x(t), y(t)), & t \in [0, 1], \quad 1 < \beta \leq 2, \end{cases} \quad (1)$$

supplemented with integral and ordinary-fractional flux boundary conditions:

$$\begin{cases} x(0) + x(1) = a \int_0^1 x(s) ds, & x'(0) = b {}^c D^\gamma x(1), \quad 0 < \gamma \leq 1, \\ y(0) + y(1) = a_1 \int_0^1 y(s) ds, & y'(0) = b_1 {}^c D^\delta y(1), \quad 0 < \delta \leq 1, \end{cases} \quad (2)$$

where ${}^c D^\alpha, {}^c D^\beta$ denote the Caputo fractional derivatives of orders α and β respectively, $f, h : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are given continuous functions and a, b, a_1, b_1 are real constants.

Fractional differential equations arise in the mathematical modeling of systems and processes occurring in many engineering and scientific disciplines such as biophysics, blood flow phenomena, control theory, aerodynamics, electrodynamics of complex medium, polymer rheology, signal and image processing to name a few [1]-[4]. The popularity of fractional order operators owes to their ability to describe the hereditary properties of various materials and processes. With this distinguished capability, fractional order models have become more realistic and practical than the corresponding classical integer order models. For some recent development on the topic, see [5]-[15] and the references therein. The investigation of coupled systems of fractional order differential equations is also very significant as such systems appear in a variety of problems of applied nature, especially in biosciences. For details and examples, the reader is referred to the papers [16]-[23] and the references cited therein.

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The paper is organized as follows. In Section 2, we recall some basic definitions of fractional calculus and present some auxiliary lemmas. The main results are presented in Section 3. We give two existence results relying on Leray-Schauder's alternative and Schauder's fixed point theorem, while the uniqueness result is established by means of Banach's contraction mapping principle. It is worthwhile to note that our results are not only new in the present configuration but also correspond to some new special results for different values of the parameters involved in the given problem.

2 Preliminaries

Before presenting an auxiliary lemma, we recall some basic definitions of fractional calculus [1, 2].

Definition 2.1 For $(n-1)$ -times absolutely continuous function $g : [0, \infty) \rightarrow \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^c D^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s) ds, \quad n-1 < q < n, n = [q] + 1,$$

where $[q]$ denotes the integer part of the real number q .

Definition 2.2 The Riemann-Liouville fractional integral of order q is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0,$$

provided the integral exists.

Lemma 2.3 (see [1], [2]) (i) If $\alpha > 0, \beta > 0, \beta > \alpha, f \in L(0, 1)$ then

$$I^\alpha I^\beta f(t) = I^{\alpha+\beta} f(t), \quad D^\alpha I^\alpha f(t) = f(t), \quad D^\alpha I^\beta f(t) = I^{\beta-\alpha} f(t).$$

$$(ii) \quad {}^c D^\alpha t^{\lambda-1} = \frac{\Gamma(\lambda)}{\Gamma(\lambda-\alpha)} t^{\lambda-\alpha-1}, \quad \lambda > [\alpha] \quad \text{and} \quad {}^c D^\alpha t^{\lambda-1} = 0, \quad \lambda < [\alpha].$$

To define the solution for the problem (1)-(2), we use the following lemma.

Lemma 2.4 Let $a \neq 2$ and $\Gamma(2-\beta) \neq b$. For $\phi \in C([0, 1], \mathbb{R})$, the integral solution of the linear problem

$$\begin{cases} {}^c D^\alpha x(t) = \phi(t), & t \in [0, 1], \quad 1 < \alpha \leq 2, \\ x(0) + x(1) = a \int_0^1 x(s) ds, \quad x'(0) = b {}^c D^\gamma x(1), \quad 0 < \gamma \leq 1, \end{cases} \quad (3)$$

is given by

$$\begin{aligned} x(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(s) ds + \frac{b(2t-1)\Gamma(2-\gamma)}{2(\Gamma(2-\gamma)-b)} \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} \phi(s) ds \\ & - \frac{1}{2-a} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(s) ds + \frac{a}{2-a} \int_0^1 \frac{(1-s)^\alpha}{\Gamma(\alpha+1)} \phi(s) ds. \end{aligned} \quad (4)$$

Proof. As argued in [2], the general solution of the fractional differential equation in (3) can be written as

$$x(t) = c_0 + c_1 t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(s) ds, \quad (5)$$

where $c_0, c_1 \in \mathbb{R}$ are arbitrary constants.

Using the boundary condition $x'(0) = b {}^c D^\gamma x(1)$ in (5), we find that

$$c_1 = \frac{b\Gamma(2-\beta)}{\Gamma(2-\beta)-b} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \phi(s) ds.$$

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In view of the condition $x(0) + x(1) = a \int_0^1 x(s)ds$, (5) yields

$$2c_0 + c_1 + \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(s)ds = a \int_0^1 \int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \phi(u)du + \frac{ac_1}{2} + ac_0,$$

which, on inserting the value of c_1 and using the composition law of Riemann-Liouville integration, gives

$$\begin{aligned} c_0 = & -\frac{1}{2} \frac{b\Gamma(2-\beta)}{[\Gamma(2-\beta)-b]} \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \phi(s)ds \\ & + \frac{a}{2-a} \int_0^1 \frac{(1-s)^\alpha}{\Gamma(\alpha+1)} \phi(s)ds - \frac{1}{2-a} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(s)ds. \end{aligned}$$

Substituting the values of c_0, c_1 in (5) yields (4). This completes the proof. \square

3 Main Results

Let us introduce the space $X_i = \{u_i(t) | u_i(t) \in C([0, 1])\}$ endowed with the norm $\|u_i\| = \sup\{|u_i(t)|, t \in [0, 1]\}$, $i = 1, 2$. Obviously $(X_i, \|\cdot\|)$ is a Banach space. In consequence, the product space $(X_1 \times X_2, \|(u_1, u_2)\|)$ is also a Banach space with norm $\|(u_1, u_2)\| = \|u_1\| + \|u_2\|$.

In view of Lemma 2.4, we define an operator $T : X_1 \times X_2 \rightarrow X_1 \times X_2$ by

$$T(u, v)(t) = \begin{pmatrix} T_1(u, v)(t) \\ T_2(u, v)(t) \end{pmatrix},$$

where

$$\begin{aligned} T_1(u, v)(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s), v(s))ds + \frac{b(2t-1)\Gamma(2-\gamma)}{2(\Gamma(2-\gamma)-b)} \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} f(s, u(s), v(s))ds \\ & + \frac{1}{2-a} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s), v(s))ds - \frac{a}{2-a} \int_0^1 \frac{(1-s)^\alpha}{\Gamma(\alpha+1)} f(s, u(s), v(s))ds, \end{aligned}$$

and

$$\begin{aligned} T_2(u, v)(t) = & \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} h(s, u(s), v(s))ds + \frac{b_1(2t-1)\Gamma(2-\delta)}{2(\Gamma(2-\delta)-b_1)} \int_0^1 \frac{(1-s)^{\beta-\delta-1}}{\Gamma(\beta-\delta)} h(s, u(s), v(s))ds \\ & + \frac{1}{2-a_1} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} h(s, u(s), v(s))ds - \frac{a_1}{2-a_1} \int_0^1 \frac{(1-s)^\beta}{\Gamma(\beta+1)} h(s, u(s), v(s))ds. \end{aligned}$$

For the sake of convenience, let us set

$$M_1 = \frac{1+|2-a|}{|2-a|\Gamma(\alpha+1)} + \frac{|b|\Gamma(2-\gamma)}{2|\Gamma(2-\gamma)-b|\Gamma(\alpha-\gamma+1)} + \frac{|a|}{|2-a|\Gamma(\alpha+2)}, \quad (6)$$

$$M_2 = \frac{1+|2-a_1|}{|2-a_1|\Gamma(\beta+1)} + \frac{|b_1|\Gamma(2-\delta)}{2|\Gamma(2-\delta)-b_1|\Gamma(\beta-\delta+1)} + \frac{|a_1|}{|2-a_1|\Gamma(\beta+2)}. \quad (7)$$

We need the following known theorems in the sequel.

Lemma 3.1 (Schauder's fixed point theorem) [24]. Let U be a closed, convex and nonempty subset of a Banach space X . Let $P : U \rightarrow U$ be a continuous mapping such that $P(U)$ is a relatively compact subset of X . Then P has at least one fixed point in U .

Lemma 3.2 (Leray-Schauder alternative) ([24] p. 4.) Let $F : E \rightarrow E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in E is compact). Let

$$\mathcal{E}(F) = \{x \in E : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}.$$

Then either the set $\mathcal{E}(F)$ is unbounded, or F has at least one fixed point.

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3.1 Existence results

Here we study the existence of solutions for the system (1)-(2) by means of Schauder's fixed point theorem and Leray-Schauder alternative.

Theorem 3.3 Assume that there exist positive constants $c_i, d_i, e_i \in (0, \infty)$, $i = 1, 2$ such that the following condition holds:

$$(H_1) \quad |f(t, x, y)| \leq c_1|x|^{\rho_1} + d_1|y|^{\sigma_1} + e_1, \text{ and} \\ |h(t, x, y)| \leq c_2|x|^{\rho_2} + d_2|y|^{\sigma_2} + e_2, \quad 0 < \rho_i, \sigma_i < 1, \quad i = 1, 2.$$

Then the system (1)-(2) has at least one solution on $[0, 1]$.

Proof. Define a ball in Banach space $X_1 \times X_2$ as $B_R = \{(u, v) : (u, v) \in X_1 \times X_2, \|(u, v)\| \leq R\}$, where

$$R \geq \max\{(6M_i c_i)^{\frac{1}{1-\rho_i}}, (6M_i d_i)^{\frac{1}{1-\sigma_i}}, 6M_i e_i, \}, \quad i = 1, 2. \quad (8)$$

Obviously B_R is a closed, bounded and convex subset of the Banach space $X_1 \times X_2$. In the first step, we show that $T : B_R \rightarrow B_R$. For $(u, v) \in B_R$. For that we have

$$\begin{aligned} |T_1(u, v)(t)| &\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, u(s), v(s))| ds \\ &\quad + \frac{|b(2t-1)|\Gamma(2-\gamma)}{2|\Gamma(2-\gamma)-b|} \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} |f(s, u(s), v(s))| ds \\ &\quad + \frac{1}{|2-a|} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, u(s), v(s))| ds + \frac{|a|}{|2-a|} \int_0^1 \frac{(1-s)^\alpha}{\Gamma(\alpha+1)} |f(s, u(s), v(s))| ds \\ &\leq (c_1 R^{\rho_1} + d_1 R^{\sigma_1} + e_1) \left\{ \frac{1}{\Gamma(\alpha+1)} + \frac{|b|\Gamma(2-\gamma)}{2|\Gamma(2-\gamma)-b|\Gamma(\alpha-\gamma+1)} \right. \\ &\quad \left. + \frac{1}{|2-a|\Gamma(\alpha+1)} + \frac{|a|}{|2-a|\Gamma(\alpha+2)} \right\}, \end{aligned}$$

which implies that

$$\|T_1(u, v)\| \leq M_1(c_1 R^{\rho_1} + d_1 R^{\sigma_1} + e_1) \leq \frac{R}{6} + \frac{R}{6} + \frac{R}{6} = \frac{R}{2}.$$

Similarly, we can obtain

$$\|T_2(u, v)\| \leq M_2(c_2 R^{\rho_2} + d_2 R^{\sigma_2} + e_2) \leq \frac{R}{6} + \frac{R}{6} + \frac{R}{6} = \frac{R}{2}.$$

Clearly

$$\|T(u, v)\| = \|T_1(u, v)\| + \|T_2(u, v)\| \leq R,$$

and in consequence we get $T : B_R \rightarrow B_R$.

Observe that continuity of f, h implies that T is continuous. Next, we shall show that for every bounded subset B_R of $X_1 \times X_2$ the family $F(B_R)$ is equicontinuous. Since f, g are continuous, we can assume that $|f(t, u(t), v(t))| \leq N_1$ and $|h(t, u(t), v(t))| \leq N_2$ for any $u, v \in B_R$ and $t \in [0, 1]$.

Now let $0 \leq t_1 < t_2 \leq 1$. Then we have

$$\begin{aligned} &|T_1(u, v)(t_2) - T_1(u, v)(t_1)| \\ &\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} f(s, u(s), v(s)) ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} f(s, u(s), v(s)) ds \right| \\ &\quad + \frac{2|b|\Gamma(2-\gamma)|t_2-t_1|}{2|\Gamma(2-\gamma)-b|} \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} |f(s, u(s), v(s))| ds \end{aligned}$$

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$$\begin{aligned}
&\leq \frac{N_1}{\Gamma(\alpha)} \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] ds + \frac{N_1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds \\
&\quad + \frac{2N_1|b|\Gamma(2-\gamma)|t_2 - t_1|}{2|\Gamma(2-\gamma) - b|} \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} ds \\
&\leq \frac{N_1}{\Gamma(\alpha+1)} (t_2^\alpha - t_1^\alpha) + \frac{2N_1|b|\Gamma(2-\gamma)|t_2 - t_1|}{2|\Gamma(2-\gamma) - b|\Gamma(\alpha-\gamma+1)}.
\end{aligned}$$

Analogously, we can have

$$|T_2(u, v)(t_2) - T_2(u, v)(t_1)| \leq \frac{N_2}{\Gamma(\beta+1)} (t_2^\beta - t_1^\beta) + \frac{2N_2|b|\Gamma(2-\delta)|t_2 - t_1|}{2|\Gamma(2-\delta) - b|\Gamma(\beta-\delta+1)}.$$

So

$$\|T_1(u, v)(t_2) - T_1(u, v)(t_1)\| \rightarrow 0, \quad \|T_2(u, v)(t_2) - T_2(u, v)(t_1)\| \rightarrow 0, \quad \text{as } t_1 \rightarrow t_2.$$

Therefore it follows that the operator $T : B_R \rightarrow B_R$ is equicontinuous and uniformly bounded. Hence, by Arzelà-Ascoli theorem, T is completely continuous operator. Thus all the conditions of Theorem 3.1 are satisfied, which in turn, implies that the problem (1) has at least one solution. This completes the proof. \square

Remark 3.4 For $\rho_i, \sigma_i > 1$ ($i = 1, 2$) in the condition (H_1) , the conclusion of Theorem 3.6 remains true with a modified value of R given by (8).

Theorem 3.5 Assume that:

(H_2) There exist real constants $k_i, \lambda_i \geq 0$ ($i = 1, 2$) and $k_0 > 0, \lambda_0 > 0$ such that $\forall x_i \in \mathbb{R}, i = 1, 2$, we have

$$|f(t, x_1, x_2)| \leq k_0 + k_1|x_1| + k_2|x_2|, \quad |h(t, x_1, x_2)| \leq \lambda_0 + \lambda_1|x_1| + \lambda_2|x_2|.$$

Then the system (1)-(2) has at least one solution, provided

$$M_1k_1 + M_2\lambda_1 < 1 \quad \text{and} \quad M_1k_2 + M_2\lambda_2 < 1,$$

where M_1 and M_2 are given by (6) and (7) respectively.

Proof. First we show that the operator $T : X_1 \times X_2 \rightarrow X_1 \times X_2$ is completely continuous. By continuity of functions f and h , the operator T is continuous.

Let $\Omega \subset X_1 \times X_2$ be bounded. Then there exist positive constants L_1 and L_2 such that

$$|f(t, u(t), v(t))| \leq L_1, \quad |h(t, u(t), v(t))| \leq L_2, \quad \forall (u, v) \in \Omega.$$

Then for any $(u, v) \in \Omega$, we have

$$\begin{aligned}
|T_1(u, v)(t)| &\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, u(s), v(s))| ds \\
&\quad + \frac{|b(2t-1)|\Gamma(2-\gamma)}{2|\Gamma(2-\gamma) - b|} \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} |f(s, u(s), v(s))| ds \\
&\quad + \frac{1}{|2-a|} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, u(s), v(s))| ds + \frac{|a|}{|2-a|} \int_0^1 \frac{(1-s)^\alpha}{\Gamma(\alpha+1)} |f(s, u(s), v(s))| ds \\
&\leq L_1 \left\{ \frac{1}{\Gamma(\alpha+1)} + \frac{|b|\Gamma(2-\gamma)}{2|\Gamma(2-\gamma) - b|\Gamma(\alpha-\gamma+1)} + \frac{1}{|2-a|\Gamma(\alpha+1)} + \frac{|a|}{|2-a|\Gamma(\alpha+2)} \right\},
\end{aligned}$$

which implies that

$$\|T_1(u, v)\| \leq L_1 \left\{ \frac{1}{\Gamma(\alpha+1)} + \frac{|b|\Gamma(2-\gamma)}{2|\Gamma(2-\gamma) - b|\Gamma(\alpha-\gamma+1)} \right\}$$

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$$+\frac{1}{|2-a|\Gamma(\alpha+1)}+\frac{|a|}{|2-a|\Gamma(\alpha+2)}\Big\}=L_1M_1.$$

Similarly, we can get

$$\begin{aligned}\|T_2(u, v)\| \leq & L_2\left\{\frac{1}{\Gamma(\beta+1)}+\frac{|b_1|\Gamma(2-\delta)}{2|\Gamma(2-\delta)-b_1|\Gamma(\beta-\delta+1)}\right. \\ & \left.+\frac{1}{|2-a_1|\Gamma(\beta+1)}+\frac{|a_1|}{|2-a_1|\Gamma(\beta+2)}\right\}=L_2M_2.\end{aligned}$$

Thus, it follows from the above inequalities that the operator T is uniformly bounded.

Next, we show that T is equicontinuous. Let $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$. Then we have

$$\begin{aligned}& |T_1(u(t_2), v(t_2)) - T_1(u(t_1), v(t_1))| \\ \leq & L_1\left|\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}ds\right| \\ & + L_1\frac{2|b|\Gamma(2-\gamma)|t_2-t_1|}{2|\Gamma(2-\gamma)-b|}\int_0^1\frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}ds \\ \leq & \frac{L_1}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]ds + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds \\ & + \frac{2|b|L_1\Gamma(2-\gamma)|t_2-t_1|}{2|\Gamma(2-\gamma)-b|}\int_0^1\frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)}ds \\ \leq & \frac{L_1}{\Gamma(\alpha+1)}(t_2^\alpha - t_1^\alpha) + \frac{2L_1|b|\Gamma(2-\gamma)|t_2-t_1|}{2|\Gamma(2-\gamma)-b|\Gamma(\alpha-\gamma+1)}.\end{aligned}$$

Analogously, we can obtain

$$|T_2(u(t_2), v(t_2)) - T_2(u(t_1), v(t_1))| \leq \frac{L_2}{\Gamma(\alpha+1)}(t_2^\alpha - t_1^\alpha) + \frac{2L_2|b_1|\Gamma(2-\delta)|t_2-t_1|}{2|\Gamma(2-\delta)-b_1|\Gamma(\beta-\delta+1)}.$$

Therefore, the operator $T(u, v)$ is equicontinuous, and thus the operator $T(u, v)$ is completely continuous.

Finally, it will be verified that the set $\mathcal{E} = \{(u, v) \in X_1 \times X_2 | (u, v) = \lambda T(u, v), 0 \leq \lambda \leq 1\}$ is bounded. Let $(u, v) \in \mathcal{E}$, then $(u, v) = \lambda T(u, v)$. For any $t \in [0, 1]$, we have

$$u(t) = \lambda T_1(u, v)(t), \quad v(t) = \lambda T_2(u, v)(t).$$

Then

$$|u(t)| \leq \left\{\frac{1+|2-a|}{|2-a|\Gamma(\alpha+1)}+\frac{|b|\Gamma(2-\gamma)}{2|\Gamma(2-\gamma)-b|\Gamma(\alpha-\gamma+1)}+\frac{|a|}{|2-a|\Gamma(\alpha+2)}\right\}(k_0+k_1\|u\|+k_2\|v\|),$$

and

$$|v(t)| \leq \left\{\frac{1+|2-a_1|}{|2-a_1|\Gamma(\alpha+1)}+\frac{|b|\Gamma(2-\delta)}{2|\Gamma(2-\delta)-b_1|\Gamma(\beta-\delta+1)}+\frac{|a_1|}{|2-a_1|\Gamma(\beta+2)}\right\}(\lambda_0+\lambda_1\|u\|+\lambda_2\|v\|).$$

Hence we have

$$\|u\| \leq M_1(k_0+k_1\|u\|+k_2\|v\|), \quad \|v\| \leq M_2(\lambda_0+\lambda_1\|u\|+\lambda_2\|v\|),$$

which imply that

$$\|u\| + \|v\| = (M_1k_0 + M_2\lambda_0) + (M_1k_1 + M_2\lambda_1)\|u\| + (M_1k_2 + M_2\lambda_2)\|v\|.$$

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Consequently,

$$\|(u, v)\| \leq \frac{M_1 k_0 + M_2 \lambda_0}{M_0},$$

for any $t \in [0, 1]$, where

$$M_0 = \min\{1 - (M_1 k_1 + M_2 \lambda_1), 1 - (M_1 k_2 + M_2 \lambda_2)\}, \quad k_i, \lambda_i \geq 0 \quad (i = 1, 2).$$

This shows that \mathcal{E} is bounded. Thus, by Lemma 3.2, the operator T has at least one fixed point. Hence the problem (1)-(2) has at least one solution. This completes the proof. \square

3.2 Uniqueness of solutions

In this subsection, we prove the uniqueness of solutions for the system (1)-(2) via Banach's contraction mapping principle.

Theorem 3.6 Assume that

(H₃) $f, h : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous functions and there exist constants m_i, n_i ($i = 1, 2$) such that for all $t \in [0, 1]$ and $u_i, v_i \in \mathbb{R}, i = 1, 2$,

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq m_1 |u_1 - v_1| + m_2 |u_2 - v_2|$$

and

$$|h(t, u_1, u_2) - h(t, v_1, v_2)| \leq n_1 |u_1 - v_1| + n_2 |u_2 - v_2|.$$

Then the system (1)-(2) has a unique solution if $M_1(m_1 + m_2) + M_2(n_1 + n_2) < 1$, where M_1 and M_2 are given by (6) and (7) respectively.

Proof. Let us fix $\sup_{t \in [0, 1]} f(t, 0, 0) = \zeta_1 < \infty$ and $\sup_{t \in [0, 1]} h(t, 0, 0) = \zeta_2 < \infty$ such that

$$r \geq \frac{\zeta_1 M_1 + \zeta_2 M_2}{1 - M_1(m_1 + m_2) - M_2(n_1 + n_2)}.$$

As a first step, we show that $TB_r \subset B_r$, where $B_r = \{(u, v) \in X \times Y : \|(u, v)\| \leq r\}$. For $(u, v) \in B_r$, we have

$$\begin{aligned} |T_1(u, v)(t)| &\leq \sup_{t \in [0, 1]} \left\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, u(s), v(s))| ds \right. \\ &\quad + \frac{|b(2t-1)|\Gamma(2-\gamma)}{2|\Gamma(2-\gamma)-b|} \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} |f(s, u(s), v(s))| ds \\ &\quad + \frac{1}{|2-a|} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, u(s), v(s))| ds + \frac{|a|}{|2-a|} \int_0^1 \frac{(1-s)^\alpha}{\Gamma(\alpha+1)} |f(s, u(s), v(s))| ds \Big\} \\ &\leq \sup_{t \in [0, 1]} \left\{ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} (|f(s, u(s), v(s)) - f(s, 0, 0)| + |f(s, 0, 0)|) ds \right. \\ &\quad + \frac{|b|\Gamma(2-\gamma)}{2|\Gamma(2-\gamma)-b|} \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} (|f(s, u(s), v(s)) - f(s, 0, 0)| + |f(s, 0, 0)|) ds \\ &\quad + \frac{1}{|2-a|} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} (|f(s, u(s), v(s)) - f(s, 0, 0)| + |f(s, 0, 0)|) ds \\ &\quad + \frac{|a|}{|2-a|} \int_0^1 \frac{(1-s)^\alpha}{\Gamma(\alpha+1)} (|f(s, u(s), v(s)) - f(s, 0, 0)| + |f(s, 0, 0)|) ds \Big\} \\ &\leq \left\{ \frac{1}{\Gamma(\alpha+1)} + \frac{|b|\Gamma(2-\gamma)}{2|\Gamma(2-\gamma)-b|\Gamma(\alpha-\gamma+1)} \right. \end{aligned}$$

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$$\begin{aligned}
 & + \frac{1}{|2-a|\Gamma(\alpha+1)} + \frac{|a|}{|2-a|\Gamma(\alpha+2)} \Big\} (m_1\|u\| + m_2\|v\| + \zeta_1) \\
 = & M_1[(m_1 + m_2)r + \zeta_1].
 \end{aligned}$$

Hence

$$\|T_1(u, v)(t)\| \leq M_1[(m_1 + m_2)r + \zeta_1].$$

In the same way, we can obtain that

$$\|T_2(u, v)(t)\| \leq M_2[(n_1 + n_2)r + \zeta_2].$$

Consequently, it follows that $\|T(u, v)(t)\| \leq r$.

Now for $(u_2, v_2), (u_1, v_1) \in X_1 \times X_2$, and for any $t \in [0, 1]$, we get

$$\begin{aligned}
 & |T_1(u_2, v_2)(t) - T_1(u_1, v_1)(t)| \\
 \leq & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, u_2(s), v_2(s)) - f(s, u_1(s), v_1(s))| ds \\
 & + \frac{|b(2t-1)|\Gamma(2-\gamma)}{2|\Gamma(2-\gamma)-b|} \int_0^1 \frac{(1-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} |f(s, u_2(s), v_2(s)) - f(s, u_1(s), v_1(s))| ds \\
 & + \frac{1}{|2-a|} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, u_2(s), v_2(s)) - f(s, u_1(s), v_1(s))| ds \\
 & + \frac{|a|}{|2-a|} \int_0^1 \frac{(1-s)^\alpha}{\Gamma(\alpha+1)} |f(s, u_2(s), v_2(s)) - f(s, u_1(s), v_1(s))| ds \\
 \leq & \left\{ \frac{1}{\Gamma(\alpha+1)} + \frac{|b|\Gamma(2-\gamma)}{2|\Gamma(2-\gamma)-b|\Gamma(\alpha-\gamma+1)} \right. \\
 & \left. + \frac{1}{|2-a|\Gamma(\alpha+1)} + \frac{|a|}{|2-a|\Gamma(\alpha+2)} \right\} (m_1\|u_2 - u_1\| + m_2\|v_2 - v_1\|) \\
 = & M_1(m_1\|u_2 - u_1\| + m_2\|v_2 - v_1\|) \\
 \leq & M_1(m_1 + m_2)(\|u_2 - u_1\| + \|v_2 - v_1\|),
 \end{aligned}$$

and consequently we obtain

$$\|T_1(u_2, v_2)(t) - T_1(u_1, v_1)(t)\| \leq M_1(m_1 + m_2)(\|u_2 - u_1\| + \|v_2 - v_1\|). \quad (9)$$

Similarly, we can get

$$\|T_2(u_2, v_2)(t) - T_2(u_1, v_1)(t)\| \leq M_2(n_1 + n_2)(\|u_2 - u_1\| + \|v_2 - v_1\|). \quad (10)$$

Clearly it follows from (9) and (10) that

$$\|T(u_2, v_2)(t) - T(u_1, v_1)(t)\| \leq [M_1(m_1 + m_2) + M_2(n_1 + n_2)](\|u_2 - u_1\| + \|v_2 - v_1\|).$$

Since $M_1(m_1 + m_2) + M_2(n_1 + n_2) < 1$, therefore T is a contraction operator. So, by Banach's fixed point theorem, the operator T has a unique fixed point, which is the unique solution of problem (1)-(2). This completes the proof. \square

Example 3.7 Consider the following problem

$$\begin{cases}
 {}^c D^{3/2} x(t) = \frac{1}{4(t+2)^2} \frac{|x(t)|}{1+|x(t)|} + 1 + \frac{1}{32} \sin^2 y(t), & t \in [0, 1], \\
 {}^c D^{5/3} y(t) = \frac{1}{32\pi} \sin(2\pi x(t)) + \frac{|y(t)|}{16(1+|y(t)|)} + \frac{1}{2}, & t \in [0, 1], \\
 x(0) + x(1) = 4 \int_0^1 x(s) ds, & x'(0) = \frac{1}{2} {}^c D^{1/2} x(1), \\
 y(0) + y(1) = \frac{1}{5} \int_0^1 y(s) ds, & y'(0) = 3 {}^c D^{1/3} y(1).
 \end{cases} \quad (11)$$

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Here $\alpha = 3/2$, $\beta = 5/3$, $\gamma = 1/2$, $\delta = 1/3$, $a = 4$, $a_1 = 1/5$, $b = 1/2$, $b_1 = 3$. Using the given data, it is found that $M_1 \approx 5.6166715$, $M_2 \approx 1.6038591$,

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq \frac{1}{16}|u_1 - u_2| + \frac{1}{16}|v_1 - v_2|,$$

$$|h(t, u_1, u_2) - h(t, v_1, v_2)| \leq \frac{1}{16}|u_1 - u_2| + \frac{1}{16}|v_1 - v_2|,$$

and $M_1(m_1 + m_2) + M_2(n_1 + n_2) \approx 0.9025662 < 1$. As all the conditions of Theorem 3.6 are satisfied, therefore its conclusion applies to the problem (11).

3.3 Special cases

We obtain some special cases of the results obtained in this paper by fixing the parameters involved in the problem (1)-(2) which are listed below.

- If $b = b_1 = 0$, then our results correspond to the boundary conditions: $x(0) + x(1) = a \int_0^1 x(s)ds$, $x'(0) = 0$; $y(0) + y(1) = a_1 \int_0^1 y(s)ds$, $y'(0) = 0$.
- We can get the results for the boundary data: $x(0) + x(1) = 0$, $x'(0) = 0$; $y(0) + y(1) = 0$, $y'(0) = 0$ by fixing $a = 0$, $a_1 = 0$, $b = 0$, $b_1 = 0$.
- In case we choose $a = 0$, $a_1 = 0$, $b \neq 0$, $b_1 \neq 0$, we get the results for the boundary conditions: $x(0) + x(1) = 0$, $x'(0) = b {}^c D^\gamma x(1)$; $y(0) + y(1) = 0$, $y'(0) = b_1 {}^c D^\delta y(1)$, $0 < \gamma, \delta \leq 1$.
- By taking $\gamma = \delta = 1$ with $b \neq 1 \neq b_1$, our results reduce to the ones for a given system of fractional differential equations with boundary conditions: $x(0) + x(1) = a \int_0^1 x(s)ds$, $x'(0) = bx'(1)$; $y(0) + y(1) = a_1 \int_0^1 y(s)ds$, $y'(0) = b_1 y'(1)$.

We emphasize that all the results obtained for different values of the parameters are new.

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Solutions of the nonlinear evolution equation via the generalized Riccati equation mapping together with the (G'/G) -expansion method

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Abstract In this article, we investigate the combined KdV-MKdV equation to obtain new exact traveling wave solutions via the generalized Riccati equation mapping together with the (G'/G) -expansion method. In this method, $G'(\theta) = h + f G(\varphi) + g G^2(\theta)$ is used with constant coefficients, as the auxiliary equation and called the generalized Riccati equation. By using this method, we obtain twenty seven exact traveling wave solutions including solitons and periodic solutions and solutions are expressed in the hyperbolic, the trigonometric and the rational functions. It is found that one of our solutions is in good agreement for a special case with the published results which validates our other results.

Keywords: The generalized Riccati equation, (G'/G) -expansion method, Exp-function method, traveling wave solutions, nonlinear evolution equations.

Mathematics Subject Classification: 35K99, 35P99, 35P05.

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1. Introduction

The enormous analysis of exact solutions of the nonlinear partial differential equations (PDEs) is one of the important and amazing research fields in all areas in science and engineering, such as, plasma physics, fluid mechanics, chemical

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physics, optical fibers, solid state physics, chemistry, biology, plasma physics and many others [1-41]. In recent years, many researchers used various methods to different nonlinear partial differential equations for constructing traveling wave solutions, for instance, the Backlund transformation [1], the inverse scattering [2,3], the Jacobi elliptic function expansion [4,5], the tanh function [6,7], the variational iteration [8], the Hirota's bilinear transformation [9], the direct algebraic [10], the Cole-Hopf transformation [11], the Exp-function [12-18] and others [19-25].

Recently, Wang *et al.* [26] presented a method, called the (G'/G) -expansion method. By using this method, they constructed exact traveling wave solutions for the nonlinear evolution equations (NLEEs). In this method, the second order linear ordinary differential equation with constant coefficients $G''(\theta) + \lambda G'(\theta) + \mu G(\theta) = 0$ is used, as an auxiliary equation. Afterwards, many researchers applied the (G'/G) -expansion method to obtain exact traveling wave solutions for the NLEEs. For example, Ozis and Aslan [27] investigated the Kawahara type equations using symbolic computation via this method. In Ref. [28] Gepreel employed this method and found exact solutions for nonlinear PDEs with variable coefficients in mathematical physics whilst Zayed and Al-Joudi [29] studied nonlinear partial differential equations by applying the same method to construct solutions. Naher *et al.* [30] investigated the Caudrey-Dodd-Gibbon equation by using the useful (G'/G) -expansion method and obtained abundant exact traveling wave solutions. Feng *et al.* [31] applied the method to the Kolmogorov-Petrovskii-Piskunov equation for constructing traveling wave solutions. In Ref. [32], Zhao *et al.* concerned about this method to obtain exact solutions for the variant Boussinesq equations while Nofel *et al.* [33] implemented the same method to the higher order KdV equation to get exact traveling wave solutions and so on.

Zhu [34] introduced the generalized Riccati equation mapping to solve the $(2+1)$ -dimensional Boiti-Leon-Pempinelle equation. In this generalized Riccati equation mapping, he employed $G'(\theta) = h + f G(\varphi) + g G^2(\theta)$ with constants co-

efficients, as the auxiliary equation. In Ref. [35], Li et al. used the Riccati equation expansion method to solve the higher dimensional NLEEs. Bekir and Cevikel [36] investigated nonlinear coupled equation in mathematical physics by applying the tanh-coth method combined with the Riccati equation. Guo *et al.* [37] studied the diffusion-reaction and the mKdV equation with variable coefficient via the extended Riccati equation mapping method whilst Li and Dai [38] implemented the generalized Riccati equation mapping with the (G'/G) -expansion method to construct traveling wave solutions for the higher dimensional Jimbo-Miwa equation. In Ref. [39,40] Salas used the projective Riccati equation method to obtain some exact solutions for the Caudrey-Dodd-Gibbon equation and the generalized Sawada-Kotera equations respectively. Many researchers utilized different methods for investigating the combined KdV-MKdV equation to construct exact traveling wave solutions, such as, Liu et al. [41] studied the equation by applying the (G'/G) -expansion method to obtain traveling wave solutions. In the (G'/G) -expansion method, they used the second order linear ordinary differential equation (LODE) with constant coefficients, as an auxiliary equation. To the best of our knowledge, the combined KdV-MKdV equation is not examined by applying the generalized Riccati equation mapping together with the (G'/G) -expansion method.

In this article, we construct twenty seven exact traveling wave solutions including solitons, periodic, and rational solutions of the combined KdV-MKdV equation involving parameters via the generalized Riccati equation mapping together with the (G'/G) -expansion method and Exp-function method.

2. The generalized Riccati equation mapping together with the (G'/G) -expansion method

Suppose the general nonlinear partial differential equation:

$$H(v, v_t, v_x, v_{xt}, v_{tt}, v_{xx}, \dots) = 0, \quad (1)$$

where $v = v(x, t)$ is an unknown function, H is a polynomial in $v = v(x, t)$ and

the subscripts indicate the partial derivatives.

The most important steps of the generalized Riccati equation mapping together with the (G'/G) -expansion method [26,34] are as follows:

Step 1. Consider the traveling wave variable:

$$v(x, t) = r(\theta), \quad \theta = x - Bt, \quad (2)$$

where B is the wave speed. Now using Eq. (2), Eq. (1) is converted into an ordinary differential equation for $r(\theta)$:

$$F(r, r', r'', r''', \dots) = 0, \quad (3)$$

where the superscripts stand for the ordinary derivatives with respect to θ .

Step 2. Eq. (3) integrates term by term one or more times according to possibility, yields constant(s) of integration. The integral constant(s) may be zero for simplicity.

Step 3. Suppose that the traveling wave solution of Eq. (3) can be expressed in the form [26,34]:

$$r(\theta) = \sum_{j=0}^n e_j \left(\frac{G'}{G} \right)^j \quad (4)$$

where e_j ($j = 0, 1, 2, \dots, n$) and $e_n \neq 0$, with $G = G(\theta)$ is the solution of the generalized Riccati equation:

$$G' = h + fG + gG^2, \quad (5)$$

where f, g, h are arbitrary constants and $g \neq 0$.

Step 4. To decide the positive integer n , consider the homogeneous balance between the nonlinear terms and the highest order derivatives appearing in Eq. (3).

Step 5. Substitute Eq. (4) along with Eq. (5) into the Eq. (3), then collect all the coefficients with the same order, the left hand side of Eq. (3) converts

into polynomials in $G^k(\theta)$ and $G^{-k}(\theta)$, ($k = 0, 1, 2, \dots$). Then equating each coefficient of the polynomials to zero, yield a set of algebraic equations for f_j ($j = 0, 1, 2, \dots, n$), f, g, h and B .

Step 6. Solve the system of algebraic equations which are found in Step 5 with the aid of algebraic software Maple and we obtain values for f_j ($j = 0, 1, 2, \dots, n$) and B . Then, substitute obtained values in Eq. (4) along with Eq. (5) with the value of n , we obtain exact solutions of Eq. (1).

In the following, we have twenty seven solutions including four different families of Eq. (5).

Family 1: When $f^2 - 4gh > 0$ and $fg \neq 0$ or $gh \neq 0$, the solutions of Eq. (5) are:

$$\begin{aligned} G_1 &= \frac{-1}{2g} \left(f + \sqrt{f^2 - 4gh} \tanh \left(\frac{\sqrt{f^2 - 4gh}}{2} \theta \right) \right), \\ G_2 &= \frac{-1}{2g} \left(f + \sqrt{f^2 - 4gh} \coth \left(\frac{\sqrt{f^2 - 4gh}}{2} \theta \right) \right), \\ G_3 &= \frac{-1}{2g} \left(f + \sqrt{f^2 - 4gh} \left(\tanh \left(\sqrt{f^2 - 4gh} \theta \right) \pm i \operatorname{sech} \left(\sqrt{f^2 - 4gh} \theta \right) \right) \right), \\ G_4 &= \frac{-1}{2g} \left(f + \sqrt{f^2 - 4gh} \left(\coth \left(\sqrt{f^2 - 4gh} \theta \right) \pm \operatorname{csc} h \left(\sqrt{f^2 - 4gh} \theta \right) \right) \right), \\ G_5 &= \frac{-1}{4g} \left(2f + \sqrt{f^2 - 4gh} \left(\tanh \left(\frac{\sqrt{f^2 - 4gh}}{4} \theta \right) + \cot h \left(\frac{\sqrt{f^2 - 4gh}}{4} \theta \right) \right) \right), \\ G_6 &= \frac{1}{2g} \left(-f + \frac{\sqrt{(X^2 + Y^2)(f^2 - 4gh)} - X\sqrt{f^2 - 4gh} \cosh \left(\sqrt{f^2 - 4gh} \theta \right)}{X \sinh \left(\sqrt{f^2 - 4gh} \theta \right) + Y} \right), \\ G_7 &= \frac{1}{2g} \left(-f - \frac{\sqrt{(Y^2 - X^2)(f^2 - 4gh)} + X\sqrt{f^2 - 4gh} \sinh \left(\sqrt{f^2 - 4gh} \theta \right)}{X \cosh \left(\sqrt{f^2 - 4gh} \theta \right) + Y} \right), \end{aligned}$$

where X and Y are two non-zero real constants and satisfies $Y^2 - X^2 > 0$.

$$G_8 = \frac{2h \cosh \left(\frac{\sqrt{f^2 - 4gh}}{2} \theta \right)}{\sqrt{f^2 - 4gh} \sinh \left(\frac{\sqrt{f^2 - 4gh}}{2} \theta \right) - f \cosh \left(\frac{\sqrt{f^2 - 4gh}}{2} \theta \right)},$$

$$G_9 = \frac{-2h \sinh\left(\frac{\sqrt{f^2-4gh}}{2}\theta\right)}{f \sinh\left(\frac{\sqrt{f^2-4gh}}{2}\theta\right) - \sqrt{f^2-4gh} \cosh\left(\frac{\sqrt{f^2-4gh}}{2}\theta\right)},$$

$$G_{10} = \frac{2h \cosh\left(\sqrt{f^2-4gh}\theta\right)}{\sqrt{f^2-4gh} \sinh\left(\sqrt{f^2-4gh}\theta\right) - f \cosh\left(\sqrt{f^2-4gh}\theta\right) \pm i\sqrt{f^2-4gh}},$$

$$G_{11} = \frac{2h \sinh\left(\sqrt{f^2-4gh}\theta\right)}{-f \sinh\left(\sqrt{f^2-4gh}\theta\right) + \sqrt{f^2-4gh} \cosh\left(\sqrt{f^2-4gh}\theta\right) \pm \sqrt{f^2-4gh}},$$

$$G_{12} = \frac{4h \sinh\left(\frac{\sqrt{f^2-4gh}}{4}\theta\right) \cosh\left(\frac{\sqrt{f^2-4gh}}{4}\theta\right)}{-2f \sinh\left(\frac{\sqrt{f^2-4gh}}{4}\theta\right) \cosh\left(\frac{\sqrt{f^2-4gh}}{4}\theta\right) + 2\sqrt{f^2-4gh} \cosh^2\left(\frac{\sqrt{f^2-4gh}}{4}\theta\right) - \sqrt{f^2-4gh}},$$

Family 2: When $f^2 - 4gh < 0$ and $fg \neq 0$ or $gh \neq 0$, the solutions of Eq. (5)

are:

$$G_{13} = \frac{1}{2g} \left(-f + \sqrt{4gh - f^2} \tan\left(\frac{\sqrt{4gh - f^2}}{2}\theta\right) \right),$$

$$G_{14} = \frac{-1}{2g} \left(f + \sqrt{4gh - f^2} \cot\left(\frac{\sqrt{4gh - f^2}}{2}\theta\right) \right),$$

$$G_{15} = \frac{1}{2g} \left(-f + \sqrt{4gh - f^2} \left(\tan\left(\sqrt{4gh - f^2}\theta\right) \pm \sec\left(\sqrt{4gh - f^2}\theta\right) \right) \right),$$

$$G_{16} = \frac{-1}{2g} \left(f + \sqrt{4gh - f^2} \left(\cot\left(\sqrt{4gh - f^2}\theta\right) \pm \csc\left(\sqrt{4gh - f^2}\theta\right) \right) \right),$$

$$G_{17} = \frac{1}{4g} \left(-2f + \sqrt{4gh - f^2} \left(\tan\left(\frac{\sqrt{4gh - f^2}}{4}\theta\right) - \cot\left(\frac{\sqrt{4gh - f^2}}{4}\theta\right) \right) \right),$$

$$G_{18} = \frac{1}{2g} \left(-f + \frac{\pm\sqrt{(X^2 - Y^2)(4gh - f^2)} - X\sqrt{4gh - f^2} \cos\left(\sqrt{4gh - f^2}\theta\right)}{X \sin\left(\sqrt{4gh - f^2}\theta\right) + Y} \right),$$

$$G_{19} = \frac{1}{2g} \left(-f - \frac{\pm\sqrt{(X^2 - Y^2)(4gh - f^2)} + X\sqrt{4gh - f^2} \cos\left(\sqrt{4gh - f^2}\theta\right)}{X \sin\left(\sqrt{4gh - f^2}\theta\right) + Y} \right),$$

where X and Y are two non-zero real constants and satisfies $X^2 - Y^2 > 0$.

$$G_{20} = \frac{-2h \cos\left(\frac{\sqrt{4gh-f^2}}{2} \theta\right)}{\sqrt{4gh-f^2} \sin\left(\frac{\sqrt{4gh-f^2}}{2} \theta\right) + f \cos\left(\frac{\sqrt{4gh-f^2}}{2} \theta\right)},$$

$$G_{21} = \frac{2h \sin\left(\frac{\sqrt{4gh-f^2}}{2} \theta\right)}{-f \sin\left(\frac{\sqrt{4gh-f^2}}{2} \theta\right) + \sqrt{4gh-f^2} \cos\left(\frac{\sqrt{4gh-f^2}}{2} \theta\right)},$$

$$G_{22} = \frac{-2h \cos\left(\sqrt{4gh-f^2} \theta\right)}{\sqrt{4gh-f^2} \sin\left(\sqrt{4gh-f^2} \theta\right) + f \cos\left(\sqrt{4gh-f^2} \theta\right) \pm \sqrt{4gh-f^2} \theta},$$

$$G_{23} = \frac{2h \sin\left(\sqrt{4gh-f^2} \theta\right)}{-f \sin\left(\sqrt{4gh-f^2} \theta\right) + \sqrt{4gh-f^2} \cos\left(\sqrt{4gh-f^2} \theta\right) \pm \sqrt{4gh-f^2} \theta},$$

$$G_{24} = \frac{4h \sin\left(\frac{\sqrt{4gh-f^2}}{4} \theta\right) \cos\left(\frac{\sqrt{4gh-f^2}}{4} \theta\right)}{-2f \sin\left(\frac{\sqrt{4gh-f^2}}{4} \theta\right) \cos\left(\frac{\sqrt{4gh-f^2}}{4} \theta\right) + 2\sqrt{4gh-f^2} \cos^2\left(\frac{\sqrt{4gh-f^2}}{4} \theta\right) - \sqrt{4gh-f^2}},$$

Family 3: when $h = 0$ and $fg \neq 0$, the solution Eq. (5) becomes:

$$G_{25} = \frac{-fb_1}{g(b_1 + \cosh(f\theta) - \sinh(f\theta))},$$

$$G_{26} = \frac{-f(\cosh(f\theta) + \sinh(f\theta))}{g(b_1 + \cosh(f\theta) + \sinh(f\theta))},$$

where b_1 is an arbitrary constant.

Family 4: when $g \neq 0$ and $h = f = 0$, the solution of Eq. (5) becomes:

$$G_{27} = \frac{-1}{g\theta + u_1},$$

where u_1 is an arbitrary constant.

2.1. Exp-function Method

Consider the general nonlinear partial differential equation of the type (1)

Using the transformation (2) in equation (1) we have equation of the type (3).

According to the Exp-function method, developed by He and Wu [12-18], we assume that the wave solutions can be expressed in the following form

$$u(\eta) = \frac{\sum_{n=-c}^d a_n \exp(n\eta)}{\sum_{m=-p}^q b_m \exp(m\eta)} \quad (6)$$

where p, q, c and d are positive integers which are to be further determined, a_n and b_m are unknown constants. We can rewrite equation (6) in the following equivalent form

$$u(\eta) = \frac{a_c \exp(c\eta) + \dots + a_{-d} \exp(-d\eta)}{b_p \exp(p\eta) + \dots + b_{-q} \exp(-q\eta)}. \quad (7)$$

To determine the value of c and p , we balance the linear term of highest order of equation (3) with the highest order nonlinear term. Similarly, to determine the value of d and q , we balance the linear term of lowest order of equation (3) with lowest order non linear term.

3. Solution procedure

By using Exp-function method and the generalized Riccati equation mapping together with the (G'/G) -expansion method, we construct new exact traveling wave solutions for the combined KdV-MKdV equation (Gardner equation) in this section.

3.1 The combined KdV-MKdV equation (Gardner equation)

We consider the combined KdV-MKdV equation with parameters followed by Liu et al. [41]:

$$u_t + p u u_x + q u^2 u_x - s u_{xxx} = 0, \quad (8)$$

where p, s are free parameters and $q \neq 0$.

Now, we use the transformation Eq. (2) into the Eq. (8), which yields:

$$-B r' + p r r' + q r^2 r' - s r''' = 0, \quad (9)$$

Eq. (9) is integrable, therefore, integrating with respect θ once yields:

$$-B r + \frac{p}{2} r^2 + \frac{q}{3} r^3 - s r'' + K = 0, \quad (10)$$

where K is an integral constant which is to be determined later.

Taking the homogeneous balance between r'' and r^3 in Eq. (10), we obtain $n = 1$.

Therefore, the solution of Eq. (10) is of the form:

$$r(\theta) = e_1 (G'/G) + e_0, \quad e_1 \neq 0. \quad (11)$$

Using Eq. (5), Eq. (11) can be re-written as:

$$r(\theta) = e_1 (f + h G^{-1} + g G) + e_0, \quad (12)$$

where f, g and h are free parameters.

By substituting Eq. (12) into Eq. (10), collecting all coefficients of G^k and G^{-k} ($k = 0, 1, 2, \dots$) and setting them equal to zero, we obtain a set of algebraic equations for e_0, e_1, f, g, h, K and B (algebraic equations are not shown, for simplicity). Solving the system of algebraic equations with the help of algebraic software Maple, we obtain

$$e_0 = \frac{\mp p \sqrt{\frac{6s}{q}} - 6sf}{\pm 2q \sqrt{\frac{6s}{q}}}, \quad e_1 = \pm \sqrt{\frac{6s}{q}}, \quad B = \frac{2sqf^2 - p^2 + 16sqgh}{4q},$$

$$K = \frac{-48psqgh \left(\pm \sqrt{\frac{6s}{q}} \right) + 6psqf^2 \left(\pm \sqrt{\frac{6s}{q}} \right) - p^3 \left(\pm \sqrt{\frac{6s}{q}} \right) + 288s^2 qfgh}{24q^2 \left(\pm \sqrt{\frac{6s}{q}} \right)},$$

where p, s are free parameters and $q \neq 0$.

Family 1: The soliton and soliton-like solutions of Eq. (6) (when $f^2 - 4gh > 0$ and $fg \neq 0$ or $gh \neq 0$) are:

$$r_1 = e_1 \frac{\Delta^2 \sec h^2 \left(\frac{\Delta}{2} \theta \right)}{2 \left(f + \Delta \tanh \left(\frac{\Delta}{2} \theta \right) \right)} + e_0,$$

where $\Delta = \sqrt{f^2 - 4gh}$, $\Delta^2 = f^2 - 4gh$, $e_0 = \frac{\mp p\sqrt{\frac{6s}{q}} - 6sf}{\pm 2q\sqrt{\frac{6s}{q}}}$, $e_1 = \pm \sqrt{\frac{6s}{q}}$ and $\theta = x - \left(\frac{2sqf^2 - p^2 + 16sqgh}{4q}\right)t$.

$$r_2 = e_1 \frac{-\Delta^2 \csc h^2 \left(\frac{\Delta}{2} \theta\right)}{2 \left(f + \Delta \coth \left(\frac{\Delta}{2} \theta\right)\right)} + e_0,$$

$$r_3 = e_1 \frac{\Delta^2 \left(\sec h^2 (\Delta \theta) \mp i \tanh (\Delta \theta) \sec h (\Delta \theta)\right)}{f + \sqrt{f^2 - 4gh} \left(\tanh (\Delta \theta) \pm i \sec h (\Delta \theta)\right)} + e_0,$$

$$r_4 = e_1 \frac{-\Delta^2 \left(\csc h^2 (\Delta \theta) \pm \coth (\Delta \theta) \csc h (\Delta \theta)\right)}{f + \Delta \left(\coth (\Delta \theta) \pm \csc h (\Delta \theta)\right)} + e_0,$$

$$r_5 = e_1 \frac{\Delta^2 \left(\sec h^2 \left(\frac{\Delta}{4} \theta\right) - \csc h^2 \left(\frac{\Delta}{4} \theta\right)\right)}{8f + 4\Delta \left(\tanh \left(\frac{\Delta}{4} \theta\right) + \coth \left(\frac{\Delta}{4} \theta\right)\right)} + e_0,$$

$$r_6 = e_1 \frac{-X \left(f^2 X - \sinh (\Delta \theta) f^2 Y - 4ghX + 4ghY \sinh (\Delta \theta) - \Delta^2 \sqrt{(X^2 + Y^2)} \cosh (\Delta \theta)\right)}{(X \sinh (\Delta \theta) + Y) \left(f X \sinh (\Delta \theta) + f Y - \Delta \sqrt{(X^2 + Y^2)} + X \Delta \cosh (\Delta \theta)\right)} + e_0,$$

$$r_7 = e_1 \frac{X \left(f^2 Y \cosh (\Delta \theta) f^2 Y - 4ghY \cosh (\Delta \theta) - \Delta^2 \sqrt{(X^2 - Y^2)} \sinh (\Delta \theta) + f^2 X - 4ghX\right)}{(X \cosh (\Delta \theta) + Y) \left(f X \cosh (\Delta \theta) + f Y + \Delta \sqrt{(X^2 - Y^2)} + X \Delta \sinh (\Delta \theta)\right)} + e_0,$$

where X and Y are two non-zero real constants and satisfies $Y^2 - X^2 > 0$.

$$r_8 = e_1 \frac{-\Delta^2}{2 \cosh \left(\frac{\Delta}{2} \theta\right) \left(\Delta \sinh \left(\frac{\Delta}{2} \theta\right) - f \cosh \left(\frac{\Delta}{2} \theta\right)\right)} + e_0,$$

$$r_9 = e_1 \frac{\Delta^2}{2 \sinh \left(\frac{\Delta}{2} \theta\right) \left(-f \sinh \left(\frac{\Delta}{2} \theta\right) + \Delta \cosh \left(\frac{\Delta}{2} \theta\right)\right)} + e_0,$$

$$r_{10} = e_1 \frac{-\Delta^2 + i f^2 \sinh (\Delta \theta) - i 4gh \sinh (\Delta \theta)}{\Delta \sinh (\Delta \theta) - f \cosh (\Delta \theta) + i \Delta \cosh (\Delta \theta)} + e_0,$$

$$r_{11} = e_1 \frac{\Delta^2 + f^2 \cosh (\Delta \theta) - 4gh \cosh (\Delta \theta)}{(-f \sinh (\Delta \theta) + \Delta \cosh (\Delta \theta) + \Delta) \sinh (\Delta \theta)} + e_0,$$

$$r_{12} = e_1 \frac{\Delta^2}{4 \sinh \left(\frac{\Delta}{4} \theta\right) \cosh \left(\frac{\Delta}{4} \theta\right) \left(-2f \sinh \left(\frac{\Delta}{4} \theta\right) \cosh \left(\frac{\Delta}{4} \theta\right) + 2\Delta \cosh^2 \left(\frac{\Delta}{4} \theta\right) - \Delta\right)} + e_0,$$

Family 2: The periodic form solutions of Eq. (8) (when $f^2 - 4gh < 0$ and

$fg \neq 0$ or $gh \neq 0$) are:

$$r_{13} = e_1 \frac{\Omega^2}{2 \cos\left(\frac{\Omega}{2}\theta\right) \left(-f \cos\left(\frac{\Omega}{2}\theta\right) + \Omega \sin\left(\frac{\Omega}{2}\theta\right)\right)} + e_0,$$

where $\Omega = \sqrt{-f^2 + 4gh}$, $\Omega^2 = 4gh - f^2$, $e_0 = \frac{\mp p \sqrt{\frac{6s}{q}} - 6sf}{\pm 2q \sqrt{\frac{6s}{q}}}$, $e_1 = \pm \sqrt{\frac{6s}{q}}$ and $\theta = x - \left(\frac{2sqf^2 - p^2 + 16sqgh}{4q}\right)t$.

$$r_{14} = e_1 \frac{\Omega^2}{2 \left(-1 + \cos^2\left(\frac{\Omega}{2}\theta\right)\right) \left(f + \Omega \cot\left(\frac{\Omega}{2}\theta\right)\right)} + e_0,$$

$$r_{15} = e_1 \frac{\Omega^2 (1 + \sin(\Omega\theta))}{\cos(\Omega\theta) (-f \cos(\Omega\theta) + \Omega \sin(\Omega\theta) + \Omega)} + e_0,$$

$$r_{16} = e_1 \frac{\Omega^2 \sin(\Omega\theta)}{\cos(\Omega\theta) f \sin(\Omega\theta) + \Omega \cos^2(\Omega\theta) - f \sin(\Omega\theta) - \Omega} + e_0,$$

$$r_{17} = e_1 \frac{-\Omega^2}{4 \cos^2\left(\frac{\Omega}{4}\theta\right) \left(-1 + \cos^2\left(\frac{\Omega}{4}\theta\right)\right) \left(-2f + \Omega \left(\tan\left(\frac{\Omega}{4}\theta\right) - \cot\left(\frac{\Omega}{4}\theta\right)\right)\right)} + e_0,$$

$$r_{18} = e_1 \frac{X.N_1}{(-X^2 + X^2 \cos^2(\Omega\theta) - 2XY \sin(\Omega\theta) - Y^2) \left(-f + \frac{\Omega \sqrt{(X^2 - Y^2)}}{X \sin(\Omega\theta) + Y} - X \Omega \cos(\Omega\theta)\right)} + e_0, \quad (13)$$

where

$$N_1 = -4ghX - 4ghY \sin(\Omega\theta) + f^2X + f^2Y \sin(\Omega\theta) +$$

$$4gh\sqrt{(X^2 - Y^2)} \cos(\Omega\theta) - f^2\sqrt{(X^2 - Y^2)} \cos(\Omega\theta)$$

$$r_{19} = e_1 \frac{X \left(-4ghX - 4ghY \sin(\Omega\theta) + f^2X + f^2Y \sin(\Omega\theta) - 4gh\sqrt{(X^2 - Y^2)} \cos(\Omega\theta) + f^2\sqrt{(X^2 - Y^2)} \cos(\Omega\theta)\right)}{(-X^2 + X^2 \cos^2(\Omega\theta) - 2XY \sin(\Omega\theta) - Y^2) \left(-f - \frac{\Omega \sqrt{(X^2 - Y^2)}}{X \sin(\Omega\theta) + Y} + X \Omega \cos(\Omega\theta)\right)} + e_0,$$

where X and Y are two non-zero real constants and satisfies $X^2 - Y^2 > 0$.

$$r_{20} = e_1 \frac{-\Omega^2 \sec\left(\frac{\Omega}{2}\theta\right) \left(\Omega \sin\left(\frac{\Omega}{2}\theta\right) + f \cos\left(\frac{\Omega}{2}\theta\right)\right)}{2 \left(4gh - 4gh \cos^2\left(\frac{\Omega}{2}\theta\right) - f^2 + 2f^2 \cos^2\left(\frac{\Omega}{2}\theta\right) + 2f\Omega \sin\left(\frac{\Omega}{2}\theta\right) \cos\left(\frac{\Omega}{2}\theta\right)\right)} + e_0,$$

$$r_{21} = e_1 \frac{-\Omega^2 \left(-f \sin\left(\frac{\Omega}{2}\theta\right) + \Omega \cos\left(\frac{\Omega}{2}\theta\right)\right)}{2 \sin\left(\frac{\Omega}{2}\theta\right) \left(-f^2 + 2f^2 \cos^2\left(\frac{\Omega}{2}\theta\right) + 2f\Omega \sin\left(\frac{\Omega}{2}\theta\right) \cos\left(\frac{\Omega}{2}\theta\right) - 4gh \cos^2\left(\frac{\Omega}{2}\theta\right)\right)} + e_0,$$

$$r_{22} = \frac{\frac{1}{2}e_1 \sec(\Omega\theta) (-\Omega^2 - 4gh \sin(\Omega\theta) + f^2 \sin(\Omega\theta)) (\Omega \sin(\Omega\theta) + f \cos(\Omega\theta) + \Omega)}{N_2} + e_0,$$

where

$$N_2 = 4gh - 2gh \cos^2(\Omega\theta) - f^2 + f^2 \cos^2(\Omega\theta) + \Omega f \sin(\Omega\theta) \cos(\Omega\theta) + \\ 4gh \sin(\Omega\theta) - f^2 \sin(\Omega\theta) + f\Omega \cos(\Omega\theta)$$

$$r_{23} = e_1 \frac{-\Omega^2(-f \sin(\Omega\theta) + \Omega \cos(\Omega\theta) + \Omega)}{2 \sin(\Omega\theta)(-2gh \cos(\Omega\theta) + f^2 \cos(\Omega\theta) + f\Omega \sin(\Omega\theta) - 2gh)} + e_0,$$

$$q_{24} = \frac{\frac{-\Omega^2}{4} e_1 \csc\left(\frac{\Omega}{4}\theta\right) \sec\left(\frac{\Omega}{4}\theta\right) (-2f \sin\left(\frac{\Omega}{4}\theta\right) \cos\left(\frac{\Omega}{4}\theta\right) + 2\Omega \cos^2\left(\frac{\Omega}{4}\theta\right) - \Omega)}{N_3} + e_0,$$

where

$$N_3 = -8f^2 \cos^2\left(\frac{\Omega}{4}\theta\right) + 8f^2 \cos^4\left(\frac{\Omega}{4}\theta\right) + 8\Omega f \cos^3\left(\frac{\Omega}{4}\theta\right) \sin\left(\frac{\Omega}{4}\theta\right) - \\ 4f\Omega \sin\left(\frac{\Omega}{4}\theta\right) \cos\left(\frac{\Omega}{4}\theta\right) - 16gh \cos^4\left(\frac{\Omega}{4}\theta\right) + 16gh \cos^2\left(\frac{\Omega}{4}\theta\right) - \Omega^2$$

Family 3: The soliton and soliton-like solutions of Eq. (6) (when $h = 0$ and $f g \neq 0$) are:

$$r_{25} = e_1 \frac{f(\cosh(f\theta) - \sinh(f\theta))}{b_1 + \cosh(f\theta) - \sinh(f\theta)} + e_0,$$

$$r_{26} = e_1 \frac{f b_1}{b_1 + \cosh(f\theta) + \sinh(f\theta)} + e_0,$$

where b_1 is an arbitrary constant, $e_0 = \frac{\mp p\sqrt{\frac{6s}{q}} - 6sf}{\pm 2q\sqrt{\frac{6s}{q}}}$, $e_1 = \pm \sqrt{\frac{6s}{q}}$ and $\theta = x - \left(\frac{2sqf^2 - p^2 + 16sqgh}{4q}\right)t$.

Family 4: The rational function solution (when $g \neq 0$ and $h = f = 0$) is:

$$r_{27} = \frac{-e_1 g}{g\theta + u_1},$$

where u_1 is an arbitrary constant, $e_1 = \pm \sqrt{\frac{6s}{q}}$ and $\theta = x - \left(\frac{2sqf^2 - p^2 + 16sqgh}{4q}\right)t$.

3.2 The combined KdV-MKdV equation (Gardner equation) using Exp-function method

We consider the combined KdV-MKdV equation (8) with parameters followed by Liu et al. [41]:

Now, we use the transformation Eq. (2) into the Eq. (8), which yields (9).

Using Exp-function Method we have following solution sets satisfy the given combined KdV-MKdV equation (8)

1st Solution set:

$$\left\{ \begin{array}{l} B = B, b_{-1} = \frac{1}{4} \frac{(p^2 + 4Bq - 2sq)}{b_1(p^2 + 4Bq + 4sq)}, b_0 = b_0, b_1 = b_1, a_{-1} = -\frac{1}{8} \frac{b_0^2(p + \sqrt{p^2 + 4Bq + 4sq})(4Bq + p^2 - 2sq)}{qb_1(p^2 + 4Bq + 4sq)}, \\ a_0 = \frac{b_0 \left(4s - 2B - \frac{1}{2} p \left(\frac{p + \sqrt{p^2 + 4Bq + 4sq}}{q} \right) \right)}{\sqrt{p^2 + 4Bq + 4sq}}, a_1 = -\frac{1}{2} \left(\frac{p + \sqrt{p^2 + 4Bq + 4sq}}{q} \right) b_1 \end{array} \right\}$$

We therefore, obtained the following generalized solitary solution

$$U(\eta) = \frac{-\frac{1}{8} \frac{b_0^2(p + \sqrt{p^2 + 4Bq + 4sq})(4Bq + p^2 - 2sq)}{qb_1(p^2 + 4Bq + 4sq)} e^{-\eta} + \frac{b_0 \left(4s - 2B - \frac{1}{2} p \left(\frac{p + \sqrt{p^2 + 4Bq + 4sq}}{q} \right) \right)}{\sqrt{p^2 + 4Bq + 4sq}} - \frac{1}{2} \left(\frac{p + \sqrt{p^2 + 4Bq + 4sq}}{q} \right) b_1 e^{\eta}}{\frac{1}{4} \frac{(p^2 + 4Bq - 2sq)}{b_1(p^2 + 4Bq + 4sq)} e^{-\eta} + b_0 + b_1 e^{\eta}}$$

$$U(x, t) = \frac{-\frac{1}{8} \frac{b_0^2(p + \sqrt{p^2 + 4Bq + 4sq})(4Bq + p^2 - 2sq)}{qb_1(p^2 + 4Bq + 4sq)} e^{-(x-Bt)} + \frac{b_0 \left(4s - 2B - \frac{1}{2} p \left(\frac{p + \sqrt{p^2 + 4Bq + 4sq}}{q} \right) \right)}{\sqrt{p^2 + 4Bq + 4sq}} - \frac{1}{2} \left(\frac{p + \sqrt{p^2 + 4Bq + 4sq}}{q} \right) b_1 e^{(x-Bt)}}{\frac{1}{4} \frac{(p^2 + 4Bq - 2sq)}{b_1(p^2 + 4Bq + 4sq)} e^{-(x-Bt)} + b_0 + b_1 e^{(x-Bt)}}$$

2nd Solution set:

$$\left\{ \begin{array}{l} B = B, b_{-1} = \frac{1}{4} \frac{(p^2 + 4Bq - 2sq)b_0^2}{b_1(p^2 + 4Bq + 4sq)}, b_0 = b_0, b_1 = b_1, a_{-1} = -\frac{1}{8} \frac{b_0^2(-p + \sqrt{p^2 + 4Bq + 4sq})(4Bq + p^2 - 2sq)}{qb_1(p^2 + 4Bq + 4sq)}, \\ a_0 = -\frac{b_0 \left(4s - 2B - \frac{1}{2} p \left(\frac{p + \sqrt{p^2 + 4Bq + 4sq}}{q} \right) \right)}{\sqrt{p^2 + 4Bq + 4sq}}, a_1 = \frac{1}{2} \left(\frac{-p + \sqrt{p^2 + 4Bq + 4sq}}{q} \right) b_1 \end{array} \right\}$$

We therefore, obtained the following generalized solitary solution

$$U(\eta) = \frac{-\frac{1}{8} \frac{b_0^2(-p + \sqrt{p^2 + 4Bq + 4sq})(4Bq + p^2 - 2sq)}{qb_1(p^2 + 4Bq + 4sq)} e^{-\eta} - \frac{b_0 \left(4s - 2B - \frac{1}{2} p \left(\frac{p + \sqrt{p^2 + 4Bq + 4sq}}{q} \right) \right)}{\sqrt{p^2 + 4Bq + 4sq}} - \frac{1}{2} \left(\frac{-p + \sqrt{p^2 + 4Bq + 4sq}}{q} \right) b_1 e^{\eta}}{\frac{1}{4} \frac{(p^2 + 4Bq - 2sq)b_0^2}{b_1(p^2 + 4Bq + 4sq)} e^{-\eta} + b_0 + b_1 e^{\eta}}$$

$$U(x, t) = \frac{-\frac{1}{8} \frac{b_0^2 \left(-p + \sqrt{p^2 + 4Bq + 4sq} \right) (4Bq + p^2 - 2sq)}{qb_1(p^2 + 4Bq + 4sq)} e^{-(x-Bt)} - \frac{b_0 \left(4s - 2B - \frac{1}{2} p \left(\frac{p + \sqrt{p^2 + 4Bq + 4sq}}{q} \right) \right)}{\sqrt{p^2 + 4Bq + 4sq}}}{-\frac{1}{2} \left(\frac{-p + \sqrt{p^2 + 4Bq + 4sq}}{q} \right) b_1 e^{(x-Bt)} + \frac{1}{4} \frac{(p^2 + 4Bq - 2sq)b_0^2}{b_1(p^2 + 4Bq + 4sq)} e^{-(x-Bt)} + b_0 + b_1 e^{(x-Bt)}}$$

4. Results and discussion

It is significance mentioning that our solution q_{27} is coincided with $u_{3,4}(x, t)$ in example 1 of section 4 of Liu et al. [41] for $s = 1, q = 1, p = 2$ and $u_1 = 0$. Moreover, it is showing that our solution q_{27} is coincided with $u_{3,4}(x, t)$ in example 2 of section 4 of Liu et al. [41] for $s = -1, q = 1, p = 2$ and $u_1 = 0$. In addition, we construct many new exact traveling wave solutions for the combined KdV-MKdV equation in this work, which have not been found in the previous literature. Furthermore, the graphical demonstrations of some of them are depicted in the following subsection in figures below.

4.1 Graphical representations of the solutions

The graphical depictions of the solutions are shown in the figures with the help of Maple:

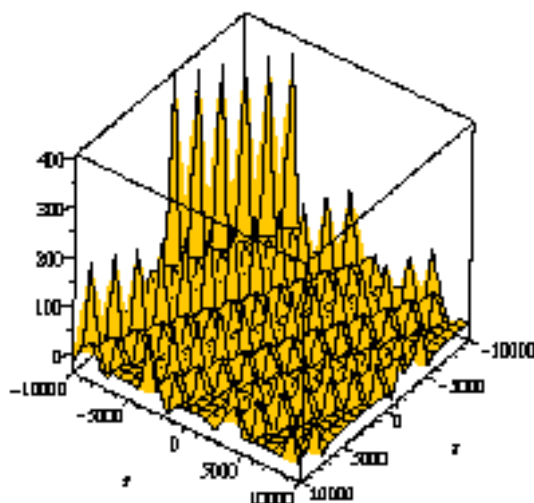


Fig. 1: Periodic solutions for $f = 5, g = 4, h = 3, p = 3, s = 2, q = 5$

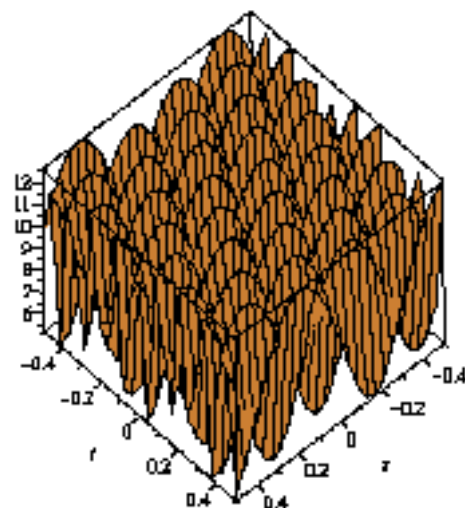


Fig. 2: Periodic solutions for $f = 9, g = 8, h = 0, p = 8, s = 6, q = 7, b_1 = 8$

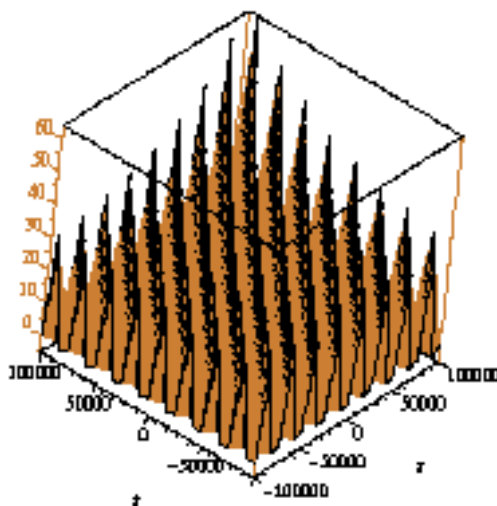


Fig. 3: Periodic solutions for $f = 5, g = 4, h = 3, p = 2, s = 5, q = 9$

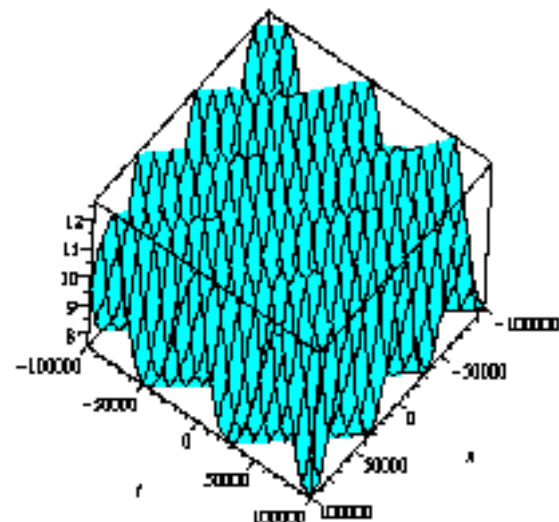


Fig. 4: Periodic solutions for $f = 3, g = 4, h = 0, p = 1, s = 3, q = 4, b_1 = 4$

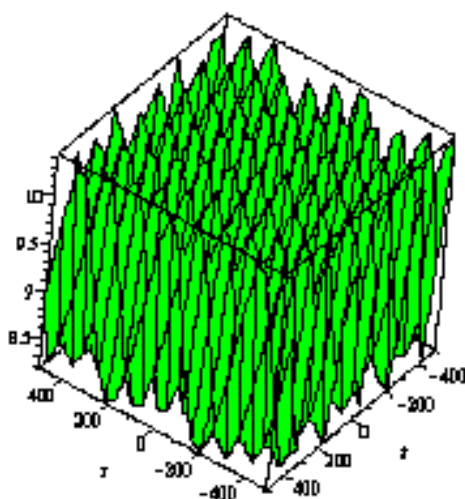


Fig. 5: Periodic solutions for $f = 3, g = 4, h = 0, p = 2, s = 5, q = 7, b_1 = 8$

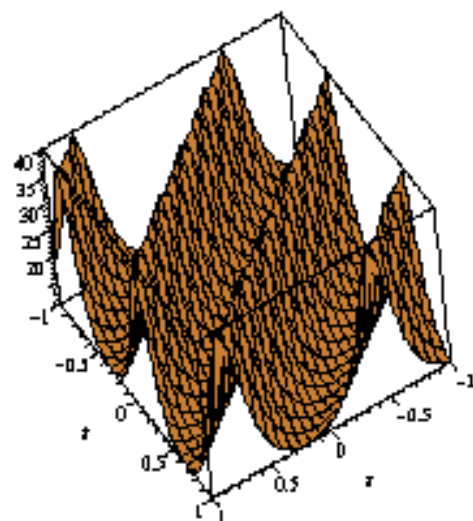


Fig. 6: Periodic solutions for $f = 5, g = 7, h = 0, p = 5, s = 5, q = 7, b_1 = 2$

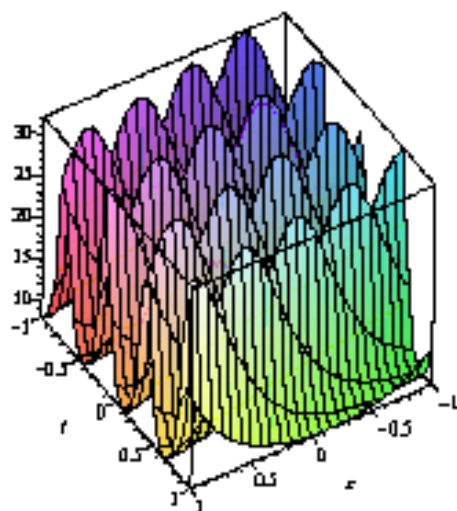


Fig. 7: Periodic solutions for $f = 2, g = 4, h = 0, p = 1, s = 3, q = 1, b_1 = 2$

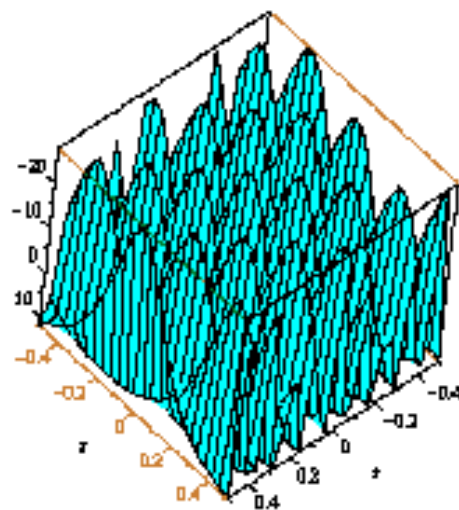


Fig. 8: Periodic solutions for $f = 5, g = 4, h = 0, p = 3, s = 4, q = 3, b_1 = 2$

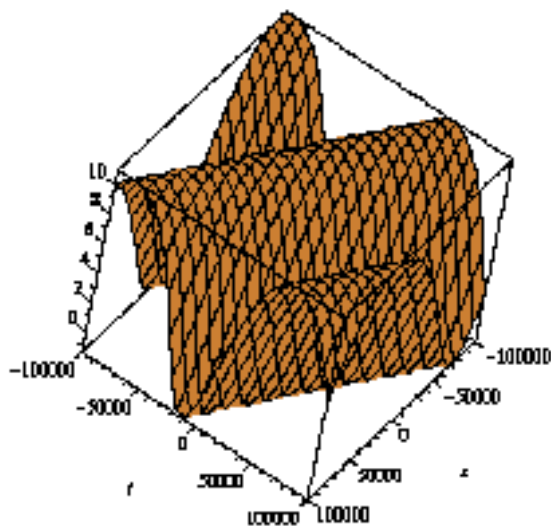


Fig. 9: Periodic solutions for $f = 7, g = 15, h = 0, p = 3, s = 4, q = 5, b_1 = 4$

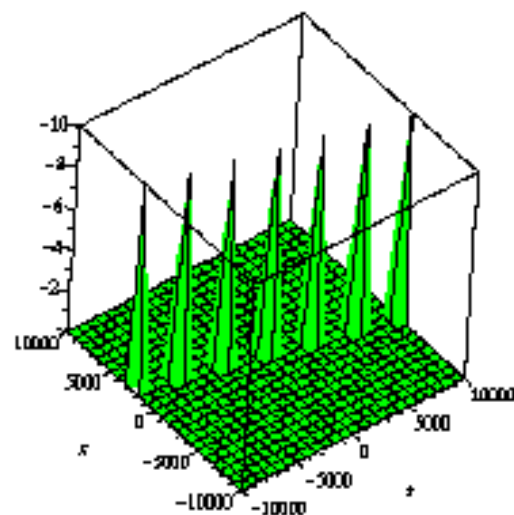


Fig. 10: Periodic solutions for $f = 0, g = 11, h = 0, p = 3, s = 5, q = 9, b_1 = 2$

5. Conclusions

In this article, we apply the Exp-function method and generalized Riccati equa-

tion mapping together with the (G'/G) -expansion method to solve the combined KdV-MKdV equation. In (G'/G) -expansion method, the generalized Riccati equation $G'(\theta) = h + f G(\varphi) + g G^2(\theta)$ is used with constant coefficients, as the auxiliary equation, instead of the second order linear ordinary differential equation with constant coefficients. By applying these methods, we obtain abundant exact traveling wave solutions including solitons and periodic solutions and solutions are expressed in terms of the hyperbolic, the trigonometric and the rational functions. The correctness of the obtained solutions is verified to compare with the published results. We hope that these useful and powerful methods can be effectively used to solve many nonlinear evolution equations which are arising in technical arena.

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STABILITY OF A LATTICE PRESERVING FUNCTIONAL EQUATION ON RIESZ SPACE: FIXED POINT ALTERNATIVE

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AND DONG YUN SHIN*

ABSTRACT. The aim of this paper is to investigate Hyers-Ulam stability of the following lattice preserving functional equation on Riesz space with fixed point method:

$$\|F(\tau x \vee \eta y) - \tau F(x) \vee \eta F(y)\| \leq \varphi(\tau x \vee \eta y, \tau x \wedge \eta y),$$

where \mathcal{X} is a Banach lattice and $\varphi : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ is a mapping such that

$$\varphi(x, y) \leq (\tau\eta)^{\frac{\alpha}{2}} \varphi\left(\frac{x}{\tau}, \frac{y}{\eta}\right)$$

for all $\tau, \eta \geq 1$ and $\alpha \in [0, \frac{1}{2})$.

1. INTRODUCTION

In 1940 Ulam [1] proposed the famous Ulam stability problem: When is it true that a function which satisfies some functional equation approximately must be close to one satisfying the equation exactly?. If the answer is affirmative, we would say that the equation is stable. In 1941, Hyers [2] solved this stability problem for additive mappings subject to the Hyers condition on approximately additive mappings. The result of Hyers was generalized by Rassias [3] for linear mapping by considering an unbounded Cauchy difference.

In 1996, Isac and Rassias [4] were the first authors to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. Some authors have considered the Hyers-Ulam stability of quadratic functional equations in random normed spaces [5, 6, 7, 8, 9, 10, 11, 12, 13]. By the fixed point method, the stability problems of several functional equations have been extensively investigated by a number of authors (see [14, 15]). We generalize the Agbeko's theorem [16] and prove it by fixed point method.

A non-empty set \mathcal{M} with a relation " \leq " is said to be an order set whenever the following conditions are satisfied:

1. $x \leq x$ for every $x \in \mathcal{M}$;
2. $x \leq y$ and $y \leq x$ implies that $x = y$;
3. $x \leq y$ and $y \leq z$ implies that $x \leq z$.

If, in addition, for two elements $x, y \in \mathcal{M}$ either $x \leq y$ or $y \leq x$, then \mathcal{M} is called a totally ordered set. Let \mathcal{A} be a subset of an ordered set \mathcal{M} . $x \in \mathcal{M}$ is called an upper bound of \mathcal{A} if $y \leq x$ for all $y \in \mathcal{A}$. $z \in \mathcal{M}$ is called a lower bound of \mathcal{A} if $y \geq z$ for all $y \in \mathcal{A}$. Moreover, if there is an upper bound of \mathcal{A} , then \mathcal{A} is said to be bounded from above. If there is a lower bound of \mathcal{A} , then \mathcal{A} is said to be bounded from below. If \mathcal{A} is bounded from above and from below, then we will briefly say that \mathcal{A} is order bounded.

An order set (\mathcal{M}, \leq) is called a lattice if any two elements $x, y \in \mathcal{M}$ have a least upper bound denoted by $x \vee y = \sup(x, y)$ and a greatest lower bound denoted by $x \wedge y = \inf(x, y)$.

A real vector space E which is also an order set is called an order vector space if the order and the vector space structure are compatible in the following sense:

1. if $x, y \in E$ such that $x \leq y$ then $x + z \leq y + z$ for all $z \in E$;

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2. if $x, y \in E$ such that $x \leq y$ then $\alpha x \leq \alpha y$ for all $\alpha \geq 0$.
 (E, \leq) is called a **Riesz space** if (E, \leq) is a lattice and order vector space.
 A norm ρ on Riesz space E , is called a lattice norm if $\rho(x) \leq \rho(y)$ whenever $|x| \leq |y|$. In the latter case $(E, \|\cdot\|)$ is called a normed Riesz space.
 $(E, \|\cdot\|)$ is called a **Banach lattice** if $(E, \|\cdot\|)$ is a Banach space, E is Riesz space and $|x| \leq |y|$ implies that $\|x\| \leq \|y\|$ for all $x, y \in E$.

Example 1.1. Suppose that \mathcal{X} is a compact Hausdorff space. We denote by $C(K)$ the Banach space of all real-valued continuous functions on \mathcal{X} . Let “ \leq ” be a point-wise order on $C(K)$, and $f \leq g$ if and only if $f(t) \leq g(t)$ for all $t \in K$. It is easy to show that $(C(K), \leq)$ is a Banach lattice.

Let E be a Riesz space, and let the positive cone E^+ of E consist of all $x \in E$ such that $x \geq 0$. For every $x \in E$ let

$$x^+ = x \vee 0 \quad x^- = -x \vee 0 \quad |x| = x \vee -x.$$

Let E be a Riesz space. For all $x, y, z \in E$ and $a \in \mathcal{R}$, the following assertions hold:

1. $x + y = x \vee y + x \wedge y$, $-(x \vee y) = -x \wedge y$.
2. $x + (y \vee z) = (x + y) \vee (x + z)$, $x + (y \wedge z) = (x + y) \wedge (x + z)$.
3. $|x| = x^+ + x^-$, $|x + y| \leq |x| + |y|$.
4. $x \leq y$ is equivalent to $x^+ \leq y^+$ and $y^- \leq x^-$.
5. $(x \vee y) \wedge z = (x \wedge y) \vee (y \wedge z)$, $(x \wedge y) \vee z = (x \vee y) \wedge (y \vee z)$.

A Riesz space E is called **Archimedean** if $x \leq 0$ holds whenever the set $\{nx : n \in \mathcal{N}\}$ is bounded from above.

Theorem 1.1. Let E be a normed Riesz space. The following assertions hold:

1. the lattice operations is continuous;
2. the positive cone E^+ is closed;
3. $\lim_{n \rightarrow \infty} x_n = \sup\{x_n : n \in \mathcal{N}\}$.

Definition 1.1. Let \mathcal{X}, \mathcal{Y} be Banach lattices. A mapping $T : \mathcal{X} \rightarrow \mathcal{Y}$ is called positive if $T(\mathcal{X}^+) = \{T(|x|) : x \in \mathcal{X}\} \subset \mathcal{Y}^+$.

Definition 1.2. Let \mathcal{X}, \mathcal{Y} be Banach lattices and let $T : \mathcal{X} \rightarrow \mathcal{Y}$ be a positive mapping. We define P_1) lattice homomorphism:

$$T(|x| \vee |y|) = T(|x|) \vee T(|y|);$$

P_2) semi-homogeneity: for all $x \in \mathcal{X}$ and all $\alpha \in \mathcal{R}^+$

$$T(\alpha|x|) = \alpha T(|x|);$$

P_3) continuity from below on the positive cone: for all increasing sequences $x_n \subset \mathcal{X}^+$

$$\lim_{n \rightarrow \infty} T(x_n) = T(\lim_{n \rightarrow \infty} x_n).$$

Observe that every lattice homomorphism $T : \mathcal{X} \rightarrow \mathcal{Y}$ is necessarily a positive operator. Indeed, if $x \in E^+$ then

$$T(x) = T(x \vee 0) = T(x) \vee T(0) = T(x)^+ \geq 0$$

holds in \mathcal{Y} . Also it is important to note that the range of a lattice homomorphism is a Riesz subspace.

Theorem 1.2. For an operator $T : \mathcal{X} \rightarrow \mathcal{Y}$ between two Riesz spaces, the following statements are equivalent:

1. T is a lattice homomorphism;
2. $T(x^+) = T(x)^+$ for all $x \in \mathcal{X}$;
3. $T(x \wedge y) = T(x) \wedge T(y)$;
4. if $x \wedge y = 0$ in \mathcal{X} , then $T(x) \wedge T(y) = 0$ holds in \mathcal{Y} ;
5. $T(|x|) = |T(x)|$.

Definition 1.3. Let \mathcal{X} be a set. A function $d : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty]$ is called a generalized metric on \mathcal{X} if d satisfies the following conditions:

- (a) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in \mathcal{X}$;
- (b) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$;
- (c) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in \mathcal{X}$.

Note that the only substantial difference of the generalized metric from the metric is that the range of generalized metric includes the infinity.

Theorem 1.3. Let (\mathcal{X}, d) be a complete generalized metric space and $\mathcal{J} : \mathcal{X} \rightarrow \mathcal{X}$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in \mathcal{X}$, either

$$d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

- (a) $d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) < \infty$ for all $n \geq n_0$;
- (b) the sequence $\{\mathcal{J}^n x\}$ converges to a fixed point y^* of \mathcal{J} ;
- (c) y^* is the unique fixed point of \mathcal{J} in the set $\mathcal{Y} = \{y \in \mathcal{X} : d(\mathcal{J}^{n_0} x, y) < \infty\}$;
- (d) $d(y, y^*) \leq \frac{1}{1-L} d(y, \mathcal{J}y)$ for all $y \in \mathcal{Y}$.

2. MAIN RESULTS

Using the fixed point method, we prove the Hyers-Ulam stability of lattice homomorphisms in Banach lattices.

Theorem 2.1. Let \mathcal{X}, \mathcal{Y} be Banach lattices. Consider a positive operator $\mathcal{F} : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$\|F(\tau x \vee \eta y) - \tau F(x) \vee \eta F(y)\| \leq \varphi(\tau x \vee \eta y, \tau x \wedge \eta y), \quad (2.1)$$

where $\varphi : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ is a mapping such that

$$\varphi(x, y) \leq (\tau\eta)^{\frac{\alpha}{2}} \varphi\left(\frac{x}{\tau}, \frac{y}{\eta}\right)$$

for all $x, y \in \mathcal{X}$, $\tau, \eta \geq 1$ and for which there is a real number $\alpha \in [0, \frac{1}{2})$. Then there is a unique positive operator $T : \mathcal{X} \rightarrow \mathcal{Y}$ satisfying the properties P_1, P_2 and the inequality

$$\|T(x) - F(x)\| \leq \frac{\tau^\alpha}{\tau - \tau^\alpha}$$

for all $x \in \mathcal{X}$.

Proof. Putting $\tau = \eta$ and $x = y$ in (2.1), we get

$$\|F(\tau x) - \tau F(x)\| \leq \varphi(\tau x, \tau x).$$

Then

$$\left\| \frac{1}{\tau} F(\tau x) - F(x) \right\| \leq \frac{1}{\tau} \varphi(\tau x, \tau x) \leq \tau^{\alpha-1} \varphi(x, x). \quad (2.2)$$

Consider the set

$$\Delta = \{g \mid g : \mathcal{X} \rightarrow \mathcal{Y} \text{ } g(0) = 0\}$$

and introduce the generalized metric on Δ

$$d(g, h) = \inf \{c \in \mathbb{R}^+, \|g(x) - h(x)\| \leq c \varphi(x, x) \text{ for all } x \in \mathcal{X}\},$$

where as usual, $\inf \emptyset = \infty$. It is easy to show that (Δ, d) is complete generalized metric space.

Now we define the operator $J : \Delta \rightarrow \Delta$ by

$$Jg(x) = \frac{1}{\tau} g(\tau x)$$

for all $x \in \mathcal{X}$. Given $g, h \in \Delta$, let $c \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq c$, that is,

$$\|g(x) - h(x)\| \leq c \varphi(x, x).$$

So we have

$$\begin{aligned} \|Jg(x) - Jh(x)\| &= \frac{1}{\tau} \|g(\tau x) - h(\tau x)\| \leq \frac{1}{\tau} c \varphi(\tau x, \tau x) \\ &\leq \frac{1}{\tau} c \tau^\alpha \varphi(x, x) = \tau^{\alpha-1} c \varphi(x, x) \end{aligned}$$

for all $x \in \mathcal{X}$, that is, $d(Jg, Jh) < \tau^{\alpha-1} c$. Thus we have

$$d(Jg, Jh) \leq \tau^{\alpha-1} d(g, h)$$

for all $g, h \in \Delta$. So J is a strictly contractive mapping with constant $\tau^{\alpha-1} < 1$ on Δ . For all $g, h \in \Delta$ and $\alpha \in [0, \frac{1}{2})$. By (2.2), we have

$$d(JF, F) \leq \tau^{\alpha-1} < \infty.$$

By Theorem 1.3, there exists a mapping $T : \mathcal{X} \rightarrow \mathcal{Y}$ satisfying the following:

1. T is a fixed point of J , i.e.,

$$T(\tau x) = \tau T(x)$$

for all $x \in \mathcal{X}$. Also the mapping T is a unique fixed point of J in the set

$$M = \{g \in \Delta : d(g, h) < \infty\}.$$

This implies that P_2 holds.

2. $d(J^n F, T) \rightarrow 0$ as $n \rightarrow \infty$. This implies the equality

$$\lim_{n \rightarrow \infty} \frac{1}{\tau^n} F(2^n x) = T(x)$$

for all $x \in \mathcal{X}$.

3. $d(F, T) \leq \frac{1}{1-L} d(F, JF)$, which implies the inequality

$$\|T(x) - F(x)\| \leq \frac{\tau^{\alpha-1}}{1 - \tau^{\alpha-1}} = \frac{\tau^\alpha}{\tau - \tau^\alpha},$$

which implies that the inequality (2.1) holds.

Now we show that T satisfies P_1 . Putting $\tau = \eta = \tau^n$ in (2.1), we get

$$\|F(\tau^n(x \vee y)) - \tau^n F(x) \vee \tau^n F(y)\| \leq \tau^{2n\alpha} \varphi(x \vee y, x \wedge y). \quad (2.3)$$

Replacing x, y by $\tau^n x$ and $\tau^n y$ in (2.3), respectively, we get

$$\begin{aligned} \|F(\tau^{2n}(x \vee y)) - \tau^n F(\tau^n x) \vee \tau^n F(\tau^n y)\| &\leq \tau^{2n\alpha} \varphi(\tau^n x \vee \tau^n y, \tau^n x \wedge \tau^n y) \\ &= \tau^{4n\alpha} (\varphi(x \vee y, x \wedge y)). \end{aligned}$$

Then

$$\left\| \frac{1}{\tau^{2n}} F(\tau^{2n}(x \vee y)) - \frac{1}{\tau^n} F(\tau^n x) \vee \frac{1}{\tau^n} F(\tau^n y) \right\| \leq \left(\tau^{2n(2\alpha-1)} \cdot \varphi(x \vee y, x \wedge y) \right).$$

Since $\alpha \in [0, \frac{1}{2})$, when $n \rightarrow \infty$, we have

$$\|T(x \vee y) - T(x) \vee T(y)\| \leq 0.$$

and so

$$T(x \vee y) = T(x) \vee T(y)$$

for all $x, y \in \mathcal{X}$. Note that the lattice operation is continuous. \square

Theorem 2.2. Let \mathcal{X}, \mathcal{Y} be Banach lattices and let a continuous function $p : [0, \infty) \rightarrow [0, \infty)$ be given. Consider a positive $T : \mathcal{X} \rightarrow \mathcal{Y}$ for which there are real numbers $\nu \in (0, \infty)$ and $0 \leq r < 1$ such that

$$\left\| T(\alpha|x| \vee \beta|y|) - \frac{\alpha p(\alpha)T(|x|) \vee \beta p(\beta)T(|y|)}{p(\alpha) \vee p(\beta)} \right\| \leq \nu (\|x\|^r + \|y\|^r) \quad (2.4)$$

for all $x, y \in \mathcal{X}$ and $\alpha, \beta \in \mathcal{R}^+$. Then there exist a unique positive mapping $F : \mathcal{X} \rightarrow \mathcal{Y}$ which satisfies the properties P_1, P_2 and the inequality

$$\|F(|x|) - T(|x|)\| \leq \frac{2\nu}{2 - 2^r}$$

for all $x \in \mathcal{X}$.

Proof. Putting $\alpha = \beta = 2$ and $x = y$ in (2.4), we get

$$\left\| T(2|x| \vee 2|x|) - \frac{2p(2)T(|x|) \vee 2p(2)T(|x|)}{p(2) \vee p(2)} \right\| \leq 2\nu\|x\|^r$$

for all $x \in \mathcal{X}$ and $r \in [0, 1)$. Thus

$$\|T(2|x|) - 2T(|x|)\| \leq 2\nu\|x\|^r$$

and so

$$\left\| \frac{1}{2}T(2|x|) - T(|x|) \right\| \leq \nu\|x\|^r \quad (2.5)$$

for all $x \in \mathcal{X}$ and $\alpha \in [0, 1)$. Consider the set

$$\Delta := \{S : S : \mathcal{X} \rightarrow \mathcal{Y}, S(0) = 0\}$$

and introduce the generalized metric on Δ

$$d(S, H) = \inf\{c \in \mathcal{R}^+, \|S(x) - H(x)\| \leq c\|x\|^r, \forall x \in \mathcal{X}\},$$

where, as usual, $\inf \emptyset = \infty$. It is known that (Δ, d) is complete. Now we define the mapping $J : \Delta \rightarrow \Delta$ by

$$JS(|x|) = \frac{1}{2}S(2|x|)$$

for all $x \in \mathcal{X}$. First we assert that J is strictly contractive with constant 2^{r-1} on Δ . Given $S, H \in \Delta$, let $c \in [0, \infty]$ be an arbitrary constant with $d(S, H) < c$, that is,

$$\|S(|x|) - H(|x|)\| \leq c\|x\|^r.$$

So we have

$$\|JS(x) - JH(x)\| = \frac{1}{2}\|S(2|x|) - H(2|x|)\| \leq \frac{1}{2}c\|2x\|^r = 2^{r-1}c\|x\|^r$$

for all $x \in \mathcal{X}$, that is, $d(JS, JH) \leq 2^{r-1}c$. Thus we have

$$d(JS, JH) \leq 2^{r-1}d(S, H)$$

for all $S, H \in \Delta$ and so J is strictly contractive with constant $2^{r-1} < 1$ on Δ . For all $S, H \in \Delta$ and $r \in [0, 1]$. By (2.5) we have

$$d(JF, F) \leq \nu < \infty$$

By Theorem 1.3, there exists a mapping $F : \mathcal{X} \rightarrow \mathcal{Y}$ satisfying the following:

1. F is a fixed point of J i. e.

$$F(2|x|) = 2F(|x|)$$

for all $x \in \mathcal{X}$. Also the mapping F is a unique fixed point of J in the set

$$M = \{S \in \Delta : d(S, H) < \infty\}.$$

2. $d(J^n T, F) \rightarrow 0$ as $n \rightarrow \infty$. This implies the equality

$$\lim_{n \rightarrow \infty} \frac{1}{2^n} T(2^n x) = F(x) \quad (2.6)$$

for all $x \in \mathcal{X}$.

3. $d(T, F) \leq \frac{1}{1-L} d(T, JT)$, which implies the inequality

$$\|F(|x|) - T(|x|)\| \leq \frac{\nu}{1 - 2^{r-1}} = \frac{2\nu}{2 - 2^r}.$$

This implies that the inequality (2.2) holds.

Now, we show that F is a lattice homomorphism. Putting $\alpha = \beta = 2^n$ in (2.4),

$$\|T(2^n(|x| \vee |y|)) - 2^n(T(|x|) \vee T(|y|))\| \leq \nu(\|x\|^r + \|y\|^r). \quad (2.7)$$

Replacing x and y by 2^x and 2^y in (2.7), respectively, we obtain

$$\|T(4^n(|x| \vee |y|)) - 2^n(T(2^n|x|) \vee T(2^n|y|))\| \leq 2^{nr}\nu(\|x\|^r + \|y\|^r)$$

and so

$$\left\| \frac{1}{4^n} T(4^n(|x| \vee |y|)) - \frac{1}{2^n} (T(2^n|x|) \vee T(2^n|y|)) \right\| \leq 2^{n(r-2)}\nu(\|x\|^r + \|y\|^r).$$

As $n \rightarrow \infty$, we have

$$\|F(|x| \vee |y|) - F(|x|) \vee F(|y|)\| \leq 0.$$

and so

$$F(|x| \vee |y|) = F(|x|) \vee F(|y|)$$

for all $x, y \in \mathcal{X}$. Next we show that $T(\alpha|x|) = \alpha T(|x|)$ for all $x \in \mathcal{X}$ and all real numbers $\alpha \in [0, \infty)$. Letting $\alpha = \beta, y = 0$ and replacing α by $2^n\alpha$ in (2.4), we get $F(0) = 0$ and so F satisfies P_1 . So $T(0) = 0$ with (2.6) and

$$\|T(2^n\alpha|x|) - 2^n\alpha T(|x|)\| \leq \nu\|x\|^r \quad (2.8)$$

for all $x \in \mathcal{X}$ and all real numbers $\alpha \in [0, \infty)$. Replacing x by $2^n x$ in (2.8),

$$\|T(4^n\alpha|x|) - 2^n\alpha T(2^n|x|)\| \leq \nu 2^{nr} \|x\|^r$$

and so

$$\left\| \frac{T(4^n\alpha|x|)}{4^n} - \alpha \frac{T(2^n|x|)}{2^n} \right\| \leq \nu 2^{n(r-2)} \|x\|^r$$

for all $x \in \mathcal{X}$. As $n \rightarrow \infty$, we obtain

$$\|F(\alpha|x|) - \alpha F(|x|)\| \leq 0$$

and so

$$F(\alpha|x|) = \alpha F(|x|).$$

for all $x \in \mathcal{X}$ and $\alpha \in [0, \infty)$. □

Corollary 2.1. *Let \mathcal{X}, \mathcal{Y} be Banach lattices. Consider a positive operator $T : \mathcal{X} \rightarrow \mathcal{Y}$ for which there are real numbers $\nu \in (0, \infty)$ and $0 \leq r < 1$ such that*

$$\|T(\alpha|x| \vee \beta|y|) - \alpha T(|x|) \vee \beta T(|y|)\| \leq \nu(\|x\|^r + \|y\|^r)$$

for all $x, y \in \mathcal{X}$ and $\alpha, \beta \in \mathcal{R}^+$. Then there exists a unique positive mapping $F : \mathcal{X} \rightarrow \mathcal{Y}$ which satisfies the properties P_1, P_2 and the inequality

$$\|F(|x|) - T(|x|)\| \leq \frac{2\nu}{2 - 2^r}.$$

for all $x \in \mathcal{X}$.

Corollary 2.2. *Let \mathcal{X}, \mathcal{Y} be Banach lattices. Consider a positive operator $T : \mathcal{X} \rightarrow \mathcal{Y}$ for which there are real numbers $\nu \in (0, \infty)$ and $0 \leq r < 1$ such that*

$$\left\| T(\alpha|x| \vee \beta|y|) - \frac{\alpha^2 T(|x|) \vee \beta^2 T(|y|)}{\alpha \vee \beta} \right\| \leq \nu(\|x\|^r + \|y\|^r)$$

for all $x, y \in \mathcal{X}$ and $\alpha, \beta \in \mathcal{R}^+$. Then there exists a unique positive mapping $F : \mathcal{X} \rightarrow \mathcal{Y}$ which satisfies the properties P_1, P_2 and the inequality

$$\|F(|x|) - T(|x|)\| \leq \frac{2\nu}{2 - 2^r}$$

for all $x \in \mathcal{X}$.

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UNIQUENESS THEOREM OF MEROMORPHIC FUNCTIONS AND THEIR k -TH DERIVATIVES SHARING SET

JUNFENG XU AND FENG LÜ

ABSTRACT. In this paper, due to the theories of normal family and complex differential equation, we consider a uniqueness problem of meromorphic functions share set $S = \{a, b\}$ with their k -th derivatives.

1. INTRODUCTION AND MAIN RESULTS

Let \mathcal{F} be a family of meromorphic functions defined in D . \mathcal{F} is said to be normal in D , in the sense of Montel, if for any sequence $f_n \in \mathcal{F}$, there exists a subsequence f_{n_j} , such that f_{n_j} converges spherically locally uniformly in D , to a meromorphic function or ∞ (see, [18]).

Let f and g be two meromorphic functions in a domain D , and let a be a complex number. If $g(z) = a$ whenever $f(z) = a$, we write $f(z) = a \Rightarrow g(z) = a$. If $f(z) = a \Rightarrow g(z) = a$ and $g(z) = a \Rightarrow f(z) = a$, we write $f(z) = a \Leftrightarrow g(z) = a$ and say that f and g share the value a IM (ignoring multiplicity). If $f - a$ and $g - a$ have the same zeros with the same multiplicities, we write $f(z) = a \Rrightarrow g(z) = a$ and say that f and g share the value a CM (counting multiplicity). Let S be a set of complex numbers. Provide that $f(z) \in S$ if and only if $g(z) \in S$ in a domain D , then we say f and g share the set S in D . It is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna theory, as found in [4, 21].

In the theory of normal family, it is meaningful to find sufficient conditions for normality(see. [1, 7, 8, 9, 10, 11, 15, 17, 20]). Recently, Y. Li [7] obtained a normal family of holomorphic functions share set with their k -th derivatives as follows.

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Theorem A. *Let \mathcal{F} be a family of holomorphic functions in a domain D , let $k(\geq 2)$ be a positive integer, and let a, b be two distinct finite complex numbers. If for each $f \in \mathcal{F}$, all the zeros of f are of multiplicity at least k , and f and f' share the set $S = \{a, b\}$, then \mathcal{F} is normal in D .*

Remark 1. In fact, for the case $ab \neq 0$, the conclusion of Theorem A still holds if the condition f and $f^{(k)}$ share the set $S = \{a, b\}$ CM is replaced by

$$f(z) \in S \Rightarrow f^{(k)}(z) \in S.$$

See Section 3.

In the uniqueness theory, an important subtopic that a meromorphic function and its derivative share some values or functions or set is well investigated. Due to Theorem A, Y. Li [7] obtained a uniqueness theorem of entire functions.

Theorem B. *Let $k(\geq 2)$ be a positive integer, and let a, b be two distinct finite complex numbers, and let f be a non-constant entire function. If all the zeros of f are of multiplicity at least k , and f and $f^{(k)}$ share the set $S = \{a, b\}$ CM, then*

- (1). $f(z) = Ce^{Dz}$, where $C \neq 0$ and D are two constants with $D^k = \pm 1$,
- (2). $f = -f^{(k)} + a + b$.

In [7], Y. Li also gave an example to show that the case (2) can not omitted.

Example 1. Let $f(z) = \cos^2 \frac{z}{2}$. Then f and f'' share set $\{0, \frac{1}{2}\}$ CM and all zeros of f are of multiplicity at least 2. Obviously, $f = -f'' + \frac{1}{2}$.

After considering Theorem B and Example 1, we naturally ask the following questions.

Question 1. What happens if f is a meromorphic function?

Question 2. Note that $k = 2$ in **Example 1**. Naturally, we ask whether Case (2) occurs for $k \neq 2$ or not?

Question 3. What's the specific form of f in Case (2)?

In the work, we focus on the above questions. Basing on the idea of Y. Li in [7] and due to the theories of normal family and complex differential equation, we further study the uniqueness problem of meromorphic functions of finitely many poles sharing a set CM with their derivatives.

Theorem 1.1. *Let $k(\geq 2)$ be a positive integer, and let a, b be two distinct finite complex numbers, and let f be a non-constant meromorphic function with finitely many poles. If all the zeros of f are of multiplicity at least k , and f and $f^{(k)}$ share the set $S = \{a, b\}$ CM, then one of the following cases must occur:*

- (1). $f(z) = Ce^{Dz}$, where $C \neq 0$ and D are two constants with $D^k = 1$, and $f = f^{(k)}$;
- (2). $f(z) = Ce^{Dz}$, where $C \neq 0$ and D are two constants with $D^k = -1$, $f = -f^{(k)}$ and $S = \{a, -a\}$;
- (3). $f(z) = A_1e^{iz} + A_2e^{-iz} + a + b$, where A_1 and A_2 are two nonzero constants with $(a + b)^2 = 4A_1A_2$, $f = -f'' + a + b$, and k must be 2.

Remark 2. For the special case that $A_1 = A_2 = \frac{1}{4}$, $a = 0$ and $b = \frac{1}{2}$, then Case (3) becomes **Example 1**.

Remark 3. We answer the **Questions 2** and find out the case (2) occurs only for $k = 2$ in Theorem B. We also answer the **Question 3** and give the form of f . We partial answer the **Question 1**.

In 2008, we considered the case of $k = 1$ and obtained a normal criteria theorem and a uniqueness theorem[10].

Theorem C. Let \mathcal{F} be a family of functions holomorphic in a domain, let a and b be two distinct finite complex numbers with $a + b \neq 0$. If for all $f \in \mathcal{F}$, f and f' share $S = \{a, b\}$ CM, then \mathcal{F} is normal in D .

Theorem D. Let a and b be two distinct complex numbers with $a + b \neq 0$, and let $f(z)$ be a nonconstant entire function. If f and f' share the set $\{a, b\}$ CM, then one and only one of the following conclusions holds: (i) $f = Ae^z$ or (ii) $f = Ae^{-z} + a + b$, where A is a nonzero constant.

By the same way to Theorem 1.1, we can obtain the following.

Theorem 1.2. Let a and b be two distinct complex numbers with $a + b \neq 0$, and let $f(z)$ be a nonconstant meromorphic function with finite poles. If f and f' share the set $\{a, b\}$ CM, then one and only one of the following conclusions holds: (i) $f = Ae^z$ or (ii) $f = Ae^{-z} + a + b$, where A is a nonzero constant.

2. SOME LEMMAS

Lemma 2.1. [15] Let \mathcal{F} be a family of functions holomorphic on a domain D , all of whose zeros have multiplicity at least k , and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$. Then if \mathcal{F} is not normal at $z_0 \in D$, for each $0 \leq \alpha \leq k$, there exist,

- (a) a number $0 < r < 1$;
- (b) points $z_n \rightarrow z_0$;
- (c) functions $f_n \in \mathcal{F}$, and
- (d) positive number $\rho_n \rightarrow \infty$ such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \xi) = g_n(\xi) \rightarrow g(\xi)$ locally uniformly, where g is a nonconstant entire function on C with order at most 1, all of whose zeros have multiplicity at least k , such that $g^\#(\xi) \leq g^\#(0) = kA + 1$.

Here, as usual,

$$g^\sharp(\xi) = \frac{|g'(\xi)|}{1 + |g(\xi)|^2}$$

is the spherical derivative.

Lemma 2.2. [3, 13] *Let f be an entire (resp. meromorphic) function, and let M be a positive number. If $f^\sharp(z) \leq M$ for any $z \in C$, then f is of order at most 1 (resp. 2).*

It is well known that it is very important of the Wiman-Valiron theory[5, 6] to investigate the property of the entire solutions of differential equations. In 1999, Zong-Xuan Chen[2] has extended the Wiman-Valiron theory from entire functions to meromorphic functions with infinitely many poles. Here we show the following form given by Jun Wang and Wei-Ran Lü[19].

Lemma 2.3. *Let $f(z) = \frac{g(z)}{d(z)}$ be a meromorphic function with $\rho(f) = \rho$, where g, d are entire functions satisfying one of the following conditions:*
(i) g being transcendental and d being polynomial;
(ii) g, d all being transcendental and $\lambda(d) = \rho(d) = \beta < \rho(g) = \rho$.
Then there exists one sequence $\{r_k\} (r_k \rightarrow \infty)$ such that

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu(r_k, g)}{z}\right)^n (1 + o(1)), \quad n \in \mathbb{N}$$

holds for enough large r_k as $|z| = r_k$ and $|g(z)| = M(r_k, g)$, where $\nu(r_k, g)$ denotes the central index of g .

Lemma 2.4. [14] *Let f be an entire function of order at most 1 and k be a positive integer, then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = o(\log r), \quad \text{as } r \rightarrow \infty.$$

Lemma 2.5. [21] *Let f be a nonconstant meromorphic function, and a_j ($j = 1, \dots, q$) be q (≥ 3) distinct constant (one of them may be ∞), then*

$$(q-2)T(r, f) \leq \sum_{j=1}^q \overline{N}\left(r, \frac{1}{f-a_j}\right) + S(r, f),$$

where

$$S(r, f) = m\left(r, \frac{f'}{f}\right) + m\left(r, \sum_{j=1}^q \frac{f'}{f-a_j}\right) + O(1).$$

Combining Lemmas 2.4 and 2.5, we have the following special case of the Nevanlinna's second fundamental theorem.

Lemma 2.6. *Let f be a nonconstant entire function of order at most 1, and a_j ($j = 1, \dots, q$) be q (≥ 3) distinct constants (one of them may be ∞), then*

$$(q-2)T(r, f) \leq \sum_{j=1}^q \overline{N}\left(r, \frac{1}{f-a_j}\right) + o(\log r).$$

3. PROOF OF THEOREM 1.1

Firstly, we will prove that the meromorphic (resp. entire) function f is of order at most 2 (resp. 1).

Suppose that the spherical derivative of f is bounded. Then by Lemma 2.2, we have meromorphic (resp. entire) function f is of order at most 2 (resp. 1). Now, we assume that the spherical derivative of f is unbounded. Then there exist a sequence $\{w_n\}$ such that $w_n \rightarrow \infty$, $f^\sharp(w_n) \rightarrow +\infty$ as $n \rightarrow \infty$.

Define $D = \{z : |z| < 1\}$ and

$$F_n(z) = f(w_n + z).$$

Since f only has finitely many poles, we can assume that all $F_n(z)$ are analytic in D . Furthermore, $F_n^\sharp(0) = f^\sharp(w_n) \rightarrow \infty$ as $n \rightarrow \infty$. It follows from Marty's criterion that $(F_n)_n$ is not normal at $z = 0$.

Obviously, for each n , F_n has zeros with multiplicities at least k , F_n and $F_n^{(k)}$ share S CM. Thus, from Theorem A, we derive that $(F_n)_n$ is normal at $z = 0$, a contradiction.

Thus, we prove that the meromorphic (resp. entire) function f is of order at most 2 (resp. 1).

Since f and $f^{(k)}$ share S CM and f has finitely many poles, we have

$$(3.1) \quad \frac{(f^{(k)} - a)(f^{(k)} - b)}{(f - a)(f - b)} = \frac{e^Q}{P},$$

where P, Q are two polynomials. Rewrite (3.1) as follows.

$$(3.2) \quad Q = \log P \frac{(\frac{f^{(k)}}{f} - \frac{a}{f})(\frac{f^{(k)}}{f} - \frac{b}{f})}{(1 - \frac{a}{f})(1 - \frac{b}{f})},$$

where $\log h$ is the principle branch of $\text{Log } h$.

If $f(z) = \frac{g(z)}{d(z)}$ is a transcendental meromorphic function, where $g(z)$ is a transcendental entire function and $d(z)$ is a polynomial. Then by Lemma 2.3, we get

$$(3.3) \quad \frac{f^{(k)}(z)}{f(z)} = \left(\frac{\nu(r_k, g)}{z}\right)^k (1 + o(1)),$$

holds for enough large r_k as $|z| = r_k$ and $|g(z)| = M(r_k, g)$. Note that f is transcendental, we have $\frac{a}{f}|_{z_r} \rightarrow 0$ and $\frac{b}{f}|_{z_r} \rightarrow 0$ as $r \rightarrow \infty$. It follows from

the fact g is of finite order that $\log \nu(r, g) = O(\log r)$. Then, we deduce that

$$|Q(z)| = \left| \log P \frac{\left(\frac{f^{(k)}}{f} - \frac{a}{f}\right)\left(\frac{f^{(k)}}{f} - \frac{b}{f}\right)}{\left(1 - \frac{a}{f}\right)\left(1 - \frac{b}{f}\right)} \right|_{z_r} = O(\log r),$$

for enough large r_k as $|z| = r_k$ and $|g(z)| = M(r_k, g)$. It implies that Q is a constant.

If $f(z)$ is a rational function, then by (3.1) we know that Q must be a constant.

Without loss of generality, we rewrite (3.1) as

$$\frac{1}{P} = \frac{(f^{(k)} - a)(f^{(k)} - b)}{(f - a)(f - b)}.$$

Next, we will prove that P is also a constant. On the contrary, suppose that P is not a constant. We know any zero of P comes from the pole of f , so $d = \deg P \geq 2k$.

From the above equation, we get

$$1 = P \frac{\left(\frac{f^{(k)}}{f} - \frac{a}{f}\right)\left(\frac{f^{(k)}}{f} - \frac{b}{f}\right)}{\left(1 - \frac{a}{f}\right)\left(1 - \frac{b}{f}\right)}.$$

In a similar way as the above, we get

$$1 = |P(z_r) \left(\frac{\nu(r, g)}{z_r}\right)^{2k} (1 + o(1))| = |\nu(r, g)^2|_{z_r}^{d-2k} = \nu(r, g)^2 r^{d-2k},$$

possibly outside a finite logarithmic measure set E , where $|g(z_r)| = M(r, g)$ and $|z| = r_k$. Since $d = \deg P \geq 2k$, it implies that $\nu(r, g)$ is bound, a contradiction. Hence, P is also a constant.

Thus, we prove that

$$A = \frac{(f^{(k)} - a)(f^{(k)} - b)}{(f - a)(f - b)},$$

where A is a nonzero constant. From the above equation, we see that f is an entire function, so the order of f is at most 1.

Set $F = f - \frac{a+b}{2}$ and $G = f^{(k)} - \frac{a+b}{2}$. Then $\frac{G^2 - \frac{(a-b)^2}{4}}{F^2 - \frac{(a-b)^2}{4}} = A$. Set $h_1 = G - \sqrt{A}F$ and $h_2 = G + \sqrt{A}F$, then we have

$$h_1 h_2 = \frac{(a-b)^2}{4} (1 - A).$$

We consider two cases.

Case 1. $A \neq 1$.

Obviously, h_1, h_2 has no zeros and poles. Then we set $h_1(z) = A_1 e^{Bz}$ and $h_2(z) = A_2 e^{-Bz}$, where A_1, A_2, B are constants. Furthermore, we have

$$\begin{aligned} f(z) &= \frac{a+b}{2} + \frac{1}{2\sqrt{A}}(-A_1 e^{Bz} + A_2 e^{-Bz}), \\ f^{(k)}(z) &= \frac{a+b}{2} + \frac{1}{2}(A_1 e^{Bz} + A_2 e^{-Bz}), \\ f'(z) &= \frac{B}{2\sqrt{A}}(-A_1 e^{Bz} - A_2 e^{-Bz}). \end{aligned}$$

The above part is based on the idea in [7]. Now, we consider two subcases again.

Case 1.1. $A_1 A_2 \neq 0$.

It follows from the form of f that f has infinitely many zeros. Noting that the zeros of f has multiplicities at least k , we have $f^{(s)}$ ($s = 0, \dots, k-2$) has multiple zeros. Clearly, f' just has simple zeros. Then, $k-2 \leq 0$, so k must equal to 2. By differentiating f' one time, we have

$$f''(z) = \frac{B^2}{2\sqrt{A}}(-A_1 e^{Bz} + A_2 e^{-Bz}).$$

Comparing it to the above form of $f^{(k)}$, we have

$$\frac{a+b}{2} + \frac{1}{2}(A_1 e^{Bz} + A_2 e^{-Bz}) = \frac{B^2}{2\sqrt{A}}(-A_1 e^{Bz} + A_2 e^{-Bz}),$$

which means that either A_1 or A_2 is zero, a contradiction.

Case 1.2. $A_1 A_2 = 0$.

Without loss of generality, we assume that $A_2 = 0$. Then we have

$$f(z) = \frac{a+b}{2} - \frac{1}{2\sqrt{A}}A_1 e^{Bz}.$$

From the form of f , it is easy to see that if f has zeros, then f just has simple zeros. It contradicts with the fact f has zeros of multiplicities at least k . So, f has no zeros and $a+b=0$. Thus, we can set

$$f(z) = C e^{Dz},$$

where C, D are two constants. By differentiating f k -times, we have $f^{(k)}(z) = C D^k e^{Dz}$. From f and $f^{(k)}$ share S CM, we have $D^k = \pm 1$.

If $D^k = 1$, then $f = f^{(k)}$, and f and $f^{(k)}$ share a, b CM.

If $D^k = -1$, then $f = -f^{(k)}$, and $b = -a$.

Case 2. $A = 1$.

Then it is easy to see that $f = f^{(k)}$ or $f^{(k)} + f = a + b$.

Suppose that $f = f^{(k)}$. Noting that f equals to $f^{(k)}$, so they share 0 CM. Moreover, from the fact that all the zeros of f has multiplicities at least k , we derive f has no zeros. Then, by the same way in **Case 1.2**, we get the same results.

Finally, by the similar way in [12], we will discuss the case of $f^{(k)} + f = a + b$.

Solving the differential equation, we have

$$(3.4) \quad f(z) = \sum_{j=0}^{k-1} C_j \exp^{\lambda_j z} + a + b,$$

where $\lambda_j = \exp^{\frac{2j\pi + \pi}{k}i}$ and C_j are constants. Since f is a non-constant, then there exist $C_j \in \{C_0, C_1, \dots, C_{k-1}\}$ such that $C_j \neq 0$. Denote the non-zero constants in $\{C_j\}$ by C_{j_m} $0 \leq j_m \leq k-1$ and $m = 0, 1, \dots, s$, $s \leq k-1$. Thus, rewrite (3.4) as

$$(3.5) \quad f(z) = \sum_{m=0}^s C_{j_m} \exp^{\lambda_{j_m} z} + a + b.$$

Differentiating (3.5) t -times yields

$$(3.6) \quad f^{(t)}(z) = \sum_{m=0}^s C_{j_m} \lambda_{j_m}^t \exp^{\lambda_{j_m} z}, \quad (t = 1, 2, \dots, k-1).$$

Suppose that f has finitely many zeros, then we can set $f(z) = P_1(z)e^{\lambda z}$, where P_1 is a polynomial. By differentiating it k times, we have

$$f^{(k)}(z) = [\lambda^k P_1 + \lambda^{k-1} P_1' + H(P_1'', P_1''', \dots, P_1^{(k)})]e^{\lambda z},$$

where $H(P_1'', P_1''', \dots, P_1^{(k)})$ is the linear combination of $P_1'', P_1''', \dots, P_1^{(k)}$. Substituting the above forms of f and $f^{(k)}$ into $f + f^{(k)} = a + b$, we derive that

$$P_1 + \lambda^k P_1 + \lambda^{k-1} P_1' + H(P_1'', P_1''', \dots, P_1^{(k)}) = 0,$$

which implies that $\lambda^k = -1$ and $P_1' = 0$. Thus, P_1 is a constant and f has no zeros. By the same way in **Subcase 1.2**, we derive the desired results.

Thus, in what follows, we assume that f has infinitely many zeros, say $z_n = r_n e^{i\theta_n}$, where $0 \leq \theta_n < 2\pi$. Without loss of generality, we may assume that $\theta_n \rightarrow \theta_0$ and $r_n \rightarrow \infty$ as $n \rightarrow \infty$. Substituting z_n into (3.5) and (3.6), we have

$$(3.7) \quad f(z_n) = \sum_{m=0}^s C_{j_m} \exp^{\lambda_{j_m} z_n} = -(a + b)$$

and

$$(3.8) \quad f^{(t)}(z_n) = \sum_{m=0}^s C_{j_m} (\lambda_{j_m})^t \exp^{\lambda_{j_m} z_n} = 0, \quad (t = 1, 2, \dots, k-1).$$

We consider two cases again.

Subcase 2.1. $s = k - 1$.

From (3.7) and (3.8), we have

$$\begin{pmatrix} a+b \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} C_{j_0} & C_{j_1} & \cdots & C_{j_{k-1}} \\ C_{j_0} \lambda_{j_0} & C_{j_1} \lambda_{j_1} & \cdots & C_{j_{k-1}} \lambda_{j_{k-1}} \\ \vdots & \vdots & \ddots & \vdots \\ C_{j_0} (\lambda_{j_0})^{k-1} & C_{j_1} (\lambda_{j_1})^{k-1} & \cdots & C_{j_{k-1}} (\lambda_{j_{k-1}})^{k-1} \end{pmatrix} \begin{pmatrix} \exp^{\lambda_{j_0} z_n} \\ \exp^{\lambda_{j_1} z_n} \\ \vdots \\ \exp^{\lambda_{j_{k-1}} z_n} \end{pmatrix}.$$

We know

$$\begin{aligned} & \det \begin{pmatrix} C_{j_0} & C_{j_1} & \cdots & C_{j_{k-1}} \\ C_{j_0} \lambda_{j_0} & C_{j_1} \lambda_{j_1} & \cdots & C_{j_{k-1}} \lambda_{j_{k-1}} \\ \vdots & \vdots & \ddots & \vdots \\ C_{j_0} (\lambda_{j_0})^{k-1} & C_{j_1} (\lambda_{j_1})^{k-1} & \cdots & C_{j_{k-1}} (\lambda_{j_{k-1}})^{k-1} \end{pmatrix} \\ &= C_{j_0} C_{j_1} \cdots C_{j_{k-1}} \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_{j_0} & \lambda_{j_1} & \cdots & \lambda_{j_{k-1}} \\ \vdots & \vdots & \ddots & \vdots \\ (\lambda_{j_0})^{k-1} & (\lambda_{j_1})^{k-1} & \cdots & (\lambda_{j_{k-1}})^{k-1} \end{pmatrix} \\ &= C_{j_0} C_{j_1} \cdots C_{j_{k-1}} \prod_{0 \leq q < p \leq k-1} (\lambda_{j_p} - \lambda_{j_q}). \end{aligned}$$

It's is a Vandermonde determinant.

Noting that $\lambda_{j_p} \neq \lambda_{j_q}$ ($0 \leq q < p \leq k-1$), we obtain that the system of linear equations of $\exp^{\lambda_{j_0} z_n}, \exp^{\lambda_{j_1} z_n}, \dots, \exp^{\lambda_{j_{k-1}} z_n}$ has a unique solution. A routine calculation leads to the solution that

$$(3.9) \quad \exp^{\lambda_{j_p} z_n} = D_p, \quad (0 \leq p \leq k-1).$$

where D_p is a constant and is of independent with z_n .

If $a + b = 0$, we see that $D_p = 0$, a contradiction. Then, we assume that $a + b \neq 0$.

Thus, as $n \rightarrow \infty$, by (3.9) we can deduce that

$$(3.10) \quad \cos(\theta_0 + \frac{2j_p\pi + \pi}{k}) = 0, \quad (0 \leq p \leq k-1).$$

Otherwise, we have $\cos(\theta_0 + \frac{2j_p\pi + \pi}{k}) > 0$ or $\cos(\theta_0 + \frac{2j_p\pi + \pi}{k}) < 0$.

If $\cos(\theta_0 + \frac{2j_p\pi + \pi}{k}) > 0$, then we can assume (for n large enough) $\cos(\theta_n + \frac{2j_p\pi + \pi}{k}) > \delta$, here δ is a small positive number. Thus, as $n \rightarrow \infty$, by (3.9)

we have

$$|D_p| = \exp^{r_n \cos(\theta_n + \frac{2j_p\pi + \pi}{k})} \rightarrow \infty,$$

a contradiction.

If $\cos(\theta_0 + \frac{2j_p\pi + \pi}{k}) < 0$, then we can assume (for n large enough) $\cos(\theta_n + \frac{2j_p\pi + \pi}{k}) < -\delta$, here δ is a small positive number. Thus, as $n \rightarrow \infty$, by (3.9) we have

$$|D_p| = \exp^{r_n \cos(\theta_n + \frac{2j_p\pi + \pi}{k})} \rightarrow 0,$$

a contradiction.

Observing that $0 \leq j_p, j_q \leq k-1$, by (3.10), we deduce

$$(3.11) \quad \left| \frac{2j_p\pi + \pi}{k} - \frac{2j_q\pi + \pi}{k} \right| = \pi, (0 \leq p \neq q \leq k-1).$$

Let $j_p = 0$ and $j_q = k-1$. Substitute them into (3.11), we have

$$2(k-1) = k,$$

that is $k = 2$. Thus, k must be 2.

Now we discuss the equation $f + f'' = a + b$. From the above discussion, we can obtain $\lambda_0 = i, \lambda_1 = -i$. Then, we have

$$f(z) = A_1 e^{iz} + A_2 e^{-iz} + a + b.$$

Noting that f has zeros of multiplicity at least 2, Then

$$(a + b)^2 = 4A_1 A_2.$$

Then, we finish the proof of this subcase.

Subcase 2.2. $s < k-1$.

Then, by (3.8), we can choose $t = 1, 2, \dots, s+1$. Then they form a system of linear equation of $\exp^{\lambda_{j_0} z_n}, \exp^{\lambda_{j_1} z_n}, \dots, \exp^{\lambda_{j_s} z_n}$. By solving it, we have

$$(3.12) \quad \exp^{\lambda_{j_p} z_n} = 0,$$

a contradiction.

Hence, we complete the proof of this theorem.

4. PROOF OF THEOREM 1.2

If Theorem A is replaced by Theorem C, by the same way to the proof of Theorem 1.1, we can also obtain the

$$A = \frac{(f' - a)(f' - b)}{(f - a)(f - b)},$$

where A is a nonzero constant. From the above equation, we see that f is an entire function. Hence we can get the conclusion by Theorem D.

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Compact adaptive aggregation multigrid method for Markov chains

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Abstract

A new adaptive aggregation-based multigrid scheme is presented for the calculation of the stationary probability vector of an irreducible Markov chain. By exploiting the experimental observation that components of vectors converge nonuniformly, we develop a new algorithm to speed up the on-the-fly adaptive multigrid method proposed by Treiter and Yavney [On-the-fly adaptive smoothed aggregation multigrid for Markov chains, *SIAM J. Sci. Comput.*, 33(2011): 2927-2949]. In our algorithm, the converged components are collected and compacted into one aggregate on the finest level, which is able to cut down the cost of coarsen operators construction and the total amount of work. In addition, we present a technique to delete the possible weak-links introduced in the process of aggregation. Several types of test cases are calculated, and experiment results show that the new adaptive method can improve the on-the-fly algorithm in terms of total execution time.

Key words: Adaptive aggregation multigrid; on-the-fly adaptive method; Markov chains; converged components

1 Introduction

This paper is concerned with a new adaptive multigrid method for the numerical calculation of the stationary probability vector of irreducible, large and sparse Markov matrices. Let $B \in \mathbb{R}^{n \times n}$ be a sparse column-stochastic matrix, which means $\mathbf{1}^T B = \mathbf{1}^T$, where $\mathbf{1}$ is the column vector of all ones, and $0 \leq b_{ij} \leq 1 \forall i, j$. We seek a vector $\mathbf{x} \in \mathbb{R}^n$ that satisfies

$$B\mathbf{x} = \mathbf{x}, \|\mathbf{x}\|_1 = 1, x_i \geq 0 \forall i. \quad (1.1)$$

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Matrix B represents the transition matrix of a Markov chain and \mathbf{x} is a stationary probability vector of this Markov chain. If B is irreducible, that is, there exists a path from each vertex i to each vertex j in its directed graph, then according to the Perron-Frobenius theorem for nonnegative matrices [1], the equation (1.1) has a unique solution \mathbf{x} , with $x_i > 0 \forall i$. This problem (1.1) is equivalent to the singular linear system

$$A\mathbf{x} = \mathbf{0}, \|\mathbf{x}\|_1 = 1, x_i \geq 0 \forall i. \quad (1.2)$$

where $A := I - B$, by $\mathbf{1}^T B = \mathbf{1}^T$, we have $\mathbf{1}^T A = \mathbf{0}$, which means the vector we seek, \mathbf{x} , is the only left null-vector of the matrix A .

Algebraic multigrid method (AMG) was developed and applied widely due to its efficiency for solving large problems arising from partial differential equations and M-matrices. Compared with geometric multigrid methods, AMG constructs the multigrid hierarchy only using the information of the given matrix, which extends the application of multigrid methods. However, it leads to the inefficiency and the lack of robustness, because the operators of these multigrid methods are constructed based on the unsatisfied assumptions made on the near null spaces of the matrices. To overcome this disadvantage, several adaptive algebraic multigrid methods were developed in [4, 26, 5]. The basic idea of these adaptive approaches was of improving multigrid methods by updating interpolation and coarsen operators to fit the slow-to-converge components of the vector. The idea was further developed in adaptive AMG [23] and adaptive SA [24, 25], where slow-to-converge components were exposed through multiscale development instead of relaxation on finest-level.

The Markov chains solver which was outlined in [13] was actually another form of adaptive AMG, because they share the same concept of updating operator to get more accurate approximation of the near null space of A . With the same idea, a multilevel adaptive aggregation [7] was suggested with aggregates updated in each step of the iteration. Based on this algorithm, a collection of Markov chains solvers were proposed recently: adaptive aggregation multigrid for Markov chains (AGG) [7], smoothed aggregation multigrid (SA) [6], AMG for Markov chains (MCAMG) [8]. Several accelerated methods were developed in [18, 10]. While all these adaptive approaches improved the algorithms robustness and accuracy by adapting coarsen operators in every cycle, they also suffered from considerable computation time for calculating the coarsen matrix [27]. The on-the-fly adaptive multigrid hierarchy for Markov chains which was developed in [19] significantly cut the cost of constructing the coarse-level operators. Here, the classical solution cycles are preferred over the adaptive cycles, under the assumption that the former is comparatively cheaper but it needs the operators provided by the latter.

The algorithm presented in this paper is inspired by the following experimental observation: when applying aggregation multigrid V-cycle to obtain approximation of stationary probability vector, the elements of the stationary probability vector do not converge uniformly. Based on this observation, we propose a compressed on-the-fly adaptive scheme to save the cost on constructing coarsen operators. The main idea is to compact the converged components into a single aggregate and rescale the coarsen operators. Also

we develop a new technique that deletes weak-links introduced by the above procedure. As the improvement of the on-the-fly adaptive aggregation method, the new algorithm adopts the same adaptive hierarchy as on-the-fly method does. It differs, however, in that the on-the-fly method uses operators supplied by SET cycles without any amendment, whereas in new algorithm, the coarsen operators are rescaled to smaller size to fit the not-converged-yet components. It is shown numerically that the new algorithm can reduce the total execution time of the on-the-fly adaptive multigrid method. New algorithm can also be applied to various adaptive multigrid Markov solvers. In this paper we apply it to the aggregation-based algebraic multigrid solver (AGG), with unsmoothed interpolation and prolongation operators.

In the next section, we give a brief description of multilevel aggregation multigrid method for Markov problems. Then we recall the on-the-fly adaptive framework in Section 3, which the new algorithm is based on. In Section 4, we outline the experimental observation as the stage for the introduction of new algorithm, and we compare the proposed algorithm with compatible relaxation method as well. Numerical tests are presented in Section 5.

2 Classical aggregation multigrid for Markov chains

In this section, we briefly recall the aggregation-based multigrid methods for Markov chains from [13, 7, 6]. The interpolation operators of aggregation multigrid are often smoothed to overcome the instinct difficulties produced by aggregation [6, 14]. In our work, we stay with the unsmoothed coarsen operators.

First, we define the multiplicative error \mathbf{e}_i by $\mathbf{x} = \text{diag}(\mathbf{x}_i)\mathbf{e}_i$, where \mathbf{x}_i is the current approximate at i th iterate. Thus we have

$$A\text{diag}(\mathbf{x}_i)\mathbf{e}_i = \mathbf{0}. \quad (2.1)$$

It is necessary to assume that all components of \mathbf{x}_i are nonzero. At convergence, $\mathbf{x}_i = \mathbf{x}$ and the fine-level error $\mathbf{e}_i = \mathbf{1}$, where $\mathbf{1}$ is the column vector with all ones.

Note that the aggregation technique used in this paper is the same as that used in [7], which is based on strength of connection in the scaled matrix $\tilde{A} = A\text{diag}(\mathbf{x}_i)$, the benefit of using the scaled matrix \tilde{A} instead of original matrix A is that the former gives more appropriate notion of weak and strong links than the latter, more details are in [7]. We consider node i is strongly connected to node j if

$$-\tilde{a}_{ij} \geq \theta \max_{k \neq i} \{-\tilde{a}_{ik}\}. \quad (2.2)$$

where $\theta \in [0, 1]$ is a strength threshold parameter, we choose $\theta = 0.8$. Aggregates based on the strength of connection are then constructed by the following procedure: choose point i with the largest value in current proximation \mathbf{x}_i from the unassigned points as the seed point of a new aggregate, then add all unassigned points j satisfies (4) to the new

aggregates. Repeat this procedure until all points are assigned to aggregates. Assuming that the n fine-level points are aggregated into m groups, then the aggregation matrix $Q \in \mathbb{R}^{n \times m}$ are formed, where $q_{ij} = 1$ indicates that fine-level point i belongs to aggregate j and $q_{ij} = 0$ the opposite[6]. Then the coarse version of (3) is given by

$$Q^T \text{Adiag}(\mathbf{x}_i) Q \mathbf{e}_c = 0, \quad (2.3)$$

where \mathbf{e}_c is the coarse-level approximation of the fine-level error \mathbf{e}_i , with $\mathbf{e}_i \approx Q \mathbf{e}_c$.

The restriction and prolongation operators, R and P are defined as follows:

$$R = Q^T, P = \text{diag}(\mathbf{x}_i) Q. \quad (2.4)$$

Then (5) can be rewritten as

$$R A P \mathbf{e}_c = 0. \quad (2.5)$$

Same as the definition of fine-level multiplicative error \mathbf{x}_i , the coarse-level error \mathbf{x}_c is given by

$$\mathbf{x}_c = \text{diag}(R \mathbf{x}_i) \mathbf{e}_c. \quad (2.6)$$

Notice that $P^T \mathbf{1} = R \mathbf{x}_i$, thus (3) can be rewritten as

$$R A P \text{diag}(P^T \mathbf{1})^{-1} \mathbf{x}_c = 0. \quad (2.7)$$

Then the coarse-level error equation (5) is equivalent to coarse-level probability equation $A_c \mathbf{x}_c = 0$, with coarsen matrix A_c defined by

$$A_c = R A P \text{diag}(P^T \mathbf{1})^{-1}. \quad (2.8)$$

When the coarsen solution \mathbf{x}_c is obtained, the next iterate, \mathbf{x}_{i+1} can be calculated according to the coarse-level correction

$$\mathbf{x}_{i+1} = P \mathbf{e}_c = P \text{diag}(P^T \mathbf{1})^{-1} \mathbf{x}_c. \quad (2.9)$$

In this paper we use weighted Jacobi method for all relaxation procedure, at each coarser level we perform ν_1 pre-relaxation and ν_2 post-relaxations. One iteration of weighted Jacobi relaxation applied to problem $A \mathbf{x} = b$ is given by

$$\mathbf{x} \leftarrow \mathbf{x} + \omega D^{-1} (b - A \mathbf{x}). \quad (2.10)$$

where D is the diagonal part of A , its relaxation parameter $\omega = 0.7$. On coarsest level we perform direct solver described in [8]. The procedure above is described in Algorithm 1, which is originally presented in [13]. The multilevel aggregation method is obtained by recursively applying Algorithm 1 to step 5.

Algorithm 1: Two-level aggregation for Markov chains $\mathbf{x} \leftarrow \text{AGG}(A, \mathbf{x}, \mu, v_1, v_2)$

Input: Initial vector: $\mathbf{x} \in \mathbb{R}^n$, operator: $A \in \mathbb{R}^{n \times n}$, cycle index: μ ,
 number of pre-relaxations: v_1 , number of post-relaxations: v_2 .

Output: New approximation to the solution of $A\mathbf{x} = \mathbf{0}$.

Algorithm:

if not at coarsest level

1. $\mathbf{x} \leftarrow \text{Relax}(A, \mathbf{x}, \mathbf{0})$ v_1 times
2. Build Q based on A and \mathbf{x}
3. Set $R \leftarrow Q^T$, $P \leftarrow \text{diag}(\mathbf{x}_i)Q$
4. Set $\mathbf{x}_c \leftarrow R\mathbf{x}$, and repeat $\mu \geq 1$ times:
 $\mathbf{x}_c \leftarrow \text{AGG}(\text{RAPdiag}(P^T \mathbf{1})^{-1}, \mathbf{x}_c, \mu, v_1, v_2)$
5. Coarse-grid correction: $\mathbf{x} \leftarrow P\text{diag}(P^T \mathbf{1})^{-1} \mathbf{x}_c$
6. $\mathbf{x} \leftarrow \text{Relax}(A, \mathbf{x}, \mathbf{0})$ v_2 times

else

7. Direct solve of $A\mathbf{x} = \mathbf{0}$

end

3 On-the-fly aggregation multigrid for Markov chains

In this section, we briefly describe the on-the-fly multigrid method developed recently in [19]. The main idea of this method is reducing the cost of expensive SET cycle such as Algorithm 1, which updating the whole multigrid hierarchy of operators in every cycle, by using classical algebraic multigrid cycles (Algorithm 2) instead, as the two algorithms are actually equivalent. In the approach, SET cycle provide classical cycle with improved operators, while classical cycle use them without adaptation and then offer SET cycle with better approximation of vector. It is obvious that the classical cycle with frozen operators are much more cheaper than the SET cycle, the advantage of this scheme is, by combining the two algorithms neatly, it speeds up the multigrid methods without sacrificing the convergence rate.

Classical algebraic multigrid method for linear systems are generally based on the following basic idea. Given the linear system

$$A\mathbf{x} = \mathbf{b}, \quad (3.1)$$

where $A \in \mathbb{R}^{n \times n}$ is a positive definite matrix. Traditional one-level iterative method for calculating \mathbf{x} , such as Power method or weighted Jacobi relaxation, converge very slowly due to only a relatively small number of components in the error, known as algebraically smooth, that approximately satisfy $A\mathbf{e} = \mathbf{0}$. To eliminate the algebraic smoothed errors, classical multigrid methods solve this problem on a coarse level with smaller size, referred to as coarse-grid correction process. It is noted that on the coarse grid, the smooth error appears to be relatively higher in frequency, which means relaxations are more effective

on coarser grid [3]. Algorithm 2 gives a typical two-level classical multigrid cycle [5], a multilevel V-cycle is obtained by recursively applying the algorithm in step 4.

Algorithm 2: Two-level additive cycle

Input: Initial vector: $\mathbf{x} \in \mathbb{R}^n$, Right-hand-side vector: $\mathbf{b} \in \mathbb{R}^n$,
operator: $A \in \mathbb{R}^{n \times n}$, $P \in \mathbb{R}^{n \times n_c}$, $R \in \mathbb{R}^{n_c \times n}$, $A_c \in \mathbb{R}^{n_c \times n_c}$.

Output: New approximation to the solution of $A\mathbf{x} = \mathbf{b}$.

Algorithm:

1. Apply pre-relaxations: $\mathbf{x} \leftarrow \text{Relax}(A, \mathbf{x}, \mathbf{b})$
 2. Define the residual: $\mathbf{r} \leftarrow \mathbf{b} - A\mathbf{x}$
 3. Restrict the residual: $\mathbf{r}_c \leftarrow R\mathbf{r}$
 4. Define \mathbf{e}_c as the solution of the coarse-grid problem: $A_c\mathbf{e}_c = \mathbf{r}_c$
 5. Prolong \mathbf{e}_c and apply coarse-grid correction: $\mathbf{x} \leftarrow \mathbf{x} + P\mathbf{e}_c$
 6. Apply post-relaxations: $\mathbf{x} \leftarrow \text{Relax}(A, \mathbf{x}, \mathbf{b})$
-

The difference between Algorithm 1 and Algorithm 2 is that, on the coarse-grid, the correction scheme of two-level additive cycle approximates the error \mathbf{e} rather than the exact solution \mathbf{x} . Moreover, the classical algorithm requires the whole hierarchy of coarsen operators in advance, while the setup schemes calculate them in every cycle. In spite of that, Algorithm 2 can be written as the form of Algorithm 1 equivalently. For the problem (2) in which $\mathbf{b} = \mathbf{0}$, the residual \mathbf{r} in step 2 and \mathbf{r}_c in step 3 of algorithm 2 are given as $\mathbf{r} = -A\mathbf{x}$ and $\mathbf{r}_c = -RAX$, the coarse-grid problem then is given by

$$A_c\mathbf{e}_c = RAP\mathbf{e}_c = -RAX, \quad (3.2)$$

then we obtain

$$RA(P\mathbf{e}_c + \mathbf{x}) = 0. \quad (3.3)$$

Since the approximation \mathbf{x} is in the range of P , there exists a vector \mathbf{x}_c satisfies $\mathbf{x} = P\mathbf{x}_c$. Then the equation above can be rewritten as $RAP(\mathbf{e}_c + \mathbf{x}_c) = 0$. Note that the \mathbf{x}_c we mentioned above is not necessary the same as \mathbf{x}_c in Step 5 in Algorithm 1. We define $\mathbf{z}_c = \mathbf{e}_c + \mathbf{x}_c$, thus $A_c\mathbf{z}_c = \mathbf{0}$, which is equivalent to the coarse-grid problem in SET cycle of Algorithm 1.

In the on-the-fly approach, an initial SET cycle is performed, followed by a SOL cycle which freezes the operators the SET cycle provided. If the convergence speed of SOL cycle is acceptable, another SOL cycle is performed. Conversely, a SET cycle is performed to yield more accurate operators. This procedure is described as follows.

Procedure: try-SOL-else-SET(γ)

1. Try a solution cycle: $\mathbf{y} = V_{sol}(\mathbf{x})$
2. If $q(\mathbf{y}) > q(\mathbf{x})$ do $\mathbf{x} \leftarrow V_{set}(\mathbf{x})$ and return
3. If $q(\mathbf{y}) > \gamma q(\mathbf{x})$ then $\mathbf{x} = \mathbf{y}$, else $\mathbf{x} \leftarrow V_{set}(\mathbf{y})$

In above procedure, V_{sol} represents a SOL cycle (Algorithm 2), V_{set} represents a SET cycle (Algorithm 1) and $\gamma \in [0, 1]$ is the scalar threshold for acceptable convergence speed of the SOL cycles. We use

$$q(\mathbf{x}) = \frac{\|A\mathbf{x}\|_1}{\|\mathbf{x}\|_1}, \quad (3.4)$$

which means the convergence factor is measured by the l_1 residual norm. The criteria $q(\mathbf{y}) > q(\mathbf{x})$ indicates that the SOL cycle increases the error and should be abandoned. The criteria $q(\mathbf{y}) > \gamma q(\mathbf{x})$ indicates that if the convergence factor of SOL cycle is better than the scalar threshold, then accept it, otherwise, perform a SET cycle instead. The on-the-fly adaptive algorithm is described as in Algorithm 3.

Algorithm 3: On-the-fly adaptive multigrid method

Input: Initial tolerance: ε_α , convergence parameter: γ , operator: $A \in \mathbb{R}^{n \times n}$,
initial guess \mathbf{x}_0 .

Output: New approximation to the solution of $A\mathbf{x} = \mathbf{0}$.

Algorithm:

1. **Initial Setup:**
 Apply a few relaxations to smooth \mathbf{x}_0
 Do an initial Setup cycle: $\mathbf{x} \leftarrow V_{set}(\mathbf{x}_0)$
 if $\|A\mathbf{x}_0\|_1 < \varepsilon_\alpha$, goto Step 4
 2. **Improve Solution Cycle:**
 while $\|A\mathbf{x}_0\|_1 > \varepsilon_\alpha$ do try-SOL-else-SET(γ)
 3. **Finalize Setup cycle:**
 $\mathbf{x} \leftarrow V_{set}(\mathbf{x})$
 4. **Solution:**
 Apply $\mathbf{x} \leftarrow V_{sol}(\mathbf{x})$ until convergence
-

4 Compact adaptive aggregation multigrid

In this section, we show how compacting the converged points into an aggregate, coupled with deleting the weak-links between them, can lead to better performance of on-the-fly method for Markov chains.

4.1 Experimental observation

We define a point has already converged as in [15]:

$$|x_i^{(v+1)} - x_i^{(v)}|/|x_i^{(v)}| < \tau_p, \quad (4.1)$$

where x_i denotes the i th element of the vector, $x_i^{(v)}$ denotes i th element at v th iterate, and τ_p is the convergence parameter. In [15], it is noted that the convergence patterns of the stationary probability vector of web matrix in the power method have a nonuniform distribution. Additional theoretical analysis in [17] has confirmed this conclusion recently. During the application of AGG on Markov problems, we have seen the similar convergence behavior that some points converge quickly while some others need more iterations before convergence. It is shown in Figure 1 that the number of the converged points increased gradually as iteration number increased.

To exploit this observation, the method outlined in [15] is that the converged components won't be recomputed so that computation cost can be reduced. The basic idea

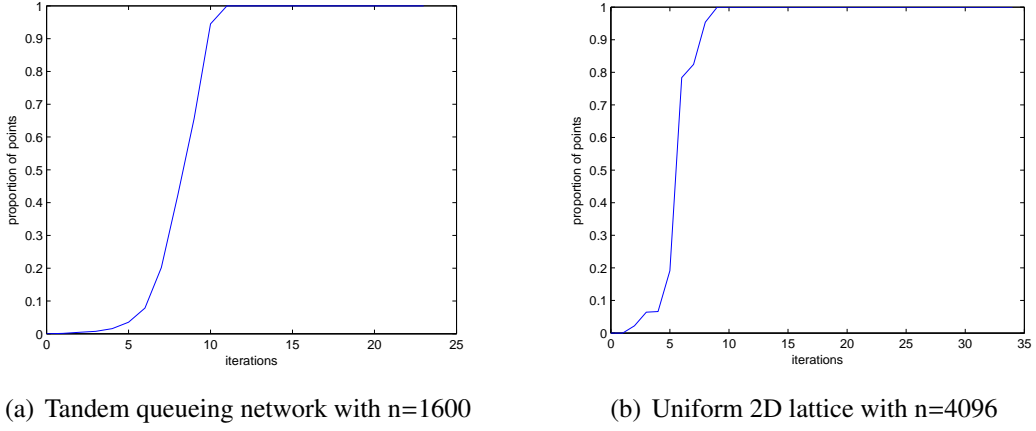


Fig. 1: (a) Tandem queueing network with n=1600, (b) Uniform 2D lattice with n=4096, where x-axis represents iterations and y-axis represents the proportion of the points that satisfy the equation (4.1).

developed there has three steps: splitting the vector into converged and not-yet-converged components, setting the submatrix $A_N \in \mathbb{R}^{m \times n}$ which corresponds to the not-yet-converged components as target matrix, and then applying the power method until convergence without recomputing converged components. More details can be seen in [15]. However, as A_N is not a $n \times n$ matrix, many algorithms including AMG can not be applied to this method. For this reason, with the similar principle but different procedures, we propose a new algorithm in this paper.

4.2 Compact adaptive aggregation multigrid

The main idea of our algorithm is reducing the computational cost by reducing the size of the coarse levels as well as the time spent on the coarse matrix construction. The new algorithm follows the same framework as the on-the-fly adaptive multigrid method does.

Consider that we have executed a setup cycle, then an approximation \mathbf{x} and the aggregation matrix Q are constructed in this cycle. Perform the try-SOL-else-SET procedure until the number of converged points meets $m > \zeta n$, where m is the number of the converged points, n is the size of the problem, $\zeta \in (0, 1)$ is the threshold parameter. The reason why we set this standard will be addressed in the following paragraphs. Let C as set of the converged points whose elements are positive integers between 1 and n , and N as set of the points have not converged yet.

Partitioning the finest-level matrix as

$$\hat{A} = \begin{pmatrix} A_{NN} & A_{NC} \\ A_{CN} & A_{CC} \end{pmatrix}. \quad (4.2)$$

Similarly, the current approximation \mathbf{x} and its multiplicative error, $\hat{\mathbf{e}}$ are reordered as

$$\hat{\mathbf{x}} = \begin{pmatrix} \mathbf{x}_N \\ \mathbf{x}_C \end{pmatrix}, \quad (4.3)$$

$$\hat{\mathbf{e}} = \begin{pmatrix} \mathbf{e}_N \\ \mathbf{e}_C \end{pmatrix}, \quad (4.4)$$

respectively. To reduce the time cost of coarse matrix construction, on-the-fly method proposed that the SOL cycle use the aggregation matrix Q which is offered by SET cycle without any modification [19]. Whereas in our method, we modify the aggregation matrix Q before we perform a SOL cycle. As to modifications, we keep the non-converged points in their aggregates and collect the converged points into a new aggregate. Then a further standard solution cycle is performed with amended operators and smaller scales.

The motivation is that we try to speed up the multigrid solvers by cutting down the cost on coarsen operators construction as well as reducing the size of coarse operators.

Now we show how to construct the new aggregation matrix \hat{Q} by modifying the aggregation matrix Q from the setup cycle. We first delete the rows of Q which belongs to C , then check for those columns with all zero elements and delete them, finally, construct \hat{Q} as given in Algorithm 3, where the length of $\mathbf{1}$ equals to that of C . The procedure is simple and inexpensive:

Procedure: Construct compact aggregation matrix \hat{Q}

1. Delete $Q(i, :)$, $i \in C$
2. Delete $Q(:, j)$ if $Q(:, j) = \mathbf{0}$
3. $\hat{Q} \leftarrow \begin{pmatrix} Q & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$, where $\mathbf{1}$ is the column vector of all ones, with length equals that of C

Now we constructed coarse operators based on aggregation matrix \hat{Q} . As the same definition in the classical AMG, the restriction and prolongation operators, R and P , are given by

$$\hat{R} = \hat{Q}^T, \quad (4.5)$$

$$\hat{P} = \text{diag}(\hat{\mathbf{x}}_C) \hat{Q}, \quad (4.6)$$

respectively. The coarse-level operator \hat{A}_c is given by

$$\hat{A}_c = \hat{R} \hat{A} \hat{P}. \quad (4.7)$$

Thus we obtain the complete hierarchy of multigrid operators the SOL cycle required, then we perform a standard SOL cycle as the final step to finish the new solution cycle, as described in Algorithm 4.

Algorithm 4: Compact Solution Cycle(C-SOL)

Input: Approximate vector: $\mathbf{x} \in \mathbb{R}^n$, operator: $A \in \mathbb{R}^{n \times n}$, the converged points set C .
aggregation matrix $Q \in \mathbb{R}^{n_c \times n}$

Output: New approximation to the solution of $A\mathbf{x} = \mathbf{0}$.

Initial setup:

1. Set $\hat{A} \leftarrow \begin{pmatrix} A_{NN} & A_{NC} \\ A_{CN} & A_{CC} \end{pmatrix}$, $\hat{\mathbf{x}} \leftarrow \begin{pmatrix} \mathbf{x}_N \\ \mathbf{x}_C \end{pmatrix}$
2. Construct compactive aggregation matrix \hat{Q} based on Q and C .
3. Set $\hat{R} = \hat{Q}^T$, $\hat{P} = \text{diag}(\hat{\mathbf{x}}_C)\hat{Q}$, Set $\hat{A}_c = \hat{R}\hat{A}\hat{P}$

Apply solution cycle:

4. Do a standard solution cycle described in Algorithm 2

In new method we prefer C-SOL cycle over SOL cycle if the former's error reduction is acceptable. The underlying assumption is that C-SOL cycles are considerably cheaper with satisfied convergence rate. However, if the ratio of m above n is too small or too big, this assumption will be ruined.

On the one hand, for most of test cases, when we put a small number ($m < 0.1n$) of the converged points into an aggregate, the C-SOL cycle is more expensive than the SOL cycle. This is because the cost on SET process in Algorithm 4 cannot be balanced out by the time saved by cutting scales of coarse-levels. On the other hand, if a large number of the converged points are compacted into an aggregate, it may lead to quite inaccurate operators in coarse-levels. Numerical experiments confirm that the resulting algorithm performs worse than the original on-the-fly method or leads to divergence for most problems. For the above reasons, we introduce the restriction for the number of converged points m : if $m < \zeta n$ we perform the procedure try-SOL-else-SET(γ), elsewhere we perform try-CSOL-else-SET(γ) instead.

Similar with on-the-fly method, the goal of our method is to fall off the time cost on reaching the accuracy $\|\mathbf{Ax}_0\|_1 < \varepsilon_\alpha$. Whereas the most distinguished difference of the new algorithm from on-the-fly adaptive multigrid is in Step 2. At Step 2 in new algorithm we initially perform the procedure try-SOL-else-SET until the number of the converged points meets the compactive condition $m \geq \zeta n$. With the converged points set C supplied by the process above and the aggregation matrix Q provided by SET cycle, we construct the C-SOL cycle, then we repeat the procedure try-CSOL-else-SET with the until the residual norm of the approximation reduced to ε_α . It is noted that in C-SOL cycle we frozen the coarsen operators as well as the converged points. The algorithm is described in Algorithm 5.

Algorithm 5: Compactive On-the-fly adaptive multigrid method

Input: Initial tolerance: ε_α , convergence criterion: τ_p , convergence factor: γ ,
size control parameter: ζ , operator: $A \in \mathbb{R}^{n \times n}$, initial guess \mathbf{x}_0 ,

Output: New approximation to the solution of $A\mathbf{x} = \mathbf{0}$.

Algorithm:

1. **Initial Setup:**

Apply a few relaxations to smooth \mathbf{x}_0

Do an initial Setup cycle: $\mathbf{x} \leftarrow V_{set}(\mathbf{x}_0)$

if $\|A\mathbf{x}_0\|_1 < \varepsilon_\alpha$ goto Step 5

2. **Improve Solution Cycle:**

$[N, C] \leftarrow \text{Detect-converged-points}(\mathbf{x}^{(1)}, \mathbf{x}^{(0)}, \tau_p)$

While $\|A\mathbf{x}_0\|_1 > \varepsilon_\alpha$ do

While $m < \zeta n$ do

try-SOL-else-SET(γ)

$[N, C] \leftarrow \text{Detect-converged-points}(\mathbf{x}^{(v+1)}, \mathbf{x}^{(v)}, \tau_p)$

end

try-CSOL-else-SOL(γ)

end

4. **Finalize Setup cycle:**

$\mathbf{x} \leftarrow V_{set}(\mathbf{x})$

5. **Solution:**

Apply $\mathbf{x} \leftarrow V_{sol}(\mathbf{x})$ until convergence

As mentioned above, compacting the converged points into an aggregate may lead to a single aggregate with a large number of points that are not strongly connected to each other. As is shown in [6], the aggregate of points that are weakly connected may result in very poor convergence of the multilevel method. The reason is that if the link between two points is weak compared to the other links in the same aggregate, the differences in the error of these two points can neither be eliminated efficiently by relaxation, nor smoothed out by coarse-level correction. Thus, although we have made the restriction for the size of the aggregate, it may still induce unsatisfied convergence.

Our next work is trying to define and delete the weak links in the aggregation of converged points to avoid the poor convergency.

4.3 Compactive on-the-fly method with correction

We illustrate with a simple example. In C-SOL cycles, the converged points are compacted into a single aggregation. Figure 2 is an example of such an aggregation. Links between the converged points and not-converged-yet points are not presented in this figure. We assume that the converged points set as $C=[4,5,9,10,14,17,18,38]$.

To capture the weak links in this aggregate, we need to determine what is meant by weak links. In the classical AMG, the strong connection is defined by formula (2.2), which indicates that if the size of the transition probability from i to j timed with the

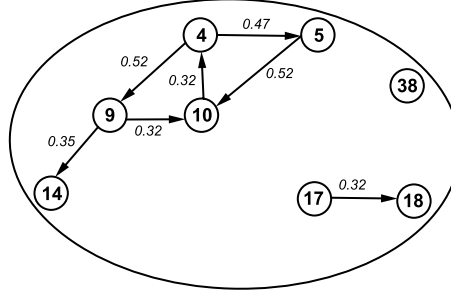


Figure 2: Single aggregate of converged points with fine-level transitions. The converged points are indicated by numbers in cycles, and the transitions are indicated by arrows with strength based on the scaled matrix $A_{CC}diag(\mathbf{x}_C)$. Connections between the converged points and those have not converged yet are not presented in this figure.

probability of residing in i is comparative large, then it is a strong link. Rather than the connection strength between two points, our attention is turned to the overall connection strength between a point and the rest points in the same aggregate, which is used to measure the importance of a point in its aggregate.

We define the connection strength of point i based on scaled matrix $A_{CC}diag(\mathbf{x}_C)$ with elements \tilde{a}_{il} by

$$S_i = - \sum_{l \neq i} (\tilde{a}_{il} + \tilde{a}_{li}). \quad (4.8)$$

This definition has a simple intuitive interpretation that the overall connection strength of a point is measured not only by the probability from other points to it but also by the probability from it to others. If a point's overall connection strength is comparative small, it cannot contribute efficiently to the elimination of errors but may lead to poor convergence. In the view of the above, we define a point is weakly connected to the others if

$$S_i \leq \delta \bar{S}_i, \quad (4.9)$$

where \bar{S}_i is the mean value of all S_i ($i \in C$), and δ is a fixed threshold parameter, whose function is as the same as θ in (2.2). Choosing $\delta > 1$ may set down all points as "not important" points especially when the number of points strong connected to the others is large. Meanwhile, it should not be taken much smaller than 1 because this may leave weak-links staying in the aggregate. The numerical results indicate that choosing $\delta < 1$ but close to 1 results in the best convergence properties for the new method. In generally we take $\delta = 0.8$.

It is easy to calculate and conclude that the points 14, 17, 18, 38 in figure 1 are weakly connected to the other points in the aggregate, thus we have the new $\bar{C} = [4, 5, 9, 10]$ to replace the original C .

5 Numerical results and discussion

In this section, we demonstrate the performance of the new algorithm for several test problems. The algorithm is applied to the two-level classical aggregation multigrid method, without smoothing operators. We compare the results of original on-the-fly adaptive aggregation multigrid algorithms (OTF) and the compactive on-the-fly adaptive aggregation multigrid algorithms (C-OTF). We start with an initial guess of unit vector with its elements all equal to $1/n$, in which n is the length of the vector. All setup cycles employ (4,1) cycles, with four prerelaxations and one postrelaxation on each level, while all compactive solution cycles and original solution cycles use (2,1) cycles. We use the stopping criteria

$$\text{stop if } v > \text{maxit} \quad \text{or} \quad \frac{\|A\mathbf{x}_v\|_1}{\|\mathbf{x}_v\|_1} < \tau \|A\mathbf{x}_0\|_1. \quad (5.1)$$

proposed in [9], where *maxit* is the upper limit of the number of iterations the algorithm will be allowed to perform, v is the current v th iteration. Here we use *maxit* = 200 and $\tau = 10^{-8}$. We also say the problem has reached global convergence if this criterion has met. Several threshold value τ_p are tested in the experiments. Through extensive simulations, we found that $\tau_p = 10^{-3}$ achieved the best performance among any others for most of test cases. For OTF algorithm we use scalar threshold $\gamma = 0.8$ and $\varepsilon_\alpha = 10^{-5}$, while for C-OTF algorithm we use $\gamma = 0.8$ and various choices of ε_α are presented in the following table. As to the AGG part in algorithms we use the aggregation strategy based on scaled matrix proposed in [7], with the strength threshold parameter $\theta = 0.25$.

In the following tables, we show the operator complexity C_{OP} and the work units WU which is defined as the cost of a single $V_{sol}(2, 1)$ [19]. WU is calculated as follows: for each problem and its size, averaging the execution time of a $V_{sol}(2, 1)$ by calculating the mean value of last five solution cycles in step 4 in Algorithm 3, the work units are the total execution time of the algorithm divided by this time. The motivation is that the execution time of the algorithm is susceptible to MATLAB's compilation time. V_{set} , V_{sol} , V_{csol} are the number of SET cycles, SOL cycles and C-SOL cycles, respectively. The experiments were performed using MATLAB R2010a with an Intel core i3 CPU with 4 GB of RAM memory.

5.1 Uniform chain

The first three test problems are generated by graphs with weighted edges[6, 20]. Their transition probabilities are determined by weights of the edge: if node i transforms to j with p weights and then its probability p_{ji} is obtained by p divided by the sum of the weights of all outgoing edges from node i . Our first test problem is the 1D uniform chain, generated by linear graphs in which each of two connected points has one outgoing edge with weight 1. The stencil of the matrix of uniform chain is given by

$$H_{UniformChain} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}. \quad (5.2)$$

Table 5.1 shows the results for the uniform chain problem using OTF-AGG algorithm (Algorithm 3) and C-OTF-AGG algorithm (Algorithm 5). When we set $\tau_p = 10^{-3}$ the new algorithm achieves the much better performance compared with $\tau_p = 10^{-4}$ and $\tau_p = 10^{-2}$.

Various choices of weak-links parameter δ are tested and it does not make too much difference when $\delta \leq 0.9$. We set $\delta = 0.8$ and the size control parameter $\zeta = 0.45$ for this test case. The experiments show that the SET cycle is significantly more expensive than the SOL cycle, while the C-SOL cycle is cheaper than SOL when the number of converged points meets $m > 0.1n$. Comparing the C-OTF and the corrected C-OTF under the same parameters, we observe a decrease in work units and the number of cycles. The results also indicate that a sufficiently small ε_α enhance the opportunities of executing C-SOL cycles, as is shown in the table 5.1, so that reduce the total execution time.

Table 5.1. Uniform chain results. t_{sol} is the average timing of a single $V_{sol}(2, 1)$ solution cycle, ε_α is the threshold parameter for performing the on-the-fly procedure at step 3 in algorithm 5, τ_p is the threshold parameter to explore the converged points in equation (12), C_{OP} is the operator complexity, V_{set} , V_{sol} , V_{csol} are the number of SET cycles, SOL cycles and C-SOL cycles, respectively. WU is the work units defined as the cost of a single $V_{sol}(2, 1)$ solution cycle. $Iter$ is the number of overall iterations.

n	t_{sol}	Algorithm ($\varepsilon_\alpha, \tau_p$)	$V_{set}, V_{sol}, V_{csol}$	C_{OP}	WU	Iter
961	0.05s	OTF ($10^{-4}, -$)	2,18,0	1.50	81	20
		C-OTF ($10^{-5}, 10^{-3}$)	2,16,4	1.22	80	> 20
		C-OTF(-cor) ($10^{-5}, 10^{-3}$)	2,15,3	1.27	80	20
		C-OTF(-cor) ($10^{-6}, 10^{-3}$)	2,12,6	1.05	73	20
4096	2.67s	OTF ($10^{-4}, -$)	2,18,0	1.50	45	20
		C-OTF($10^{-8}, 10^{-3}$)	2,15,16	0.77	42	> 21
		C-OTF(-cor) ($10^{-8}, 10^{-3}$)	2,9,10	0.79	36	21
		C-OTF(-cor) ($10^{-10}, 10^{-3}$)	2,3,16	0.35	30	21
13225	339.25s	OTF ($10^{-4}, -$)	2,18,0	1.54	54	20
		C-OTF ($10^{-8}, 10^{-3}$)	2,18,11,	0.82	56	> 20
		C-OTF(-cor) ($10^{-8}, 10^{-3}$)	2,9,9	0.82	44	20
		C-OTF(-cor) ($10^{-10}, 10^{-3}$)	2,4,15	0.43	37	21

5.2 Uniform chain with two weak links

The next test problem is a chain with uniform weights, except for two weak links with weight ϵ in the middle of the chain [6]. The stencil matrix is given by

$$H_{TwoWeakLinks} = \begin{pmatrix} \frac{1}{2} & \frac{1}{1+\epsilon} & 0 & \frac{\epsilon}{1+\epsilon} & \frac{1}{2} \end{pmatrix}. \quad (5.3)$$

where $\epsilon = 10^{-3}$ same as in [6]. As the same as the first case, we set the weak-links parameter $\delta = 0.8$ and the size control parameter $\zeta = 0.45$ here. The experiments show that the convergence criterion parameter $\tau_p = 10^{-3}$ is the best choice for this case. Results in Table 5.2 show again that the corrected C-OTF method is competitive compared with OTF and C-OTF without corrections.

5.3 Uniform 2D lattice

The next test problem is a 2D lattice with uniform weights [6, 20]. The stencil matrix is given by

$$H_{Uniform2D} = \frac{1}{4} \begin{pmatrix} & 1 & \\ 1 & 0 & 1 \\ & 1 & \end{pmatrix}. \quad (5.4)$$

We set the weak-links parameter $\delta = 0.8$ and the size control parameter $\zeta = 0.45$ for this test case. Table 5.3 shows numerical results for this problem.

For the small scale $n = 4096$ of this case, the choice of $\tau_p = 10^{-2}$ performs better than $\tau_p = 10^{-3}$ because the components of the prototype vector converge comparative slowly. In the larger case $n = 13225$, when we set $\tau_p = 10^{-3}$, the new algorithm fails to reduce the work units of OTF, largely due to the poor convergency of C-SOL cycles. To be specific, if the convergence rate of C-SOL cycle is unacceptable, we perform a SOL cycle instead. This procedure costs more time than a single SOL cycle and thus results in the worse performance than that of OTF.

Table 5.2. Uniform chain with two weak links results.

n	t_{sol}	Algorithm (ϵ_a, τ_p)	$V_{set}, V_{sol}, V_{csol}$	C_{OP}	WU	Iter
962	0.05	OTF ($10^{-4}, -$)	2,18,0	1.50	79	20
		C-OTF ($10^{-5}, 10^{-3}$)	2,17,2	1.40	80	21
		C-OTF(-cor) ($10^{-5}, 10^{-3}$)	2,17,2	1.41	80	21
		C-OTF(-cor) ($10^{-6}, 10^{-3}$)	2,13,5	1.26	78	21
4096	2.63s	OTF ($10^{-4}, -$)	2,20,0	1.50	47	22
		C-OTF($10^{-8}, 10^{-4}$)	2,15,11	0.91	42	> 22
		C-OTF(-cor) ($10^{-8}, 10^{-3}$)	2,8,11	0.71	36	21
		C-OTF(-cor) ($10^{-10}, 10^{-3}$)	2,14,17	0.31	29	21
13224	426.50	OTF ($10^{-4}, -$)	2,19,0	1.54	53	21
		C-OTF ($10^{-8}, 10^{-3}$)	2,19,12,	0.77	43	> 21
		C-OTF(-cor) ($10^{-8}, 10^{-3}$)	2,9,10,	0.79	35	21
		C-OTF(-cor) ($10^{-10}, 10^{-3}$)	2,3,16	0.35	26	21

Table 5.3. Uniform 2D lattice results.

n	t_{sol}	Algorithm ($\varepsilon_\alpha, \tau_p$)	$V_{set}, V_{sol}, V_{csol}$	C_{OP}	WU	cycles
961	0.05s	OTF ($10^{-4}, -$)	2,31,0	1.63	97	33
		C-OTF ($10^{-5}, 10^{-2}$)	2,32,5	1.35	98	> 33
		C-OTF(-cor) ($10^{-5}, 10^{-2}$)	2,29,8	1.30	94	> 33
		C-OTF(-cor) ($10^{-6}, 10^{-2}$)	2,31,12	1.15	98	> 33
4096	2.48s	OTF ($10^{-4}, -$)	2,33,0	1.66	61	35
		C-OTF($10^{-8}, 10^{-3}$)	2,41,15	0.99	60	> 35
		C-OTF(-cor) ($10^{-8}, 10^{-2}$)	2,28,24	0.82	51	> 35
		C-OTF(-cor) ($10^{-6}, 10^{-2}$)	2,22,16	1.06	53	> 35
13225	522.11s	OTF ($10^{-4}, -$)	2,24,0	1.70	44	26
		C-OTF ($10^{-8}, 10^{-3}$)	2,37,23	0.96	49	> 26
		C-OTF(-cor) ($10^{-8}, 10^{-3}$)	2,36,21	0.97	48	> 26
		C-OTF(-cor) ($10^{-10}, 10^{-3}$)	2,37,25	0.92	48	> 26

Table 5.4. Tandem queueing network results.

n	t_{sol}	Algorithm ($\varepsilon_\alpha, \tau_p$)	$V_{set}, V_{sol}, V_{csol}$	C_{OP}	WU	Iter
961	0.04	OTF ($10^{-4}, -$)	2,23,0	1.57	97	25
		C-OTF ($10^{-5}, 10^{-3}$)	2,22,0	1.57	99	24
		C-OTF(-cor) ($10^{-5}, 10^{-3}$)	2,22,0	1.57	99	24
		C-OTF(-cor) ($10^{-6}, 10^{-3}$)	2,22,2	1.42	97	26
4096	2.40s	OTF ($10^{-4}, -$)	2,23,0	1.60	52	25
		C-OTF($10^{-8}, 10^{-3}$)	2,15,5	1.20	50	> 25
		C-OTF(-cor) ($10^{-6}, 10^{-3}$)	2,21,1	1.59	51	24
		C-OTF(-cor) ($10^{-8}, 10^{-3}$)	2,15,5	1.22	50	> 25
13225	570.38s	OTF ($10^{-4}, -$)	2,23,0	1.66	49	25
		C-OTF ($10^{-8}, 10^{-3}$)	2,25,3,	1.30	47	> 25
		C-OTF(-cor) ($10^{-8}, 10^{-3}$)	2,25,3	1.30	44	> 25
		C-OTF(-cor) ($10^{-10}, 10^{-3}$)	2,26,6	1.10	36	>25

5.4 Tandem queueing network

The next test problem is a tandem queueing network appeared in [2, 6, 9, 20], which has two finite single-server queues placed in tandem. Customers arrive in Poisson distribution with rate μ , and two server stations' service time distribution is Poisson with rates μ_1 and μ_2 respectively. The stencil matrix of tandem queueing work is given by

$$H_{TandemQueue} = \frac{1}{\mu + \mu_1 + \mu_2} \begin{pmatrix} \mu & 0 & \mu_1 \\ & \mu_2 & \end{pmatrix}, \quad (5.5)$$

where we use $\mu = 10, \mu_1 = 11, \mu_2 = 10$ as in [2, 6, 9, 20]. Table 5.4 shows numerical results for this problem.

In this case we set the weak-links parameter $\delta = 0.8$, the size control parameter $\zeta = 0.45$ and convergence parameter $\tau_p = 10^{-3}$. We also try using more strict convergency parameter $\tau_p = 10^{-4}$. Results show that the algorithm fails to expose the converged points and the number of C-SOL cycle is equal to 0. Similar with the previous problems, several choices of ε_α are tested. Experiments show that with a sufficiently small ε_α , new algorithm improves the performance of OTF in terms of the total execution time, but suffers from an unsatisfied convergence rate, which increases the number of iterations. For the reason that the operators of C-SOL cycles are less accurate than that of SOL cycles, they have a probability to lead to poor convergence rate. To achieve the same accuracy ε_α , more C-SOL cycles are needed. Whereas, the total execution time is reduced because C-SOL cycles are comparative cheaper than SOL cycles.

Table 5.5. Random walk on unstructured planar graph results.

n	t_{sol}	Algorithm ($\varepsilon_\alpha, \tau_p$)	$V_{set}, V_{sol}, V_{csol}$	C_{OP}	WU	Iter
961	2	OTF ($10^{-4}, -$)	2,27,0	1.20	182	29
		C-OTF ($10^{-5}, 10^{-3}$)	2,29,3	1.07	186	> 29
		C-OTF(-cor) ($10^{-5}, 10^{-3}$)	2,29,3	1.12	186	> 29
		C-OTF(-cor) ($10^{-6}, 10^{-3}$)	2,40,2	0.95	183	> 29
4096	0.23s	OTF ($10^{-4}, -$)	2,29,0	1.21	162	31
		C-OTF($10^{-6}, 10^{-3}$)	2,32,6	0.99	162	> 31
		C-OTF(-cor) ($10^{-6}, 10^{-3}$)	2,18,6	0.99	157	> 31
		C-OTF(-cor) ($10^{-8}, 10^{-3}$)	2,18,6	0.94	158	> 31
13225	6.16s	OTF ($10^{-4}, -$)	2,28,0	1.21	122	30
		C-OTF ($10^{-6}, 10^{-3}$)	2,29,8,	0.98	130	> 30
		C-OTF(-cor) ($10^{-6}, 10^{-3}$)	2,30,5	0.94	120	> 30
		C-OTF(-cor) ($10^{-9}, 10^{-3}$)	2,41,20	0.57	116	> 30

5.5 Random walk on unstructured planar graph

The next test problem is random walks on graphs, which have significant applications in many fields, one of the well-known examples is Google's pagerank algorithm. Here we consider an unstructured planar graph, which is generated by choosing n random points in the unit square, and triangulating them by Delaunay triangulation. The transition probability from point i to point j is given by the reciprocal of the number of edges incident on point i .

In this test case, when $m < 0.6n$, a single C-SOL cycle costs more time than SOL cycle does, thus we use the size control parameter $\zeta = 0.6$ here. We set the weak-links parameter $\delta = 0.8$. Experiments show that convergence parameter $\tau_p = 10^{-3}$ is the best choice among any others. The performance of corrected C-OTF method is moderate. However, the work units, which indicates the total execution time, is still smaller than that of OTF and C-OTF without corrections. Table 5.5 shows numerical results for this problem.

6 Conclusions

This paper proposes a compact on-the-fly adaptive aggregation multigrid method for Markov chain problems. As is known, adaptive multigrid methods suffer from the common defect that considerable computation cost is spent on coarsen operators construction. The reason is that they update the entire multigrid hierarchy of operators in every cycles. We consider distributing the converged points into an aggregate and reducing the scale of the coarsen operators to decrease this cost. Meanwhile, a simple technique is proposed to delete the possible weak-links introduced by the procedure above. According to numerical results, for most of test cases, the corrected algorithm leads to better performance than on-the-fly adaptive aggregation multigrid algorithm in terms of total execution time. New algorithm can also be applied to various adaptive multigrid Markov solvers. One future work may be to study how to improve the convergence rate of compacted solution cycles.

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Travelling Solitary Wave Solutions for Stochastic Kadomtsev-Petviashvili Equation.

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Abstract. In this paper, generalized Wick-type stochastic Kadomtsev-Petviashvili equations are investigated. Abundant white noise functional solutions for Wick-type generalized stochastic Kadomtsev-Petviashvili equations are obtained. By using white noise analysis, Hermite transform, modified Riccati equation and modified tanh-coth method many exact travelling wave solutions are given. Detailed computations and implemented examples for the investigated model are explicitly provided .

Keywords: White noise; Stochastic ; Wick product; Kadomtsev-Petviashvili equations.

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1 Introduction

In this paper we investigate the generalized variable coefficient Kadomtsev-Petviashvili (KP) equation:

$$u_t + \frac{\partial}{\partial x}(\phi(t)u \frac{\partial u}{\partial x} + \psi(t) \frac{\partial^3 u}{\partial x^3}) + \theta(t) \frac{\partial^2 u}{\partial y^2} = 0, \quad (x, y, t) \in \mathbb{R}^2 \times \mathbb{R}_+ \quad (1.1)$$

where u is a stochastic process on $\mathbb{R}^2 \times \mathbb{R}_+$ and $\phi(t), \psi(t)$ and $\theta(t)$ are bounded measurable or integrable functions on \mathbb{R}_+ . Equation (1.1) plays a significant role in many scientific applications such as solid state physics, nonlinear optics, chemical kinetics, etc. The KP equations[1-2] are universal models(normal forms) for the propagation of long, dispersive, weakly nonlinear waves that travel predominantly in the x direction, with weak transverse effects. The notion of well-posed-ness will be the usual one in the context of nonlinear dispersive equations, that is, it includes existence, uniqueness, persistence property, and continuous dependence upon the data. Recently, many researchers pay more attention to study of random waves, which are important subjects of stochastic partial differential equation (SPDE). Wadati [3] first answered the interesting question, How does external noise affect the motion of solitons? and studied the diffusion of soliton of the KdV equation under Gaussian noise, which satisfies a diffusion equation in transformed coordinates.

Wadati and Akutsu also studied the behaviors of solitons under the Gaussian white noise of the stochastic KdV equations with and without damping [4]. Wadati [3] first answered the interesting question, “How does external noise affect the motion of solitons?” and studied the diffusion of soliton of the KdV equation under Gaussian noise, which satisfies a diffusion equation in transformed coordinates. The stochastic PDEs was discussed by many authors, e.g., de Bouard and Debussche [6, 7], Debussche and Printems [8, 9], Printems [17] and Ghany and Hyder [13]. On the basis of white noise functional analysis [5], Ghany et al. [10-16] studied more intensely the white noise functional solutions for some nonlinear stochastic PDEs. This paper is mainly concerned to investigate the white noise functional solutions for the generalized Wick-type stochastic Kadomstev-Petviashvili (KP) equation:

$$U_t + \Phi(t) \diamond U_x \diamond U_x + \Psi(t) \diamond U \diamond U_{xx} + \Psi(t) \diamond U_{xxx} + \Theta(t) \diamond U_{yy} = 0. \quad (1.2)$$

where “ \diamond ” is the Wick product on the Kondratiev distribution space $(\mathcal{S})_{-1}$ and $\Phi(t)$, $\Psi(t)$ and $\Theta(t)$ are $(\mathcal{S})_{-1}$ -valued functions [5]. It is well known that the solitons are stable against mutual collisions and behave like particles. In this sense, it is very important to study the nonlinear equations in random environment. However, variable coefficients nonlinear equations, as well as constant coefficients equations, cannot describe the realistic physical phenomena exactly. The rest of this paper is organized as follows: In Section 2, we recall the definition and some properties of white noise analysis. In Section 3, we apply some method to explore exact travelling wave solutions for Eq.(1.1). In Section 4, we use the Hermite transform and [5, Theorem 4.1.1] to obtain white noise functional solutions for Eq.(1.2). In Section 5, we give illustrative examples for the investigated model. The last section is devoted to summary and discussion.

2 Preliminaries

Suppose that $S(\mathbb{R}^d)$ and $S'(\mathbb{R}^d)$ are the Hida test function space and the Hida distribution space on \mathbb{R}^d , respectively. Let $h_n(x)$ be Hermite polynomials and put

$$\zeta_n = e^{-x^2} h_n(\sqrt{2}x) / ((n-1)!\pi)^{\frac{1}{2}}, \quad n \geq 1. \quad (2.1)$$

then, the collection $\{\zeta_n\}_{n \geq 1}$ constitutes an orthogonal basis for $L_2(\mathbb{R})$.

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ denote d-dimensional multi-indices with $\alpha_1, \alpha_2, \dots, \alpha_d \in \mathbb{N}$. The family of tensor products

$$\zeta_\alpha := \zeta_{(\alpha_1, \alpha_2, \dots, \alpha_d)} = \zeta_{\alpha_1} \otimes \zeta_{\alpha_2} \otimes \dots \otimes \zeta_{\alpha_d} \quad (2.2)$$

forms an orthogonal basis for $L_2(\mathbb{R}^d)$.

Suppose that $\alpha^{(i)} = (\alpha_1^{(i)}, \alpha_2^{(i)}, \dots, \alpha_d^{(i)})$ is the i-th multi-index number in some fixed ordering of all d-dimensional multi-indices α . We can, and will, assume that this ordering has the property that

$$i < j \Rightarrow \alpha_1^{(i)} + \alpha_2^{(i)} + \dots + \alpha_d^{(i)} < \alpha_1^{(j)} + \alpha_2^{(j)} + \dots + \alpha_d^{(j)} \quad (2.3)$$

i.e., the $\{\alpha^{(j)}\}_{j=1}^\infty$ occurs in an increasing order. Now

Define

$$\eta_i := \zeta_{\alpha_1^{(i)}} \otimes \zeta_{\alpha_2^{(i)}} \otimes \dots \otimes \zeta_{\alpha_d^{(i)}}, \quad i \geq 1. \quad (2.4)$$

We need to consider multi-indices of arbitrary length. For simplification of notation, we regard multi-indices as elements of the space $(\mathbb{N}_0^{\mathbb{N}})_c$ of all sequences $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d)$ with elements $\alpha_i \in \mathbb{N}_0$ and with compact support, i.e., with only finitely many $\alpha_i \neq 0$. We write $J = (\mathbb{N}_0^{\mathbb{N}})_c$, for $\alpha \in J$,

Define

$$H_\alpha(\omega) := \prod_{i=1}^{\infty} h_{\alpha_i}(\langle \omega, \eta_i \rangle), \quad \omega = (\omega_1, \omega_2, \dots, \omega_d) \in S'(\mathbb{R}^d) \quad (2.5)$$

For a fixed $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$, suppose the space $(S)_1^n$ consists of those $f(\omega) = \sum_{\alpha} c_{\alpha} H_{\alpha}(\omega) \in \bigoplus_{k=1}^n L_2(\mu)$ with $c_{\alpha} \in \mathbb{R}^n$ such that

$$\|f\|_{1,k}^2 = \sum_{\alpha} c_{\alpha}^2 (\alpha!)^2 (2\mathbb{N})^{k\alpha} < \infty \quad (2.6)$$

where, $c_{\alpha}^2 = |c_{\alpha}|^2 = \sum_{k=1}^n (c_{\alpha}^{(k)})^2$ if $c_{\alpha} = (c_{\alpha}^{(1)}, c_{\alpha}^{(2)}, \dots, c_{\alpha}^{(n)}) \in \mathbb{R}^n$ and μ is the white noise measure on $(S'(\mathbb{R}), B(S'(\mathbb{R})))$, $\alpha! = \prod_{k=1}^{\infty} \alpha_k!$ and $(2\mathbb{N})^{\alpha} = \prod_j (2j)^{\alpha_j}$ for $\alpha \in J$.

The space $(S)_{-1}^n$ consists of all formal expansions $F(\omega) = \sum_{\alpha} b_{\alpha} H_{\alpha}(\omega)$ with $b_{\alpha} \in \mathbb{R}^n$ such that $\|f\|_{-1,-q} = \sum_{\alpha} b_{\alpha}^2 (2\mathbb{N})^{-q\alpha} < \infty$ for some $q \in \mathbb{N}$. The family of seminorms $\|f\|_{1,k}, k \in \mathbb{N}$ gives rise to a topology on $(S)_1^n$, and we can regard $(S)_{-1}^n$ as the dual of $(S)_1^n$ by the action

$$\langle F, f \rangle = \sum_{\alpha} (b_{\alpha}, c_{\alpha}) \alpha! \quad (2.7)$$

where (b_{α}, c_{α}) is the inner product in \mathbb{R}^n .

The Wick product $f \diamond F$ of two elements $f = \sum_{\alpha} a_{\alpha} H_{\alpha}, F = \sum_{\beta} b_{\beta} H_{\beta} \in (S)_{-1}^n$ with $a_{\alpha}, b_{\beta} \in \mathbb{R}^n$, is defined by

$$f \diamond F = \sum_{\alpha, \beta} (a_{\alpha}, b_{\beta}) H_{\alpha+\beta} \quad (2.8)$$

The spaces $(S)_1^n, (S)_{-1}^n, S(\mathbb{R}^d)$ and $S'(\mathbb{R}^d)$ are closed under Wick products.

For $F = \sum_{\alpha} b_{\alpha} H_{\alpha} \in (S)_{-1}^n$, with $b_{\alpha} \in \mathbb{R}^n$, the Hermite transformation of F , is defined by

$$\mathcal{H}F(z) = \tilde{F}(z) = \sum_{\alpha} b_{\alpha} z^{\alpha} \in \mathbb{C}^N \quad (2.9)$$

where $z = (z_1, z_2, \dots) \in \mathbb{C}^N$ (the set of all sequences of complex numbers) and $z^{\alpha} = z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}$, if $\alpha \in J$, where $z_j^0 = 1$.

For $F, G \in (S)_{-1}^n$ we have

$$\widetilde{F \diamond G}(z) = \tilde{F}(z) \cdot \tilde{G}(z) \quad (2.10)$$

for all z such that $\tilde{F}(z)$ and $\tilde{G}(z)$ exist. The product on the right-hand side of the above formula is the complex bilinear product between two elements of \mathbb{C}^N defined by $(z_1^1, z_2^1, \dots, z_n^1) \cdot (z_1^2, z_2^2, \dots, z_n^2) = \sum_{k=1}^n z_k^1 z_k^2$.

Let $X = \sum_{\alpha} a_{\alpha} H_{\alpha}$, then the vector $c_0 = \tilde{X}(0) \in \mathbb{R}^N$ is called the generalized expectation

of X which denoted by $E(X)$. Suppose that $g : U \longrightarrow \mathbb{C}^M$ is an analytic function, where U is a neighborhood of $E(X)$. Assume that the Taylor series of g around $E(X)$ have coefficients in \mathbb{R}^M . Then the Wick version $g^\diamond(X) = \mathcal{H}^{-1}(g \circ \tilde{X}) \in (S)_{-1}^M$. In other words, if g has the power series expansion $g(z) = \sum a_\alpha (z - E(X))^\alpha$, with $a_\alpha \in \mathbb{R}^M$, then $g^\diamond(z) = \sum a_\alpha (z - E(X))^{\diamond\alpha} \in (S)_{-1}^M$.

3 Exact travelling wave solutions

In this section, we will give exact solutions of Eq.(1.1). Taking the Hermite transform of Eq.(1.2), we get:

$$\begin{aligned} & \widetilde{U}_t(t, x, y, z) + \widetilde{\Phi}(t) \cdot \widetilde{U}_x(t, x, y, z) \cdot \widetilde{U}_x(t, x, y, z) + \widetilde{\Psi}(t) \cdot \widetilde{U}(t, x, y, z) \cdot \widetilde{U}_{xx}(t, x, y, z) \\ & + \widetilde{\Psi}(t) \cdot \widetilde{U}_{xxx}(t, x, y, z) + \widetilde{\Theta}(t) \cdot \widetilde{U}_{yy}(t, x, y, z) = 0 \end{aligned} \quad (3.1)$$

where $z = (z_1, z_2, \dots) \in C^{\mathbb{N}}$ is a parameter. To look for the travelling wave solution of Eq.(3.1), we make the transformations $u(t, x, y, z) := \tilde{U}(t, x, y, z) = \varphi(\xi(t, x, y, z))$ with

$$\xi(t, x, y, z) := k_1 x + k_2 y + s \int_0^t l(\tau, z) d\tau + c$$

where k_1, k_2, s, c are arbitrary constants which satisfy $k_1 k_2 s \neq 0$, $l(\tau, z)$ is a non zero functions of indicated variables to be determined. So, Eq.(3.1) can be changing into the form:

$$\begin{aligned} & s l u'(t, x, z) + k_1^2 \Phi u'(t, x, z) u'(t, x, z) + k_1^2 \Psi u(t, x, z) u''(t, x, z) + \\ & k_1^4 \Psi u'''(t, x, z) + k_2^2 \Theta u''(t, x, z) = 0 \end{aligned} \quad (3.2)$$

The solution can be proposed by the tanh method as a finite power series in Y in the form:

$$u(\mu\zeta) = S(Y) = \sum_{k=0}^M a_k Y^k, \quad (3.3)$$

limiting them to solitary and shock wave profiles. However, the extended tanh method admits the use of the finite expansion

$$u(\mu\zeta) = S(Y) = \sum_{k=0}^M a_k Y^k + \sum_{k=1}^M b_k Y^{-k}, \quad (3.4)$$

where M is a positive integer, in most cases, that will be determined. Expansion (3.4) reduces to the standard tanh method [4-6], where $Y(\xi)$ satisfies the Riccati equation

$$Y' = c_0 + c_1 Y + c_2 Y^2, \quad (3.5)$$

and c_0, c_1, c_2 are constant to be prescribed later. By virtue of (3.3) and (3.4) with observation of the linear independence of $Y^n (n = -6, -5, \dots, 6)$ and using Mathematica Eqn.(3.2) implies the

following nonlinear algebraic system of equations:

$$\left\{ \begin{aligned}
 &sl\alpha_{1,0} + k_1^4\Psi\alpha_{4,0} + k_2^2\Theta\alpha_{2,0} + k_1^2[\Phi(\alpha_{1,0}^2 + 2\alpha_{1,1}\alpha_{1,-1} + 2\alpha_{1,2}\alpha_{1,-2} + 2\alpha_{1,3}\alpha_{1,-3}) \\
 &\quad + \Psi(a_0\alpha_{2,0} + a_1\alpha_{2,-1} + a_2\alpha_{2,-2} + b_1\alpha_{2,1} + b_2\alpha_{2,2})] = 0, \\
 &sl\alpha_{1,1} + k_1^4\Psi\alpha_{4,1} + k_2^2\Theta\alpha_{2,1} + k_1^2[\Phi(2\alpha_{1,0}\alpha_{1,1} + 2\alpha_{1,-1}\alpha_{1,2} + 2\alpha_{1,-2}\alpha_{1,3}) \\
 &\quad + \Psi(a_0\alpha_{2,1} + a_1\alpha_{2,0} + a_2\alpha_{2,-1} + b_1\alpha_{2,2} + b_2\alpha_{2,3})] = 0, \\
 &sl\alpha_{1,-1} + k_1^4\Psi\alpha_{4,-1} + k_2^2\Theta\alpha_{2,-1} + k_1^2[\Phi(2\alpha_{1,0}\alpha_{1,-1} + 2\alpha_{1,1}\alpha_{1,-2} + 2\alpha_{1,2}\alpha_{1,-3}) \\
 &\quad + \Psi(a_0\alpha_{2,-1} + a_1\alpha_{2,0} + a_2\alpha_{2,-2} + b_1\alpha_{2,1} + b_2\alpha_{2,2})] = 0, \\
 &sl\alpha_{1,2} + k_1^4\Psi\alpha_{4,2} + k_2^2\Theta\alpha_{2,2} + k_1^2[\Phi(\alpha_{1,1}^2 + 2\alpha_{1,0}\alpha_{1,2} + 2\alpha_{1,-1}\alpha_{1,3}) \\
 &\quad + \Psi(a_0\alpha_{2,2} + a_1\alpha_{2,1} + a_2\alpha_{2,0} + b_1\alpha_{2,3} + b_2\alpha_{2,4})] = 0, \\
 &sl\alpha_{1,-2} + k_1^4\Psi\alpha_{4,-2} + k_2^2\Theta\alpha_{2,-2} + k_1^2[\Phi(\alpha_{1,-1}^2 + 2\alpha_{1,0}\alpha_{1,-2} + 2\alpha_{1,1}\alpha_{1,-3}) \\
 &\quad + \Psi(a_0\alpha_{2,-2} + a_1\alpha_{2,-3} + a_2\alpha_{2,-4} + b_1\alpha_{2,-1} + b_2\alpha_{2,0})] = 0, \\
 &sl\alpha_{1,3} + k_1^4\Psi\alpha_{4,3} + k_2^2\Theta\alpha_{2,3} + k_1^2[\Phi(2\alpha_{1,0}\alpha_{1,3} + 2\alpha_{1,1}\alpha_{1,2}) \\
 &\quad + \Psi(a_0\alpha_{2,3} + a_1\alpha_{2,2} + a_2\alpha_{2,1} + b_1\alpha_{2,4})] = 0, \\
 &sl\alpha_{1,-3} + k_1^4\Psi\alpha_{4,-3} + k_2^2\Theta\alpha_{2,-3} + k_1^2[\Phi(2\alpha_{1,0}\alpha_{1,-3} + 2\alpha_{1,-1}\alpha_{1,-2}) \\
 &\quad + \Psi(a_0\alpha_{2,-3} + a_1\alpha_{2,-4} + b_1\alpha_{2,-2} + b_2\alpha_{2,-1})] = 0, \\
 &k_1^4\Psi\alpha_{4,4} + k_2^2\Theta\alpha_{2,4} + k_1^2[\Phi(2\alpha_{1,1}\alpha_{1,3} + \alpha_{1,2}^2) + \Psi(a_0\alpha_{2,4} + a_1\alpha_{2,3} + a_2\alpha_{2,2})] = 0, \\
 &k_1^4\Psi\alpha_{4,-4} + k_2^2\Theta\alpha_{2,-4} + k_1^2[\Phi(2\alpha_{1,-1}\alpha_{1,-3} + \alpha_{1,-2}^2) + \Psi(a_0\alpha_{2,-4} + b_1\alpha_{2,-3} + b_2\alpha_{2,-2})] = 0, \\
 &k_1^4\Psi\alpha_{4,5} + 2k_1^2\alpha_{1,2}\alpha_{1,3} + k_1^2(a_1\alpha_{2,4} + a_2\alpha_{2,3}) = 0, \\
 &k_1^4\Psi\alpha_{4,-5} + 2k_1^2\alpha_{1,-2}\alpha_{1,-3} + k_1^2(b_1\alpha_{2,-4} + b_2\alpha_{2,-3}) = 0, \\
 &k_1^4\Psi\alpha_{4,6} + k_1^2\Phi\alpha_{1,3}^2 + k_1^2\Psi a_2\alpha_{2,4} = 0, \\
 &k_1^4\Psi\alpha_{4,-6} + k_1^2\Phi\alpha_{1,-3}^2 + k_1^2\Psi b_2\alpha_{2,-4} = 0,
 \end{aligned} \right. \tag{3.6}$$

where

$$\left\{ \begin{array}{l} \alpha_{1,0} = a_1 c_0 - b_1 c_2, \quad \alpha_{1,1} = a_1 c_1 + 2a_2 c_0, \quad \alpha_{1,2} = a_1 c_2 + 2a_2 c_1, \quad \alpha_{1,3} = 2a_2 c_2, \\ \alpha_{1,-2} = -(b_1 c_0 + 2b_2 c_1), \quad \alpha_{1,-1} = -(b_1 c_1 + 2b_2 c_2), \quad \alpha_{2,0} = \alpha_{1,1} c_0 - \alpha_{1,-1} c_2, \\ \alpha_{1,-3} = -2b_2 c_0, \quad \alpha_{2,1} = \alpha_{1,1} c_1 + 2\alpha_{1,2} c_0, \quad \alpha_{2,2} = \alpha_{1,1} c_2 + 2\alpha_{1,2} c_1 + 3\alpha_{1,3} c_0, \\ \alpha_{2,3} = 2\alpha_{1,2} c_2 + 3\alpha_{1,3} c_1, \quad \alpha_{2,4} = 3\alpha_{1,3} c_2, \quad \alpha_{2,-1} = -(\alpha_{1,-1} c_1 + 2\alpha_{1,-2} c_2), \\ \alpha_{2,-2} = -(\alpha_{1,-1} c_0 + 2\alpha_{1,-2} c_1 + 3\alpha_{1,-3} c_2), \quad \alpha_{2,-3} = -(2\alpha_{1,-2} c_0 + 3\alpha_{1,-3} c_1), \\ \alpha_{2,-4} = -3\alpha_{1,-3} c_0, \quad \alpha_{3,0} = \alpha_{2,1} c_0 - \alpha_{2,-1} c_2, \quad \alpha_{3,1} = \alpha_{2,1} c_1 + 2\alpha_{2,2} c_0, \\ \alpha_{3,2} = \alpha_{2,1} c_2 + 2\alpha_{2,2} c_1 + 3\alpha_{2,3} c_2, \quad \alpha_{3,3} = 2\alpha_{2,2} c_2 + 3\alpha_{2,3} c_1 + 4\alpha_{2,4} c_0, \\ \alpha_{3,4} = 3\alpha_{2,3} c_2 + 4\alpha_{2,4} c_1, \quad \alpha_{3,5} = 4\alpha_{2,4} c_2, \quad \alpha_{3,-1} = -(\alpha_{2,-1} c_1 + 2\alpha_{2,-2} c_2), \\ \alpha_{3,-2} = -(\alpha_{2,-1} c_0 + 2\alpha_{2,-2} c_1 + \alpha_{2,-3} c_2), \quad \alpha_{3,-4} = -(3\alpha_{2,-3} c_0 + 4\alpha_{2,-4} c_1), \\ \alpha_{3,-3} = -(2\alpha_{2,-2} c_0 + 3\alpha_{2,-3} c_1 + 4\alpha_{2,-4} c_2), \quad \alpha_{3,-5} = -(4\alpha_{2,-4} c_0), \\ \alpha_{4,0} = \alpha_{3,1} c_0 - \alpha_{3,-1} c_2, \quad \alpha_{4,1} = \alpha_{3,1} c_1 + 2\alpha_{3,2} c_0, \quad \alpha_{4,6} = 5\alpha_{3,5} c_2, \\ \alpha_{4,2} = \alpha_{3,1} c_2 + 2\alpha_{3,2} c_1 + 3\alpha_{3,3} c_0, \quad \alpha_{4,3} = 2\alpha_{3,2} c_2 + 3\alpha_{3,3} c_1 + 4\alpha_{3,4} c_0, \\ \alpha_{4,4} = 3\alpha_{3,3} c_2 + 4\alpha_{3,4} c_1 + 5\alpha_{3,5} c_0, \quad \alpha_{4,5} = 4\alpha_{3,4} c_2 + 5\alpha_{3,5} c_0, \\ \alpha_{4,-1} = -(\alpha_{3,-1} c_1 + 2\alpha_{3,-2} c_2), \quad \alpha_{4,-3} = -(2\alpha_{3,-2} c_0 + 3\alpha_{3,-3} c_1 + 4\alpha_{3,-4} c_2), \\ \alpha_{4,-2} = -(\alpha_{3,-1} c_0 + 2\alpha_{3,-2} c_1 + 3\alpha_{3,-3} c_2), \quad \alpha_{4,-6} = -5\alpha_{3,-5} c_0, \\ \alpha_{4,-4} = -(3\alpha_{3,-3} c_0 + 4\alpha_{3,-4} c_1 + 5\alpha_{3,-5} c_2), \quad \alpha_{4,-5} = -(4\alpha_{3,-4} c_0 + 5\alpha_{3,-5} c_2). \end{array} \right.$$

At the rest of this section we will discuss and solve our problem for some particular cases for the Riccati equation as follows:

A. $c_0 = c_1 = 1, c_2 = 0$.

For this choice of the constants, the Riccati equation has the solution:

$$Y_1(\xi) = \exp(\xi) - 1 \quad (3.7)$$

By the aid of Mathematica, the above system of equations (3.6) can be solved for the following cases:

Case 1:

$a_1 = a_2 = 0, \alpha_{i,j} = 0$ for all $i, j > 0$; $a_0 = \frac{1}{k_1^2 \tilde{\Psi}} \{sl - k_1^4 \tilde{\Psi} - k_2^2 \tilde{\Theta}\}$; $b_1 = 12k_1^2 \frac{3\tilde{\Psi}}{5\tilde{\Phi}}$; $b_2 = -12k_1^2 \frac{\tilde{\Psi}}{\tilde{\Phi}}$.

According to (3.2),(3.6) and (3.7), Eq.(3.1) has the solution

$$u_1(t, x, y, z) = \frac{1}{k_1^2 \tilde{\Psi}} \{sl - k_1^4 \tilde{\Psi} - k_2^2 \tilde{\Theta}\} + \frac{36k_1^2 \tilde{\Psi}}{5\tilde{\Phi}} (\exp(\xi) - 1)^{-1} - \frac{12k_1^2 \tilde{\Psi}}{\tilde{\Phi}} (\exp(\xi) - 1)^{-2}. \quad (3.8)$$

where,

$$\xi = k_1 x + k_2 y - 11.4k_1^4 \int_0^t \tilde{\Psi}(\tau, z) d\tau \quad (3.9)$$

Case 2:

$$a_2 = b_2 = 0; \quad a_0 = 25k_1^2 \frac{\tilde{\Phi} + 3\tilde{\Psi}}{\tilde{\Phi} + 2\tilde{\Psi}} - 25k_1^2 - \left(\frac{k_2^2}{k_1^2}\right)^2 \frac{\tilde{\Theta}}{\tilde{\Psi}}; \quad b_1 = -50k_1^2 \frac{\tilde{\Psi}}{\tilde{\Phi} + 2\tilde{\Psi}}; \quad a_1 = -2k_1^2 \frac{\tilde{\Psi}}{\tilde{\Phi} + \tilde{\Psi}}.$$

According to (3.2), (3.6) and (3.7), Eq.(3.1) has the solution

$$u_2(t, x, y, z) = 25k_1^2 \frac{\tilde{\Phi} + 3\tilde{\Psi}}{\tilde{\Phi} + 2\tilde{\Psi}} - 25k_1^2 - \left(\frac{k_2^2}{k_1^2}\right)^2 \frac{\tilde{\Theta}}{\tilde{\Psi}} - \frac{2k_1^2 \tilde{\Psi}}{\tilde{\Phi} + \tilde{\Psi}} (\exp(\xi) - 1) - \frac{50k_1^2 \tilde{\Psi}}{\tilde{\Phi} + 2\tilde{\Psi}} (\exp(\xi) - 1)^{-1} \quad (3.10)$$

where,

$$\xi = k_1 x + k_2 y + k_1^4 \int_0^t \frac{11\tilde{\Psi}^2(\tau, z) - 12\tilde{\Phi}^2(\tau, z) + 12\tilde{\Phi}(\tau, z)\tilde{\Psi}(\tau, z)}{(\tilde{\Phi}(\tau, z) + \tilde{\Psi}(\tau, z))(\tilde{\Phi}(\tau, z) + 3\tilde{\Psi}(\tau, z))} \tilde{\Psi}(\tau, z) d\tau \quad (3.11)$$

B. $c_0 = -c_2 = 0.5, c_1 = 0$.

For this choice of the constants, the Riccati equation has the solution:

$$Y_2(\xi) = \tanh(\xi) \pm \operatorname{sech}(\xi) \quad (3.12)$$

or

$$Y_3(\xi) = \coth(\xi) \pm \operatorname{csch}(\xi) \quad (3.13)$$

By the aid of Mathematica, the above system of equations (3.6) can be solved for the following case:

Case 3:

$$a_2 = b_1 = b_2 = 0; \quad a_0 = 1.25k_1^2 - \left(\frac{k_2}{k_1}\right)^2 \frac{\tilde{\Theta}}{\tilde{\Psi}} - 7.5k_1^2 \frac{\tilde{\Phi}}{2\tilde{\Phi} + 3\tilde{\Psi}} - 3.75k_1^2 \frac{\tilde{\Psi}}{2\tilde{\Phi} + 3\tilde{\Psi}}; \quad a_2 = -15k_1^2 \frac{\tilde{\Psi}}{2\tilde{\Phi} + 3\tilde{\Psi}}.$$

According to (3.2), (3.6) and (3.7), Eq.(3.1) has the solution

$$u_i(t, x, y, z) = 1.25k_1^2 - \left(\frac{k_2}{k_1}\right)^2 \frac{\tilde{\Theta}}{\tilde{\Psi}} - 7.5k_1^2 \frac{\tilde{\Phi}}{2\tilde{\Phi} + 3\tilde{\Psi}} - 3.75k_1^2 \frac{\tilde{\Psi}}{2\tilde{\Phi} + 3\tilde{\Psi}} - 15k_1^2 \frac{\tilde{\Psi}}{2\tilde{\Phi} + 3\tilde{\Psi}} Y_{i-1}^2(\xi), \quad i = 3, 4. \quad (3.14)$$

where,

$$\xi = k_1 x + k_2 y. \quad (3.15)$$

C. $c_2 = 4c_0 = 1, c_1 = 0$.

For this choice of the constants, the Riccati equation has the solution:

$$Y_4(\xi) = 0.5 \tan(2\xi) \quad (3.16)$$

or

$$Y_5(\xi) = 0.5 \cot(2\xi) \quad (3.17)$$

By the aid of Mathematica, the above system of equations (3.6) can be solved for the following case:

Case 4:

$$a_1 = a_2 = b_1 = 0; \quad a_0 = -16k_1^2 - \left(\frac{k_2}{k_1}\right)^2 \frac{\tilde{\Theta}}{\tilde{\Psi}}; \quad b_2 = -120k_1^2 \frac{\tilde{\Psi}}{4\tilde{\Phi} + 6\tilde{\Psi}}.$$

According to (3.2), (3.6) and (3.7), Eq.(3.1) has the solution

$$u_i(t, x, y, z) = -16k_1^2 - \left(\frac{k_2}{k_1}\right)^2 \frac{\tilde{\Theta}}{\tilde{\Psi}} - 120k_1^2 \frac{\tilde{\Psi}}{4\tilde{\Phi} + 6\tilde{\Psi}} Y_{i-1}^{-2}(\xi), \quad i = 5, 6. \quad (3.18)$$

where,

$$\xi = k_1 x + k_2 y. \quad (3.19)$$

At the end of this section we should remark that, there exists infinitely number of solutions for Eqn.(1.1) these solution coming from solving the system (3.6) with regarding the Riccati equation (3.5). The above mentioned cases are just to clarify how far my technique is applicable.

4 White noise functional solutions

The main aim of the rest of this paper is to obtain white noise functional solutions of Eqs.(1.2). As pointed out from Xie [16], we will use Theorem 2.1 of for $d = 2$. The properties of hyperbolic functions yield that there exists a bounded open set $\mathbf{S} \subset \mathbb{R}_+ \times \mathbb{R}^2, m > 0$ and $n > 0$ such that $u(x, y, t, z), u_{xt}(x, y, t, z)$ are uniformly bounded for all $(t, x, y, z) \in \mathbf{S} \times \mathbb{K}_m(n)$, continuous with respect to $(t, x, y) \in \mathbf{S}$ for all $z \in \mathbb{K}_m(n)$ and analytic with respect to $z \in \mathbb{K}_m(n)$ for all $(t, x, y) \in \mathbf{S}$. Using Theorem 2.1 of Xie [16], there exists a stochastic process $U(t, x, y)$ such that the Hermite transformation of $U(t, x, y)$ is $u(t, x, y, z)$ for all $\mathbf{S} \times \mathbb{K}_m(n)$, and $U(t, x, y)$ is the solution of (1.2). This implies that $U(t, x, y)$ is the inverse Hermite transformation of $u(t, x, y, z)$. Hence, for $\Phi(t)\Psi(t)\Theta(t) \neq 0$ the white noise functional solutions of Eqs.(1.2) can be written as follows:

$$U_1(t, x, y) = \frac{1}{k_1^2 \Psi(t)} \{sl - k_1^4 \Psi(t) - k_2^2 \Theta(t)\} + \frac{36k_1^2 \Psi(t)}{5\Phi(t)(\exp^\diamond(\Xi_1(t, x, y)) - 1)} - \frac{12k_1^2 \Psi(t)}{\Phi(t)(\exp^\diamond(\Xi_1(t, x, y)) - 1)^{\diamond 2}} \quad (4.1)$$

where,

$$\Xi_1 = k_1x + k_2y - 11.4k_1^4 \int_0^t \Psi(\tau) d\tau \quad (4.2)$$

and,

$$\begin{aligned} U_2(t, x, y) = & 25k_1^2 \frac{\Phi(t) + 3\Psi(t)}{\Phi(t) + 2\Psi(t)} - 25k_1^2 - \left(\frac{k_2}{k_1}\right)^2 \frac{\Theta(t)}{\Psi(t)} - 2k_1^2 \frac{\Psi(t)}{\Phi(t) + \Psi(t)} Y_1^\diamond(\Xi_2(t, x, y)) \\ & - 50k_1^2 \frac{\Psi(t)}{\Phi(t) + 2\Psi(t)} Y_1^{-\diamond}(\Xi_2(t, x, y)) \end{aligned} \quad (4.3)$$

where,

$$Y_1^\diamond(\Xi_2(t, x, y)) = \exp^\diamond(\Xi_2(t, x, y)) - 1$$

and,

$$\Xi_2 = k_1x + k_2y + k_1^4 \int_0^t \frac{11\Psi^2(\tau) - 12\Phi^2(\tau) + 12\Phi(\tau)\Psi(\tau)}{(\Phi(\tau) + \Psi(\tau))(\Phi(\tau) + 3\Psi(\tau))} \Psi(\tau) d\tau \quad (4.4)$$

$$\begin{aligned} U_i(t, x, y) = & 1.25k_1^2 - \left(\frac{k_2}{k_1}\right)^2 \frac{\Theta(t)}{\Psi(t)} - 7.5k_1^2 \frac{\Phi(t)}{2\Phi(t) + 3\Psi(t)} - 3.75k_1^2 \frac{\Psi(t)}{2\Phi(t) + 3\Psi(t)} \\ & - 15k_1^2 \frac{\Psi(t)}{2\Phi(t) + 3\Psi(t)} Y_{i-1}^{\diamond^2}(\Xi_3(x, y)), \quad i = 3, 4. \end{aligned} \quad (4.5)$$

where

$$Y_2^\diamond(\Xi_3(x, y)) = \tanh^\diamond(\Xi_3(x, y)) \pm \operatorname{isech}^\diamond(\Xi_3(x, y))$$

or

$$Y_3^\diamond(\Xi_3(x, y)) = \coth^\diamond(\Xi_3(x, y)) \pm \operatorname{csch}^\diamond(\Xi_3(x, y))$$

$$U_i(t, x, y) = -16k_1^2 - \left(\frac{k_2}{k_1}\right)^2 \frac{\Theta(t)}{\Psi(t)} - 120k_1^2 \frac{\Psi(t)}{(4\Phi(t) + 6\Psi(t))Y_{i-1}^{\diamond^2}(\Xi_3(x, y))}, \quad i = 5, 6. \quad (4.6)$$

where,

$$Y_4^\diamond(\Xi_3(x, y)) = 0.5\tan^\diamond(2\Xi_3(x, y))$$

or

$$Y_5^\diamond(\Xi_3(x, y)) = 0.5\cot^\diamond(2\Xi_3(x, y))$$

and,

$$\Xi_3(x, y) = k_1x + k_2y \quad (4.7)$$

5 Discussions

Our first interest in present work being in implementing the extended tanh-coth method, Hermite transform and white noise analysis to stress its power in handling nonlinear equations so that one can apply it to models of various types of nonlinearity. The next interest is in the determination of exact travelling wave solutions for modified KP equations. Also, we have presented Riccati equation expansion method and applied it to the modified KP equations. As a result, some new exact travelling wave solutions of the modified KP equation are obtained because of more special solutions of Eq.(2.1). The method which we have proposed in this letter is standard, direct and computerized method, which allow us to do complicated and tedious algebraic calculation. It is shown that the algorithm can be also applied to other NLPDEs in mathematical physics such as KdV-Burgers, Modified KdV-Burgers, Zhiber-Shabat equations (specially: Liouville equation, Sinh-Gordon equation, Dodd-Bullough-Mikhailov equation, Dodd-Bullough-Mikhailov equation and Tzitzeica-Dodd-Bullough equation) and Benjamin-Bona-Mahony equations. Also, we remark that, since the Riccati equation has other solution if select other values of c_0, c_1 and c_2 , there are many other exact solutions of variable coefficient and wick-type stochastic modified KP equation.

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Global Dynamics and Bifurcations of Two Quadratic Fractional Second Order Difference Equations

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Abstract. We investigate the bifurcations and the global asymptotic stability of the following two difference equation

$$\begin{aligned}x_{n+1} &= \frac{\beta x_n x_{n-1} + \gamma x_{n-1}}{Ax_n^2 + Bx_n x_{n-1}}, \quad x_0 + x_{-1} > 0, \quad A + B > 0 \\x_{n+1} &= \frac{\alpha x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}}{Ax_n^2}, \quad x_0 > 0, \quad A > 0\end{aligned}$$

where all parameters and initial conditions are positive.

Keywords. asymptotic stability, attractivity, bifurcation, difference equation, global, local stability, period two;

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1 Introduction and Preliminaries

We investigate global behavior of the equations:

$$x_{n+1} = \frac{\beta x_n x_{n-1} + \gamma x_{n-1}}{Ax_n^2 + Bx_n x_{n-1}}, \quad n = 0, 1, \dots \quad (1)$$

$$x_{n+1} = \frac{\alpha x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}}{Ax_n^2}, \quad n = 0, 1, \dots \quad (2)$$

where the parameters $\alpha, \beta, \gamma, A, B$ and the initial conditions x_{-1}, x_0 are positive numbers. Equations (1), (2) are the special cases of equations

$$x_{n+1} = \frac{\alpha x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}}{Ax_n^2 + Bx_n x_{n-1} + Cx_{n-1}}, \quad n = 0, 1, 2, \dots \quad (3)$$

and

$$x_{n+1} = \frac{Ax_n^2 + Bx_n x_{n-1} + Cx_{n-1}^2 + Dx_n + Ex_{n-1} + F}{ax_n^2 + bx_n x_{n-1} + cx_{n-1}^2 + dx_n + ex_{n-1} + f}, \quad n = 0, 1, 2, \dots \quad (4)$$

Some special cases of equation (4) have been considered in the series of papers [3, 4, 12, 13, 20, 22]. Some special second order quadratic fractional difference equations have appeared in analysis of competitive and anti-competitive systems of linear fractional difference equations in the plane, see [5, 8, 7, 9, 18, 19]. Local stability analysis of the equilibrium solutions of equation (3) was performed in [11].

Describing the global dynamics of equation (4) is a formidable task as this equation contains as a special cases many equations with complicated dynamics, such as the linear fractional difference equation

$$x_{n+1} = \frac{Dx_n + Ex_{n-1} + F}{dx_n + ex_{n-1} + f}, \quad n = 0, 1, 2, \dots \quad (5)$$

The special cases considered so far shows that all kind of dynamics are possible including conservative and non-conservative chaos, Naimark-Sacker bifurcation, period-doubling bifurcation, exchange of stability bifurcation, etc. In this paper we use the theory of monotone maps developed in [16, 17] to describe precisely the basins of attraction of all attractors of this equation as well as bifurcations. Equations (1) and (2) exhibit essentially one period doubling bifurcation with different outcomes. Equation (1) allows the coexistence of the unique minimal period-two solution, which is a saddle point and the equilibrium but only the equilibrium solution and the degenerate period-two solution $(0, \infty)$ and $(\infty, 0)$ have substantial basins of attraction. In one region of parameters, Equation (2) also allows the coexistence of the unique minimal period-two solution, which is locally asymptotically stable and the equilibrium, but the period-two solution attracts all solutions outside the global stable manifold of the equilibrium. In the complementary region of parameters every solution is either attracted to the equilibrium or to the degenerate period-two solution $(1, \infty)$ and $(\infty, 1)$.

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Our results will be based on the following theorem for a general second order difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \quad n = 0, 1, 2, \dots, \quad (6)$$

see [2].

Theorem 1 *Let I be a set of real numbers and $f : I \times I \rightarrow I$ be a function which is non-increasing in the first variable and non-decreasing in the second variable. Then, for every solution $\{x_n\}_{n=-1}^{\infty}$ of the equation*

$$x_{n+1} = f(x_n, x_{n-1}), \quad x_{-1}, x_0 \in I, \quad n = 0, 1, 2, \dots \quad (7)$$

the subsequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ of even and odd terms of the solution do exactly one of the following:

- (i) *Eventually they are both monotonically increasing.*
- (ii) *Eventually they are both monotonically decreasing.*
- (iii) *One of them is monotonically increasing and the other is monotonically decreasing.*

The consequence of Theorem 1 is that every bounded solution of (7) converges to either equilibrium or period-two solution or to the point on the boundary, and most important question becomes determining the basins of attraction of these solutions as well as the unbounded solutions. The answer to this question follows from an application of theory of monotone maps in the plane which will be presented for the sake of completeness.

We now give some basic notions about monotone maps in the plane.

Consider a partial ordering \preceq on \mathbb{R}^2 . Two points $x, y \in \mathbb{R}^2$ are said to be related if $x \preceq y$ or $x \succeq y$. Also, a strict inequality between points may be defined as $x \prec y$ if $x \preceq y$ and $x \neq y$. A stronger inequality may be defined as $x = (x_1, x_2) \ll y = (y_1, y_2)$ if $x \preceq y$ with $x_1 \neq y_1$ and $x_2 \neq y_2$.

A map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ is a continuous function $T : \mathcal{R} \rightarrow \mathcal{R}$. The map T is monotone if $x \preceq y$ implies $T(x) \preceq T(y)$ for all $x, y \in \mathcal{R}$, and it is strongly monotone on \mathcal{R} if $x \prec y$ implies that $T(x) \ll T(y)$ for all $x, y \in \mathcal{R}$. The map is strictly monotone on \mathcal{R} if $x \prec y$ implies that $T(x) \prec T(y)$ for all $x, y \in \mathcal{R}$. Clearly, being related is invariant under iteration of a strongly monotone map.

Throughout this paper we shall use the *North-East ordering* (NE) for which the positive cone is the first quadrant, i.e. this partial ordering is defined by $(x_1, y_1) \preceq_{ne} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ and the *South-East* (SE) ordering defined as $(x_1, y_1) \preceq_{se} (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \geq y_2$.

A map T on a nonempty set $\mathcal{R} \subset \mathbb{R}^2$ which is monotone with respect to the North-East ordering is called *cooperative* and a map monotone with respect to the South-East ordering is called *competitive*.

If T is differentiable map on a nonempty set \mathcal{R} , a sufficient condition for T to be strongly monotone with respect to the SE ordering is that the Jacobian matrix at all points x has the sign configuration

$$\text{sign}(J_T(\mathbf{x})) = \begin{bmatrix} + & - \\ - & + \end{bmatrix}, \quad (8)$$

provided that \mathcal{R} is open and convex.

For $x \in \mathbb{R}^2$, define $Q_\ell(x)$ for $\ell = 1, \dots, 4$ to be the usual four quadrants based at x and numbered in a counterclockwise direction, for example, $Q_1(x) = \{y \in \mathbb{R}^2 : x_1 \leq y_1, x_2 \leq y_2\}$. Basin of attraction of a fixed point (\bar{x}, \bar{y}) of a map T , denoted as $\mathcal{B}((\bar{x}, \bar{y}))$, is defined as the set of all initial points (x_0, y_0) for which the sequence of iterates $T^n((x_0, y_0))$ converges to (\bar{x}, \bar{y}) . Similarly, we define a basin of attraction of a periodic point of period p . The next five results, from [17, 16], are useful for determining basins of attraction of fixed points of competitive maps. Related results have been obtained by H. L. Smith in [21, 22].

Theorem 2 *Let T be a competitive map on a rectangular region $\mathcal{R} \subset \mathbb{R}^2$. Let $\bar{x} \in \mathcal{R}$ be a fixed point of T such that $\Delta := \mathcal{R} \cap \text{int}(Q_1(\bar{x}) \cup Q_3(\bar{x}))$ is nonempty (i.e., \bar{x} is not the NW or SE vertex of \mathcal{R}), and T is strongly competitive on Δ . Suppose that the following statements are true.*

- a. *The map T has a C^1 extension to a neighborhood of \bar{x} .*
- b. *The Jacobian $J_T(\bar{x})$ of T at \bar{x} has real eigenvalues λ, μ such that $0 < |\lambda| < \mu$, where $|\lambda| < 1$, and the eigenspace E^λ associated with λ is not a coordinate axis.*

Then there exists a curve $C \subset \mathcal{R}$ through \bar{x} that is invariant and a subset of the basin of attraction of \bar{x} , such that C is tangential to the eigenspace E^λ at \bar{x} , and C is the graph of a strictly increasing continuous function of the first coordinate on an interval. Any endpoints of C in the interior of \mathcal{R} are either fixed points or minimal period-two points. In the latter case, the set of endpoints of C is a minimal period-two orbit of T .

We shall see in Theorem 4 that the situation where the endpoints of C are boundary points of \mathcal{R} is of interest. The following result gives a sufficient condition for this case.

Theorem 3 *For the curve C of Theorem 2 to have endpoints in $\partial\mathcal{R}$, it is sufficient that at least one of the following conditions is satisfied.*

- i. *The map T has no fixed points nor periodic points of minimal period two in Δ .*
- ii. *The map T has no fixed points in Δ , $\det J_T(\bar{x}) > 0$, and $T(x) = \bar{x}$ has no solutions $x \in \Delta$.*
- iii. *The map T has no points of minimal period-two in Δ , $\det J_T(\bar{x}) < 0$, and $T(x) = \bar{x}$ has no solutions $x \in \Delta$.*

For maps that are strongly competitive near the fixed point, hypothesis b. of Theorem 2 reduces just to $|\lambda| < 1$. This follows from a change of variables [22] that allows the Perron-Frobenius Theorem to be applied. Also, one can show that in such case no associated eigenvector is aligned with a coordinate axis. The next result is useful for determining basins of attraction of fixed points of competitive maps.

Theorem 4 *Assume the hypotheses of Theorem 2, and let C be the curve whose existence is guaranteed by Theorem 2. If the endpoints of C belong to $\partial\mathcal{R}$, then C separates \mathcal{R} into two connected components, namely*

$$\mathcal{W}_- := \{x \in \mathcal{R} \setminus C : \exists y \in C \text{ with } x \preceq_{se} y\} \quad \text{and} \quad \mathcal{W}_+ := \{x \in \mathcal{R} \setminus C : \exists y \in C \text{ with } y \preceq_{se} x\}, \quad (9)$$

such that the following statements are true.

- (i) \mathcal{W}_- is invariant, and $\text{dist}(T^n(x), Q_2(\bar{x})) \rightarrow 0$ as $n \rightarrow \infty$ for every $x \in \mathcal{W}_-$.
- (ii) \mathcal{W}_+ is invariant, and $\text{dist}(T^n(x), Q_4(\bar{x})) \rightarrow 0$ as $n \rightarrow \infty$ for every $x \in \mathcal{W}_+$.
- (B) If, in addition to the hypotheses of part (A), \bar{x} is an interior point of \mathcal{R} and T is C^2 and strongly competitive in a neighborhood of \bar{x} , then T has no periodic points in the boundary of $Q_1(\bar{x}) \cup Q_3(\bar{x})$ except for \bar{x} , and the following statements are true.
 - (iii) For every $x \in \mathcal{W}_-$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \text{int } Q_2(\bar{x})$ for $n \geq n_0$.
 - (iv) For every $x \in \mathcal{W}_+$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \text{int } Q_4(\bar{x})$ for $n \geq n_0$.

If T is a map on a set \mathcal{R} and if \bar{x} is a fixed point of T , the stable set $\mathcal{W}^s(\bar{x})$ of \bar{x} is the set $\{x \in \mathcal{R} : T^n(x) \rightarrow \bar{x}\}$ and unstable set $\mathcal{W}^u(\bar{x})$ of \bar{x} is the set

$$\left\{ x \in \mathcal{R} : \text{there exists } \{x_n\}_{n=-\infty}^0 \subset \mathcal{R} \text{ s.t. } T(x_n) = x_{n+1}, x_0 = x, \text{ and } \lim_{n \rightarrow -\infty} x_n = \bar{x} \right\}$$

When T is non-invertible, the set $\mathcal{W}^s(\bar{x})$ may not be connected and made up of infinitely many curves, or $\mathcal{W}^u(\bar{x})$ may not be a manifold. The following result gives a description of the stable and unstable sets of a saddle point of a competitive map. If the map is a diffeomorphism on \mathcal{R} , the sets $\mathcal{W}^s(\bar{x})$ and $\mathcal{W}^u(\bar{x})$ are the stable and unstable manifolds of \bar{x} .

Theorem 5 *In addition to the hypotheses of part (B) of Theorem 4, suppose that $\mu > 1$ and that the eigenspace E^μ associated with μ is not a coordinate axis. If the curve C of Theorem 2 has endpoints in $\partial\mathcal{R}$, then C is the stable set $\mathcal{W}^s(\bar{x})$ of \bar{x} , and the unstable set $\mathcal{W}^u(\bar{x})$ of \bar{x} is a curve in \mathcal{R} that is tangential to E^μ at \bar{x} and such that it is the graph of a strictly decreasing function of the first coordinate on an interval. Any endpoints of $\mathcal{W}^u(\bar{x})$ in \mathcal{R} are fixed points of T .*

Remark 1 We say that $f(u, v)$ is strongly decreasing in the first argument and strongly increasing in the second argument if it is differentiable and has first partial derivative $D_1 f$ negative and first partial derivative $D_2 f$ positive in a considered set. The connection between the theory of monotone maps and the asymptotic behavior of equation (7) follows from the fact that if f is strongly decreasing in the first argument and strongly increasing in the second argument, then the second iterate of a map associated to equation (7) is a strictly competitive map on $I \times I$, see [17].

Set $x_{n-1} = u_n$ and $x_n = v_n$ in Eq.(7) to obtain the equivalent system

$$\begin{aligned} u_{n+1} &= v_n \\ v_{n+1} &= f(v_n, u_n) \end{aligned} \quad , \quad n = 0, 1, \dots$$

Let $T(u, v) = (v, f(v, u))$. The second iterate T^2 is given by

$$T^2(u, v) = (f(v, u), f(f(v, u), v))$$

and it is strictly competitive on $I \times I$, see [17].

Remark 2 The characteristic equation of Eq.(7) at an equilibrium point (\bar{x}, \bar{x}) :

$$\lambda^2 - D_1 f(\bar{x}, \bar{x})\lambda - D_2 f(\bar{x}, \bar{x}) = 0, \quad (10)$$

has two real roots λ, μ which satisfy $\lambda < 0 < \mu$, and $|\lambda| < \mu$, whenever f is strictly decreasing in first and increasing in second variable. Thus the applicability of Theorems 2-5 depends on the nonexistence of minimal period-two solution.

There are several global attractivity results for Eq. (7). Some of these results give the sufficient conditions for all solutions to approach a unique equilibrium and they were used efficiently in [14].

The next result is from [6]. See also [1].

Theorem 6 *Consider Eq. (7) where $f : I \times I \rightarrow I$ is a continuous function and f is decreasing in the first argument and increasing in the second argument. Assume that \bar{x} is a unique equilibrium point which is locally asymptotically stable and assume that (φ, ψ) and (ψ, φ) are minimal period-two solutions which are saddle points such that*

$$(\varphi, \psi) \preceq_{se} (\bar{x}, \bar{x}) \preceq_{se} (\psi, \varphi).$$

Then, the basin of attraction $\mathcal{B}((\bar{x}, \bar{x}))$ of (\bar{x}, \bar{x}) is the region between the global stable sets $\mathcal{W}^s((\varphi, \psi))$ and $\mathcal{W}^s((\psi, \varphi))$. More precisely

$$\mathcal{B}((\bar{x}, \bar{x})) = \{(x, y) : \exists y_u, y_l : y_u < y < y_l, (x, y_l) \in \mathcal{W}^s((\varphi, \psi)), (x, y_u) \in \mathcal{W}^s((\psi, \varphi))\}.$$

The basins of attraction $\mathcal{B}((\varphi, \psi)) = \mathcal{W}^s((\varphi, \psi))$ and $\mathcal{B}((\psi, \varphi)) = \mathcal{W}^s((\psi, \varphi))$ are exactly the global stable sets of (φ, ψ) and (ψ, φ) .

If $(x_{-1}, x_0) \in \mathcal{W}_+((\psi, \varphi))$ or $(x_{-1}, x_0) \in \mathcal{W}_-((\varphi, \psi))$, then $T^n((x_{-1}, x_0))$ converges to the other equilibrium point or to the other minimal period-two solutions or to the boundary of the region $I \times I$.

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4

2 Equation $x_{n+1} = \frac{\beta x_n^2 + \gamma x_{n-1}}{Ax_n^2 + Bx_n x_{n-1}}$

In this section we present the global dynamics of Eq. (11).

2.1 Local stability analysis

By substitution $x_n = \frac{\beta}{A}y_n$, this equation is reduced to the equation

$$y_{n+1} = \frac{y_n y_{n-1} + \frac{\gamma A}{\beta^2} y_{n-1}}{y_n^2 + \frac{B}{A} y_n y_{n-1}}, \quad n = 0, 1, \dots$$

Thus we consider the following equation

$$x_{n+1} = \frac{x_n x_{n-1} + \gamma x_{n-1}}{x_n^2 + Bx_n x_{n-1}}, \quad n = 0, 1, \dots \quad (11)$$

Equation (11) has the unique positive equilibrium \bar{x} given by

$$\bar{x} = \frac{1 + \sqrt{1 + 4\gamma(1+B)}}{2(1+B)}.$$

The partial derivatives associated to the Eq(11) at equilibrium \bar{x} are

$$f'_x = \frac{-x^2 y - 2\gamma x y - B\gamma y^2}{(x^2 + Bxy)^2} \Big|_{\bar{x}} = \frac{-2(1+2(1+B)(2+B)\gamma\sqrt{1+4(1+B)\gamma})}{(1+B)(1+\sqrt{1+4(1+B)\gamma})^2}, \quad f'_y = \frac{x+\gamma}{(x+B\gamma)^2} \Big|_{\bar{x}} = \frac{1}{1+B}.$$

Characteristic equation associated to the Eq.(11) at equilibrium is

$$\lambda^2 + \frac{2(1+2(1+B)(2+B)\gamma\sqrt{1+4(1+B)\gamma})}{(1+B)(1+\sqrt{1+4(1+B)\gamma})^2} \lambda - \frac{1}{1+B} = 0.$$

By applying the linearized stability Theorem [14, 15] we obtain the following result.

Theorem 7 *The unique positive equilibrium point $\bar{x} = \frac{1+\sqrt{1+4\gamma(1+B)}}{2(1+B)}$ of equation (11) is:*

- i) *locally asymptotically stable when $B > 4\gamma + 1$;*
- ii) *a saddle point when $B < 4\gamma + 1$;*
- iii) *a nonhyperbolic point (with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = \frac{1}{2+4\gamma}$) when $B = 4\gamma + 1$.*

Lemma 1 *If*

$$B > 1 + 4\gamma$$

then Eq.(11) possesses a unique minimal period-two solution $\{P(\phi, \psi), Q(\psi, \phi)\}$ where

$$\phi = \frac{1}{2} - \frac{\sqrt{B-1-4\gamma}}{2\sqrt{B-1}} \quad \text{and} \quad \psi = \frac{1}{2} + \frac{\sqrt{B-1-4\gamma}}{2\sqrt{B-1}}.$$

The minimal period-two solution $\{P(\phi, \psi), Q(\psi, \phi)\}$ is a saddle point.

Proof. Periodic solution $\phi, \psi, \phi, \psi, \dots$ is the positive solution of the following system

$$\begin{cases} (B-1)y - \gamma = 0 \\ -xy + y = 0. \end{cases} \quad (12)$$

where $\phi + \psi = x$ and $\phi\psi = y$. We have that solution of system (12) is

$$x = 1 \quad \text{and} \quad y = \frac{\gamma}{B-1}.$$

Since

$$x^2 - 4y = \frac{B-1-4\gamma}{B-1} > 0$$

if and only if $B > 1 + 4\gamma$, we have a unique minimal period-two solution $\{P(\phi, \psi), Q(\psi, \phi)\}$ where

$$\phi = \frac{1}{2} - \frac{\sqrt{B-1-4\gamma}}{2\sqrt{B-1}} \quad \text{and} \quad \psi = \frac{1}{2} + \frac{\sqrt{B-1-4\gamma}}{2\sqrt{B-1}}.$$

Set

$$u_n = x_{n-1} \quad \text{and} \quad v_n = x_n, \quad \text{for } n = 0, 1, \dots$$

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and write equation (11) in the equivalent form

$$\begin{aligned} u_{n+1} &= v_n \\ v_{n+1} &= \frac{u_n v_n + \gamma u_n}{v_n^2 + B u_n v_n}, \quad n = 0, 1, \dots \end{aligned}$$

Let T be the function on $(0, \infty) \times (0, \infty)$ defined by

$$T \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \frac{uv + \gamma u}{v^2 + Buv} \end{pmatrix}.$$

By a straightforward calculation we find that

$$T^2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} g(u, v) \\ h(u, v) \end{pmatrix}$$

where

$$g(u, v) = \frac{uv + \gamma u}{v^2 + Buv}, \quad h(u, v) = \frac{v^2(Bu + v)(v^2\gamma + u(v + \gamma + Bv\gamma))}{u(v + \gamma)(Bv^3 + u(v + B^2v^2 + \gamma))}.$$

We have

$$J_{T^2} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} g'_u(\phi, \psi) & g'_v(\phi, \psi) \\ h'_u(\phi, \psi) & h'_v(\phi, \psi) \end{pmatrix},$$

where

$$\begin{aligned} g'_u &= \frac{v + \gamma}{(Bu + v)^2}, \\ g'_v &= -\frac{u(v^2 + B\gamma u + 2\gamma v)}{v^2(Bu + v)^2}, \\ h'_u &= -\frac{v^3(B\gamma v^5 + 2u\gamma v^2(v + \gamma + B^2v^2) + u^2(v^2 + v(2 + Bv(2 + B^2v))\gamma + (1 + 2Bv)\gamma^2))}{u^2(v + \gamma)(Bv^3 + u(v + B^2v^2 + \gamma))^2}, \\ h'_v &= \frac{v(B^4u^3v^3\gamma^2 + B^3u^2v^4\gamma(v + 4\gamma) + B(v + 2\gamma)(v^6\gamma + 4u^2v^2\gamma(v + \gamma) + u^3(v + \gamma)^2))}{u^2(v + \gamma)(Bv^3 + u(v + B^2v^2 + \gamma))^2} \\ &\quad + \frac{B^2uv\gamma(u^2(v + \gamma)(v + 3\gamma) + v^4(2v + 5\gamma)) + uv(v + \gamma)(u(v + \gamma)(2v + 3\gamma) + v^2\gamma(3v + 5\gamma))}{u^2(v + \gamma)(Bv^3 + u(v + B^2v^2 + \gamma))^2}. \end{aligned}$$

Set

$$S = g'_u(\phi, \psi) + h'_v(\phi, \psi), \quad \mathcal{D} = g'_u(\phi, \psi)h'_v(\phi, \psi) - g'_v(\phi, \psi)h'_u(\phi, \psi).$$

After some lengthy calculation one can see that

$$S = \frac{1 + 6\gamma + B(-3 - 6\gamma + B(2 + \gamma))}{(B - 1)(B + (B - 1)\gamma)} \quad \text{and} \quad \mathcal{D} = \frac{\gamma}{(B - 1)(B + (B - 1)\gamma)}.$$

We have that

$$|S| > |1 + \mathcal{D}| \quad \text{if and only if} \quad B > 1 + 4\gamma.$$

By applying the linearized stability Theorem we obtain that a unique prime period-two solution $\{P(\phi, \psi), Q(\psi, \phi)\}$ of Eq.(11) is a saddle point if and only if $B > 1 + 4\gamma$. ■

2.2 Global results and basins of attraction

In this section we present global dynamics results for equation (11).

Theorem 8 *If $B > 4\gamma + 1$ then equation (11) has a unique equilibrium point $E(\bar{x}, \bar{x})$ which is locally asymptotically stable and there exists the minimal period-two solution $\{P(\phi, \psi), Q(\psi, \phi)\}$, where*

$$\phi = \frac{1}{2} - \frac{\sqrt{B-1-4\gamma}}{2\sqrt{B-1}} \quad \text{and} \quad \psi = \frac{1}{2} + \frac{\sqrt{B-1-4\gamma}}{2\sqrt{B-1}}$$

which is a saddle point.

Furthermore, the global stable manifold of the periodic solution $\{P, Q\}$ is given by $\mathcal{W}^s(\{P, Q\}) = \mathcal{W}^s(P) \cup \mathcal{W}^s(Q)$ where $\mathcal{W}^s(P)$ and $\mathcal{W}^s(Q)$ are continuous increasing curves, that divide the first quadrant into two connected components, namely

$$\begin{aligned} \mathcal{W}_1^+ &:= \{x \in \mathcal{R} \setminus \mathcal{W}^s(P) : \exists y \in \mathcal{W}^s(P) \text{ with } y \preceq_{se} x\}, \quad \mathcal{W}_1^- := \{x \in \mathcal{R} \setminus \mathcal{W}^s(P) : \exists y \in \mathcal{W}^s(P) \text{ with } x \preceq_{se} y\}, \\ \text{and} \\ \mathcal{W}_2^+ &:= \{x \in \mathcal{R} \setminus \mathcal{W}^s(Q) : \exists y \in \mathcal{W}^s(Q) \text{ with } y \preceq_{se} x\}, \quad \mathcal{W}_2^- := \{x \in \mathcal{R} \setminus \mathcal{W}^s(Q) : \exists y \in \mathcal{W}^s(Q) \text{ with } x \preceq_{se} y\} \end{aligned}$$

respectively such that the following statements are true.

- i) If $(u_0, v_0) \in \mathcal{W}^s(P)$ then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to P and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to Q .
- ii) If $(u_0, v_0) \in \mathcal{W}^s(Q)$ then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to Q and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P .

- iii) If $(u_0, v_0) \in \mathcal{W}_1^-$ (the region above $\mathcal{W}^s(P)$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ tends to $(0, \infty)$ and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ tends to $(\infty, 0)$.
- iv) If $(u_0, v_0) \in \mathcal{W}_2^+$ (the region below $\mathcal{W}^s(Q)$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ tends to $(\infty, 0)$ and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ tends to $(0, \infty)$.
- v) If $(u_0, v_0) \in \mathcal{W}_1^+ \cap \mathcal{W}_2^-$ (the region between $\mathcal{W}^s(P)$ and $\mathcal{W}^s(Q)$) then the sequence $\{(u_n, v_n)\}$ is attracted to $E(\bar{x}, \bar{x})$.

Proof. From Theorem 7 Eq.(11) has a unique equilibrium point $E(\bar{x}, \bar{x})$, which is locally asymptotically stable. Theorem 1 implies that the periodic solution $\{P, Q\}$ is a saddle point. The map $T^2(u, v) = T(T(u, v))$ is competitive on $\mathcal{R} = \mathbb{R}^2 \setminus \{(0, 0)\}$ and strongly competitive on $\text{int}(\mathcal{R})$. It follows from the Perron-Frobenius Theorem and a change of variables that at each point the Jacobian matrix of a strongly competitive map has two real and distinct eigenvalues, the larger one in absolute value being positive and that corresponding eigenvectors may be chosen to point in the direction of the second and first quadrant, respectively, see [16, 17]. Also, as is well known [16, 17] if the map is strongly competitive then no eigenvector is aligned with a coordinate axis.

- i) By Theorem 4 we have that if $(u_0, v_0) \in \mathcal{W}^s(P)$ then $(u_{2n}, v_{2n}) = T^{2n}(u_0, v_0) \rightarrow P$ as $n \rightarrow \infty$, which implies that $(u_{2n+1}, v_{2n+1}) = T(T^{2n}(u_0, v_0)) \rightarrow T(P) = Q$ as $n \rightarrow \infty$, which implies the statement i).
- ii) The proof of the statement ii) is similar to the proof of the statement i) and will be omitted.
- iii) A straightforward calculation shows that $(\phi, \psi) \preceq_{se} (\bar{x}, \bar{x}) \preceq_{se} (\psi, \phi)$. Since Eq.(11) has no the other equilibrium point or the other minimal-period two solution from Theorem 6 we have if $(x_{-1}, x_0) \in \mathcal{W}_1^-$, then

$$(u_{2n}, v_{2n}) = T^{2n}((u_0, v_0)) \rightarrow (0, \infty) \text{ and } (u_{2n+1}, v_{2n+1}) = T^{2n+1}((u_0, v_0)) \rightarrow (\infty, 0).$$

and hence if $(x_{-1}, x_0) \in \mathcal{W}_1^-$, then

$$\lim_{n \rightarrow \infty} x_{2n} = \infty \text{ and } \lim_{n \rightarrow \infty} x_{2n+1} = 0.$$

- iv) If $(x_{-1}, x_0) \in \mathcal{W}_2^+$, then

$$(u_{2n}, v_{2n}) = T^{2n}((u_0, v_0)) \rightarrow (\infty, 0) \text{ and } (u_{2n+1}, v_{2n+1}) = T^{2n+1}((u_0, v_0)) \rightarrow (0, \infty).$$

and hence if $(x_{-1}, x_0) \in \mathcal{W}_2^+$, then

$$\lim_{n \rightarrow \infty} x_{2n} = 0 \text{ and } \lim_{n \rightarrow \infty} x_{2n+1} = \infty.$$

- v) If $(x_{-1}, x_0) \in \mathcal{W}_1^+ \cap \mathcal{W}_2^-$, then

$$\lim_{n \rightarrow \infty} x_n = \frac{1 + \sqrt{1 + 4\gamma(1+B)}}{2(1+B)}.$$

■

Theorem 9 If $B < 4\gamma + 1$ then equation (11) has a unique equilibrium point $E(\bar{x}, \bar{x})$ which is a saddle point.

The global stable manifold $\mathcal{W}^s(E)$ which is a continuous increasing curve divides the first quadrant such that the following holds:

- i) Every initial point (u_0, v_0) in $\mathcal{W}^s(E)$ is attracted to E .
- ii) If $(u_0, v_0) \in \mathcal{W}^+(E)$ (the region below $\mathcal{W}^s(E)$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ tends to $(\infty, 0)$ and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ tends to $(0, \infty)$.
- iii) If $(u_0, v_0) \in \mathcal{W}^-(E)$ (the region above $\mathcal{W}^s(E)$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ tends to $(0, \infty)$ and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ tends to $(\infty, 0)$.

Proof. From Theorem 7 equation (11) has a unique equilibrium point $E(\bar{x}, \bar{x})$, which is a saddle point. The map T has no fixed points or periodic points of minimal period-two in $\Delta = \mathcal{R} \cap \text{int}(Q_1(\bar{x}) \cup Q_3(\bar{x}))$. It is immediate to see that $\det J_T(E) < 0$ and $T(x) = \bar{x}$ only for $x = \bar{x}$. Since the map T is anti-competitive, see [10] and T^2 is strongly competitive we have that all conditions of Theorem 10 in [10] are satisfied from which the proof follows. ■

Theorem 10 If $B = 4\gamma + 1$ then Eq.(11) has a unique equilibrium point $E(\bar{x}, \bar{x}) = (\frac{1}{2}, \frac{1}{2})$ which is a nonhyperbolic point.

There exists a continuous increasing curve \mathcal{C}_E which is a subset of the basin of attraction of E and it divides the first quadrant such that the following holds:

- i) Every initial point (u_0, v_0) in \mathcal{C}_E is attracted to E .
- ii) If $(u_0, v_0) \in \mathcal{W}^-(E)$ (the region above \mathcal{C}_E) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ tends to $(0, \infty)$ and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ tends to $(\infty, 0)$.
- iii) If $(u_0, v_0) \in \mathcal{W}^+(E)$ (the region below \mathcal{C}_E) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ tends to $(\infty, 0)$ and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ tends to $(0, \infty)$.

Proof. From Theorem 7 equation (11) has a unique equilibrium point $E(\bar{x}, \bar{x}) = (\frac{1}{2}, \frac{1}{2})$, which is nonhyperbolic. All conditions of Theorem 4 are satisfied, which yields the existence of a continuous increasing curve \mathcal{C}_E which is a subset of the basin of attraction of E and for every $x \in \mathcal{W}^-(E)$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \text{int}Q_2(\bar{x})$ for $n \geq n_0$ and for every $x \in \mathcal{W}^+(E)$ there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \text{int}Q_4(\bar{x})$ for $n \geq n_0$.

Set

$$U(t) = \frac{1 - (4\gamma + 1)t + \sqrt{(1 - (4\gamma + 1)t)^2 + 4\gamma}}{2}.$$

It is easy to see that $(t, U(t)) \preceq_{se} E$ if $t < \bar{x}$ and $E \preceq_{se} (t, U(t))$ if $t > \bar{x}$. One can show that

$$T^2(t, U(t)) = \left(t, \frac{2\gamma(t + \gamma)}{t(-t + t^2 + 2\gamma + 4t^2\gamma + 8\gamma^2 + t\sqrt{4\gamma + (-1 + t + 4t\gamma)^2})} \right).$$

Now we have that

$$T^2(t, U(t)) \preceq_{se} (t, U(t)) \text{ if } t < \bar{x}$$

and

$$(t, U(t)) \preceq_{se} T^2(t, U(t)) \text{ if } t > \bar{x}.$$

By monotonicity if $t < \bar{x}$ we obtain that $T^{2n}(t, U(t)) \rightarrow (0, \infty)$ as $n \rightarrow \infty$ and if $t > \bar{x}$ then we have that $T^{2n}(t, U(t)) \rightarrow (\infty, 0)$ as $n \rightarrow \infty$.

If $(u', v') \in \text{int}Q_2(\bar{x})$ then there exists t_1 such that $(u', v') \preceq_{se} (t_1, U(t_1)) \preceq_{se} E$. By monotonicity of the map T^2 we obtain that $T^{2n}(u', v') \preceq_{se} T^{2n}(t_1, U(t_1)) \preceq_{se} E$ which implies that $T^{2n}(u', v') \rightarrow (0, \infty)$ and $T^{2n+1}(u', v') \rightarrow T(0, \infty) = (\infty, 0)$ as $n \rightarrow \infty$ which proves the statement ii).

If $(u'', v'') \in \text{int}Q_4(\bar{x})$ then there exists t_2 such that $E \preceq_{se} (t_2, U(t_2)) \preceq_{se} (u'', v'')$. By monotonicity of the map T^2 we obtain that $E \preceq_{se} T^{2n}(t_2, U(t_2)) \preceq_{se} T^{2n}(u'', v'')$ which implies that $T^{2n}(u'', v'') \rightarrow (\infty, 0)$ and $T^{2n+1}(u'', v'') \rightarrow T(\infty, 0) = (0, \infty)$ as $n \rightarrow \infty$ which proves the statement iii). This completes the proof of Theorem. ■

Remark 3 Theorems 8, 9 and 10 show new type of period doubling bifurcation. When $B \leq 4\gamma + 1$ all solutions outside the global stable manifold are asymptotic to $(0, \infty)$ or to $(\infty, 0)$, and when $B > 4\gamma + 1$ all solutions are either asymptotic to $(0, \infty)$ or to $(\infty, 0)$ or to the minimal period-two solution $\{P, Q\}$ or a unique equilibrium E . In the second case each attractor has a substantial basin of attraction.

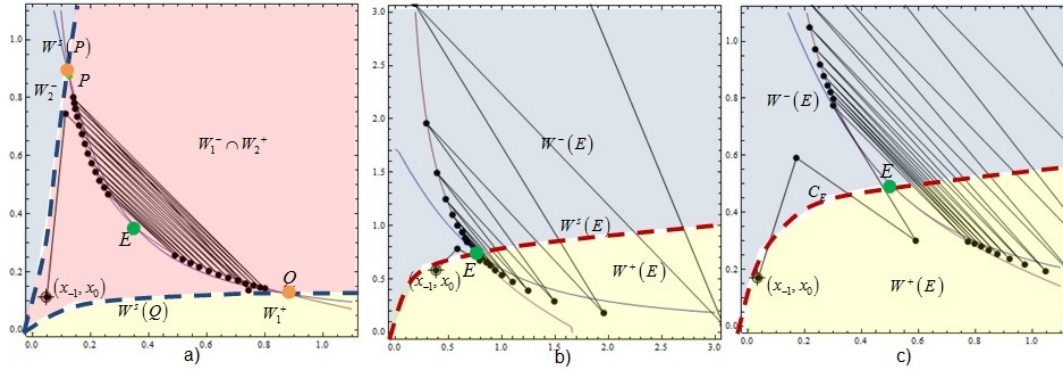


Figure 1: Visual illustration of Theorems 8, 9 and 10 . Figures are generated by *Dynamica 3*, [15].

3 Equation $x_{n+1} = \frac{\alpha x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}}{A x_n^2}$

In this section we present the global dynamics and bifurcation analysis of Equation (13).

3.1 Local stability analysis

This equation is reduced to the equation

$$x_{n+1} = \frac{x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}}{x_n^2}, \quad n = 0, 1, \dots \quad (13)$$

Equation (13) has the unique positive equilibrium \bar{x} given by

$$\bar{x} = \frac{1 + \beta + \sqrt{(1 + \beta)^2 + 4\gamma}}{2}.$$

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The partial derivatives associated to equation (13) at equilibrium \bar{x} are

$$f'_x = \left. \frac{-xy\beta - 2\gamma y}{x^3} \right|_{\bar{x}} = -\frac{2(4\gamma + \beta(1+\beta + \sqrt{(1+\beta)^2 + 4\gamma}))}{(1+\beta + \sqrt{(1+\beta)^2 + 4\gamma})^2}, \quad f'_y = \left. \frac{\beta x + \gamma}{x^2} \right|_{\bar{x}} = \frac{2(2\gamma + \beta(1+\beta + \sqrt{(1+\beta)^2 + 4\gamma}))}{(1+\beta + \sqrt{(1+\beta)^2 + 4\gamma})^2}.$$

Characteristic equation associated to the Eq.(13) at equilibrium is

$$\lambda^2 + \frac{2(4\gamma + \beta(1+\beta + \sqrt{(1+\beta)^2 + 4\gamma}))}{(1+\beta + \sqrt{(1+\beta)^2 + 4\gamma})^2} \lambda - \frac{2(2\gamma + \beta(1+\beta + \sqrt{(1+\beta)^2 + 4\gamma}))}{(1+\beta + \sqrt{(1+\beta)^2 + 4\gamma})^2} = 0.$$

By applying the linearized stability Theorem we obtain the following result.

Theorem 11 *The unique positive equilibrium point $\bar{x} = \frac{1+\beta + \sqrt{(1+\beta)^2 + 4\gamma}}{2}$ of equation (13) is*

- i) *locally asymptotically stable when $4\gamma + 2\beta + \beta^2 < 3$;*
- ii) *a saddle point when $4\gamma + 2\beta + \beta^2 > 3$;*
- iii) *a nonhyperbolic point (with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = \frac{\beta+1}{\beta+3}$) when $4\gamma + 2\beta + \beta^2 = 3$.*

Lemma 2 *Equation (13) has the minimal period-two solution $\{P(\phi, \psi), Q(\psi, \phi)\}$ where*

$$\phi = \frac{-\gamma + \beta\gamma - \gamma\sqrt{-3+2\beta+\beta^2+4\gamma}}{2(-1+\beta+\gamma)} \text{ and } \psi = \frac{-\gamma + \beta\gamma + \gamma\sqrt{-3+2\beta+\beta^2+4\gamma}}{2(-1+\beta+\gamma)}$$

if and only if

$$\beta < 1 \text{ and } \frac{3-2\beta-\beta^2}{4} < \gamma < 1-\beta.$$

The minimal period-two solution $\{P(\phi, \psi), Q(\psi, \phi)\}$ is locally asymptotically stable.

Proof. Period-two solution is a positive solution of the following systems

$$\begin{cases} x - y - \gamma = 0 \\ x^2 - xy + (\beta - 1)y = 0. \end{cases} \quad (14)$$

where $\phi + \psi = x$ and $\phi\psi = y$. We have that only one solution of system (14) is

$$x = \frac{(\beta - 1)\gamma}{\beta + \gamma - 1}, \quad y = \frac{-\gamma^2}{\beta + \gamma - 1},$$

Since

$$x^2 - 4y = \frac{\gamma^2(-3 + 2\beta + \beta^2 + 4\gamma)}{(\beta + \gamma - 1)^2} > 0$$

if and only if

$$\frac{3-2\beta-\beta^2}{4} < \gamma$$

and $x, y > 0$ if and only if $\beta < 1$ and $\gamma < 1 - \beta$, we have that ϕ and ψ are solution of the equation

$$t^2 - \frac{(\beta - 1)\gamma}{\beta + \gamma - 1}t + \frac{-\gamma^2}{\beta + \gamma - 1} = 0$$

if and only if

$$\beta < 1 \text{ and } \frac{3-2\beta-\beta^2}{4} < \gamma < 1-\beta.$$

The second iterate of the map T is

$$T^2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} g(u, v) \\ h(u, v) \end{pmatrix}$$

where

$$g(u, v) = \frac{v^2 + \beta uv + \gamma u}{v^2}, \quad h(u, v) = \frac{v^4 \left(1 + v(\beta + \gamma) + \frac{u(2 + v\beta)(v\beta + \gamma)}{v^2} + \frac{u^2(v\beta + \gamma)^2}{v^4} \right)}{(v^2 + \beta uv + \gamma u)^2}.$$

We have

$$J_{T^2} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} g'_u(\phi, \psi) & g'_v(\phi, \psi) \\ h'_u(\phi, \psi) & h'_v(\phi, \psi) \end{pmatrix}$$

where

$$\begin{aligned} g'_u &= \frac{v\beta + \gamma}{v^2}, \\ g'_v &= -\frac{u(v\beta + 2\gamma)}{v^3}, \\ h'_u &= -\frac{v^3(v\beta + \gamma)(uv\beta^2 + u\beta\gamma + v^2(\beta + 2\gamma))}{(v^2 + \beta uv + \gamma u)^3}, \end{aligned}$$

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$$h'_v = \frac{v^2(5u^2v\beta^2\gamma + 3u^2\beta\gamma^2 + v^4(\beta + \gamma) + 3uv^3\beta(\beta + \gamma) + uv^2(2u\beta^3 + \gamma(u\beta + 5\gamma)))}{(v^2 + \beta uv + \gamma u)^3}.$$

Set

$$\mathcal{S} = g'_u(\phi, \psi) + h'_v(\phi, \psi) \quad \text{and} \quad \mathcal{D} = g'_u(\phi, \psi)h'_v(\phi, \psi) - g'_v(\phi, \psi)h'_u(\phi, \psi).$$

After some lengthy calculation one can see that

$$\mathcal{S} = \frac{4 + \beta(-6 + \beta + \beta^2) - 9\gamma + \beta(7 + \beta)\gamma + 6\gamma^2}{\gamma^2}, \quad \mathcal{D} = \frac{(-1 + \gamma)(-1 + \beta + \gamma)}{\gamma^2}.$$

Applying the linearized stability Theorem we obtain that a unique prime period-two solution $\{P(\phi, \psi), Q(\psi, \phi)\}$ of Eq(13) is locally asymptotically stable when

$$\beta < 1 \text{ and } \frac{3 - 2\beta - \beta^2}{4} < \gamma < 1 - \beta.$$

■

3.2 Global results and basins of attraction

In this section we present global dynamics results for Eq.(13).

Theorem 12 *If $4\gamma + 2\beta + \beta^2 < 3$ then Eq. (13) has a unique equilibrium point $E(\bar{x}, \bar{x})$ which is globally asymptotically stable.*

Proof. From Theorem 11 equation (13) has a unique equilibrium point $E(\bar{x}, \bar{x})$, which is locally asymptotically stable. Every solution of equation (13) is bounded from above and from below by positive constants. If $4\gamma + 2\beta + \beta^2 < 3$ then $\beta + \gamma < 1$ and we have

$$x_{n+1} = \frac{x_n^2 + \beta x_n x_{n-1} + \gamma x_{n-1}}{x_n^2} \geq 1$$

and

$$\begin{aligned} x_{n+1} &= 1 + \frac{\beta x_{n-1}}{x_n} + \frac{\gamma x_{n-1}}{x_n^2} \leq 1 + \beta x_{n-1} + \gamma x_{n-1} = 1 + (\beta + \gamma)x_{n-1}. \\ x_{2n} &\leq 1 + (\beta + \gamma)[1 + (\beta + \gamma)x_{2n-4}] \leq \dots \\ &\leq 1 + (\beta + \gamma) + (\beta + \gamma)^2 + \dots + (\beta + \gamma)^n x_0, \\ &< \frac{1}{1 - \alpha - \beta} + (\beta + \gamma)^n x_0, \\ x_{2n-1} &\leq 1 + (\beta + \gamma)[1 + (\beta + \gamma)x_{2n-5}] \leq \dots \\ &\leq 1 + (\beta + \gamma) + (\beta + \gamma)^2 + \dots + (\beta + \gamma)^n x_{-1} \\ &< \frac{1}{1 - \alpha - \beta} + (\beta + \gamma)^n x_{-1}. \end{aligned}$$

Thus $x_n \leq \frac{1}{1 - \alpha - \beta} + \varepsilon$, for some $\varepsilon > 0$ and $n \geq N$ and so every solution is bounded. Equation (13) has no other equilibrium points or period two points and using Theorem 1 we have that equilibrium point $E(\bar{x}, \bar{x})$ is globally asymptotically stable. ■

Theorem 13 *If $4\gamma + 2\beta + \beta^2 > 3$ and $\beta + \gamma < 1$ then equation (13) has a unique equilibrium point $E(\bar{x}, \bar{x})$ which is a saddle point and the minimal period-two solution $\{P(\phi, \psi), Q(\psi, \phi)\}$ which is locally asymptotically stable, where*

$$\phi = \frac{-\gamma + \beta\gamma - \gamma\sqrt{-3 + 2\beta + \beta^2 + 4\gamma}}{2(-1 + \beta + \gamma)}, \quad \psi = \frac{-\gamma + \beta\gamma + \gamma\sqrt{-3 + 2\beta + \beta^2 + 4\gamma}}{2(-1 + \beta + \gamma)}.$$

The global stable manifold $\mathcal{W}^s(E)$ which is a continuous increasing curve, divides the first quadrant such that the following holds:

- i) Every initial point (u_0, v_0) in $\mathcal{W}^s(E)$ is attracted to E .
- ii) If $(u_0, v_0) \in \mathcal{W}^+(E)$ (the region below $\mathcal{W}^s(E)$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to Q and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to P .
- iii) If $(u_0, v_0) \in \mathcal{W}^-(E)$ (the region above $\mathcal{W}^s(E)$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ is attracted to P and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ is attracted to Q .

Proof. From Theorem 11 equation (13) has a unique equilibrium point $E(\bar{x}, \bar{x})$, which is a saddle point. The map T has no fixed points or periodic points of minimal period-two in $\Delta = \mathcal{R} \cap \text{int}(Q_1(\bar{x}) \cup Q_3(\bar{x}))$. A straightforward calculation shows that $\det J_T(E) < 0$ and $T(x) = \bar{x}$ only for $x = \bar{x}$. Since the map T is anti-competitive and T^2 is strongly competitive we have that all conditions of Theorem 10 in [10] are satisfied from which the proof follows. ■

Theorem 14 *If $4\gamma + 2\beta + \beta^2 > 3$ and $\beta + \gamma \geq 1$ then Eq. (13) has a unique equilibrium point $E(\bar{x}, \bar{x})$ which is a saddle point.*

The global stable manifold $\mathcal{W}^s(E)$, which is a continuous increasing curve divides the first quadrant such that the following holds:

- i) Every initial point (u_0, v_0) in $\mathcal{W}^s(E)$ is attracted to E .
- ii) If $(u_0, v_0) \in \mathcal{W}^+(E)$ (the region below $\mathcal{W}^s(E)$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ tends to $(\infty, 1)$ and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ tends to $(1, \infty)$.
- iii) If $(u_0, v_0) \in \mathcal{W}^-(E)$ (the region above $\mathcal{W}^s(E)$) then the subsequence of even-indexed terms $\{(u_{2n}, v_{2n})\}$ tends to $(1, \infty)$ and the subsequence of odd-indexed terms $\{(u_{2n+1}, v_{2n+1})\}$ tends to $(\infty, 1)$.

Proof. The proof is similar to the proof of the previous theorem using the fact that every solution of equation (13) is bounded from below by 1. ■

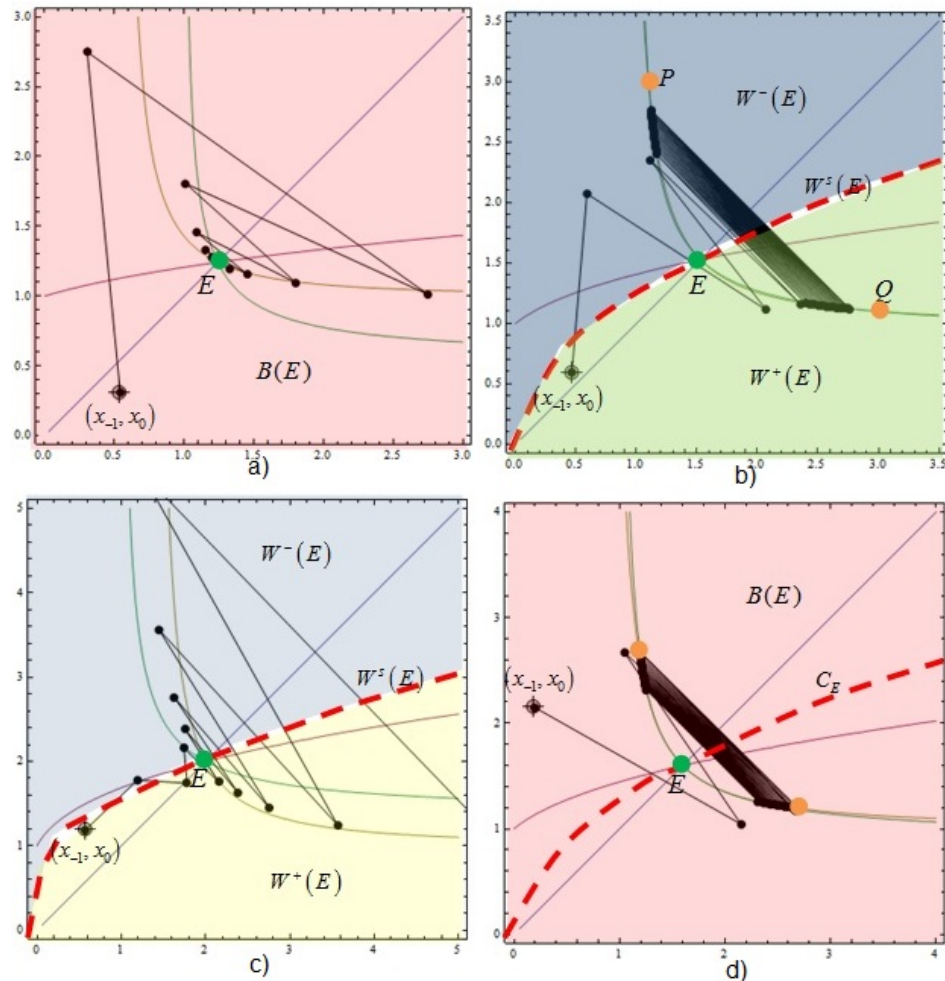


Figure 2: Visual illustration of Theorems 12, 13, 14 and 15 . Figures are generated by *Dynamica 3*, [15].

Theorem 15 If $4\gamma + 2\beta + \beta^2 = 3$ then Eq. (13) has a unique equilibrium point $E(\bar{x}, \bar{x})$ which is a nonhyperbolic point and a global attractor.

Proof. From Theorem 11 Eq.(13) has a unique equilibrium point $E(\bar{x}, \bar{x})$, which is non-hyperbolic. All conditions of Theorem 4 are satisfied, which yields the existence a continuous increasing curve C_E which is a subset of the basin of attraction of E and for every $x \in \mathcal{W}^-(E)$ (the region above C_E) there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \text{int}Q_2(\bar{x})$ for $n \geq n_0$ and for every $x \in \mathcal{W}^+(E)$ (the region below C_E) there exists $n_0 \in \mathbb{N}$ such that $T^n(x) \in \text{int}Q_4(\bar{x})$ for $n \geq n_0$.

Set

$$U(t) = \frac{\beta t + \sqrt{\beta^2 t^2 + (3 - 2\beta - \beta^2)(t^2 - t)}}{2(t - 1)}.$$

It is easy to see that $(t, U(t)) \preceq_{se} E$ if $t < \bar{x}$ and $E \preceq_{se} (t, U(t))$ if $t > \bar{x}$. One can show that

$$T^2(t, U(t)) = \left(t, \frac{(t\beta + s)^4(8t^2 + \frac{t\beta + (3+(-2+4t-\beta)\beta)}{t-1} + \frac{(3+(-2+4t-\beta)\beta)s}{t-1})}{8t^4(-3+t(3+(-2+4t-\beta)\beta) + \beta(2+\beta+2s))^2} \right),$$

where

$$s = \sqrt{t(t(3-2\beta) + (\beta-1)(\beta+3))}.$$

Now we have that

$$T^2(t, U(t)) \preceq_{se} (t, U(t)) \text{ if } t > \bar{x}$$

and

$$(t, U(t)) \preceq_{se} T^2(t, U(t)) \text{ if } t < \bar{x}$$

since

$$\frac{(t\beta + s)^4(8t^2 + \frac{t\beta + (3+(-2+4t-\beta)\beta)}{t-1} + \frac{(3+(-2+4t-\beta)\beta)s}{t-1})}{8t^4(-3+t(3+(-2+4t-\beta)\beta) + \beta(2+\beta+2s))^2} - \frac{\beta t + \sqrt{\beta^2 t^2 + (3-2\beta-\beta^2)(t^2-t)}}{2(t-1)} > 0,$$

if and only if $t > \bar{x}$. By monotonicity if $t < \bar{x}$ then we obtain that $T^{2n}(t, U(t)) \rightarrow E$ as $n \rightarrow \infty$ and if $t > \bar{x}$ then we have that $T^{2n}(t, U(t)) \rightarrow E$ as $n \rightarrow \infty$.

If $(u', v') \in \text{int}Q_2(\bar{x})$ then there exists t_1 such that $(t_1, U(t_1)) \preceq_{se} (u', v') \preceq_{se} E$. By monotonicity of the map T^2 we obtain that $T^{2n}(t_1, U(t_1)) \preceq_{se} T^{2n}(u', v') \preceq_{se} E$ which implies that $T^{2n}(u', v') \rightarrow E$ and $T^{2n+1}(u', v') \rightarrow T(E) = E$, as $n \rightarrow \infty$ which proves the statement ii).

If $(u'', v'') \in \text{int}Q_4(\bar{x})$ then there exists t_2 such that $E \preceq_{se} (u'', v'') \preceq_{se} (t_2, U(t_2))$. By monotonicity of the map T^2 we obtain that $E \preceq_{se} T^{2n}(u'', v'') \preceq_{se} T^{2n}(t_2, U(t_2))$ which implies that $T^{2n}(u'', v'') \rightarrow E$ and $T^{2n+1}(u'', v'') \rightarrow T(E) = E$ as $n \rightarrow \infty$ which proves the statement iii), which completes the proof of the Theorem. ■

Remark 4 Theorems 12, 13, 14 and 15 show another type of period doubling bifurcation. When $4\gamma + 2\beta + \beta^2 \leq 3$ all solutions are asymptotic to the unique equilibrium E . When $4\gamma + 2\beta + \beta^2 > 3$ and $\beta + \gamma < 1$ all solutions which starts off the global stable manifold of the unique equilibrium E are asymptotic to the unique minimal period-two solution $\{P, Q\}$. Finally, when $4\gamma + 2\beta + \beta^2 > 3$ and $\beta + \gamma \geq 1$ all solutions which starts off the global stable manifold of the unique equilibrium E are asymptotic to $(1, \infty)$ or to $(\infty, 1)$.

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Some New Inequalities of Hermite–Hadamard Type for Geometrically Mean Convex Functions on the Co-ordinates

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Abstract

In the paper, the authors introduce a new concept geometrically mean convex function on co-ordinates and establish some new integral inequalities of Hermite–Hadamard type for geometrically mean convex functions of two variables on the co-ordinates.

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1 Introduction

The following definitions are well known in the literature.

Definition 1.1 ([3, 4]). A function $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ is said to be convex on the co-ordinates on Δ with $a < b$ and $c < d$ if the partial mappings

$$f_y : [a, b] \rightarrow \mathbb{R}, \quad f_y(u) = f(u, y) \quad \text{and} \quad f_x : [c, d] \rightarrow \mathbb{R}, \quad f_x(v) = f(x, v)$$

are convex for all $x \in (a, b)$ and $y \in (c, d)$.

Definition 1.2 ([3, 4]). A function $f : \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R}$ is said to be convex on the co-ordinates on Δ with $a < b$ and $c < d$ if

$$f(tx + (1-t)z, \lambda y + (1-\lambda)w) \leq t\lambda f(x, y) + t(1-\lambda)f(x, w) + (1-t)\lambda f(z, y) + (1-t)(1-\lambda)f(z, w)$$

holds for all $t, \lambda \in [0, 1]$, $(x, y), (z, w) \in \Delta$.

Definition 1.3 ([1]). A function $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}_+$ is called co-ordinated log-convex on Δ with $a < b$ and $c < d$ for all $t, \lambda \in [0, 1]$ and $(x, y), (z, w) \in \Delta$, if

$$f(tx + (1-t)z, \lambda y + (1-\lambda)w) \leq [f(x, y)]^{t\lambda} [f(x, w)]^{t(1-\lambda)} [f(z, y)]^{(1-t)\lambda} [f(z, w)]^{(1-t)(1-\lambda)}.$$

In recent years, the following integral inequalities of Hermite–Hadamard type for the above kinds of convex functions were published.

Theorem 1.1 ([3, Theorem 2.2] and [4, Theorem 2.2]). *Let $f : \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R}$ be convex on the co-ordinates on Δ with $a < b$ and $c < d$. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{2} \left[\frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \right] \\ &\leq \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) dx dy \\ &\leq \frac{1}{4} \left[\frac{1}{b-a} \left(\int_a^b f(x, c) dx + \int_a^b f(x, d) dx \right) + \frac{1}{d-c} \left(\int_c^d f(a, y) dy + \int_c^d f(b, y) dy \right) \right] \\ &\leq \frac{1}{4} [f(a, c) + f(b, c) + f(a, d) + f(b, d)]. \end{aligned}$$

Theorem 1.2 ([7, Theorem 2.3]). *Let $f : \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R}$ be a partial differentiable function on Δ . If $\left| \frac{\partial^2 f}{\partial x \partial y} \right|$ is convex on the co-ordinates on Δ , then*

$$\begin{aligned} &\left| \frac{1}{9} \left[f\left(a, \frac{c+d}{2}\right) + f\left(b, \frac{c+d}{2}\right) + 4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + f\left(\frac{a+b}{2}, c\right) + f\left(\frac{a+b}{2}, d\right) \right] \right. \\ &\quad \left. + \frac{1}{36} [f(a, c) + f(a, d) + f(b, c) + f(b, d)] + \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) dx dy - A \right| \\ &\leq \left(\frac{5}{72} \right)^2 (b-a)(d-c) \left\{ \left| \frac{\partial^2 f(a, c)}{\partial x \partial y} \right| + \left| \frac{\partial^2 f(a, d)}{\partial x \partial y} \right| + \left| \frac{\partial^2 f(b, c)}{\partial x \partial y} \right| + \left| \frac{\partial^2 f(b, d)}{\partial x \partial y} \right| \right\}, \end{aligned}$$

where

$$\begin{aligned} A = \frac{1}{b-a} \int_a^b \left\{ \frac{1}{6} \left[f(x, c) + 4f\left(x, \frac{c+d}{2}\right) + f(x, d) \right] \right\} dx \\ + \frac{1}{d-c} \int_c^d \left\{ \frac{1}{6} \left[f(a, y) + 4f\left(\frac{a+b}{2}, y\right) + f(b, y) \right] \right\} dy. \end{aligned}$$

Theorem 1.3 ([6, Theorem 2]). *Let $f : \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R}$ be a partial differentiable function on Δ . If $\left| \frac{\partial^2 f}{\partial x \partial y} \right|$ is convex on the co-ordinates on Δ , then*

$$\begin{aligned} &\left| \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) dx dy + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A \right| \\ &\leq \frac{(b-a)(d-c)}{16} \left[\frac{\left| \frac{\partial^2 f(a, c)}{\partial x \partial y} \right| + \left| \frac{\partial^2 f(a, d)}{\partial x \partial y} \right| + \left| \frac{\partial^2 f(b, c)}{\partial x \partial y} \right| + \left| \frac{\partial^2 f(b, d)}{\partial x \partial y} \right|}{4} \right], \end{aligned}$$

where

$$A = \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy.$$

Theorem 1.4 ([1, Corollary 3.1]). *Suppose that $f : \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R}_+$ is log-convex on the co-ordinates on Δ . Then*

$$\begin{aligned} \ln f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b \ln f(x, y) \, dx \, dy \\ &\leq \frac{\ln f(a, c) + \ln f(b, c) + \ln f(a, d) + \ln f(b, d)}{4}. \end{aligned}$$

In the papers [2, 5, 8, 9, 10, 11], there are also some new results on this topic.

2 A definition and lemmas

In this section, we introduce the notion “geometrically mean convex function” and establish an integral identity.

Definition 2.1. A function $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ is said to be geometrically mean convex on the co-ordinates on Δ with $a < b$ and $c < d$, if

$$f(x^t z^{1-t}, y^\lambda w^{1-\lambda}) \leq [f(x, y)]^{[t+\lambda]/4} [f(x, w)]^{[t+(1-\lambda)]/4} [f(z, y)]^{[(1-t)+\lambda]/4} [f(z, w)]^{[(1-t)+(1-\lambda)]/4}$$

holds for all $t, \lambda \in [0, 1]$ and $(x, y), (z, w) \in \Delta$.

In order to prove our main results, we need the following integral identity.

Lemma 2.1. *Let $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}_+^2 \rightarrow \mathbb{R}$ have partial derivatives of the second order with $a < b$ and $c < d$. If $\frac{\partial^2 f}{\partial x \partial y} \in L_1(\Delta)$, where $L_1(\Delta)$ denotes the set of all Lebesgue integrable functions on Δ , then*

$$\begin{aligned} S(f) &\triangleq \frac{4}{(\ln b - \ln a)(\ln d - \ln c)} \left[\frac{f(a, c) + f(b, c) + f(a, d) + f(b, d)}{4} \right. \\ &\quad \left. - A + \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b \frac{f(x, y)}{xy} \, dx \, dy \right] \\ &= \int_0^1 \int_0^1 t \lambda a^t b^{1-t} c^\lambda d^{1-\lambda} \frac{\partial^2}{\partial x \partial y} f(a^t b^{1-t}, c^\lambda d^{1-\lambda}) \, dt \, d\lambda \\ &\quad - \int_0^1 \int_0^1 t \lambda a^t b^{1-t} c^{1-\lambda} d^\lambda \frac{\partial^2}{\partial x \partial y} f(a^t b^{1-t}, c^{1-\lambda} d^\lambda) \, dt \, d\lambda \\ &\quad - \int_0^1 \int_0^1 t \lambda a^{1-t} b^t c^\lambda d^{1-\lambda} \frac{\partial^2}{\partial x \partial y} f(a^{1-t} b^t, c^\lambda d^{1-\lambda}) \, dt \, d\lambda \\ &\quad + \int_0^1 \int_0^1 t \lambda a^{1-t} b^t c^{1-\lambda} d^\lambda \frac{\partial^2}{\partial x \partial y} f(a^{1-t} b^t, c^{1-\lambda} d^\lambda) \, dt \, d\lambda, \end{aligned}$$

where

$$A = \frac{1}{2(\ln b - \ln a)} \int_a^b \left[\frac{f(x, c)}{x} + \frac{f(x, d)}{x} \right] dx + \frac{1}{2(\ln d - \ln c)} \int_c^d \left[\frac{f(a, y)}{y} + \frac{f(b, y)}{y} \right] dy.$$

Proof. Let $x = a^t b^{1-t}$ and $y = c^\lambda d^{1-\lambda}$ for $0 \leq t, \lambda \leq 1$. Using integration by parts, we have

$$\begin{aligned}
 & \int_0^1 \int_0^1 t \lambda a^t b^{1-t} c^\lambda d^{1-\lambda} \frac{\partial^2}{\partial x \partial y} f(a^t b^{1-t}, c^\lambda d^{1-\lambda}) \, dt \, d\lambda \\
 &= \frac{1}{\ln a - \ln b} \int_0^1 \lambda c^\lambda d^{1-\lambda} \left[t \frac{\partial}{\partial y} f(a^t b^{1-t}, c^\lambda d^{1-\lambda}) \Big|_0^1 - \int_0^1 \frac{\partial}{\partial y} f(a^t b^{1-t}, c^\lambda d^{1-\lambda}) \, dt \right] d\lambda \\
 &= \frac{1}{\ln a - \ln b} \left[\int_0^1 \lambda c^\lambda d^{1-\lambda} \frac{\partial}{\partial y} f(a, c^\lambda d^{1-\lambda}) \, d\lambda - \int_0^1 \int_0^1 \lambda c^\lambda d^{1-\lambda} \frac{\partial}{\partial y} f(a^t b^{1-t}, c^\lambda d^{1-\lambda}) \, dt \, d\lambda \right] \\
 &= \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \left[f(a, c) - \int_0^1 f(a, c^\lambda d^{1-\lambda}) \, d\lambda \right. \\
 &\quad \left. - \int_0^1 f(a^t b^{1-t}, c) \, dt + \int_0^1 \int_0^1 f(a^t b^{1-t}, c^\lambda d^{1-\lambda}) \, dt \, d\lambda \right] \\
 &= \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \left[f(a, c) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x, c)}{x} \, dx \right. \\
 &\quad \left. - \frac{1}{\ln d - \ln c} \int_c^d \frac{f(a, y)}{y} \, dy + \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b \frac{f(x, y)}{xy} \, dx \, dy \right].
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 & \int_0^1 \int_0^1 t \lambda a^t b^{1-t} c^{1-\lambda} d^\lambda \frac{\partial^2}{\partial x \partial y} f(a^t b^{1-t}, c^{1-\lambda} d^\lambda) \, dt \, d\lambda \\
 &= -\frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \left[f(a, d) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x, d)}{x} \, dx \right. \\
 &\quad \left. - \frac{1}{\ln d - \ln c} \int_c^d \frac{f(a, y)}{y} \, dy + \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b \frac{f(x, y)}{xy} \, dx \, dy \right], \\
 & \int_0^1 \int_0^1 t \lambda a^{1-t} b^t c^\lambda d^{1-\lambda} \frac{\partial^2}{\partial x \partial y} f(a^{1-t} b^t, c^\lambda d^{1-\lambda}) \, dt \, d\lambda \\
 &= -\frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \left[f(b, c) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x, c)}{x} \, dx \right. \\
 &\quad \left. - \frac{1}{\ln d - \ln c} \int_c^d \frac{f(b, y)}{y} \, dy + \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b \frac{f(x, y)}{xy} \, dx \, dy \right],
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_0^1 \int_0^1 t \lambda a^{1-t} b^t c^{1-\lambda} d^\lambda \frac{\partial^2}{\partial x \partial y} f(a^{1-t} b^t, c^{1-\lambda} d^\lambda) \, dt \, d\lambda \\
 &= \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \left[f(b, d) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x, d)}{x} \, dx \right. \\
 &\quad \left. - \frac{1}{\ln d - \ln c} \int_c^d \frac{f(b, y)}{y} \, dy + \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b \frac{f(x, y)}{xy} \, dx \, dy \right].
 \end{aligned}$$

Lemma 2.1 is proved. \square

Lemma 2.2. *Let $u, v > 0$. Then*

$$F(u, v) \triangleq \int_0^1 t u^t v^{1-t} dt = \begin{cases} \frac{L(u, v) - u}{\ln v - \ln u}, & u \neq v, \\ \frac{1}{2}u, & u = v, \end{cases} \quad (2.1)$$

where $L(u, v)$ is logarithmic mean defined by

$$L(u, v) \triangleq \int_0^1 u^t v^{1-t} dt = \begin{cases} \frac{v - u}{\ln v - \ln u}, & u \neq v, \\ u, & u = v. \end{cases}$$

Proof. The proof is straightforward. \square

3 Some integral inequalities of Hermite–Hadamard type

In this section, we prove some new inequalities of Hermite–Hadamard type for geometrically mean convex functions.

Theorem 3.1. *Let $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}_+^2 \rightarrow \mathbb{R}$ be a partial differentiable function on Δ with $a < b$, $c < d$ and $\frac{\partial^2 f}{\partial x \partial y} \in L_1(\Delta)$. If $\left| \frac{\partial^2 f}{\partial x \partial y} \right|^q$ is geometrically mean convex functions on the co-ordinates on Δ for $q \geq 1$, then*

$$\begin{aligned} |S(f)| &\leq [F(a, b)F(c, d)]^{1-1/q} [F(M_q(a, a), M_q(b, b))F(N_q(c, c), N_q(d, d))]^{1/q} \\ &\quad + [F(a, b)F(d, c)]^{1-1/q} [F(M_q(a, a), M_q(b, b))F(N_q(d, d), N_q(c, c))]^{1/q} \\ &\quad + [F(b, a)F(c, d)]^{1-1/q} [F(M_q(b, b), M_q(a, a))F(N_q(c, c), N_q(d, d))]^{1/q} \\ &\quad + [F(b, a)F(d, c)]^{1-1/q} [F(M_q(b, b), M_q(a, a))F(N_q(d, d), N_q(c, c))]^{1/q}, \end{aligned} \quad (3.1)$$

where $F(u, v)$ is defined by (2.1),

$$M_q(u^r, u) = u^r \left[\left| \frac{\partial^2 f(u, c)}{\partial x \partial y} \right| \left| \frac{\partial^2 f(u, d)}{\partial x \partial y} \right| \right]^{q/4}, \quad N_q(v^r, v) = v^r \left[\left| \frac{\partial^2 f(a, v)}{\partial x \partial y} \right| \left| \frac{\partial^2 f(b, v)}{\partial x \partial y} \right| \right]^{q/4} \quad (3.2)$$

for $r \geq 0$.

Proof. Since $\left| \frac{\partial^2 f}{\partial x \partial y} \right|^q$ is geometrically mean convex on coordinates Δ , using Lemma 2.1 and by Hölder's integral inequality, we have

$$\begin{aligned} |S(f)| &\leq \int_0^1 \int_0^1 t \lambda a^t b^{1-t} c^\lambda d^{1-\lambda} \left| \frac{\partial^2}{\partial x \partial y} f(a^t b^{1-t}, c^\lambda d^{1-\lambda}) \right| dt d\lambda \\ &\quad + \int_0^1 \int_0^1 t \lambda a^t b^{1-t} c^{1-\lambda} d^\lambda \left| \frac{\partial^2}{\partial x \partial y} f(a^t b^{1-t}, c^{1-\lambda} d^\lambda) \right| dt d\lambda \\ &\quad + \int_0^1 \int_0^1 t \lambda a^{1-t} b^t c^\lambda d^{1-\lambda} \left| \frac{\partial^2}{\partial x \partial y} f(a^{1-t} b^t, c^\lambda d^{1-\lambda}) \right| dt d\lambda \\ &\quad + \int_0^1 \int_0^1 t \lambda a^{1-t} b^t c^{1-\lambda} d^\lambda \left| \frac{\partial^2}{\partial x \partial y} f(a^{1-t} b^t, c^{1-\lambda} d^\lambda) \right| dt d\lambda \end{aligned}$$

$$\begin{aligned}
&\leq \left(\int_0^1 \int_0^1 t \lambda a^t b^{1-t} c^\lambda d^{1-\lambda} dt d\lambda \right)^{1-1/q} \left[\int_0^1 \int_0^1 t \lambda a^t b^{1-t} c^\lambda d^{1-\lambda} \left| \frac{\partial^2 f(a, c)}{\partial x \partial y} \right|^{q[t+\lambda]/4} \right. \\
&\quad \times \left| \frac{\partial^2 f(a, d)}{\partial x \partial y} \right|^{q[t+(1-\lambda)]/4} \left| \frac{\partial^2 f(b, c)}{\partial x \partial y} \right|^{q[(1-t)+\lambda]/4} \left| \frac{\partial^2 f(b, d)}{\partial x \partial y} \right|^{q[(1-t)+(1-\lambda)]/4} dt d\lambda \left. \right]^{1/q} \\
&\quad + \left(\int_0^1 \int_0^1 t \lambda a^t b^{1-t} c^{1-\lambda} d^\lambda dt d\lambda \right)^{1-1/q} \left[\int_0^1 \int_0^1 t \lambda a^t b^{1-t} c^{1-\lambda} d^\lambda \left| \frac{\partial^2 f(a, c)}{\partial x \partial y} \right|^{q[t+(1-\lambda)]/4} \right. \\
&\quad \times \left| \frac{\partial^2 f(a, d)}{\partial x \partial y} \right|^{q[t+\lambda]/4} \left| \frac{\partial^2 f(b, c)}{\partial x \partial y} \right|^{q[(1-t)+(1-\lambda)]/4} \left| \frac{\partial^2 f(b, d)}{\partial x \partial y} \right|^{q[(1-t)+\lambda]/4} dt d\lambda \left. \right]^{1/q} \\
&\quad + \left(\int_0^1 \int_0^1 t \lambda a^{1-t} b^t c^\lambda d^{1-\lambda} dt d\lambda \right)^{1-1/q} \left[\int_0^1 \int_0^1 t \lambda a^{1-t} b^t c^\lambda d^{1-\lambda} \left| \frac{\partial^2 f(a, c)}{\partial x \partial y} \right|^{q[(1-t)+\lambda]/4} \right. \\
&\quad \times \left| \frac{\partial^2 f(a, d)}{\partial x \partial y} \right|^{q[(1-t)+(1-\lambda)]/4} \left| \frac{\partial^2 f(b, c)}{\partial x \partial y} \right|^{q[t+\lambda]/4} \left| \frac{\partial^2 f(b, d)}{\partial x \partial y} \right|^{q[t+(1-\lambda)]/4} dt d\lambda \left. \right]^{1/q} \\
&\quad + \left(\int_0^1 \int_0^1 t \lambda a^{1-t} b^t c^{1-\lambda} d^\lambda dt d\lambda \right)^{1-1/q} \left[\int_0^1 \int_0^1 t \lambda a^{1-t} b^t c^{1-\lambda} d^\lambda \left| \frac{\partial^2 f(a, c)}{\partial x \partial y} \right|^{q[(1-t)+(1-\lambda)]/4} \right. \\
&\quad \times \left| \frac{\partial^2 f(a, d)}{\partial x \partial y} \right|^{q[(1-t)+\lambda]/4} \left| \frac{\partial^2 f(b, c)}{\partial x \partial y} \right|^{q[t+(1-\lambda)]/4} \left| \frac{\partial^2 f(b, d)}{\partial x \partial y} \right|^{q[t+\lambda]/4} dt d\lambda \left. \right]^{1/q}.
\end{aligned} \tag{3.3}$$

Also by Lemma 2.2, we have

$$\int_0^1 \int_0^1 t \lambda a^t b^{1-t} c^\lambda d^{1-\lambda} dt d\lambda = F(a, b)F(c, d) \tag{3.4}$$

and

$$\begin{aligned}
&\int_0^1 \int_0^1 t \lambda a^t b^{1-t} c^\lambda d^{1-\lambda} \left| \frac{\partial^2 f(a, c)}{\partial x \partial y} \right|^{q[t+\lambda]/4} \left| \frac{\partial^2 f(a, d)}{\partial x \partial y} \right|^{q[t+(1-\lambda)]/4} \\
&\quad \times \left| \frac{\partial^2 f(b, c)}{\partial x \partial y} \right|^{q[(1-t)+\lambda]/4} \left| \frac{\partial^2 f(b, d)}{\partial x \partial y} \right|^{q[(1-t)+(1-\lambda)]/4} dt d\lambda \\
&= \int_0^1 \int_0^1 t \lambda [M_q(a, a)]^t [M_q(b, b)]^{1-t} [N_q(c, c)]^\lambda [N_q(d, d)]^{1-\lambda} dt d\lambda \\
&= F(M_q(a, a), M_q(b, b))F(N_q(c, c), N_q(d, d)).
\end{aligned}$$

By simple computation,

$$\begin{aligned}
&\int_0^1 \int_0^1 t \lambda a^t b^{1-t} c^{1-\lambda} d^\lambda dt d\lambda = F(a, b)F(d, c), \quad \int_0^1 \int_0^1 t \lambda a^{1-t} b^t c^\lambda d^{1-\lambda} dt d\lambda = F(b, a)F(c, d), \\
&\int_0^1 \int_0^1 t \lambda a^{1-t} b^t c^{1-\lambda} d^\lambda dt d\lambda = F(b, a)F(d, c), \\
&\int_0^1 \int_0^1 t \lambda a^t b^{1-t} c^{1-\lambda} d^\lambda \left| \frac{\partial^2 f(a, c)}{\partial x \partial y} \right|^{q[t+(1-\lambda)]/4} \left| \frac{\partial^2 f(a, d)}{\partial x \partial y} \right|^{q[t+\lambda]/4} \left| \frac{\partial^2 f(b, c)}{\partial x \partial y} \right|^{q[(1-t)+(1-\lambda)]/4}
\end{aligned}$$

$$\begin{aligned} & \times \left| \frac{\partial^2 f(b, d)}{\partial x \partial y} \right|^{q[(1-t)+\lambda]/4} dt d\lambda = F(M_q(a, a), M_q(b, b)) F(N_q(d, d), N_q(c, c)), \\ & \int_0^1 \int_0^1 t\lambda a^{1-t} b^t c^\lambda d^{1-\lambda} \left| \frac{\partial^2 f(a, c)}{\partial x \partial y} \right|^{q[(1-t)+\lambda]/4} \left| \frac{\partial^2 f(a, d)}{\partial x \partial y} \right|^{q[(1-t)+(1-\lambda)]/4} \left| \frac{\partial^2 f(b, c)}{\partial x \partial y} \right|^{q[t+\lambda]/4} \\ & \times \left| \frac{\partial^2 f(b, d)}{\partial x \partial y} \right|^{q[t+(1-\lambda)]/4} dt d\lambda = F(M_q(b, b), M_q(a, a)) F(N_q(c, c), N_q(d, d)) \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 \int_0^1 t\lambda a^{1-t} b^t c^{1-\lambda} d^\lambda \left| \frac{\partial^2 f(a, c)}{\partial x \partial y} \right|^{q[(1-t)+(1-\lambda)]/4} \left| \frac{\partial^2 f(a, d)}{\partial x \partial y} \right|^{q[(1-t)+\lambda]/4} \left| \frac{\partial^2 f(b, c)}{\partial x \partial y} \right|^{q[t+(1-\lambda)]/4} \\ & \times \left| \frac{\partial^2 f(b, d)}{\partial x \partial y} \right|^{q[t+\lambda]/4} dt d\lambda = F(M_q(b, b), M_q(a, a)) F(N_q(d, d), N_q(c, c)). \quad (3.5) \end{aligned}$$

Substituting equalities (3.4) to (3.5) into the inequality (3.3) and rearranging yield the inequality (3.1). Theorem 3.1 is proved. \square

Corollary 3.1.1. *Under the conditions of Theorem 3.1, when $q = 1$, we have*

$$\begin{aligned} |S(f)| & \leq F(M_1(a, a), M_1(b, b)) F(N_1(c, c), N_1(d, d)) + F(M_1(a, a), M_1(b, b)) F(N_1(d, d), N_1(c, c)) \\ & + F(M_1(b, b), M_1(a, a)) F(N_1(c, c), N_1(d, d)) + F(M_1(b, b), M_1(a, a)) F(N_1(d, d), N_1(c, c)), \end{aligned}$$

where $F(u, v)$ is defined by (2.1), and $M_q(u^r, u)$ and $N_q(v^r, v)$ are defined by (3.2).

Theorem 3.2. *Let $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}_+^2 \rightarrow \mathbb{R}$ be a partial differentiable function on Δ with $a < b$, $c < d$ and $\frac{\partial^2 f}{\partial x \partial y} \in L_1(\Delta)$. If $\left| \frac{\partial^2 f}{\partial x \partial y} \right|^q$ is geometrically mean convex functions on the co-ordinates on Δ for $q > 1$ and $q \geq r \geq 0$, then*

$$\begin{aligned} |S(f)| & \leq [F(a^{(q-r)/(q-1)}, b^{(q-r)/(q-1)}) F(c^{(q-r)/(q-1)}, d^{(q-r)/(q-1)})]^{1-1/q} \\ & \times [F(M_q(a^r, a), M_q(b^r, b)) F(N_q(c^r, c), N_q(d^r, d))]^{1/q} \\ & + [F(a^{(q-r)/(q-1)}, b^{(q-r)/(q-1)}) F(d^{(q-r)/(q-1)}, c^{(q-r)/(q-1)})]^{1-1/q} \\ & \times [F(M_q(a^r, a), M_q(b^r, b)) F(N_q(d^r, d), N_q(c^r, c))]^{1/q} \\ & + [F(b^{(q-r)/(q-1)}, a^{(q-r)/(q-1)}) F(c^{(q-r)/(q-1)}, d^{(q-r)/(q-1)})]^{1-1/q} \\ & \times [F(M_q(b^r, b), M_q(a^r, a)) F(N_q(c^r, c), N_q(d^r, d))]^{1/q} \\ & + [F(b^{(q-r)/(q-1)}, a^{(q-r)/(q-1)}) F(d^{(q-r)/(q-1)}, c^{(q-r)/(q-1)})]^{1-1/q} \\ & \times [F(M_q(b^r, b), M_q(a^r, a)) F(N_q(d^r, d), N_q(c^r, c))]^{1/q}, \end{aligned}$$

where $F(u, v)$ is defined by (2.1), and $M_q(u^r, u)$ and $N_q(v^r, v)$ are defined by (3.2).

Proof. From Lemma 2.1, we have

$$\begin{aligned}
 |S(f)| &\leq \int_0^1 \int_0^1 t\lambda a^t b^{1-t} c^\lambda d^{1-\lambda} \left| \frac{\partial^2}{\partial x \partial y} f(a^t b^{1-t}, c^\lambda d^{1-\lambda}) \right| dt d\lambda \\
 &\quad + \int_0^1 \int_0^1 t\lambda a^t b^{1-t} c^{1-\lambda} d^\lambda \left| \frac{\partial^2}{\partial x \partial y} f(a^t b^{1-t}, c^{1-\lambda} d^\lambda) \right| dt d\lambda \\
 &\quad + \int_0^1 \int_0^1 t\lambda a^{1-t} b^t c^\lambda d^{1-\lambda} \left| \frac{\partial^2}{\partial x \partial y} f(a^{1-t} b^t, c^\lambda d^{1-\lambda}) \right| dt d\lambda \\
 &\quad + \int_0^1 \int_0^1 t\lambda a^{1-t} b^t c^{1-\lambda} d^\lambda \left| \frac{\partial^2}{\partial x \partial y} f(a^{1-t} b^t, c^{1-\lambda} d^\lambda) \right| dt d\lambda.
 \end{aligned} \tag{3.6}$$

Using Hölder's integral inequality, and by the geometrically mean convexity of $\left| \frac{\partial^2 f}{\partial x \partial y} \right|^q$ on Δ and Lemma 2.2, it is easy to observe that

$$\begin{aligned}
 &\int_0^1 \int_0^1 t\lambda a^t b^{1-t} c^\lambda d^{1-\lambda} \left| \frac{\partial^2}{\partial x \partial y} f(a^t b^{1-t}, c^\lambda d^{1-\lambda}) \right| dt d\lambda \\
 &\leq \left(\int_0^1 \int_0^1 t\lambda (a^t b^{1-t} c^\lambda d^{1-\lambda})^{(q-r)/(q-1)} dt d\lambda \right)^{1-1/q} \\
 &\quad \times \left[\int_0^1 \int_0^1 t\lambda a^{rt} b^{r(1-t)} c^{r\lambda} d^{r(1-\lambda)} \left| \frac{\partial^2 f(a, c)}{\partial x \partial y} \right|^{q[t+\lambda]/4} \left| \frac{\partial^2 f(a, d)}{\partial x \partial y} \right|^{q[t+(1-\lambda)]/4} \right. \\
 &\quad \times \left. \left| \frac{\partial^2 f(b, c)}{\partial x \partial y} \right|^{q[(1-t)+\lambda]/4} \left| \frac{\partial^2 f(b, d)}{\partial x \partial y} \right|^{q[(1-t)+(1-\lambda)]/4} dt d\lambda \right]^{1/q} \\
 &= [F(a^{(q-r)/(q-1)}, b^{(q-r)/(q-1)}) F(c^{(q-r)/(q-1)}, d^{(q-r)/(q-1)})]^{1-1/q} \\
 &\quad \times [F(M_q(a^r, a), M_q(b^r, b)) F(N_q(c^r, c), N_q(d^r, d))]^{1/q}.
 \end{aligned} \tag{3.7}$$

Similarly, we can show that

$$\begin{aligned}
 &\int_0^1 \int_0^1 t\lambda a^t b^{1-t} c^{1-\lambda} d^\lambda \left| \frac{\partial^2}{\partial x \partial y} f(a^t b^{1-t}, c^{1-\lambda} d^\lambda) \right| dt d\lambda \\
 &\leq [F(a^{(q-r)/(q-1)}, b^{(q-r)/(q-1)}) F(d^{(q-r)/(q-1)}, c^{(q-r)/(q-1)})]^{1-1/q} \\
 &\quad \times [F(M_q(a^r, a), M_q(b^r, b)) F(N_q(d^r, d), N_q(c^r, c))]^{1/q}, \\
 &\int_0^1 \int_0^1 t\lambda a^{1-t} b^t c^\lambda d^{1-\lambda} \left| \frac{\partial^2}{\partial x \partial y} f(a^{1-t} b^t, c^\lambda d^{1-\lambda}) \right| dt d\lambda \\
 &= [F(b^{(q-r)/(q-1)}, a^{(q-r)/(q-1)}) F(c^{(q-r)/(q-1)}, d^{(q-r)/(q-1)})]^{1-1/q} \\
 &\quad \times [F(M_q(b^r, b), M_q(a^r, a)) F(N_q(c^r, c), N_q(d^r, d))]^{1/q},
 \end{aligned}$$

and

$$\begin{aligned}
 &\int_0^1 \int_0^1 t\lambda a^{1-t} b^t c^{1-\lambda} d^\lambda \left| \frac{\partial^2}{\partial x \partial y} f(a^{1-t} b^t, c^{1-\lambda} d^\lambda) \right| dt d\lambda \\
 &\leq [F(b^{(q-r)/(q-1)}, a^{(q-r)/(q-1)}) F(d^{(q-r)/(q-1)}, c^{(q-r)/(q-1)})]^{1-1/q} \\
 &\quad \times [F(M_q(b^r, b), M_q(a^r, a)) F(N_q(d^r, d), N_q(c^r, c))]^{1/q}.
 \end{aligned} \tag{3.8}$$

Using the inequalities (3.7) to (3.8) in the inequality (3.6), we conclude the required inequality. The proof is completed. \square

Corollary 3.2.1. *Under the conditions of Theorem 3.2,*

1. *when $r = 0$, we deduce*

$$\begin{aligned} |S(f)| &\leq [F(a^{q/(q-1)}, b^{q/(q-1)})F(c^{q/(q-1)}, d^{q/(q-1)})]^{1-1/q} \\ &\quad \times [F(M_q(1, a), M_q(1, b))F(N_q(1, c), N_q(1, d))]^{1/q} \\ &\quad + [F(a^{q/(q-1)}, b^{q/(q-1)})F(d^{q/(q-1)}, c^{q/(q-1)})]^{1-1/q} \\ &\quad \times [F(M_q(1, a), M_q(1, b))F(N_q(1, d), N_q(1, c))]^{1/q} \\ &\quad + [F(b^{q/(q-1)}, a^{q/(q-1)})F(c^{q/(q-1)}, d^{q/(q-1)})]^{1-1/q} \\ &\quad \times [F(M_q(1, b), M_q(1, a))F(N_q(1, c), N_q(1, d))]^{1/q} \\ &\quad + [F(b^{q/(q-1)}, a^{q/(q-1)})F(d^{q/(q-1)}, c^{q/(q-1)})]^{1-1/q} \\ &\quad \times [F(M_q(1, b), M_q(1, a))F(N_q(1, d), N_q(1, c))]^{1/q}; \end{aligned}$$

2. *when $r = q$, we have*

$$\begin{aligned} |S(f)| &\leq \left(\frac{1}{4}\right)^{1-1/q} \left\{ [F(M_q(a^q, a), M_q(b^q, b))F(N_q(c^q, c), N_q(d^q, d))]^{1/q} \right. \\ &\quad + [F(M_q(a^q, a), M_q(b^q, b))F(N_q(d^q, d), N_q(c^q, c))]^{1/q} \\ &\quad + [F(M_q(b^q, b), M_q(a^q, a))F(N_q(c^q, c), N_q(d^q, d))]^{1/q} \\ &\quad \left. + [F(M_q(b^q, b), M_q(a^q, a))F(N_q(d^q, d), N_q(c^q, c))]^{1/q} \right\}, \end{aligned}$$

where $F(u, v)$ is defined by (2.1), and $M_q(u^r, u)$ and $N_q(v^r, v)$ are defined by (3.2).

Proof. This follows from letting $r = 0$ and $r = q$ respectively in Theorem 3.2. \square

Theorem 3.3. *Let $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ be integrable on Δ with $a < b$, $c < d$. If f is geometrically mean convex on Δ , then*

$$\begin{aligned} f(\sqrt{ab}, \sqrt{cd}) &\leq \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b \frac{[f(x, y)f(x, \frac{cd}{y})f(\frac{ab}{x}, y)f(\frac{ab}{x}, \frac{cd}{y})]^{1/4}}{xy} dx dy \\ &\leq [f(a, c)f(a, d)f(b, c)f(b, d)]^{1/4}. \end{aligned}$$

Proof. Taking $x = a^t b^{1-t}$ and $y = c^\lambda d^{1-\lambda}$ for $0 \leq t, \lambda \leq 1$ and using the geometrically mean convexity of f , we have

$$f(\sqrt{ab}, \sqrt{cd}) = \int_0^1 \int_0^1 f([a^t b^{1-t}]^{1/2} [a^{1-t} b^t]^{1/2}, [c^\lambda d^{1-\lambda}]^{1/2} [c^{1-\lambda} d^\lambda]^{1/2}) dt d\lambda$$

$$\begin{aligned} &\leq \int_0^1 \int_0^1 \left[f(a^t b^{1-t}, c^\lambda d^{1-\lambda}) f(a^t b^{1-t}, c^{1-\lambda} d^\lambda) f(a^{1-t} b^t, c^\lambda d^{1-\lambda}) f(a^{1-t} b^t, c^{1-\lambda} d^\lambda) \right]^{1/4} dt d\lambda \\ &= \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b \frac{[f(x, y) f(x, \frac{cd}{y}) f(\frac{ab}{x}, y) f(\frac{ab}{x}, \frac{cd}{y})]^{1/4}}{xy} dx dy \end{aligned}$$

and

$$\begin{aligned} &\int_0^1 \int_0^1 [f(a^t b^{1-t}, c^\lambda d^{1-\lambda}) f(a^t b^{1-t}, c^{1-\lambda} d^\lambda) f(a^{1-t} b^t, c^\lambda d^{1-\lambda}) f(a^{1-t} b^t, c^{1-\lambda} d^\lambda)]^{1/4} dt d\lambda \\ &\leq \int_0^1 \int_0^1 \{ [f(a, c)]^{t+\lambda} [f(a, d)]^{t+(1-\lambda)} [f(b, c)]^{(1-t)+\lambda} [f(b, d)]^{(1-t)+(1-\lambda)} \\ &\quad \times [f(a, d)]^{t+\lambda} [f(a, c)]^{t+(1-\lambda)} [f(b, d)]^{(1-t)+\lambda} [f(b, c)]^{(1-t)+(1-\lambda)} \\ &\quad \times [f(b, c)]^{t+\lambda} [f(b, d)]^{t+(1-\lambda)} [f(a, c)]^{(1-t)+\lambda} [f(a, d)]^{(1-t)+(1-\lambda)} \\ &\quad \times [f(b, d)]^{t+\lambda} [f(b, c)]^{t+(1-\lambda)} [f(a, d)]^{(1-t)+\lambda} [f(a, c)]^{(1-t)+(1-\lambda)} \}^{1/16} dt d\lambda \\ &= \int_0^1 \int_0^1 [f(a, c) f(a, d) f(b, c) f(b, d)]^{1/4} dt d\lambda = [f(a, c) f(a, d) f(b, c) f(b, d)]^{1/4}. \end{aligned}$$

The proof of Theorem 3.3 is complete. \square

Theorem 3.4. Let $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ be integrable on Δ with $a < b$, $c < d$. If f is geometrically mean convex on Δ , then

$$\begin{aligned} &\frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b \frac{f(x, y)}{xy} dx dy \\ &\leq L([f(a, c) f(a, d)]^{1/4}, [f(b, c) f(b, d)]^{1/4}) L([f(a, c) f(b, c)]^{1/4}, [f(a, d) f(b, d)]^{1/4}), \end{aligned}$$

where $L(u, v)$ is the logarithmic mean.

Proof. Putting $x = a^t b^{1-t}$ and $y = c^\lambda d^{1-\lambda}$ for $0 \leq t, \lambda \leq 1$, from the geometrically mean convexity of f , we obtain

$$\begin{aligned} &\frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b \frac{f(x, y)}{xy} dx dy = \int_0^1 \int_0^1 f(a^t b^{1-t}, c^\lambda d^{1-\lambda}) dt d\lambda \\ &\leq \int_0^1 \int_0^1 \{ [f(a, c)]^{t+\lambda} [f(a, d)]^{t+(1-\lambda)} [f(b, c)]^{(1-t)+\lambda} [f(b, d)]^{(1-t)+(1-\lambda)} \}^{1/4} dt d\lambda \\ &= L([f(a, c) f(a, d)]^{1/4}, [f(b, c) f(b, d)]^{1/4}) L([f(a, c) f(b, c)]^{1/4}, [f(a, d) f(b, d)]^{1/4}). \end{aligned}$$

The proof of Theorem 3.4 is complete. \square

Theorem 3.5. Let $f : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ be integrable on Δ with $a < b$, $c < d$. If f is co-ordinated geometrically mean convex on Δ , then

$$\begin{aligned} &\frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b f(x, y) dx dy \\ &\leq L([f(a, c) f(a, d)]^{1/4}, b[f(b, c) f(b, d)]^{1/4}) L(c[f(a, c) f(b, c)]^{1/4}, d[f(a, d) f(b, d)]^{1/4}), \end{aligned}$$

where $L(u, v)$ is the logarithmic mean.

Proof. Similar to the proof of Theorem 3.4, by the geometrically mean convexity of f , we drive

$$\begin{aligned} & \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b f(x, y) \, dx \, dy = \int_0^1 \int_0^1 a^t b^{1-t} c^\lambda d^{1-\lambda} f(a^t b^{1-t}, c^\lambda d^{1-\lambda}) \, dt \, d\lambda \\ & \leq \int_0^1 \int_0^1 a^t b^{1-t} c^\lambda d^{1-\lambda} \{ [f(a, c)]^{t+\lambda} [f(a, d)]^{t+(1-\lambda)} [f(b, c)]^{(1-t)+\lambda} [f(b, d)]^{(1-t)+(1-\lambda)} \}^{1/4} \, dt \, d\lambda \\ & = L(a[f(a, c)f(a, d)]^{1/4}, b[f(b, c)f(b, d)]^{1/4}) L(c[f(a, c)f(b, c)]^{1/4}, d[f(a, d)f(b, d)]^{1/4}). \end{aligned}$$

The proof of Theorem 3.5 is complete. \square

We proceed similarly as in the proof of Theorem 3.3 to Theorem 3.5, we can get

Theorem 3.6. Let $f, g : \Delta = [a, b] \times [c, d] \subseteq \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ be integrable on Δ with $a < b, c < d$. If f and g are co-ordinated geometrically mean convex on Δ , then

$$\begin{aligned} f(\sqrt{ab}, \sqrt{cd}) g(\sqrt{ab}, \sqrt{cd}) & \leq \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \\ & \times \int_c^d \int_a^b \frac{[f(x, y)g(x, y)f(x, \frac{cd}{y})g(x, \frac{cd}{y})f(\frac{ab}{x}, y)g(\frac{ab}{x}, y)f(\frac{ab}{x}, \frac{cd}{y})g(\frac{ab}{x}, \frac{cd}{y})]^{1/4}}{xy} \, dx \, dy \\ & \leq [f(a, c)g(a, c)f(a, d)g(a, d)f(b, c)g(b, c)f(b, d)g(b, d)]^{1/4}. \end{aligned}$$

Theorem 3.7. Under the conditions of Theorem 3.6, we have

$$\begin{aligned} & \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b \frac{f(x, y)g(x, y)}{xy} \, dx \, dy \\ & \leq L([f(a, c)g(a, c)f(a, d)g(a, d)]^{1/4}, [f(b, c)g(b, c)f(b, d)g(b, d)]^{1/4}) \\ & \quad \times L([f(a, c)g(a, c)f(b, c)g(b, c)]^{1/4}, [f(a, d)g(a, d)f(b, d)g(b, d)]^{1/4}) \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{(\ln b - \ln a)(\ln d - \ln c)} \int_c^d \int_a^b f(x, y)g(x, y) \, dx \, dy \\ & \leq L(a[f(a, c)g(a, c)f(a, d)g(a, d)]^{1/4}, b[f(b, c)g(b, c)f(b, d)g(b, d)]^{1/4}) \\ & \quad \times L(c[f(a, c)g(a, c)f(b, c)g(b, c)]^{1/4}, d[f(a, d)g(a, d)f(b, d)g(b, d)]^{1/4}). \end{aligned}$$

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STABILITY OF THE GENERALAIZED VERSION OF EULER-LAGRANGE TYPE QUADRATIC EQUATION

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ABSTRACT. In this paper, we consider the generalized Hyers-Ulam stability for the following quadratic functional equation.

$$f(ax + by) + f(ax - by) + G_f(x, y) = 2a^2 f(x) + 2b^2 f(y)$$

Here G_f is a functional operator of f . We consider some sufficient conditions on G_f which can be applied easily for the generalized Hyers-Ulam stability, and illustrate some new functional equations by using them.

1. INTRODUCTION

In 1940, Ulam proposed the following stability problem (See [17]):

“Let G_1 be a group and G_2 a metric group with the metric d . Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \rightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \rightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?”

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In 1941, Hyers [7] answered this problem under the assumption that the groups are Banach spaces. Aoki [1] and Rassias [13] generalized the result of Hyers. Th. M. Rassias [13] solved the generalized Hyers-Ulam stability of the functional inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for some $\epsilon \geq 0$ and p with $p < 1$ and for all $x, y \in X$, where $f : X \rightarrow Y$ is a function between Banach spaces. The paper of Rassias [13] has provided a lot of influence in the development of what we call *the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Rassias' theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation

$$(1.1) \quad f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called *a quadratic functional equation* and a solution of a quadratic functional equation is called *quadratic*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [16] for mappings $f : X \rightarrow Y$, where X is a normed space and Y is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [3] proved the generalized Hyers-Ulam stability for the quadratic functional equation and Park [12] proved the generalized Hyers-Ulam stability of the quadratic functional equation in Banach modules over a C^* -algebra.

Rassias [14] investigated the following Euler-Lagrange functional equation

$$f(ax+by) + f(bx-ay) = 2(a^2+b^2)[f(x)+f(y)]$$

and Gordji and Khodaei [6] investigated other Euler-Lagrange functional equations

$$(1.2) \quad \begin{aligned} f(ax+by) + f(ax-by) &= \frac{b(a+b)}{2}f(x+y) \\ &+ \frac{b(a+b)}{2}f(x-y) + (2a^2-ab-b^2)f(x) + (b^2-ab)f(y) \end{aligned}$$

for fixed integers a, b with $b \neq a, -a, -3a$ and

$$(1.3) \quad f(ax+by) + f(ax-by) = 2a^2f(x) + 2b^2f(y)$$

for fixed integers a, b with $a^2 \neq b^2$ and $ab \neq 0$.

In this paper, we are interested in what kind of terms can be added to the quadratic functional equation

$$f(ax + by) + f(ax - by) = 2a^2f(x) + 2b^2f(y)$$

while the generalized Hyers-Ulam stability still holds for the new functional equation.

We denote the added term by $G_f(x, y)$ which can be regarded as a functional operator depending on the variables x, y , and function f . Then the new functional equation can be written as

$$f(ax + by) + f(ax - by) + G_f(x, y) = 2a^2f(x) + 2b^2f(y)$$

for some rational numbers a, b with $ab \neq 0$ and $a^2 \neq b^2$. The precise definition of G_f is given in section 2. In fact, the functional operator $G_f(x, y)$ was introduced and considered in the case of the additive functional equations with somewhat different point of view by the authors([11]).

The new observation in this article makes possible to prove many previous problems on quadratic functional equations more easily and provides methods to construct new ones. So we can have a larger class of functional equations related with quadratic functions for the generalized Hyers-Ulam stability. We illustrate some new functional equations in section 3 in order to see how our observation works for the generalized Hyers-Ulam stability.

2. QUADRATIC FUNCTIONAL EQUATIONS WITH GENERAL TERMS

Let X be a real normed linear space and Y a real Banach space. For given $l \in \mathbb{N}$ and any $i \in \{1, 2, \dots, l\}$, let $\sigma_i : X \times X \longrightarrow X$ be a binary operation such that

$$\sigma_i(rx, ry) = r\sigma_i(x, y)$$

for all $x, y \in X$ and all $r \in \mathbb{R}$. It is clear that $\sigma_i(0, 0) = 0$.

Also let $F : Y^l \longrightarrow Y$ be a linear, continuous function. For a map $f : X \longrightarrow Y$, define

$$G_f(x, y) = F(f(\sigma_1(x, y)), f(\sigma_2(x, y)), \dots, f(\sigma_l(x, y))).$$

Here, G_f is a functional operator on the function space $\{f|f : X \longrightarrow Y\}$. In this paper, for an appropriate function $\phi : X^2 \longrightarrow [0, \infty)$, we consider the functional inequality

$$(2.1) \quad \|f(ax + by) + f(ax - by) + G_f(x, y) - 2a^2f(x) - 2b^2f(y)\| \leq \phi(x, y)$$

for fixed non-zero rational numbers a, b with $a^2 \neq b^2$, where the functional operator G_f satisfies

$$(2.2) \quad G_f(x, 0) \equiv \lambda[f(ax) - a^2f(x)]$$

for some $\lambda(\lambda \neq -2)$. Here, \equiv means that $G_f(x, 0) = \lambda[f(ax) - a^2f(x)]$ holds for all $x \in X$ and all $f : X \longrightarrow Y$.

In fact, as we shall see in Theorem 2.2, for a function f with $f(0) = 0$ satisfying the equation

$$(2.3) \quad f(ax + by) + f(ax - by) + G_f(x, y) = 2a^2f(x) + 2b^2f(y),$$

f is quadratic if and only if $G_f(x, 0) = \lambda[f(ax) - a^2f(x)]$ and $G_f(x, y) = G_f(y, x)$. So the condition (2.2) is reasonable for the stability problem of (2.1). From now on, we assume that the functional operator G_f satisfies the condition (2.2) unless otherwise stated. We denote

$$H_f(x, y) = f(x + y) + f(x - y) - 2f(x) - 2f(y).$$

The following lemma is proved in the authors' previous paper [10].

Lemma 2.1. [10] *Consider the following functional equation.*

$$(2.4) \quad f(ax + by) + f(ax - by) + cH_f(x, y) = 2a^2f(x) + 2b^2f(y)$$

for fixed non-zero rational numbers a, b with $a^2 \neq b^2$ and a real number c . Then if $f : X \longrightarrow Y$ satisfies (2.4) and $f(0) = 0$, f is quadratic.

By using Lemma 2.1, we can examine the properties of a solution function of the equation (2.3).

Theorem 2.2. *Suppose the equation (2.3) holds. Then the following conditions are equivalent :*

- (1) f is quadratic.
- (2) $G_f(x, y) = G_f(y, x)$ for all $x, y \in X$, and $f(0) = 0$

(3) *There are non-zero rational numbers m, n, δ such that $a^2m^2 \neq b^2n^2$ and*

$$(2.5) \quad G_f(mx, ny) = \delta H_f(x, y), \quad f(mx) = m^2 f(x), \quad f(nx) = n^2 f(x)$$

for all $x, y \in X$.

Proof. We prove the theorem by showing $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$

$(1) \Rightarrow (2)$ Since f is quadratic, $f(0) = 0$ and $G_f(x, y) = 0$ for all $x, y \in X$. So $G_f(x, y)$ satisfies (2) with $\lambda = 0$ in (2.2).

$(2) \Rightarrow (3)$ Putting $y = 0$ in (2.3) and by (2.2), we have

$$(2.6) \quad (2 + \lambda)[f(ax) - a^2 f(x)] = 0$$

for all $x \in X$ and since $f(0) = 0$, $G_f(x, 0) = 2(a^2 f(x) - f(ax)) = 0$. From the condition $G_f(x, 0) = G_f(0, x)$, we have

$$f(bx) + f(-bx) = 2b^2 f(x)$$

for all $x \in X$ and so we have

$$(2.7) \quad b^2 f(x) = b^2 f(-x)$$

for all $x \in X$. Since $b \neq 0$, by (2.7), f is even and hence $f(bx) = b^2 f(x)$ for all $x \in X$. Thus (2.3) becomes

$$H_f(ax, by) + G_f(x, y) = 0,$$

and from the condition $G_f(x, y) = G_f(y, x)$ we have

$$(2.8) \quad G_f(x, y) = -H_f(ay, bx)$$

for all $x, y \in X$. Replacing x and y by ax and by respectively in (2.8), we have

$$\begin{aligned} G_f(ax, by) &= -H_f(aby, abx) \\ &= -a^2 b^2 H_f(y, x) \\ &= -a^2 b^2 H_f(x, y) \end{aligned}$$

for all $x, y \in X$. The last equality comes from the fact that f is even. Note that $a^4 \neq b^4$. So we have (3) with $m = a$, $n = b$, $\delta = -a^2b^2$.

(3) \Rightarrow (1) By (2.3) and (3),

$$\begin{aligned} & \delta H_f(x, y) - G_f(mx, ny) \\ &= \delta H_f(x, y) + f(amx + bny) + f(amx - bny) - 2a^2f(mx) - 2b^2f(ny) \\ &= \delta H_f(x, y) + f(amx + bny) + f(amx - bny) - 2a^2m^2f(x) - 2b^2n^2f(y) \\ &= 0 \end{aligned}$$

for all $x, y \in X$. Since $a^2m^2 \neq b^2n^2$, by Lemma 2.1, f is quadratic. \square

Now we prove the following stability theorem.

Theorem 2.3. *Let $\phi : X^2 \rightarrow [0, \infty)$ be a function such that*

$$(2.9) \quad \sum_{n=0}^{\infty} a^{-2n} \phi(a^n x, a^n y) < \infty$$

for all $x, y \in X$. Assume that $G_f(x, y)$ satisfies one of the conditions in Theorem 2.2 when the equation (2.3) holds, and let $f : X \rightarrow Y$ be a mapping such that

$$(2.10) \quad \|f(ax + by) + f(ax - by) + G_f(x, y) - 2a^2f(x) - 2b^2f(y)\| \leq \phi(x, y)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$(2.11) \quad \|Q(x) - f(x) - f(0)\| \leq \frac{1}{|\lambda + 2|} \sum_{n=0}^{\infty} a^{-2(n+1)} \phi(a^n x, 0)$$

for all $x \in X$.

Proof. By the standard argument, we may assume that $f(0) = 0$.

Setting $y = 0$ in (2.10), we have

$$\|f(ax) + 2^{-1}G_f(x, 0) - a^2f(x)\| \leq 2^{-1}\phi(x, 0)$$

for all $x \in X$ and by (2.2), we have

$$(2.12) \quad \|f(x) - a^{-2}f(ax)\| \leq \frac{1}{|\lambda + 2|} a^{-2}\phi(x, 0)$$

for all $x \in X$. Replacing x by $a^n x$ in (2.12) and dividing (2.12) by a^{2n} , we have

$$\|a^{-2n}f(a^n x) - a^{-2(n+1)}f(a^{n+1}x)\| \leq \frac{1}{|\lambda + 2|} a^{-2(n+1)}\phi(a^n x, 0)$$

for all $x \in X$ and all non-negative integer n . For $m, n \in \mathbb{N} \cup \{0\}$ with $0 \leq m < n$,

$$(2.13) \quad \begin{aligned} & \|a^{-2m}f(a^m x) - a^{-2n}f(a^n x)\| \\ &= a^{-2m} \|f(a^m x) - a^{-2(n-m)}f(a^{n-m}(a^m x))\| \\ &\leq \frac{1}{|\lambda + 2|} \sum_{k=m}^{n-1} a^{-2(k+1)}\phi(a^k x, 0) \end{aligned}$$

for all $x \in X$. By (2.13), $\{a^{-2n}f(a^n x)\}$ is a Cauchy sequence in Y and since Y is a Banach space, there exists a mapping $Q : X \rightarrow Y$ such that

$$Q(x) = \lim_{n \rightarrow \infty} a^{-2n}f(a^n x)$$

for all $x \in X$ and

$$\|Q(x) - f(x)\| \leq \frac{1}{|\lambda + 2|} \sum_{n=0}^{\infty} a^{-2(n+1)}\phi(a^n x, 0)$$

for all $x \in X$. Replacing x and y by $a^n x$ and $a^n y$ respectively in (2.10) and dividing (2.10) by a^{2n} , we have

$$\begin{aligned} & \|a^{-2n}f(a^n(ax + by)) + a^{-2n}f(a^n(ax - by)) + a^{-2n}G_f(a^n x, a^n y) \\ & - 2 \cdot a^2 \cdot a^{-2n}f(a^n x) - 2 \cdot b^2 \cdot a^{-2n}f(a^n y)\| \leq a^{-2n}\phi(a^n x, a^n y) \end{aligned}$$

for all $x, y \in X$ and letting $n \rightarrow \infty$ in the above inequality, we have

$$(2.14) \quad \begin{aligned} & Q(ax + by) + Q(ax - by) \\ & + \lim_{n \rightarrow \infty} a^{-2n}G_f(a^n x, a^n y) - 2a^2Q(x) - 2b^2Q(y) = 0 \end{aligned}$$

for all $x, y \in X$. Since F is continuous, we have

$$\begin{aligned}
& \lim_{n \rightarrow \infty} a^{-2n} G_f(a^n x, a^n y) \\
&= \lim_{n \rightarrow \infty} F(a^{-2n} f(a^n \sigma_1(x, y)), a^{-2n} f(a^n \sigma_2(x, y)), \dots, a^{-2n} f(a^n \sigma_l(x, y))) \\
&= F(Q(\sigma_1(x, y)), Q(\sigma_2(x, y)), \dots, Q(\sigma_l(x, y))) \\
&= G_Q(x, y)
\end{aligned}$$

for all $x, y \in X$. Hence by (2.14), we have

$$(2.15) \quad Q(ax + by) + Q(ax - by) + G_Q(x, y) = 2a^2 Q(x) + 2b^2 Q(y)$$

for all $x, y \in X$. Since Q satisfies (2.3), Q is quadratic by Theorem 2.2.

Now, we show the uniqueness of Q . Suppose that Q_0 is a quadratic mapping with (2.11). Then we have

$$\begin{aligned}
& \|Q(x) - Q_0(x)\| \\
&= a^{-2k} \|Q(a^k x) - Q_0(a^k x)\| \\
&\leq \frac{2}{|\lambda + 2|} \sum_{n=k}^{\infty} a^{-2(n+1)} \phi(a^n x, 0)
\end{aligned}$$

for all $x \in X$. Hence, letting $k \rightarrow \infty$ in the above inequality, we have

$$Q(x) = Q_0(x)$$

for all $x \in X$. □

Theorem 2.4. Assume that G_f satisfies all of the conditions in Theorem 2.3. Let $\phi : X^2 \rightarrow [0, \infty)$ be a function such that

$$\sum_{n=0}^{\infty} a^{2n} \phi(a^{-n} x, a^{-n} y) < \infty$$

for all $x, y \in X$. Let $f : X \rightarrow Y$ be a mapping such that

$$\|f(ax + by) + f(ax - by) + G_f(x, y) - 2a^2 f(x) - 2b^2 f(y)\| \leq \phi(x, y)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\|Q(x) - f(x) - f(0)\| \leq \frac{1}{|\lambda + 2|} \sum_{n=0}^{\infty} a^{2(n+1)} \phi(a^{-n} x, 0)$$

for all $x \in X$.

As examples of $\phi(x, y)$ in Theorem 2.3 and Theorem 2.4, we can take $\phi(x, y) = \epsilon(\|x\|^p \|y\|^p + \|x\|^{2p} + \|y\|^{2p})$ which is appeared in [11]. Then we can formulate the following corollary

Corollary 2.5. *Assume that all of the conditions in Theorem 2.3 are satisfied.*

Let p be a real number with $p \neq 1$. Let $f : X \rightarrow Y$ be a mapping such that

$$\begin{aligned} & \|f(ax + by) + f(ax - by) - 2a^2 f(x) - 2b^2 f(y) + G_f(x, y)\| \\ & \leq \epsilon(\|x\|^p \|y\|^p + \|x\|^{2p} + \|y\|^{2p}) \end{aligned}$$

for fixed non-zero rational numbers a, b with $a^2 \neq b^2$, a fixed positive real number ϵ , and all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\begin{aligned} \|Q(x) - f(x) - f(0)\| & \leq \frac{\epsilon \|x\|^{2p}}{a^2 |\lambda + 2| [1 - a^{2(p-1)}]} \\ & (p < 1 \text{ and } |a| > 1, \text{ or } p > 1 \text{ and } |a| < 1) \end{aligned}$$

and

$$\begin{aligned} \|Q(x) - f(x) - f(0)\| & \leq \frac{a^2 \epsilon \|x\|^{2p}}{|\lambda + 2| [1 - a^{2(1-p)}]} \\ & (p > 1 \text{ and } |a| > 1, \text{ or } p < 1 \text{ and } |a| < 1) \end{aligned}$$

for all $x \in X$.

3. APPLICATIONS

In this section we illustrate how the theorems in section 2 work well for the generalized Hyers-Ulam stability of various quadratic functional equations. By applying the results in this article, we can construct many concrete members in our calss of functional equations easily.

First, we consider the following functional equation related with Theorem 2.3.

$$\begin{aligned} & f(ax + by) + f(ax - by) + f(x + y) + f(x - y) \\ (3.1) \quad & + f(y - x) - f(-x) - f(-y) = 2(a^2 + 1)f(x) + 2(b^2 + 1)f(y) \end{aligned}$$

for fixed non-zero rational numbers a, b with $a^2 \neq b^2$.

Using Theorem 2.3, we can prove the stability for (3.1).

Theorem 3.1. *Let $\phi : X^2 \rightarrow [0, \infty)$ be a function with (2.9) and $f : X \rightarrow Y$ a mapping such that*

$$(3.2) \quad \begin{aligned} & \|f(ax + by) + f(ax - by) + f(x + y) + f(x - y) + f(y - x) \\ & - f(-x) - f(-y) - 2(a^2 + 1)f(x) - 2(b^2 + 1)f(y)\| \leq \phi(x, y). \end{aligned}$$

for fixed non-zero rational numbers a, b with $a^2 \neq b^2$, and all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that Q satisfies (3.1) and

$$(3.3) \quad \|Q(x) - f(x) - f(0)\| \leq \frac{1}{2} \sum_{n=0}^{\infty} a^{-2(n+1)} \phi(a^n x, 0)$$

for all $x \in X$.

Proof. In this case, $G_f(x, y) = f(x + y) + f(x - y) + f(y - x) - f(-x) - f(-y) - 2f(x) - 2f(y)$. So $G_f(x, 0) = 0$. Hence G_f and f satisfies all the conditions in Theorem 2.3. and the functional inequality (3.2) can be rewritten as the functional inequality

$$\|f(ax + by) + f(ax - by) + G_f(x, y) - 2a^2 f(x) - 2b^2 f(y)\| \leq \phi(x, y).$$

By Theorem 2.3, we get the result. \square

When the equation (2.3) holds, $G_f(x, y)$ can be represented as different forms. In some cases, these forms together help us to analyze a solution. Especially the following case happens often in some interesting equations. We will give an example later.

Lemma 3.2. *Suppose when the equation (2.3) with $a^2 \neq b^4$ holds, $G_f(x, y)$ can be represented as both of the followings.*

$$(3.4) \quad G_f(0, y) = k[f(y) - f(-y)]$$

$$(3.5) \quad G_f(x, by) = k_1 H_f(x, y)$$

for all $x, y \in X$ and a fixed positive real number k with $k \neq b^2$, and a fixed real number k_1 .

Then G_f satisfies the condition (3) in Theorem 2.2.

Proof. Suppose that (2.3) holds. Then we have

$$G_f(0, y) - G_f(0, -y) = 2b^2[f(y) - f(-y)]$$

for all $y \in X$. Also by (3.4), we have

$$G_f(0, y) - G_f(0, -y) = 2k[f(y) - f(-y)]$$

Since $k \neq b^2$, f is even and so $f(bx) = b^2f(x)$. By (3.5) and $a^2 \neq b^4$, we get the result with $m = 1$, $n = b$, and $k_1 = \delta$. \square

Note that Lemma 3.2 is still valid if we does not impose the condition (2.2) on G_f . By Lemma 3.2 and Theorem 2.3, we can formulate the following proposition.

Proposition 3.3. *Let ϕ be a function in Theorem 2.2 and suppose that $G_f(x, y)$ satisfies the condition in Lemma 3.2 when the equation (2.3) holds. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that*

$$\|Q(x) - f(x) - f(0)\| \leq \frac{1}{|\lambda + 2|} \sum_{n=0}^{\infty} a^{-2(n+1)} \phi(a^n x, 0)$$

for all $x \in X$.

Now, we consider the following functional equation related with Proposition 3.3.

$$\begin{aligned} & f(ax + by) + f(ax - by) - f(bx + y) - f(bx - y) + 2f(bx) \\ (3.6) \quad & = 2a^2 f(x) + 2(b^2 - 1)f(y) \end{aligned}$$

for fixed non-zero rational numbers a, b with $a^2 \neq b^2$ and $a^2 \neq b^4$.

Theorem 3.4. *Let $\phi : X^2 \rightarrow [0, \infty)$ be a function such that*

$$\sum_{n=0}^{\infty} a^{-2n} \phi(a^n x, a^n y) < \infty$$

for all $x, y \in X$. Let $f : X \rightarrow Y$ be a mapping such that

$$(3.7) \quad \begin{aligned} & \|f(ax + by) + f(ax - by) - f(bx + y) - f(bx - y) \\ & + 2f(bx) - 2a^2f(x) - 2(b^2 - 1)f(y)\| \leq \phi(x, y). \end{aligned}$$

Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that Q satisfies (3.6) and

$$\|Q(x) - f(x) - f(0)\| \leq \frac{1}{2} \sum_{n=0}^{\infty} a^{-2(n+1)} \phi(a^n x, 0)$$

for all $x \in X$.

Proof. It is enough to show with the condition $f(0) = 0$. In this case, $G_f(x, y) = -f(bx + y) - f(bx - y) + 2f(bx) + 2f(y)$. First we can check $G_f(x, 0) = 0$ as a functional operator. Now suppose that f satisfies (3.6). Then $G_f(0, y) = f(y) - f(-y)$ for all $y \in X$ and hence $f(by) = b^2f(y)$. So $G_f(x, by) = -b^2H_f(x, y)$. Since all the conditions in Proposition 3.3 are satisfied, we have the result. \square

Similar to Proposition 3.3, we have the following proposition :

Proposition 3.5. *Suppose that $f(0) = 0$ and $G_f(x, y)$ satisfies*

$$(3.8) \quad G_f(ax, y) = k_2 H_f(x, y)$$

for all $x, y \in X$ and a fixed real number k_2 when the equation (2.3) holds. Let ϕ be a function in Theorem 2.3. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\|Q(x) - f(x) - f(0)\| \leq \frac{1}{|\lambda + 2|} \sum_{n=0}^{\infty} a^{-2(n+1)} \phi(a^n x, 0)$$

for all $x \in X$.

Finally, we consider the following functional equation related with Proposition 3.5.

$$(3.9) \quad 2f(2x + y) + f(2x - y) + f(x - 2y) = 13f(x) + 6f(y) + f(-y)$$

for all $x, y \in X$.

Theorem 3.6. Let $\phi : X^2 \longrightarrow [0, \infty)$ be a function such that

$$\sum_{n=0}^{\infty} a^{-2n} \phi(a^n x, a^n y) < \infty$$

for all $x, y \in X$. Let $f : X \longrightarrow Y$ be a mapping such that

$$(3.10) \quad \begin{aligned} & \|2f(2x+y) + f(2x-y) + f(x-2y) - 13f(x) - 6f(y) - f(-y)\| \\ & \leq \phi(x, y) \end{aligned}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \longrightarrow Y$ such that Q satisfies (3.6) and

$$\|Q(x) - f(x) - f(0)\| \leq \frac{1}{3} \sum_{n=0}^{\infty} a^{-2(n+1)} \phi(a^n x, 0)$$

for all $x \in X$.

Proof. In this case $G_f(x, y) = f(2x+y) + f(x-2y) - 5f(x) - 4f(y) - f(-y)$, so $G_f(x, 0) = f(2x) - 4f(x)$ under the condition $f(0) = 0$. Now suppose f satisfies (3.9). Since $G_f(0, y) = 3[f(-y) - f(y)]$ for all $y \in X$, by the argument in Lemma 3.2, f is even. Also, the functional equation (3.9) implies

$$(3.11) \quad f(2x+y) + f(2x-y) + \bar{G}_f(x, y) = 8f(x) + 2f(y),$$

where $\bar{G}_f(x, y) = \frac{1}{3}f(x+2y) + \frac{1}{3}f(x-2y) - \frac{2}{3}f(x) - \frac{1}{3}f(y) - \frac{7}{3}f(-y)$.

Since f is even, we have $\bar{G}_f(2x, y) = \frac{4}{3}H_f(x, y)$ for all $x, y \in X$. By (3.10), we have

$$(3.12) \quad \begin{aligned} & \|f(2x+y) + f(2x-y) + \bar{G}_f(x, y) - 8f(x) - 2f(y)\| \\ & \leq \frac{1}{3}[\phi(x, y) + \phi(x, -y)] \end{aligned}$$

for all $x, y \in X$ and so by Proposition 3.5, we have the result. \square

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A FIXED POINT APPROACH TO THE STABILITY OF EULER-LAGRANGE SEXTIC (a, b) -FUNCTIONAL EQUATIONS IN ARCHIMEDEAN AND NON-ARCHIMEDEAN BANACH SPACES

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ABSTRACT. In this paper, we present a fixed point method to prove the Hyers-Ulam stability of the system of Euler-Lagrange quadratic-quartic functional equations

$$\begin{cases} f(ax_1 + bx_2, y) + f(bx_1 + ax_2, y) + abf(x_1 - x_2, y) \\ \quad = (a^2 + b^2)[f(x_1, y) + f(x_2, y)] + 4abf(\frac{x_1 + x_2}{2}, y), \\ f(x, ay_1 + by_2) + f(x, by_1 + ay_2) + \frac{1}{2}ab(a - b)^2 f(x, y_1 - y_2) \\ \quad = (a^2 - b^2)^2[f(x, y_1) + f(x, y_2)] + 8abf(x, \frac{y_1 + y_2}{2}) \end{cases} \quad (0.1)$$

for all numbers a and b with $a + b \notin \{0, \pm 1\}$, $ab + 2 \neq 2(a + b)^2$ and $ab(a - b)^2 + 4 \neq 4(a + b)^4$ in Archimedean and non-Archimedean Banach spaces and we show that the approximation in non-Archimedean Banach spaces is better than the approximation in (Archimedean) Banach spaces.

1. Introduction

The *stability problem* of functional equations started with the following question concerning stability of group homomorphisms proposed by Ulam [69] during a talk before a Mathematical Colloquium at the University of Wisconsin. In 1941, Hyers [32] gave a first affirmative answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Rassias [61] for linear mappings by considering an unbounded Cauchy difference, respectively. In 1994, a generalization of the Rassias theorem was obtained by Găvruta [28] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. For more details about the results concerning such problems, the reader refer to [2, 5, 8, 11, 14, 15, 24, 27, 29, 33, 34, 35, 36, 40, 41, 42, 44], [52]–[67] and [71, 72, 73].

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \quad (1.1)$$

is related to a symmetric bi-additive mapping [1, 43]. It is natural that this equation is called a *quadratic functional equation*. In particular, every solution of the quadratic equation (1.1) is called a *quadratic mapping*. The Hyers-Ulam stability problem for the quadratic functional equation was solved by Skof [68]. In [14], Czerwik proved the Hyers-Ulam stability of the equation (1.1). Eshaghi Gordji and Khodaei [25] obtained the general solution and the Hyers-Ulam stability of the following quadratic functional equation: for all $a, b \in \mathbb{Z} \setminus \{0\}$ with $a \neq \pm 1, \pm b$,

$$f(ax + by) + f(ax - by) = 2a^2 f(x) + 2b^2 f(y). \quad (1.2)$$

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Jun and Kim [38] introduced the following cubic functional equation:

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x) \quad (1.3)$$

and they established the general solution and the Hyers-Ulam stability for the functional equation (1.3). Jun et al. [39] investigated the solution and the Hyers-Ulam stability for the cubic functional equation

$$f(ax + by) + f(ax - by) = ab^2(f(x + y) + f(x - y)) + 2a(a^2 - b^2)f(x), \quad (1.4)$$

where $a, b \in \mathbb{Z} \setminus \{0\}$ with $a \neq \pm 1, \pm b$. For other cubic functional equations, see [50].

Lee et. al. [48] considered the following functional equation:

$$f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y) \quad (1.5)$$

In fact, they proved that a mapping f between two real vector spaces X and Y is a solution of the equation (1.5) if and only if there exists a unique symmetric bi-quadratic mapping $B_2 : X \times X \rightarrow Y$ such that $f(x) = B_2(x, x)$ for all $x \in X$. The bi-quadratic mapping B_2 is given by

$$B_2(x, y) = \frac{1}{12}(f(x + y) + f(x - y) - 2f(x) - 2f(y)).$$

Obviously, the function $f(x) = cx^4$ satisfies the functional equation (1.5), which is called the *quartic functional equation*. For other quartic functional equations, see [13].

Ebadian et al. [16] proved the Hyers-Ulam stability of the following systems of the additive-quartic functional equation:

$$\begin{cases} f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y), \\ f(x, 2y_1 + y_2) + f(x, 2y_1 - y_2) \\ \quad = 4f(x, y_1 + y_2) + 4f(x, y_1 - y_2) + 24f(x, y_1) - 6f(x, y_2) \end{cases} \quad (1.6)$$

and the quadratic-cubic functional equation:

$$\begin{cases} f(x, 2y_1 + y_2) + f(x, 2y_1 - y_2) \\ \quad = 2f(x, y_1 + y_2) + 2f(x, y_1 - y_2) + 12f(x, y_1), \\ f(x, y_1 + y_2) + f(x, y_1 - y_2) = 2f(x, y_1) + 2f(x, y_2). \end{cases} \quad (1.7)$$

For more details about the results concerning mixed type functional equations, the readers refer to [18, 20, 21] and [23].

Recently, Ghaemi et. al. [30] and Cho et. al. [10] investigated the the stability of the following systems of the quadratic-cubic functional equation:

$$\begin{cases} f(ax_1 + bx_2, y) + f(ax_1 - bx_2, y) = 2a^2f(x_1, y) + 2b^2f(x_2, y), \\ f(x, ay_1 + by_2) + f(x, ay_1 - by_2) \\ \quad = ab^2(f(x, y_1 + y_2) + f(x, y_1 - y_2)) + 2a(a^2 - b^2)f(x, y_1) \end{cases} \quad (1.8)$$

and the additive-quadratic-cubic functional equation:

$$\begin{cases} f(ax_1 + bx_2, y, z) + f(ax_1 - bx_2, y, z) = 2af(x_1, y, z), \\ f(x, ay_1 + by_2, z) + f(x, ay_1 - by_2, z) = 2a^2f(x, y_1, z) + 2b^2f(x, y_2, z), \\ f(x, y, az_1 + bz_2) + f(x, y, az_1 - bz_2) \\ \quad = ab^2(f(x, y, z_1 + z_2) + f(x, y, z_1 - z_2)) + 2a(a^2 - b^2)f(x, y, z_1) \end{cases} \quad (1.9)$$

STABILITY OF EULER-LAGRANGE SEXTIC FUNCTIONAL EQUATIONS

in PN -spaces and PM -spaces, where $a, b \in \mathbb{Z} \setminus \{0\}$ with $a \neq \pm 1, \pm b$. The function $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x, y) = cx^2y^3$ is a solution of the system (1.8). In particular, letting $y = x$, we get a quintic function $g : \mathbb{R} \rightarrow \mathbb{R}$ in one variable given by $g(x) := f(x, x) = cx^5$.

In this paper, we present a fixed point method to prove the Hyers-Ulam stability of the following system of the Euler-Lagrange quadratic-quartic (a, b) -functional equation:

$$\begin{cases} f(ax_1 + bx_2, y) + f(bx_1 + ax_2, y) + abf(x_1 - x_2, y) \\ \quad = (a^2 + b^2)[f(x_1, y) + f(x_2, y)] + 4abf(\frac{x_1+x_2}{2}, y), \\ f(x, ay_1 + by_2) + f(x, by_1 + ay_2) + \frac{1}{2}ab(a-b)^2f(x, y_1 - y_2) \\ \quad = (a^2 - b^2)^2[f(x, y_1) + f(x, y_2)] + 8abf(x, \frac{y_1+y_2}{2}) \end{cases} \quad (1.10)$$

for all numbers a and b with $a + b \notin \{0, \pm 1\}$, $ab + 2 \neq 2(a + b)^2$ and $ab(a - b)^2 + 4 \neq 4(a + b)^4$ in Archimedean and non-Archimedean Banach spaces. For details about the results concerning such problems in non-Archimedean normed spaces, the reader refer to [9, 12, 17, 20, 26, 37, 46, 47, 55, 72]. It is easy to see that the function $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x, y) = cx^2y^4$ is a solution of the system (1.10). In particular, letting $x = y$, we get a sextic function $h : \mathbb{R} \rightarrow \mathbb{R}$ in one variable given by $h(x) := f(x, x) = cx^6$.

The proof of the following propositions is evident.

Proposition 1.1. *Let X and Y be real linear spaces. If a mapping $f : X \times X \rightarrow Y$ satisfies the system (1.10), then $f(\lambda x, \mu y) = \lambda^2\mu^4f(x, y)$ for all $x, y \in X$ and rational numbers λ, μ .*

In this paper, we investigate the Hyers-Ulam stability of a sextic mapping from linear spaces into Archimedean and non-Archimedean Banach spaces. Hensel [31] has introduced a normed space which does not have the Archimedean property. During the last three decades theory of non-Archimedean spaces has gained the interest of physicists for their research in particular in problems coming from quantum physics, p -adic strings and superstrings [45]. Although many results in the classical normed space theory have a non-Archimedean counterpart, their proofs are different and require a rather new kind of intuition [4, 22, 51, 54, 70]. One may note that $|n| \leq 1$ in each valuation field, every triangle is isosceles and there may be no unit vector in a non-Archimedean normed space; cf. [51]. These facts show that the non-Archimedean framework is of special interest.

Definition 1.2. Let \mathbb{K} be a field. A valuation mapping on \mathbb{K} is a function $|\cdot| : \mathbb{K} \rightarrow \mathbb{R}$ such that for any $a, b \in \mathbb{K}$ we have

- (i) $|a| \geq 0$ and equality holds if and only if $a = 0$,
- (ii) $|ab| = |a||b|$,
- (iii) $|a + b| \leq |a| + |b|$.

A field endowed with a valuation mapping will be called a valued field. If the condition (iii) in the definition of a valuation mapping is replaced with

$$(iii)' \quad |a + b| \leq \max\{|a|, |b|\}$$

then the valuation $|\cdot|$ is said to be non-Archimedean. The condition (iii)' is called the strict triangle inequality. By (ii), we have $|1| = |-1| = 1$. Thus, by induction, it follows from (iii)' that $|n| \leq 1$ for each integer n . We always assume in addition that $|\cdot|$ is non trivial, i.e., that there is an $a_0 \in \mathbb{K}$ such that $|a_0| \notin \{0, 1\}$. The most important examples of non-Archimedean spaces are p -adic numbers.

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Example 1.3. Let p be a prime number. For any non-zero rational number $a = p^r \frac{m}{n}$ such that m and n are coprime to the prime number p , define the p -adic absolute value $|a|_p = p^{-r}$. Then $|\cdot|$ is a non-Archimedean norm on \mathbb{Q} . The completion of \mathbb{Q} with respect to $|\cdot|$ is denoted by \mathbb{Q}_p and is called the p -adic number field.

Definition 1.4. Let X be a linear space over a scalar field \mathbb{K} with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : X \rightarrow \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

- (NA1) $\|x\| = 0$ if and only if $x = 0$;
- (NA2) $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in X$;
- (NA3) the strong triangle inequality (ultrametric); namely,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\} \quad (x, y \in X).$$

Then $(X, \|\cdot\|)$ is called a non-Archimedean normed space.

Definition 1.5. (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X . Then the sequence $\{x_n\}$ is called Cauchy if for a given $\varepsilon > 0$ there is a positive integer N such that

$$\|x_n - x_m\| < \varepsilon$$

for all $n, m \geq N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X . Then the sequence $\{x_n\}$ is called convergent if for a given $\varepsilon > 0$ there are a positive integer N and an $x \in X$ such that

$$\|x_n - x\| < \varepsilon$$

for all $n \geq N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n \rightarrow \infty} x_n = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a non-Archimedean Banach space.

In 2003, Radu [60] proposed a new method for obtaining the existence of exact solutions and error estimations, based on the fixed point alternative (see also [6, 7]). Our aim is based on the following fixed point approach:

Let (X, d) be a generalized metric space. An operator $T : X \rightarrow X$ satisfies a Lipschitz condition with Lipschitz constant L if there exists a constant $L \geq 0$ such that $d(Tx, Ty) \leq Ld(x, y)$ for all $x, y \in X$. If the Lipschitz constant L is less than 1, then the operator T is called a strictly contractive operator. Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity. We recall the following theorem by Margolis and Diaz.

Theorem 1.6. ([49, 60]) Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \rightarrow \Omega$ with Lipschitz constant L . Then for each given $x \in \Omega$, either

$$d(T^m x, T^{m+1} x) = \infty \quad \text{for all } m \geq 0,$$

or there exists a natural number m_0 such that

$$\bullet \quad d(T^m x, T^{m+1} x) < \infty \quad \text{for all } m \geq m_0;$$

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- the sequence $\{T^m x\}$ is convergent to a fixed point y^* of T ;
- y^* is the unique fixed point of T in $\Lambda = \{y \in \Omega : d(T^{m_0} x, y) < \infty\}$;
- $d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Lambda$.

2. Sextic functional inequalities in non-Archimedean Banach spaces

Throughout this section, we will assume that X is a non-Archimedean Banach space. In this section, we establish the conditional stability of sextic functional equations in non-Archimedean Banach spaces.

Theorem 2.1. *Let $s \in \{-1, 1\}$ be fixed. Let E be a real or complex linear space and let X be a non-Archimedean Banach space. Suppose $f : E \times E \rightarrow X$ satisfies the condition $f(x, 0) = f(0, y) = 0$ and inequalities of the form*

$$\begin{aligned} & \|f(ax_1 + bx_2, y) + f(bx_1 + ax_2, y) + abf(x_1 - x_2, y) \\ & - (a^2 + b^2)[f(x_1, y) + f(x_2, y)] - 4abf\left(\frac{x_1 + x_2}{2}, y\right)\| \leq \phi(x_1, x_2, y), \end{aligned} \quad (2.1)$$

$$\begin{aligned} & \|f(x, ay_1 + by_2) + f(x, by_1 + ay_2) + \frac{1}{2}ab(a - b)^2 f(x, y_1 - y_2) \\ & - (a^2 - b^2)^2[f(x, y_1) + f(x, y_2)] - 8abf\left(x, \frac{y_1 + y_2}{2}\right)\| \leq \psi(x, y_1, y_2), \end{aligned} \quad (2.2)$$

where $\phi, \psi : E \times E \times E \rightarrow [0, \infty)$ is given functions such that

$$\begin{aligned} \phi((a+b)^s x_1, (a+b)^s x_2, (a+b)^s y) & \leq |a+b|^{6s} L \phi(x_1, x_2, y), \\ \psi((a+b)^s x, (a+b)^s y_1, (a+b)^s y_2) & \leq |a+b|^{6s} L \psi(x, y_1, y_2), \end{aligned} \quad (2.3)$$

and have the properties

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| (a+b)^{-6sn} \right| \phi((a+b)^{sn} x_1, (a+b)^{sn} x_2, (a+b)^{sn} y) & = 0, \\ \lim_{n \rightarrow \infty} \left| (a+b)^{-6sn} \right| \psi((a+b)^{sn} x, (a+b)^{sn} y_1, (a+b)^{sn} y_2) & = 0, \end{aligned} \quad (2.4)$$

for all $x, x_1, x_2, y, y_1, y_2 \in E$ and a constant $0 < L < 1$. Then there exists a unique sextic mapping $T : E \times E \rightarrow X$ satisfying the system (1.10) and

$$\|T(x, y) - f(x, y)\| \leq \frac{1}{1-L} \Phi(x, y), \quad (2.5)$$

where

$$\begin{aligned} \Phi(x, y) & := \left| \frac{1}{2} \right| \max \left\{ \left| (a+b)^{-3s+1} \right| \phi \left((a+b)^{\frac{s-1}{2}} x, (a+b)^{\frac{s-1}{2}} x, (a+b)^{\frac{s-1}{2}} y \right), \right. \\ & \quad \left. \left| (a+b)^{-3s-3} \right| \psi \left((a+b)^{\frac{s+1}{2}} x, (a+b)^{\frac{s-1}{2}} y, (a+b)^{\frac{s-1}{2}} y \right) \right\} \end{aligned}$$

for all $x, y \in E$.

Proof. We denote $A := a + b$. Putting $x_1 = x_2 = x$ in (2.1), we get

$$\|f(Ax, y) - A^2 f(x, y)\| \leq \left| \frac{1}{2} \right| \phi(x, x, y) \quad (2.6)$$

for all $x, y \in E$. Putting $y_1 = y_2 = y$ and replacing x by Ax in (2.2), we get

$$\|f(Ax, Ay) - A^4 f(Ax, y)\| \leq \left| \frac{1}{2} \right| \psi(Ax, y, y) \quad (2.7)$$

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for all $x, y \in E$. Thus by (2.6) and (2.7) we have

$$\|f(Ax, Ay) - A^6 f(x, y)\| \leq \left| \frac{1}{2} \right| \max \{ |A^4| \phi(x, x, y), \psi(Ax, y, y) \},$$

for all $x, y \in E$. By last inequality we get

$$\|A^{-6} f(Ax, Ay) - f(x, y)\| \leq \left| \frac{1}{2} \right| \max \{ |A^{-2}| \phi(x, x, y), |A^{-6}| \psi(Ax, y, y) \}, \quad (2.8)$$

$$\|A^6 f\left(\frac{x}{A}, \frac{y}{A}\right) - f(x, y)\| \leq \left| \frac{1}{2} \right| \max \left\{ |A^4| \phi\left(\frac{x}{A}, \frac{x}{A}, \frac{y}{A}\right), \psi\left(x, \frac{y}{A}, \frac{y}{A}\right) \right\}, \quad (2.9)$$

for all $x, y \in E$. Therefore

$$\left\| \frac{1}{A^{6s}} f(A^s x, A^s y) - f(x, y) \right\| \leq \Phi(x, y), \quad (2.10)$$

for all $x, y \in E$. We now consider the set

$$\mathcal{S} = \{h : E \times E \rightarrow X, \quad h(x, 0) = h(0, x) = 0 \text{ for all } x \in E\}$$

and introduce the generalized metric on \mathcal{S} as follows:

$$d(h, k) = \inf \left\{ \alpha \in \mathbb{R}^+ : \|h(x, y) - k(x, y)\| \leq \alpha \Phi(x, y), \quad \forall x, y \in E \right\}$$

where, as usual, $\inf \emptyset = +\infty$. The proof of the fact that (\mathcal{S}, d) is a complete generalized metric space, can be found in [6]. Now we consider the mapping $J : \mathcal{S} \rightarrow \mathcal{S}$ defined by

$$Jh(x, y) := A^{-6s} h(A^s x, A^s y)$$

for all $h \in \mathcal{S}$ and $x, y \in E$. Let $f, g \in \mathcal{S}$ such that $d(f, g) < \varepsilon$. Then

$$\begin{aligned} \|Jg(x, y) - Jf(x, y)\| &= \|A^{-6s} g(A^s x, A^s y) - A^{-6s} f(A^s x, A^s y)\| \\ &= |A^{-6s}| \|g(A^s x, A^s y) - f(A^s x, A^s y)\| \\ &\leq |A^{-6s}| \varepsilon \Phi(A^s x, A^s y) \\ &\leq L \varepsilon \Phi(x, y), \end{aligned}$$

that is, if $d(f, g) < \varepsilon$, then we have $d(Jf, Jg) \leq L \varepsilon$. This means that

$$d(Jf, Jg) \leq L d(f, g)$$

for all $f, g \in \mathcal{S}$, that is, J is a strictly contractive self-mapping on \mathcal{S} with the Lipschitz constant L . It follows from (2.10) that

$$\|Jf(x, y) - f(x, y)\| \leq \Phi(x, y)$$

for all $x, y \in E$ which implies that $d(Jf, f) \leq 1$. Due to Theorem 1.6, there exists a unique mapping $T : E \times E \rightarrow X$ such that T is a fixed point of J , i.e., $T(A^s x, A^s y) = A^{6s} T(x, y)$ for all $x, y \in E$. Also, $d(J^m f, T) \rightarrow 0$ as $m \rightarrow \infty$, which implies the equality

$$\lim_{m \rightarrow \infty} A^{-6sm} f(A^{sm} x, A^{sm} y) = T(x, y)$$

for all $x, y \in E$.

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It follows from (2.1), (2.2) and (2.4) that

$$\begin{aligned}
& \|T(ax_1 + bx_2, y) + T(bx_1 + ax_2, y) + abT(x_1 - x_2, y) - (a^2 + b^2)[T(x_1, y) + T(x_2, y)] \\
& - 4abT(\frac{x_1 + x_2}{2}, y)\| = \\
& \lim_{n \rightarrow \infty} \left\| \frac{f(A^{ns}(ax_1 + bx_2), A^{ns}y)}{A^{6ns}} + \frac{f(A^{ns}(bx_1 + ax_2), A^{ns}y)}{A^{6ns}} + ab \frac{f(A^{ns}(x_1 - x_2), A^{ns}y)}{A^{6ns}} \right. \\
& \left. - (a^2 + b^2) \frac{f(A^{ns}x_1, A^{ns}y) + f(A^{ns}x_2, A^{ns}y)}{A^{6ns}} - 4ab \frac{f(A^{ns}\{x_1 + x_2/2\}, A^{ns}y)}{A^{6ns}} \right\| = \\
& \lim_{n \rightarrow \infty} |A^{-6ns}| \|f(A^{ns}(ax_1 + bx_2), A^{ns}y) + f(A^{ns}(bx_1 + ax_2), A^{ns}y) + abf(A^{ns}(x_1 - x_2), A^{ns}y) \\
& - (a^2 + b^2)[f(A^{ns}x_1, A^{ns}y) + f(A^{ns}x_2, A^{ns}y)] - 4abf(A^{ns}\{x_1 + x_2/2\}, A^{ns}y)\| \\
& \leq \lim_{n \rightarrow \infty} |A^{-6ns}| \phi(A^{ns}x_1, A^{ns}x_2, A^{ns}y) = 0,
\end{aligned} \tag{2.11}$$

for all $x_1, x_2, y \in E$, and

$$\begin{aligned}
& \|T(x, ay_1 + by_2) + T(x, by_1 + ay_2) + \frac{1}{2}ab(a - b)^2T(x, y_1 - y_2) - (a^2 - b^2)^2[T(x, y_1) + T(x, y_2)] \\
& - 8abT(x, \frac{y_1 + y_2}{2})\| = \\
& \lim_{n \rightarrow \infty} \left\| \frac{f(A^{ns}x, A^{ns}(ay_1 + by_2))}{A^{6ns}} + \frac{f(A^{ns}x, A^{ns}(by_1 + ay_2))}{A^{6ns}} \right. \\
& \left. + \frac{1}{2}ab(a - b)^2 \frac{f(A^{ns}x, A^{ns}(y_1 - y_2))}{A^{6ns}} \right\| = \\
& - (a^2 - b^2)^2 \frac{f(A^{ns}x, A^{ns}y_1) + f(A^{ns}x, A^{ns}y_2)}{A^{6ns}} - 8ab(a + b)^2 \frac{f(A^{ns}x, A^{ns}\{y_1 + y_2/2\})}{A^{6ns}} \\
& \leq \lim_{n \rightarrow \infty} |A^{6ns}| \psi(A^{ns}x, A^{ns}y_1, A^{ns}y_2) = 0,
\end{aligned} \tag{2.12}$$

for all $x, y_1, y_2 \in E$. It follows from (2.11) and (2.12) that T satisfies (1.10), that is, T is sextic.

According to the fixed point alternative, since T is the unique fixed point of J in the set $\Omega = \{g \in \mathcal{S} : d(f, g) < \infty\}$, T is the unique mapping such that

$$\|f(x, y) - T(x, y)\| \leq \Phi(x, y)$$

for all $x, y \in E$. Using the fixed point alternative, we obtain that

$$d(f, T) \leq \frac{1}{1 - L} d(f, Jf) \leq \frac{1}{1 - L} \Phi(x, y),$$

for all $x, y \in E$, which implies the inequality (2.5). \square

Corollary 2.2. *Let $s \in \{-1, 1\}$ be fixed. Let E be a normed space and let F be a non-Archimedean Banach space. Suppose $f : E \times E \rightarrow F$ is a mapping with $f(x, 0) = f(0, y) = 0$ and there exist constants $\theta, \vartheta \geq 0$ and non-negative real number p such that $ps < 6s$ and*

$$\begin{aligned}
& \|f(ax_1 + bx_2, y) + f(bx_1 + ax_2, y) + abf(x_1 - x_2, y) \\
& - (a^2 + b^2)[f(x_1, y) + f(x_2, y)] - 4abf(\frac{x_1 + x_2}{2}, y)\| \leq \theta(\|x_1\|^p + \|x_2\|^p + \|y\|^p), \\
& \|f(x, ay_1 + by_2) + f(x, by_1 + ay_2) + \frac{1}{2}ab(a - b)^2f(x, y_1 - y_2) \\
& - (a^2 - b^2)^2[f(x, y_1) + f(x, y_2)] - 8abf(x, \frac{y_1 + y_2}{2})\| \leq \vartheta(\|x\|^p + \|y_1\|^p + \|y_2\|^p),
\end{aligned}$$

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for all $x, x_1, x_2, y, y_1, y_2 \in E$, where norms of the left-hand side of last inequalities is the non-Archimedean norm on F . Then there exists a unique sextic mapping $T : E \times E \rightarrow F$ such that

$$\|f(x, y) - T(x, y)\| \leq \max \left\{ \frac{\theta(2\|x\|^p + \|y\|^p)}{2s|a+b|^2 - 2s|a+b|^{p-4}}, \frac{\vartheta(\|a+b\|^p + 2\|y\|^p)}{2s|a+b|^6 - 2s|a+b|^p} \right\},$$

for all $x, y \in E$.

Proof. Defining

$$\phi(x_1, x_2, y) = \theta(\|x_1\|^p + \|x_2\|^p + \|y\|^p), \quad \psi(x, y_1, y_2) = \vartheta(\|x\|^p + \|y_1\|^p + \|y_2\|^p),$$

and applying Theorem 2.1, we get the desired result. \square

3. Sextic functional inequalities in (Archimedean) Banach spaces

Throughout this section, we will assume that X is a (Archimedean) Banach space. In this section, we establish the conditional stability of sextic functional equations.

Theorem 3.1. *Let $s \in \{-1, 1\}$ be fixed. Let E be a real or complex linear space and let X be a (Archimedean) Banach space. Suppose $f : E \times E \rightarrow X$ satisfies the condition $f(x, 0) = f(0, y) = 0$ and inequalities of (2.1) and (2.2), where $\phi, \psi : E \times E \times E \rightarrow [0, \infty)$ are given functions which satisfy (2.3) and have the properties (2.4) for all $x, x_1, x_2, y, y_1, y_2 \in E$ and a constant $0 < L < 1$. Then there exists a unique sextic mapping $T : E \times E \rightarrow X$ satisfying the system (1.10) and*

$$\|T(x, y) - f(x, y)\| \leq \frac{1}{1-L} \tilde{\Phi}(x, y), \quad (3.1)$$

where

$$\begin{aligned} \tilde{\Phi}(x, y) := & \left| \frac{1}{2} \right| \left\{ |(a+b)^{-3s+1}| \phi \left((a+b)^{\frac{s-1}{2}} x, (a+b)^{\frac{s-1}{2}} x, (a+b)^{\frac{s-1}{2}} y \right) \right. \\ & \left. + |(a+b)^{-3s-3}| \psi \left((a+b)^{\frac{s+1}{2}} x, (a+b)^{\frac{s-1}{2}} y, (a+b)^{\frac{s-1}{2}} y \right) \right\} \end{aligned}$$

for all $x, y \in E$.

Proof. We denote $A := a + b$. Putting $x_1 = x_2 = x$ in (2.1), we get

$$\|f(Ax, y) - A^2 f(x, y)\| \leq \left| \frac{1}{2} \right| \phi(x, x, y) \quad (3.2)$$

for all $x, y \in E$. Putting $y_1 = y_2 = y$ and replacing x by Ax in (2.2), we get

$$\|f(Ax, Ay) - A^4 f(Ax, y)\| \leq \left| \frac{1}{2} \right| \psi(Ax, y, y) \quad (3.3)$$

for all $x, y \in E$. Thus by (3.2) and (3.3) we have

$$\|f(Ax, Ay) - A^6 f(x, y)\| \leq \left| \frac{1}{2} \right| \left\{ |A^4| \phi(x, x, y) + \psi(Ax, y, y) \right\},$$

for all $x, y \in E$. By last inequality we get

$$\|A^{-6} f(Ax, Ay) - f(x, y)\| \leq \left| \frac{1}{2} \right| \left\{ |A^{-2}| \phi(x, x, y) + |A^{-6}| \psi(Ax, y, y) \right\}, \quad (3.4)$$

$$\|A^6 f\left(\frac{x}{A}, \frac{y}{A}\right) - f(x, y)\| \leq \left| \frac{1}{2} \right| \left\{ |A^4| \phi\left(\frac{x}{A}, \frac{x}{A}, \frac{y}{A}\right) + \psi\left(x, \frac{y}{A}, \frac{y}{A}\right) \right\}, \quad (3.5)$$

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for all $x, y \in E$. Therefore

$$\left\| \frac{1}{A^{6s}} f(A^s x, A^s y) - f(x, y) \right\| \leq \tilde{\Phi}(x, y),$$

for all $x, y \in E$.

The rest of the proof is similar to the proof of Theorem 2.1. \square

Corollary 3.2. *Let $s \in \{-1, 1\}$ be fixed. Let E be a normed space and let F be a (Archimedean) Banach space. Suppose $f : E \times E \rightarrow F$ is a mapping with $f(x, 0) = f(0, y) = 0$ and there exist constants $\theta, \vartheta \geq 0$ and non-negative real number p such that $ps < 6s$ and*

$$\begin{aligned} & \|f(ax_1 + bx_2, y) + f(bx_1 + ax_2, y) + abf(x_1 - x_2, y) \\ & \quad - (a^2 + b^2)[f(x_1, y) + f(x_2, y)] - 4abf\left(\frac{x_1 + x_2}{2}, y\right)\| \leq \theta(\|x_1\|^p + \|x_2\|^p + \|y\|^p), \\ & \|f(x, ay_1 + by_2) + f(x, by_1 + ay_2) + \frac{1}{2}ab(a - b)^2 f(x, y_1 - y_2) \\ & \quad - (a^2 - b^2)^2[f(x, y_1) + f(x, y_2)] - 8abf\left(x, \frac{y_1 + y_2}{2}\right)\| \leq \vartheta(\|x\|^p + \|y_1\|^p + \|y_2\|^p), \end{aligned}$$

for all $x, x_1, x_2, y, y_1, y_2 \in E$. Then there exists a unique sextic mapping $T : E \times E \rightarrow F$ such that

$$\|f(x, y) - T(x, y)\| \leq \frac{\theta(2\|x\|^p + \|y\|^p)}{2s|a + b|^2 - 2s|a + b|^{p-4}} + \frac{\vartheta(\|a + b\|^p + 2\|y\|^p)}{2s|a + b|^6 - 2s|a + b|^p},$$

for all $x, y \in E$.

Proof. Defining

$$\phi(x_1, x_2, y) = \theta(\|x_1\|^p + \|x_2\|^p + \|y\|^p), \quad \psi(x, y_1, y_2) = \vartheta(\|x\|^p + \|y_1\|^p + \|y_2\|^p),$$

and applying Theorem 3.1, we get the desired result. \square

Remark 3.3. Comparison of (2.5) and (3.1) shows that the approximation in non-Archimedean Banach spaces is better than the approximation in (Archimedean) Banach spaces.

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Dynamical behaviors of a nonlinear virus infection model with latently infected cells and immune response

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Abstract

In this paper, we study the global stability of a mathematical model that describes the virus dynamics under the effect of antibody immune response. The model is a modification of some of the existing virus dynamics models by considering the latently infected cells and nonlinear incidence rate for virus infections. We show that the global dynamics of the model is completely determined by two threshold values R_0 , the corresponding reproductive number of viral infection and R_1 , the corresponding reproductive number of antibody immune response, respectively. Using Lyapunov method, we have proven that, if $R_0 \leq 1$, then the uninfected steady state is globally asymptotically stable (GAS), if $R_1 \leq 1 < R_0$, then the infected steady state without antibody immune response is GAS, and if $R_1 > 1$, then the infected steady state with antibody immune response is GAS.

Keywords: Virus infection; Global stability; Immune response; Lyapunov function; nonlinear infection rate.

1 Introduction

Recently, mathematical modeling and analysis of viral infections such as hepatitis C virus (HCV) [1]-[3], hepatitis B virus (HBV) [4]-[5], human immunodeficiency virus (HIV) [6]-[15] human T cell leukemia (HTLV) [16] have attracted the interest several researchers. In 1996, Nowak and Bangham [7] has proposed the basic viral infection model which contains three compartments, the uninfected target cells, infected cells and free virus particles. This model does not take into consideration the latently infected cells which is due to the delay between the moment of infection and the moment when the infected cell becomes active to produce infectious viruses. Latently infected cells have been incorporated into the basic viral infection model in several papers (see e.g. [18], [19] and [20]). The

basic viral infection model which takes into account the latently infected cells is given by [20]:

$$\dot{x} = \rho - dx - \eta xv, \quad (1)$$

$$\dot{w} = \eta xv - (e + b)w, \quad (2)$$

$$\dot{y} = bw - ay, \quad (3)$$

$$\dot{v} = ky - cv, \quad (4)$$

where x , w , y and v representing the populations of the uninfected target cells, latently infected cells, actively infected cells and free virus particles, respectively. Parameters ρ and k represent, respectively, the rate at which new uninfected cells are generated from the source within the body, and the generation rate constant of free viruses produced from the actively infected cells. Parameters d , e , a and c are the natural death rate constants of the uninfected cells, latently infected cells, actively infected cells and free virus particles, respectively. Parameter η is the infection rate constant. Eq. (2) describes the population dynamics of the latently infected cells and show that they are converted to actively infected cells with rate constant b . All the parameters given in model (1)-(4) are positive.

We observe that in model (1)-(4), the immune response has been neglected. To provide more accurate modelling for the viral infection, the effect of immune response has to be considered. The antibody immune response which is based on the antibodies that are produced by the B cells plays an important role in controlling the disease [17]. In the literature, several mathematical models have been formulated to consider the antibody immune response into the viral infection models (see e.g., [21]-[24]). However, in [21]-[24], it was assumed that all the infected cells are active which is an unrealistic assumption. The aim of this paper is to propose a viral infection model with antibody immune response taking into consideration both latently and actively infected cells and investigate its basic and global properties. The incidence rate is given by nonlinear function which is more general than the bilinear incidence rate given in model (1)-(4). Using Lyapunov functions, we prove that the global dynamics of the model is determined by two threshold parameters, the basic reproductive number of viral infection R_0 and the the basic reproductive number of antibody immune response R_1 . If $R_0 \leq 1$, then the infection-free steady state is globally asymptotically stable (GAS), if $R_1 \leq 1 < R_0$, then the chronic-infection steady state without antibody immune response is GAS, and if $R_1 > 1$, then the chronic-infection steady state with antibody immune response is GAS.

2 The mathematical model

In this section, we propose a viral infection model with latently infected cells and antibody immune response. The incidence rate is given by a nonlinear infection rate.

$$\dot{x} = \rho - dx - \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)}, \quad (5)$$

$$\dot{w} = (1 - \xi) \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} - (e + b)w, \quad (6)$$

$$\dot{y} = \xi \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} + bw - ay, \quad (7)$$

$$\dot{v} = ky - cv - rvz, \quad (8)$$

$$\dot{z} = gvz - \mu z, \quad (9)$$

where, z is the population of the antibody immune cells. Here δ , θ and n are positive constants. The fractions $(1 - \xi)$ and ξ with $0 < \xi < 1$ are the probabilities that upon infection, an uninfected cell will become either latently infected or actively infected. Parameters r , g and μ are the removal rate constant of the virus due to the antibodies, the proliferation rate constant of antibody immune cells and the natural death rate constant of the antibody immune cells, respectively.

2.1 Positive invariance

We note that, model (5)-(9) is biologically acceptable in the sense that no population goes negative. It is straightforward to check the positive invariance of the non-negative orthant $\mathbb{R}_{\geq 0}^5$ by model (5)-(9). In the following, we show the boundedness of the solution of model (5)-(9).

Proposition 1. There exist positive numbers $L_i, i = 1, 2, 3$ such that the compact set $\Omega = \{(x, w, y, v, z) \in \mathbb{R}_{\geq 0}^5 : 0 \leq x, w, y \leq L_1, 0 \leq v \leq L_2, 0 \leq z \leq L_3\}$ is positively invariant.

Proof. Let $X_1(t) = x(t) + w(t) + y(t)$, then

$$\dot{X}_1 = \rho - dx - ew - ay \leq \rho - s_1 X_1,$$

where $s_1 = \min\{d, a, e\}$. Hence $X_1(t) \leq L_1$, if $X_1(0) \leq L_1$, where $L_1 = \frac{\rho}{s_1}$. Since $x(t) > 0$, $w(t) \geq 0$ and $y(t) \geq 0$, then $0 \leq x(t), w(t), y(t) \leq L_1$ if $0 \leq x(0) + w(0) + y(0) \leq L_1$. On the other hand, let $X_2(t) = v(t) + \frac{r}{g}z(t)$, then

$$\dot{X}_2 = ky - cv - \frac{r\mu}{g}z \leq kL_1 - s_2 \left(v + \frac{r}{g}z \right) = kL_1 - s_2 X_2,$$

where $s_2 = \min\{c, \mu\}$. Hence $X_2(t) \leq L_2$, if $X_2(0) \leq L_2$, where $L_2 = \frac{kL_1}{s_2}$. Since $v(t) \geq 0$ and $z(t) \geq 0$, then $0 \leq v(t) \leq L_2$ and $0 \leq z(t) \leq L_3$ if $0 \leq v(0) + \frac{r}{g}z(0) \leq L_2$, where $L_3 = \frac{gL_2}{r}$.

2.2 Steady states

In this subsection, we calculate the steady states of model (5)-(9) and derive two thresholds parameters.

The steady states of model (5)-(9) satisfy the following equations:

$$\rho - dx - \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} = 0, \quad (10)$$

$$(1 - \xi) \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} - (e + b)w = 0, \quad (11)$$

$$\xi \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} + bw - ay = 0, \quad (12)$$

$$ky - cv - rvz = 0, \quad (13)$$

$$(gv - \mu)z = 0. \quad (14)$$

Equation (14) has two possible solutions, $z = 0$ or $v = \mu/g$. If $z = 0$, then from Eqs. (11) and (12) we obtain w and y as:

$$w = \frac{(1 - \xi)}{e + b} \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)}, \quad y = \frac{(e\xi + b)}{a(e + b)} \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)}. \quad (15)$$

Substituting Eq. (15) into Eq. (13), we obtain

$$\frac{k(e\xi + b)}{a(e + b)} \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} - cv = 0. \quad (16)$$

Equation (16) has two possibilities, $v = 0$ or $v \neq 0$. If $v = 0$, then $w = y = 0$ and $x = \frac{\rho}{d}$ which leads to the uninfected steady state $E_0 = (x_0, 0, 0, 0, 0)$, where $x_0 = \frac{\rho}{d}$. If $v \neq 0$, then from Eqs. (10) and (16) we obtain

$$v = \frac{k(e\xi + b)}{ac(e + b)} \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} = \frac{k(e\xi + b)(\rho - dx)}{ac(e + b)} \quad (17)$$

$$\Rightarrow x = x_0 - \frac{ac(e + b)}{dk(e\xi + b)} v. \quad (18)$$

From Eq. (18) into Eq. (16) we get

$$\frac{k(e\xi + b)}{a(e + b)} \frac{\eta \left(x_0 - \frac{ac(e + b)}{dk(e\xi + b)} v \right)^n}{\delta^n + \left(x_0 - \frac{ac(e + b)}{dk(e\xi + b)} v \right)^n} \frac{v}{(\theta + v)} - cv = 0.$$

Let us define a function Ψ_1 as

$$\Psi_1(v) = \frac{k(e\xi + b)}{a(e + b)} \frac{\eta \left(x_0 - \frac{ac(e + b)}{dk(e\xi + b)} v \right)^n}{\delta^n + \left(x_0 - \frac{ac(e + b)}{dk(e\xi + b)} v \right)^n} \frac{v}{(\theta + v)} - cv = 0.$$

It is clear that, $\Psi_1(0) = 0$, and when $v = \bar{v} = \frac{x_0 dk(e\xi + b)}{ac(e + b)} > 0$, then $\Psi_1(\bar{v}) = -c\bar{v} < 0$. Since $\Psi_1(v)$ is continuous for all $v \geq 0$, then we have

$$\Psi_1'(0) = c \left(\frac{k(e\xi + b)}{ac\theta(e + b)} \frac{\eta x_0^n}{\delta^n + x_0^n} - 1 \right).$$

Therefore, if $\Psi_1'(0) > 0$ i.e.

$$\frac{k(e\xi + b)}{ac\theta(e + b)} \frac{\eta x_0^n}{\delta^n + x_0^n} > 1,$$

then there exist a $v_1 \in (0, \bar{v})$ such that $\Psi_1(v_1) = 0$. From Eq. (13) we obtain $y_1 = \frac{c}{k}v_1 > 0$ and from Eq. (10) we define a function Ψ_2 as:

$$\Psi_2(x) = \rho - dx - \frac{\eta x^n v_1}{(\delta^n + x^n)(\theta + v_1)} = 0.$$

We have $\Psi_2(0) = \rho > 0$ and $\Psi_2(x_0) = -\frac{\eta x_0^n v_1}{(\delta^n + x_0^n)(\theta + v_1)} < 0$. Since $f(x) = \frac{x^n}{\delta^n + x^n}$ is a strictly increasing function of x , for all $n, \delta > 0$, then Ψ_2 is a strictly decreasing function of x , and there exist a unique $x_1 \in (0, x_0)$ such that $\Psi_2(x_1) = 0$. It follows that, $w_1 = \frac{(1-\xi)}{e+b} \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} > 0$ and $y_1 = \frac{(e\xi+b)}{a(e+b)} \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} > 0$. It means that, an infected steady state without antibody immune response $E_1(x_1, w_1, y_1, v_1, 0)$ exists when $\frac{k(e\xi+b)}{ac\theta(e+b)} \frac{\eta x_0^n}{\delta^n + x_0^n} > 1$. Then we can define the basic reproductive number of viral infection as:

$$R_0 = \frac{k(e\xi + b)}{ac\theta(e + b)} \frac{\eta x_0^n}{\delta^n + x_0^n}.$$

The parameter R_0 determines whether a chronic-infection can be established.

The other possibility of Eq. (14) is $v_2 = \frac{\mu}{g}$. Inserting v_2 in Eq. (10) and defining a function Ψ_3 as:

$$\Psi_3(x) = \rho - dx - \frac{\eta x^n v_2}{(\delta^n + x^n)(\theta + v_2)} = 0.$$

Note that, Ψ_3 is a strictly decreasing function of x . Clearly, $\Psi_3(0) = \rho > 0$ and $\Psi_3(x_0) = -\frac{\eta x_0^n v_2}{(\delta^n + x_0^n)(\theta + v_2)} < 0$. Thus, there exists a unique $x_2 \in (0, x_0)$ such that $\Psi_3(x_2) = 0$. It follows from Eqs. (11)-(13) that,

$$w_2 = \frac{(1-\xi)}{e+b} \frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)}, \quad y_2 = \frac{(e\xi+b)}{a(e+b)} \frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)},$$

$$z_2 = \frac{c}{r} \left[\frac{k(e\xi+b)}{ac(e+b)} \frac{\eta x_2^n}{(\delta^n + x_2^n)(\theta + v_2)} - 1 \right].$$

Thus $w_2, y_2 > 0$, and if $\frac{k(e\xi+b)}{ac(e+b)} \frac{\eta x_2^n}{(\delta^n + x_2^n)(\theta + v_2)} > 1$, then $z_2 > 0$. Now we define the basic reproductive number of antibody immune response:

$$R_1 = \frac{k(e\xi + b)}{ac(e + b)} \frac{\eta x_2^n}{(\delta^n + x_2^n)(\theta + v_2)},$$

which determines whether a persistent antibody immune response can be established. Hence, z_2 can be rewritten as $z_2 = \frac{c}{r}(R_1 - 1)$. It follows that, there exists an infected steady state with antibody immune response $E_2(x_2, w_2, y_2, v_2, z_2)$ when $R_1 > 1$. Since $x_1 < x_0$ and $v_2 > 0$, then

$$R_1 = \frac{k(e\xi + b)}{ac(e + b)} \frac{\eta x_2^n}{(\delta^n + x_2^n)(\theta + v_2)} < \frac{k(e\xi + b)}{ac\theta(e + b)} \frac{\eta x_0^n}{\delta^n + x_0^n} = R_0.$$

From above we have the following result.

- Lemma 1** (i) if $R_0 \leq 1$, then there exists only one positive steady state E_0 ,
(ii) if $R_1 \leq 1 < R_0$, then there exist two positive steady states E_0 and E_1 , and
(iii) if $R_1 > 1$, then there exist three positive steady states E_0, E_1 and E_2 .

3 Main results

In this section, we investigate the global stability of steady states E_0 , E_1 and E_2 employing the direct Lyapunov method and LaSalle's invariance principle.

3.1 Global stability of the uninfected steady state E_0

Theorem 1. If $R_0 \leq 1$, then E_0 is globally asymptotically stable (GAS).

Proof. Define a Lyapunov functional W_0 as follows:

$$W_0 = x - x_0 - \int_{x_0}^x \frac{x_0^n(\delta^n + s^n)}{s^n(\delta^n + x_0^n)} ds + \frac{b}{e\xi + b}w + \frac{e+b}{e\xi + b}y + \frac{a(e+b)}{k(e\xi + b)}v + \frac{ar(e+b)}{kg(e\xi + b)}z.$$

Calculating $\frac{dW_0}{dt}$ along the trajectories of (5)-(9) as:

$$\begin{aligned} \frac{dW_0}{dt} &= \left(1 - \frac{x_0^n(\delta^n + x^n)}{x^n(\delta^n + x_0^n)}\right) \left(\rho - dx - \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)}\right) + \frac{b}{e\xi + b} \left((1 - \xi) \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} - (e + b)w\right) \\ &+ \frac{e+b}{e\xi + b} \left(\xi \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} + bw - ay\right) + \frac{a(e+b)}{k(e\xi + b)} (ky - cv - rvz) + \frac{ar(e+b)}{kg(e\xi + b)} (gvz - \mu z) \\ &= \rho \left(1 - \frac{x_0^n(\delta^n + x^n)}{x^n(\delta^n + x_0^n)}\right) \left(1 - \frac{x}{x_0}\right) + \frac{\eta x_0^n v}{(\delta^n + x_0^n)(\theta + v)} - \frac{ac(e+b)}{k(e\xi + b)}v - \frac{ar\mu(e+b)}{kg(e\xi + b)}z \\ &= \rho \left(1 - \frac{x_0^n(\delta^n + x^n)}{x^n(\delta^n + x_0^n)}\right) \left(1 - \frac{x}{x_0}\right) + \frac{ac(e+b)}{k(e\xi + b)} \left(\frac{k(e\xi + b)}{ac(e+b)} \frac{\eta x_0^n}{(\delta^n + x_0^n)(\theta + v)} - 1\right) v - \frac{ar\mu(e+b)}{kg(e\xi + b)}z \\ &= \rho \left(1 - \frac{x_0^n(\delta^n + x^n)}{x^n(\delta^n + x_0^n)}\right) \left(1 - \frac{x}{x_0}\right) + \frac{ac(e+b)}{k(e\xi + b)} \left(R_0 \frac{\theta}{\theta + v} - 1\right) v - \frac{ar\mu(e+b)}{kg(e\xi + b)}z \\ &= \rho \left(1 - \frac{x_0^n(\delta^n + x^n)}{x^n(\delta^n + x_0^n)}\right) \left(1 - \frac{x}{x_0}\right) + \frac{ac(e+b)}{k(e\xi + b)} \left(R_0 \frac{\theta}{\theta + v} - 1\right) v - \frac{ar\mu(e+b)}{kg(e\xi + b)}z \\ &= \frac{d\delta^n(x^n - x_0^n)(x_0 - x)}{x^n(\delta^n + x_0^n)} + \frac{ac(e+b)}{k(e\xi + b)}(R_0 - 1)v - \frac{ac(e+b)R_0}{k(e\xi + b)} \frac{v^2}{\theta + v} - \frac{ar\mu(e+b)}{kg(e\xi + b)}z. \end{aligned} \quad (19)$$

We have $(x^n - x_0^n)(x_0 - x) \leq 0$ for all $x, n > 0$. Thus if $R_0 \leq 1$ then $\frac{dW_0}{dt} \leq 0$ for all $x, v, z > 0$. Thus, the solutions of system (5)-(9) limited to M , the largest invariant subset of $\left\{\frac{dW_0}{dt} = 0\right\}$ [25]. Clearly, it follows from Eq. (19) that $\frac{dW_0}{dt} = 0$ if and only if $x(t) = x_0$, $v(t) = 0$ and $z(t) = 0$. The set M is invariant and for any element belongs to M satisfies $v(t) = 0$ and $z(t) = 0$, then $\dot{v}(t) = 0$. We can see from Eq. (8) that, $0 = \dot{v}(t) = ky(t)$, and thus $y(t) = 0$. Moreover, from Eq. (7) we get $w(t) = 0$. Hence, $\frac{dW_0}{dt} = 0$ if and only if $x(t) = x_0$, $w(t) = 0$, $y(t) = 0$, $v(t) = 0$ and $z(t) = 0$. From LaSalle's invariance principle, E_0 is GAS.

3.2 Global stability of the infected steady state without antibody immune response E_1

Theorem 2. If $R_1 \leq 1 < R_0$, then E_1 is GAS.

Proof. We construct the following Lyapunov functional

$$W_1 = x - x_1 - \int_{x_1}^x \frac{x_1^n(\delta^n + s^n)}{s^n(\delta^n + x_1^n)} ds + \frac{b}{e\xi + b} w_1 H\left(\frac{w}{w_1}\right) \\ + \frac{e+b}{e\xi + b} y_1 H\left(\frac{y}{y_1}\right) + \frac{a(e+b)}{k(e\xi + b)} v_1 H\left(\frac{v}{v_1}\right) + \frac{ar(e+b)}{kg(e\xi + b)} z.$$

The time derivative of W_1 along the trajectories of (5)-(9) is given by

$$\begin{aligned} \frac{dW_1}{dt} = & \left(1 - \frac{x_1^n(\delta^n + x^n)}{x^n(\delta^n + x_1^n)}\right) \left(\rho - dx - \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)}\right) \\ & + \frac{b}{e\xi + b} \left(1 - \frac{w_1}{w}\right) \left((1 - \xi) \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} - (e + b)w\right) \\ & + \frac{e+b}{e\xi + b} \left(1 - \frac{y_1}{y}\right) \left(\xi \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} + bw - ay\right) \\ & + \frac{a(e+b)}{k(e\xi + b)} \left(1 - \frac{v_1}{v}\right) (ky - cv - rvz) + \frac{ar(e+b)}{kg(e\xi + b)} (gvz - \mu z). \end{aligned} \quad (20)$$

Applying $\rho = dx_1 + \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)}$ and collecting terms of Eq. (20) we get

$$\begin{aligned} \frac{dW_1}{dt} = & \left(1 - \frac{x_1^n(\delta^n + x^n)}{x^n(\delta^n + x_1^n)}\right) (dx_1 - dx) + \frac{\eta x_1^n v}{(\delta^n + x_1^n)(\theta + v)} \\ & + \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} \left(1 - \frac{x_1^n(\delta^n + x^n)}{x^n(\delta^n + x_1^n)}\right) \\ & - \frac{b(1 - \xi)}{e\xi + b} \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} \frac{w_1}{w} + \frac{b(e+b)}{e\xi + b} w_1 - \frac{(e+b)\xi}{e\xi + b} \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} \frac{y_1}{y} \\ & - \frac{(e+b)b}{e\xi + b} \frac{y_1 w}{y} + \frac{e+b}{e\xi + b} ay_1 - \frac{ac(e+b)}{k(e\xi + b)} v - \frac{a(e+b)}{(e\xi + b)} \frac{y v_1}{v} + \frac{ac(e+b)}{k(e\xi + b)} v_1 \\ & + \frac{ar(e+b)}{k(e\xi + b)} v_1 z - \frac{ar\mu(e+b)}{kg(e\xi + b)} z. \end{aligned}$$

Using the equilibrium conditions for E_1 :

$$(1 - \xi) \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} = (e + b)w_1, \quad y_1 = \frac{(e\xi + b)}{a(e + b)} \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)}, \quad cv_1 = ky_1,$$

we obtain

$$\begin{aligned} \frac{dW_1}{dt} = & dx_1 \left(1 - \frac{x_1^n(\delta^n + x^n)}{x^n(\delta^n + x_1^n)}\right) \left(1 - \frac{x}{x_1}\right) + \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} \left[\frac{v(\theta + v_1)}{v_1(\theta + v)} - \frac{v}{v_1}\right] \\ & + \left(\frac{b(1 - \xi)}{e\xi + b} + \frac{(e+b)\xi}{e\xi + b}\right) \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} \left(1 - \frac{x_1^n(\delta^n + x^n)}{x^n(\delta^n + x_1^n)}\right) \\ & - \frac{b(1 - \xi)}{e\xi + b} \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} \frac{x^n(\delta^n + x_1^n)(\theta + v_1)v w_1}{x_1^n(\delta^n + x^n)(\theta + v)v_1 w} \\ & + \frac{b(1 - \xi)}{e\xi + b} \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} - \frac{(e+b)\xi}{e\xi + b} \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} \frac{x^n(\delta^n + x_1^n)(\theta + v_1)v y_1}{x_1^n(\delta^n + x^n)(\theta + v)v_1 y} \\ & - \frac{b(1 - \xi)}{e\xi + b} \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} \frac{y_1 w}{y w_1} + \left(\frac{b(1 - \xi)}{e\xi + b} + \frac{(e+b)\xi}{e\xi + b}\right) \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} \\ & - \left(\frac{b(1 - \xi)}{e\xi + b} + \frac{(e+b)\xi}{e\xi + b}\right) \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} \frac{y v_1}{y_1 v} \\ & + \left(\frac{b(1 - \xi)}{e\xi + b} + \frac{(e+b)\xi}{e\xi + b}\right) \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} + \frac{ar(e+b)}{k(e\xi + b)} \left(v_1 - \frac{\mu}{g}\right) z. \end{aligned}$$

Collecting terms we get

$$\begin{aligned} \frac{dW_1}{dt} = & -\frac{d\delta^n(x^n - x_1^n)(x - x_1)}{x^n(\delta^n + x_1^n)} - \frac{\eta x_1^n \theta (v - v_1)^2}{(\delta^n + x_1^n)(\theta + v)(\theta + v_1)^2} + \frac{ar(e + b)}{k(e\xi + b)} \left(v_1 - \frac{\mu}{g} \right) z \\ & + \frac{b(1 - \xi)}{(e\xi + b)} \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} \left[5 - \frac{x_1^n(\delta^n + x^n)}{x^n(\delta^n + x_1^n)} - \frac{x^n(\delta^n + x_1^n)(\theta + v_1)vw_1}{x_1^n(\delta^n + x^n)(\theta + v)v_1w} - \frac{y_1w}{yw_1} - \frac{yv_1}{y_1v} - \frac{\theta + v}{\theta + v_1} \right] \\ & + \frac{(e + b)\xi}{(e\xi + b)} \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} \left[4 - \frac{x_1^n(\delta^n + x^n)}{x^n(\delta^n + x_1^n)} - \frac{x^n(\delta^n + x_1^n)(\theta + v_1)vy_1}{x_1^n(\delta^n + x^n)(\theta + v)v_1y} - \frac{yv_1}{y_1v} - \frac{\theta + v}{\theta + v_1} \right]. \end{aligned} \quad (21)$$

Clearly, the first two terms of Eq. (21) are less than or equal zero. Because the geometrical mean is less than or equal to the arithmetical mean, then the last two terms of Eq. (21) are less than or equal zero. Now we show that if $R_1 \leq 1$ then $v_1 \leq \frac{\mu}{r} = v_2$. This can be achieved if we show that

$$\text{sgn}(x_2 - x_1) = \text{sgn}(v_1 - v_2) = \text{sgn}(R_1 - 1).$$

We have

$$(x_2^n - x_1^n)(x_2 - x_1) > 0, \quad \text{for all } n > 0 \quad (22)$$

Suppose that, $\text{sgn}(x_2 - x_1) = \text{sgn}(v_2 - v_1)$. Using the conditions of the steady states E_1 and E_2 we have

$$\begin{aligned} (\rho - dx_2) - (\rho - dx_1) &= \frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)} - \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} \\ &= \frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)} - \frac{\eta x_2^n v_1}{(\delta^n + x_2^n)(\theta + v_1)} + \frac{\eta x_2^n v_1}{(\delta^n + x_2^n)(\theta + v_1)} - \frac{\eta x_1^n v_1}{(\delta^n + x_1^n)(\theta + v_1)} \\ &= \frac{\eta x_2^n}{\delta^n + x_2^n} \frac{\theta(v_2 - v_1)}{(\theta + v_2)(\theta + v_1)} + \frac{\eta v_1}{\theta + v_1} \left(\frac{\delta^n(x_2^n - x_1^n)}{(\delta^n + x_2^n)(\delta^n + x_1^n)} \right) \end{aligned}$$

and from inequalities (22) we get:

$$\text{sgn}(x_1 - x_2) = \text{sgn}(x_2 - x_1),$$

which leads to contradiction. Thus, $\text{sgn}(x_2 - x_1) = \text{sgn}(v_1 - v_2)$. Using the steady state conditions for E_1 we have $\frac{k(e\xi + b)}{ac(e + b)} \frac{\eta x_1^n}{(\delta^n + x_1^n)(\theta + v_1)} = 1$, then

$$\begin{aligned} R_1 - 1 &= \frac{k(e\xi + b)}{ac(e + b)} \left[\frac{\eta x_2^n}{(\delta^n + x_2^n)(\theta + v_2)} - \frac{\eta x_1^n}{(\delta^n + x_1^n)(\theta + v_1)} \right] \\ &= \frac{k(e\xi + b)}{ac(e + b)} \left[\frac{\eta x_2^n}{(\delta^n + x_2^n)(\theta + v_2)} - \frac{\eta x_2^n}{(\delta^n + x_2^n)(\theta + v_1)} \right. \\ &\quad \left. + \frac{\eta x_2^n}{(\delta^n + x_2^n)(\theta + v_1)} - \frac{\eta x_1^n}{(\delta^n + x_1^n)(\theta + v_1)} \right] \\ &= \frac{k(e\xi + b)}{ac(e + b)} \left[\frac{\eta x_2^n(v_1 - v_2)}{(\delta^n + x_2^n)(\theta + v_1)(\theta + v_2)} + \frac{\eta \delta^n(x_2^n - x_1^n)}{(\delta^n + x_2^n)(\delta^n + x_1^n)(\theta + v_1)} \right]. \end{aligned}$$

From inequality (22) we get:

$$\text{sgn}(R_1 - 1) = \text{sgn}(v_1 - v_2).$$

It follows that, if $R_1 \leq 1$ then $v_1 \leq \frac{\mu}{r} = v_2$. Therefore, if $R_1 \leq 1$ then $\frac{dW_1}{dt} \leq 0$ for all $x, w, y, v, z > 0$, where the equality occurs at the steady state E_1 . LaSalle's invariance principle implies the global stability of E_1 .

3.3 Global stability of the infected steady state with antibody immune response

E_2

Theorem 3. If $R_1 > 1$, then E_2 is GAS.

Proof. We construct the following Lyapunov functional

$$W_2 = x - x_2 - \int_{x_2}^x \frac{x_2^n(\delta^n + s^n)}{s^n(\delta^n + x_2^n)} ds + \frac{b}{e\xi + b} w_2 H\left(\frac{w}{w_2}\right) + \frac{e+b}{e\xi + b} y_2 H\left(\frac{y}{y_2}\right) + \frac{a(e+b)}{k(e\xi + b)} v_2 H\left(\frac{v}{v_2}\right) + \frac{ar(e+b)}{kg(e\xi + b)} z_2 H\left(\frac{z}{z_2}\right).$$

We calculate the time derivative of W_2 along the trajectories of (5)-(9) as:

$$\begin{aligned} \frac{dW_2}{dt} &= \left(1 - \frac{x_2^n(\delta^n + x^n)}{x^n(\delta^n + x_2^n)}\right) \left(\rho - dx - \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)}\right) \\ &+ \frac{b}{e\xi + b} \left(1 - \frac{w_2}{w}\right) \left((1 - \xi) \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} - (e + b)w\right) \\ &+ \frac{e+b}{e\xi + b} \left(1 - \frac{y_2}{y}\right) \left(\xi \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} + bw - ay\right) \\ &+ \frac{a(e+b)}{k(e\xi + b)} \left(1 - \frac{v_2}{v}\right) (ky - cv - rvz) + \frac{ar(e+b)}{kg(e\xi + b)} \left(1 - \frac{z_2}{z}\right) (gvz - \mu z). \end{aligned} \quad (23)$$

Applying $\rho = dx_2 + \frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)}$ and collecting terms of Eq. (23) we get

$$\begin{aligned} \frac{dW_2}{dt} &= \left(1 - \frac{x_2^n(\delta^n + x^n)}{x^n(\delta^n + x_2^n)}\right) (dx_2 - dx) + \frac{\eta x_2^n v}{(\delta^n + x_2^n)(\theta + v)} \\ &+ \frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)} \left(1 - \frac{x_2^n(\delta^n + x^n)}{x^n(\delta^n + x_2^n)}\right) - \frac{b(1 - \xi)}{e\xi + b} \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} \frac{w_2}{w} + \frac{b(e+b)}{e\xi + b} w_2 \\ &- \frac{(e+b)\xi}{e\xi + b} \frac{\eta x^n v}{(\delta^n + x^n)(\theta + v)} \frac{y_2}{y} - \frac{(e+b)b}{e\xi + b} \frac{y_2 w}{y} + \frac{e+b}{e\xi + b} ay_2 - \frac{ac(e+b)}{k(e\xi + b)} v - \frac{a(e+b)}{(e\xi + b)} \frac{y v_2}{v} \\ &+ \frac{ac(e+b)}{k(e\xi + b)} v_2 + \frac{ar(e+b)}{k(e\xi + b)} v_2 z - \frac{ar\mu(e+b)}{kg(e\xi + b)} z - \frac{ar(e+b)}{k(e\xi + b)} z_2 v + \frac{ar\mu(e+b)}{kg(e\xi + b)} z_2. \end{aligned}$$

Using the steady state conditions for E_2

$$(1 - \xi) \frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)} = (e + b)w_2, \quad \xi \frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)} + bw_2 = ay_2, \quad ky_2 = cv_2 + rv_2 z_2, \quad \mu = gv_2,$$

we get

$$\begin{aligned}
\frac{dW_2}{dt} &= dx_2 \left(1 - \frac{x_2^n(\delta^n + x^n)}{x^n(\delta^n + x_2^n)} \right) \left(1 - \frac{x}{x_2} \right) + \frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)} \left[\frac{v(\theta + v_2)}{v_2(\theta + v)} - \frac{v}{v_2} \right] \\
&+ \left(\frac{b(1-\xi)}{e\xi + b} + \frac{(e+b)\xi}{e\xi + b} \right) \frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)} \left(1 - \frac{x_2^n(\delta^n + x^n)}{x^n(\delta^n + x_2^n)} \right) \\
&- \frac{b(1-\xi)}{e\xi + b} \frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)} \frac{x^n(\delta^n + x_2^n)(\theta + v_2)vw_2}{x_2^n(\delta^n + x^n)(\theta + v)v_2w} \\
&+ \frac{b(1-\xi)}{e\xi + b} \frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)} - \frac{(e+b)\xi}{e\xi + b} \frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)} \frac{x^n(\delta^n + x_2^n)(\theta + v_2)vy_2}{x_2^n(\delta^n + x^n)(\theta + v)v_2y} \\
&- \frac{b(1-\xi)}{e\xi + b} \frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)} \frac{y_2w}{y_2v} + \left(\frac{b(1-\xi)}{e\xi + b} + \frac{(e+b)\xi}{e\xi + b} \right) \frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)} \\
&- \left(\frac{b(1-\xi)}{e\xi + b} + \frac{(e+b)\xi}{e\xi + b} \right) \frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)} \frac{yv_2}{y_2v} + \left(\frac{b(1-\xi)}{e\xi + b} + \frac{(e+b)\xi}{e\xi + b} \right) \frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)} \\
&= - \frac{d\delta^n(x^n - x_2^n)(x - x_2)}{x^n(\delta^n + x_2^n)} - \frac{\eta x_2^n \theta (v - v_2)^2}{(\delta^n + x_2^n)(\theta + v)(\theta + v_2)^2} \\
&+ \frac{b(1-\xi)}{(e\xi + b)} \frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)} \left[5 - \frac{x_2^n(\delta^n + x^n)}{x^n(\delta^n + x_2^n)} - \frac{x^n(\delta^n + x_2^n)(\theta + v_2)vw_2}{x_2^n(\delta^n + x^n)(\theta + v)v_2w} - \frac{y_2w}{y_2v} - \frac{yv_2}{y_2v} - \frac{\theta + v}{\theta + v_2} \right] \\
&+ \frac{(e+b)\xi}{(e\xi + b)} \frac{\eta x_2^n v_2}{(\delta^n + x_2^n)(\theta + v_2)} \left[4 - \frac{x_2^n(\delta^n + x^n)}{x^n(\delta^n + x_2^n)} - \frac{x^n(\delta^n + x_2^n)(\theta + v_2)vy_2}{x_2^n(\delta^n + x^n)(\theta + v)v_2y} - \frac{yv_2}{y_2v} - \frac{\theta + v}{\theta + v_2} \right].
\end{aligned} \tag{24}$$

Thus, if $R_1 > 1$ then x_2, w_2, y_2, v_2 and $z_2 > 0$. Clearly, $\frac{dW_2}{dt} \leq 0$ and $\frac{dW_2}{dt} = 0$ if and only if $x(t) = x_2, w(t) = w_2$ and $v(t) = v_2$. From Eq. (8), if $v(t) = v_2$ and $y(t) = y_2$, then $\dot{v}(t) = 0$ and $0 = ky_2 - cv_2 - rv_2z(t)$, which yields $z(t) = z_2$ and hence $\frac{dW_2}{dt}$ equal to zero at E_2 . LaSalle's invariance principle implies global stability of E_2 .

4 Conclusion

In this paper, we have proposed and analyzed a virus dynamics model with antibody immune response. The model is a five dimensional that describe the interaction between the uninfected target cells, latently infected cells, actively infected cells, free virus particles and antibody immune cells. The incidence rate has been represented by nonlinear function. We have derived two threshold parameters, the basic reproductive number of viral infection R_0 and the basic reproductive number of antibody immune response R_1 which completely determined the basic and global properties of the virus dynamics model. Using Lyapunov method and applying LaSalle's invariance principle we have proven that, if $R_0 \leq 1$, then the uninfected steady state is GAS, if $R_1 \leq 1 < R_0$, then the infected steady state without antibody immune response is GAS, and if $R_1 > 1$, then the infected steady state with antibody immune response is GAS.

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On the symmetric properties for the generalized twisted (h, q) -tangent numbers and polynomials associated with p -adic integral on \mathbb{Z}_p

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Abstract : In this paper, we study the symmetry for the generalized twisted (h, q) -tangent numbers $T_{n,\chi,q,\zeta}^{(h)}$ and polynomials $T_{n,\chi,q,\zeta}^{(h)}(x)$. We obtain some interesting identities of the power sums and the generalized twisted polynomials $T_{n,\chi,q,\zeta}^{(h)}(x)$ using the symmetric properties for the p -adic invariant integral on \mathbb{Z}_p .

Key words : Symmetric properties, power sums, the tangent numbers and polynomials, the generalized twisted (h, q) -tangent numbers and polynomials, p -adic integral on \mathbb{Z}_p .

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1. Introduction

Recently, many mathematicians have studied different kinds of the Euler, Bernoulli, Genocchi, Tangent numbers and polynomials (see [1-10]). These numbers and polynomials play important roles in many different areas of mathematics such as number theory, combinatorics, special function and analysis. The purpose of this paper is to obtain some interesting identities of the power sums and generalized twisted (h, q) -tangent polynomials $T_{n,\chi,q,\zeta}^{(h)}(x)$ using the symmetric properties for the p -adic invariant integral on \mathbb{Z}_p .

Throughout this paper, we always make use of the following notations: \mathbb{N} denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, \mathbb{C} denotes the set of complex numbers, \mathbb{Z}_p denotes the ring of p -adic rational integers, \mathbb{Q}_p denotes the field of p -adic rational numbers, and \mathbb{C}_p denotes the completion of algebraic closure of \mathbb{Q}_p . Let ν_p be the normalized exponential valuation of \mathbb{C}_p with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q -extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p -adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on \mathbb{Z}_p . For $g \in UD(\mathbb{Z}_p)$ the fermionic p -adic invariant q -integral on \mathbb{Z}_p is defined by Kim as follows:

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x)(-q)^x, \text{ see [1, 2, 3].}$$

Note that

$$\lim_{q \rightarrow 1} I_{-q}(g) = I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x). \quad (1.1)$$

If we take $g_n(x) = g(x + n)$ in (1.1), then we see that

$$I_{-1}(g_n) = (-1)^n I_{-1}(g) + 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} g(l). \quad (1.2)$$

Let a fixed positive integer d with $(p, d) = 1$, set

$$\begin{aligned} X &= X_d = \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}), \quad X_1 = \mathbb{Z}_p, \\ X^* &= \bigcup_{\substack{0 < a < dp \\ (a, p) = 1}} a + dp\mathbb{Z}_p, \\ a + dp^N\mathbb{Z}_p &= \{x \in X \mid x \equiv a \pmod{dp^N}\}, \end{aligned}$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$.

It is easy to see that

$$I_{-1}(g) = \int_X g(x) d\mu_{-1}(x) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x). \quad (1.3)$$

We assume that $h \in \mathbb{Z}$. Let $T_p = \cup_{N \geq 1} C_{p^N} = \lim_{N \rightarrow \infty} C_{p^N}$, where $C_{p^N} = \{\zeta \mid \zeta^{p^N} = 1\}$ is the cyclic group of order p^N . For $\zeta \in T_p$, we denote by $\phi_\zeta : \mathbb{Z}_p \rightarrow \mathbb{C}_p$ the locally constant function $x \mapsto \zeta^x$.

First, we introduce the tangent numbers and tangent polynomials. In [5], we investigated the zeros of the tangent polynomials $T_n(x)$. The tangent numbers T_n are defined by the generating function:

$$F(t) = \frac{2}{e^{2t} + 1} = \sum_{n=0}^{\infty} T_n \frac{t^n}{n!} \quad (|t| < \frac{\pi}{2}), \text{ cf. [5]}$$

where we use the technique method notation by replacing T^n by $T_n (n \geq 0)$ symbolically. We consider the tangent polynomials $T_n(x)$ as follows:

$$F(x, t) = \left(\frac{2}{e^{2t} + 1} \right) e^{xt} = \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!}. \quad (1.4)$$

Note that $T_n(x) = \sum_{k=0}^n \binom{n}{k} T_k x^{n-k}$. In the special case $x = 0$, we define $T_n(0) = T_n$.

In [8], we introduced the generalized twisted (h, q) -tangent numbers $T_{n, \chi, q, \zeta}^{(h)}$ and polynomials $T_{n, \chi, q, \zeta}^{(h)}(x)$ attached to χ . Let χ be Dirichlet's character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$. The generalized twisted (h, q) -tangent numbers $T_{n, \chi, q, \zeta}^{(h)}$ attached to χ are defined by the generating function:

$$\frac{2 \sum_{a=0}^{d-1} \chi(a) (-1)^a \zeta^a q^{ha} e^{2at}}{\zeta^d q^{hd} e^{2dt} + 1} = \sum_{n=0}^{\infty} T_{n, \chi, q, \zeta}^{(h)} \frac{t^n}{n!}, \text{ cf. [8]}. \quad (1.5)$$

We consider the generalized twisted (h, q) -tangent polynomials $T_{n, \chi, q, w}^{(h)}(x)$ attached to χ as follows:

$$\left(\frac{2 \sum_{a=0}^{d-1} \chi(a) (-1)^a \zeta^a q^{ha} e^{2at}}{\zeta^d q^{hd} e^{2dt} + 1} \right) e^{xt} = \sum_{n=0}^{\infty} T_{n, \chi, q, \zeta}^{(h)}(x) \frac{t^n}{n!}. \quad (1.6)$$

Let $g(y) = \chi(y) \phi_\zeta(y) q^{hy} e^{(2y+x)t}$. By (1.3), we derive

$$\begin{aligned} I_{-1} \left(\chi(y) \phi_\zeta(y) q^{hy} e^{(2y+x)t} \right) &= \int_X \chi(y) \phi_\zeta(y) q^{hy} e^{(2y+x)t} d\mu_{-1}(y) \\ &= \left(\frac{2 \sum_{a=0}^{d-1} \chi(a) (-1)^a \zeta^a q^{ha} e^{2at}}{\zeta^d q^{hd} e^{2dt} + 1} \right) e^{xt} \\ &= \sum_{n=0}^{\infty} T_{n, \chi, q, \zeta}^{(h)}(x) \frac{t^n}{n!}. \end{aligned} \quad (1.7)$$

By using Taylor series of $e^{(2y+x)t}$ in the above equation (1.7), we obtain

$$\sum_{n=0}^{\infty} \left(\int_X \chi(y) \phi_\zeta(y) q^{hy} (2y+x)^n d\mu_{-1}(y) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} T_{n, \chi, q, \zeta}^{(h)}(x) \frac{t^n}{n!}.$$

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we have the Witt formula for the generalized twisted (h, q) -tangent polynomials attached to χ as follows:

Theorem 1. For positive integers $n, \zeta \in T_p$, and $h \in \mathbb{Z}$, we have

$$T_{n,\chi,q,\zeta}^{(h)}(x) = \int_X \chi(y) \phi_\zeta(y) q^{hy} (2y+x)^n d\mu_{-1}(y). \quad (1.8)$$

If we take $x = 0$ in Theorem 1, we also have the following corollary.

Corollary 2. For positive integers $n, \zeta \in T_p$, and $h \in \mathbb{Z}$, we have

$$T_{n,\chi,q,\zeta}^{(h)} = \int_X \chi(y) \phi_\zeta(y) q^{hy} (2y)^n d\mu_{-1}(y). \quad (1.9)$$

By (1.8) and (1.9), we have the following theorem.

Theorem 3. For positive integers $n, \zeta \in T_p$, and $h \in \mathbb{Z}$, we have

$$T_{n,\chi,q,\zeta}^{(h)}(x) = \sum_{l=0}^n \binom{n}{l} T_{l,\chi,q,\zeta}^{(h)} x^{n-l}.$$

2. Symmetry for the generalized twisted (h, q) -tangent polynomials

In this section, we assume that $q \in \mathbb{C}_p$ and $\zeta \in T_p$. By using the symmetric properties for the p -adic invariant integral on \mathbb{Z}_p , we obtain some interesting identities of the power sums and the generalized twisted polynomials $T_{n,\chi,q,\zeta}^{(h)}(x)$. If n is odd from (1.2), we obtain

$$I_{-1}(g_n) + I_{-1}(g) = 2 \sum_{k=0}^{n-1} (-1)^k g(k) \text{ (see [1], [2], [3], [5])}. \quad (2.1)$$

It will be more convenient to write (2.1) as the equivalent integral form

$$\int_{\mathbb{Z}_p} g(x+n) d\mu_{-1}(x) + \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = 2 \sum_{k=0}^{n-1} (-1)^k g(k). \quad (2.2)$$

Substituting $g(x) = \chi(x) \zeta^x q^{hx} e^{2xt}$ into the above, we obtain

$$\begin{aligned} & \int_X \chi(x+n) \zeta^{x+n} q^{h(x+n)} e^{(2x+2n)t} d\mu_{-1}(x) + \int_X \chi(x) \zeta^x q^{hx} e^{2xt} d\mu_{-1}(x) \\ &= 2 \sum_{j=0}^{n-1} (-1)^j \chi(j) \zeta^j q^{hj} e^{2jt}. \end{aligned} \quad (2.3)$$

For $k \in \mathbb{Z}_+$, let us define the power sums $\mathcal{T}_{k,\chi,q,\zeta}^{(h)}(n)$ as follows:

$$\mathcal{T}_{k,\chi,q,\zeta}^{(h)}(n) = \sum_{l=0}^n (-1)^l \chi(l) \zeta^l q^{hl} (2l)^k. \quad (2.4)$$

After some elementary calculations, we have

$$\begin{aligned} \int_X \chi(x) \zeta^x q^{hx} e^{2xt} d\mu_{-1}(x) &= \frac{2 \sum_{a=0}^{d-1} \chi(a) (-1)^a \zeta^a q^{ha} e^{2at}}{\zeta^d q^{hd} e^{2dt} + 1}, \\ \int_X \chi(x) \zeta^{x+n} q^{h(x+n)} e^{(2x+2n)t} d\mu_{-1}(x) &= \zeta^n q^{hn} e^{2nt} \frac{2 \sum_{a=0}^{d-1} \chi(a) (-1)^a \zeta^a q^{ha} e^{2at}}{\zeta^d q^{hd} e^{2dt} + 1}. \end{aligned} \quad (2.5)$$

By using (2.5), we have

$$\begin{aligned} & \int_X \chi(x) \zeta^{x+nd} q^{h(x+nd)} e^{(2x+2nd)t} d\mu_{-1}(x) + \int_X \chi(x) \zeta^x q^{hx} e^{2xt} d\mu_{-1}(x) \\ &= (1 + \zeta^{nd} q^{hnd} e^{2ndt}) \frac{2 \sum_{a=0}^{d-1} \chi(a) (-1)^a \zeta^a q^{ha} e^{2at}}{\zeta^d q^{hd} e^{2dt} + 1}. \end{aligned}$$

From the above, we get

$$\begin{aligned} & \int_X \chi(x) \zeta^{x+nd} q^{h(x+nd)} e^{(2x+2nd)t} d\mu_{-1}(x) + \int_X \chi(x) \zeta^x q^{hx} e^{2xt} d\mu_{-1}(x) \\ &= \frac{2 \int_X \chi(x) \zeta^x q^{hx} e^{2xt} d\mu_{-1}(x)}{\int_X \zeta^{ndx} q^{hndx} e^{2ndtx} d\mu_{-1}(x)}. \end{aligned} \quad (2.6)$$

By substituting Taylor series of e^{2xt} into (2.3), we obtain

$$\begin{aligned} & \sum_{m=0}^{\infty} \left(\int_X \chi(x) \zeta^{x+nd} q^{h(x+nd)} (2x+2nd)^m d\mu_{-1}(x) + \int_X \chi(x) \zeta^x q^{hx} (2x)^m d\mu_{-1}(x) \right) \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left(2 \sum_{j=0}^{nd-1} (-1)^j \chi(j) \zeta^j q^{hj} (2j)^m \right) \frac{t^m}{m!} \end{aligned}$$

By comparing coefficients $\frac{t^m}{m!}$ in the above equation, we obtain

$$\begin{aligned} & \zeta^{nd} q^{hnd} \sum_{k=0}^m \binom{m}{k} (2nd)^{m-k} \int_X \chi(x) \zeta^x q^{hx} (2x)^k d\mu_{-1}(x) + \int_X \chi(x) \zeta^x q^{hx} (2x)^m d\mu_{-1}(x) \\ &= 2 \sum_{j=0}^{nd-1} (-1)^j \chi(j) \zeta^j q^{hj} (2j)^m \end{aligned}$$

By using (2.4), we have

$$\begin{aligned} & \zeta^{nd} q^{hnd} \sum_{k=0}^m \binom{m}{k} (2nd)^{m-k} \int_X \chi(x) \zeta^x q^{hx} (2x)^k d\mu_{-1}(x) + \int_X \chi(x) \zeta^x q^{hx} (2x)^m d\mu_{-1}(x) \\ &= 2 \mathcal{T}_{m,\chi,q,\zeta}^{(h)}(nd-1). \end{aligned} \quad (2.7)$$

By using (2.6) and (2.7), we arrive at the following theorem:

Theorem 4. Let n be odd positive integer. Then we obtain

$$\frac{2 \int_X \chi(x) \zeta^x q^{hx} e^{2xt} d\mu_{-1}(x)}{\int_X \zeta^{ndx} q^{hndx} e^{2ndtx} d\mu_{-1}(x)} = \sum_{m=0}^{\infty} \left(2 \mathcal{T}_{m,\chi,q,\zeta}^{(h)}(nd-1) \right) \frac{t^m}{m!}.$$

Let w_1 and w_2 be odd positive integers. Then we set

$$\begin{aligned} S(w_1, w_2) &= \\ &= \frac{\int_X \int_X \chi(x_1) \chi(x_2) \zeta^{(w_1 x_1 + w_2 x_2)} q^{h(w_1 x_1 + w_2 x_2)} e^{(2w_1 x_1 + 2w_2 x_2 + w_1 w_2 x) t} d\mu_{-1}(x_1) d\mu_{-1}(x_2)}{\int_X \zeta^{w_1 w_2 dx} q^{h w_1 w_2 dx} e^{2w_1 w_2 dx t} d\mu_{-1}(x)}. \end{aligned} \quad (2.8)$$

By Theorem 4 and (2.8), after elementary calculations, we obtain

$$\begin{aligned} S(w_1, w_2) &= \left(\frac{1}{2} \int_X \chi(x_1) \zeta^{w_1 x_1} q^{h w_1 x_1} e^{(2w_1 x_1 + w_1 w_2 x) t} d\mu_{-1}(x_1) \right) \\ &\quad \times \left(\frac{2 \int_X \chi(x_2) \zeta^{w_2 x_2} q^{h w_2 x_2} e^{2w_2 x_2 t} d\mu_{-1}(x_2)}{\int_X \zeta^{w_1 w_2 dx} q^{h w_1 w_2 dx} e^{2w_1 w_2 dx t} d\mu_{-1}(x)} \right) \\ &= \left(\frac{1}{2} \sum_{m=0}^{\infty} \mathcal{T}_{m,\chi,q^{w_1},\zeta^{w_1}}^{(h)}(w_2 x) w_1^m \frac{t^m}{m!} \right) \left(2 \sum_{m=0}^{\infty} \mathcal{T}_{m,\chi,q^{w_2},\zeta^{w_2}}^{(h)}(w_1 d - 1) w_2^m \frac{t^m}{m!} \right). \end{aligned} \quad (2.9)$$

By using Cauchy product in the above, we have

$$S(w_1, w_2) = \sum_{m=0}^{\infty} \left(\sum_{j=0}^m \binom{m}{j} T_{j, \chi, q^{w_1}, \zeta^{w_1}}^{(h)}(w_2 x) w_1^j \mathcal{T}_{m-j, \chi, q^{w_2}, \zeta^{w_2}}^{(h)}(w_1 d - 1) w_2^{m-j} \right) \frac{t^m}{m!}. \quad (2.10)$$

From the symmetry of $S(w_1, w_2)$ in w_1 and w_2 , we also see that

$$\begin{aligned} S(w_1, w_2) &= \left(\frac{1}{2} \int_X \chi(x_2) \zeta^{w_2 x_2} q^{h w_2 x_2} e^{(2w_2 x_2 + w_1 w_2 x) t} d\mu_{-1}(x_2) \right) \\ &\quad \times \left(\frac{2 \int_X \chi(x_1) \zeta^{w_1 x_1} q^{h w_1 x_1} e^{2x_1 w_1 t} d\mu_{-1}(x_1)}{\int_X \zeta^{w_1 w_2 dx} q^{h w_1 w_2 dx} e^{2w_1 w_2 dx} d\mu_{-1}(x)} \right) \\ &= \left(\frac{1}{2} \sum_{m=0}^{\infty} T_{m, \chi, q^{w_2}, \zeta^{w_2}}^{(h)}(w_1 x) w_2^m \frac{t^m}{m!} \right) \left(2 \sum_{m=0}^{\infty} \mathcal{T}_{m, \chi, q^{w_1}, \zeta^{w_1}}^{(h)}(w_2 d - 1) w_1^m \frac{t^m}{m!} \right). \end{aligned}$$

Thus we have

$$S(w_1, w_2) = \sum_{m=0}^{\infty} \left(\sum_{j=0}^m \binom{m}{j} T_{j, \chi, q^{w_2}, \zeta^{w_2}}^{(h)}(w_1 x) w_2^j \mathcal{T}_{m-j, \chi, q^{w_1}, \zeta^{w_1}}^{(h)}(w_2 d - 1) w_1^{m-j} \right) \frac{t^m}{m!} \quad (2.11)$$

By comparing coefficients $\frac{t^m}{m!}$ in the both sides of (2.10) and (2.11), we arrive at the following theorem:

Theorem 5. Let w_1 and w_2 be odd positive integers. Then we obtain

$$\begin{aligned} &\sum_{j=0}^m \binom{m}{j} w_1^{m-j} w_2^j T_{j, \chi, q^{w_2}, \zeta^{w_2}}^{(h)}(w_1 x) \mathcal{T}_{m-j, \chi, q^{w_1}, \zeta^{w_1}}^{(h)}(w_2 d - 1) \\ &= \sum_{j=0}^m \binom{m}{j} w_1^j w_2^{m-j} T_{j, \chi, q^{w_1}, \zeta^{w_1}}^{(h)}(w_2 x) \mathcal{T}_{m-j, \chi, q^{w_2}, \zeta^{w_2}}^{(h)}(w_1 d - 1), \end{aligned}$$

where $T_{k, \chi, q, \zeta}^{(h)}(x)$ and $\mathcal{T}_{m, \chi, q, \zeta}^{(h)}(k)$ denote the generalized twisted (h, q) -tangent polynomials and the alternating sums of powers of consecutive (h, q) -integers, respectively.

By Theorem 3, we have the following corollary.

Corollary 6. Let w_1 and w_2 be odd positive integers. Then we obtain

$$\begin{aligned} &\sum_{j=0}^m \sum_{k=0}^j \binom{m}{j} \binom{j}{k} w_1^{m-k} w_2^j x^{j-k} T_{k, \chi, q^{w_2}, \zeta^{w_2}}^{(h)} \mathcal{T}_{m-j, \chi, q^{w_1}, \zeta^{w_1}}^{(h)}(w_2 d - 1) \\ &= \sum_{j=0}^m \sum_{k=0}^j \binom{m}{j} \binom{j}{k} w_1^j w_2^{m-k} x^{j-k} T_{k, \chi, q^{w_1}, \zeta^{w_1}}^{(h)} \mathcal{T}_{m-j, \chi, q^{w_2}, \zeta^{w_2}}^{(h)}(w_1 d - 1). \end{aligned}$$

Now we will derive another interesting identities for the generalized twisted (h, q) -tangent polynomials using the symmetric property of $S(w_1, w_2)$.

$$\begin{aligned}
S(w_1, w_2) &= \left(\frac{1}{2} \int_X \chi(x_1) \zeta^{w_1 x_1} q^{h w_1 x_1} e^{(2w_1 x_1 + w_1 w_2 x) t} d\mu_{-1}(x_1) \right) \\
&\quad \times \left(\frac{2 \int_X \chi(x_2) \zeta^{w_2 x_2} q^{h w_2 x_2} e^{2x_2 w_2 t} d\mu_{-1}(x_2)}{\int_X \zeta^{w_1 w_2 dx} q^{h w_1 w_2 dx} e^{2w_1 w_2 dx} d\mu_{-1}(x)} \right) \\
&= \left(\frac{1}{2} e^{w_1 w_2 x t} \int_X \chi(x_1) \zeta^{w_1 x_1} q^{h w_1 x_1} e^{2x_1 w_1 t} d\mu_{-1}(x_1) \right) \\
&\quad \times \left(2 \sum_{j=0}^{w_1 d-1} (-1)^j \chi(j) \zeta^{w_2 j} q^{w_2 h j} e^{2j w_2 t} \right) \\
&= \sum_{j=0}^{w_1 d-1} (-1)^j \chi(j) \zeta^{w_2 j} q^{w_2 h j} \int_X \chi(x_1) \zeta^{w_1 x_1} q^{h w_1 x_1} e^{\left(2x_1 + w_2 x + \frac{2j w_2}{w_1}\right) (w_1 t)} d\mu_{-1}(x_1) \\
&= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{w_1 d-1} (-1)^j \chi(j) \zeta^{w_2 j} q^{w_2 h j} T_{n, \chi, q^{w_1}, \zeta^{w_1}}^{(h)} \left(w_2 x + \frac{2j w_2}{w_1} \right) w_1^n \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.12}$$

By using the symmetry property in (2.12), we also have

$$\begin{aligned}
S(w_1, w_2) &= \left(\frac{1}{2} e^{w_1 w_2 x t} \int_X \chi(x_2) \zeta^{w_2 x_2} q^{h w_2 x_2} e^{2x_2 w_2 t} d\mu_{-1}(x_2) \right) \\
&\quad \times \left(\frac{2 \int_X \chi(x_1) \zeta^{w_1 x_1} q^{h w_1 x_1} e^{2x_1 w_1 t} d\mu_{-1}(x_1)}{\int_X \zeta^{w_1 w_2 dx} q^{h w_1 w_2 dx} e^{2w_1 w_2 dx} d\mu_{-1}(x)} \right) \\
&= \left(\frac{1}{2} e^{w_1 w_2 x t} \int_X \chi(x_2) \zeta^{w_2 x_2} q^{h w_2 x_2} e^{2x_2 w_2 t} d\mu_{-1}(x_2) \right) \\
&\quad \times \left(2 \sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) \zeta^{w_1 j} q^{w_1 h j} e^{2j w_1 t} \right) \\
&= \sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) \zeta^{w_1 j} q^{w_1 h j} \int_X \chi(x_2) \zeta^{w_2 x_2} q^{h w_2 x_2} e^{\left(2x_2 + w_1 x + \frac{2j w_1}{w_2}\right) (w_2 t)} d\mu_{-1}(x_2) \\
&= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) \zeta^{w_1 j} q^{w_1 h j} T_{n, \chi, q^{w_2}, \zeta^{w_2}}^{(h)} \left(w_1 x + \frac{2j w_1}{w_2} \right) w_2^n \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.13}$$

By comparing coefficients $\frac{t^n}{n!}$ in the both sides of (2.12) and (2.13), we have the following theorem.

Theorem 7. Let w_1 and w_2 be odd positive integers. Then we obtain

$$\begin{aligned}
&\sum_{j=0}^{w_1 d-1} (-1)^j \chi(j) \zeta^{w_2 j} q^{w_2 h j} T_{n, \chi, q^{w_1}, \zeta^{w_1}}^{(h)} \left(w_2 x + \frac{2j w_2}{w_1} \right) w_1^n \\
&= \sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) \zeta^{w_1 j} q^{w_1 h j} T_{n, \chi, q^{w_2}, \zeta^{w_2}}^{(h)} \left(w_1 x + \frac{2j w_1}{w_2} \right) w_2^n.
\end{aligned} \tag{2.14}$$

If we take $x = 0$ in Theorem 7, we also derive the interesting identity for the generalized twisted (h, q) -tangent numbers as follows:

$$\begin{aligned}
&\sum_{j=0}^{w_1 d-1} (-1)^j \chi(j) \zeta^{w_2 j} q^{w_2 h j} T_{n, \chi, q^{w_1}, \zeta^{w_1}}^{(h)} \left(\frac{2j w_2}{w_1} \right) w_1^n \\
&= \sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) \zeta^{w_1 j} q^{w_1 h j} T_{n, \chi, q^{w_2}, \zeta^{w_2}}^{(h)} \left(\frac{2j w_1}{w_2} \right) w_2^n.
\end{aligned}$$

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8. Mathematical notation must be typeset. Equations should be numbered consecutively with Arabic numerals in parentheses placed flush right, and should be thusly referred to in the text [such as Eqs.(2) and (5)]. The running title must be placed at the top of even numbered pages and the first author's name, et al., must be placed at the top of the odd numbered pages.

9. Illustrations (photographs, drawings, diagrams, and charts) are to be numbered in one consecutive series of Arabic numerals. The captions for illustrations should be typed double space. All illustrations, charts, tables, etc., must be embedded in the body of the manuscript in proper, final, print position. In particular, manuscript, source, and PDF file version must be at camera ready stage for publication or they cannot be considered.

Tables are to be numbered (with Roman numerals) and referred to by number in the text. Center the title above the table, and type explanatory footnotes (indicated by superscript lowercase letters) below the table.

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1. H.H.Gonska, Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

Book

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea, New York, 1986.

Contribution to a Book

3. M.K.Khan, Approximation properties of beta operators, in (title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus, eds.), Academic Press, New York, 1991, pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

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Existence and uniqueness of fuzzy solutions for the nonlinear second-order fuzzy Volterra integrodifferential equations

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Abstract. Formulation of uncertainty Volterra integrodifferential equations (VIDEs) is very important issue in applied sciences and engineering; whilst the natural way to model such dynamical systems is to use the fuzzy approach. In this work, we present and prove the existence and uniqueness of four solutions of fuzzy VIDEs based on the Hausdorff distance under the assumption of strongly generalized differentiability for the fuzzy-valued mappings of a real variable whose values are normal, convex, upper semicontinuous, and compactly supported fuzzy sets in \mathbb{R} . In addition to that, we utilize and prove the characterization theorem for solutions of fuzzy VIDEs which allow us to translate a fuzzy VIDE into a system of crisp equations. The proof methodology is based on the assumption of the generalized Lipchitz property for each nonlinear term appears in the fuzzy equation subject to the specific metric used, while the main tools employed in the analysis are founded on the applications of the Banach fixed point theorem and a certain integral inequality with explicit estimate. An efficient computational algorithm is provided to guarantee the procedure and to confirm the performance of the proposed approach.

Keywords: Fuzzy VIDE; Banach fixed point theorem; Existence and uniqueness

AMS Subject Classification: 26E50; 46S40; 34A07

1. Introduction

There is an inexhaustible supply of applications of VIDEs, especially, in characterizing many social, physical, biological, and engineering problems. On the other aspect as well, since many real-world problems are too complex to be defined in precise terms, uncertainty is often involved in any real-world design process. Fuzzy sets provide a widely appreciated tool to introduce uncertain parameters into mathematical applications. In many applications, at least some of the parameters of the model should be represented by fuzzy rather than crisp numbers. Thus, it is immensely important to develop appropriate and applicable definitions and theorems to accomplish the mathematical construction that would appropriately treat fuzzy VIDEs and solve them.

In this work we are interested in the following main questions; firstly, under what conditions can we be sure that solutions of fuzzy VIDE exist; secondly, under what conditions can we be sure that there are four unique solutions; one solution for each lateral derivative; to fuzzy VIDE, thirdly under what conditions can we be sure that fuzzy VIDE is equivalent into system of crisp VIDEs. Anyhow, in this paper we will answered the aforementioned questions and present an efficient computational algorithm to guarantee the procedure and to confirm the performance of the proposed approach. More precisely, we consider the following second-order fuzzy VIDE under the assumption of strongly generalized differentiability of the general form:

$$x''(t) = f(t, x(t), x'(t)) + \int_0^t g(t, \tau, x(\tau), x'(\tau)) d\tau, 0 \leq \tau < t \leq 1, \quad (1.1)$$

subject to the fuzzy initial conditions

$$x(0) = \alpha, x'(0) = \beta, \quad (1.2)$$

where $f: [0,1] \times \mathbb{R}_{\mathcal{F}}^2 \rightarrow \mathbb{R}_{\mathcal{F}}$ and $g: [0,1]^2 \times \mathbb{R}_{\mathcal{F}}^2 \rightarrow \mathbb{R}_{\mathcal{F}}$ are continuous fuzzy-valued functions that satisfy a generalized Lipchitz condition and $\alpha, \beta \in \mathbb{R}_{\mathcal{F}}$.

The topics of fuzzy VIDEs which is growing interest for some time, in particular in relation to fuzzy control, fuzzy population growth model, fuzzy oscillating magnetic fields, have been rapidly developed in recent years. Anyhow, in

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this work, we are focusing our attention on second-order fuzzy VIDEs subject to given fuzzy initial conditions. At the beginning, approaches to fuzzy IDEs and other fuzzy equations can be of three types. The first approach assumes that even if only the initial values are fuzzy, the solution is a fuzzy function, and consequently the derivatives in the IDE must be considered as fuzzy derivatives [1,2]. These can be done by the use of the Hukuhara derivative for fuzzy-valued functions. Generally, this approach has a drawback; the solution becomes fuzzier as time goes, hence, the fuzzy solution behaves quite differently from the crisp solution. In the second approach, the fuzzy IDE is transformed to a crisp one by interpreting it as a family of differential inclusions [3,4]. The main shortcoming of using differential inclusions is that we do not have a derivative of a fuzzy-valued function. The third approach based on the Zadeh's extension principle, where the associated crisp problem is solved and in the solution the initial fuzzy values are substituted instead of the real constants, and in the final solution, arithmetic operations are considered to be operations on fuzzy numbers [5,6]. The weakness of this approach is the need to rewrite the solution in the fuzzy setting which in turn makes the methods of solution are not user-friendly and more restricted with more computation steps. As a conclusion, to overcome the above-mentioned shortcoming, the concept of a strongly generalized differentiability was developed and investigated in [7-14]. Anyhow, using the strongly generalized differentiability, the fuzzy IDE has locally four solutions. Indeed, with this approach, we can find solutions for a larger class of fuzzy IDEs than using other types of differentiability.

The solvability analysis of fuzzy VIDEs has been studied by several researchers by using the strongly generalized differentiability, the Hukuhara derivative, or the Zadeh's extension principle for the fuzzy-valued mappings of a real variable whose values are normal, convex, upper semicontinuous, and compactly supported fuzzy sets in \mathbb{R} . The reader is asked to refer to [15-22] in order to know more details about these analyzes, including their kinds and history, their modifications and conditions for use, their scientific applications, their importance and characteristics, and their relationship including the differences. But on the other aspect as well, more details about characterization theorem can be found in [23,24].

The organization of the paper is as follows. In the next section, we present some necessary definitions and preliminary results from the fuzzy calculus theory. The procedure of solving fuzzy VIDEs is presented in section 3. In section 4, existence and uniqueness of four solutions are introduced. In section 5, we utilize the characterization theorem for the solution of fuzzy VIDEs. This article ends in section 6 with some concluding remarks.

2. Excerpts of fuzzy calculus theory

Fuzzy calculus is the study of theory and applications of integrals and derivatives of uncertain functions. This branch of mathematical analysis, extensively investigated in the recent years, has emerged as an effective and powerful tool for the mathematical modeling of several engineering and scientific phenomena. In this section, we present some necessary definitions from fuzzy calculus theory and preliminary results. For the concept of fuzzy derivative, we will adopt strongly generalized differentiability, which is a modification of the Hukuhara differentiability and has the advantage of dealing properly with fuzzy VIDEs.

Let X be a nonempty set. A fuzzy set u in X is characterized by its membership function $u: X \rightarrow [0,1]$. Thus, $u(s)$ is interpreted as the degree of membership of an element s in the fuzzy set u for each $s \in X$. A fuzzy set u on \mathbb{R} is called convex if for each $s, t \in \mathbb{R}$ and $\lambda \in [0,1]$, $u(\lambda s + (1-\lambda)t) \geq \min\{u(s), u(t)\}$, is called upper semicontinuous if $\{s \in \mathbb{R}: u(s) > r\}$ is closed for each $r \in [0,1]$, and is called normal if there is $s \in \mathbb{R}$ such that $u(s) = 1$. The support of a fuzzy set u is defined as $\{s \in \mathbb{R}: u(s) > 0\}$.

Definition 2.1. [25] A fuzzy number u is a fuzzy subset of the real line with a normal, convex, and upper semicontinuous membership function of bounded support.

For each $r \in (0,1]$, set $[u]^r = \{s \in \mathbb{R}: u(s) \geq r\}$ and $[u]^0 = \overline{\{s \in \mathbb{R}: u(s) > 0\}}$, where $\overline{\{\cdot\}}$ denote the closure of $\{\cdot\}$. Then, it easily to establish that u is a fuzzy number if and only if $[u]^r$ is compact convex subset of \mathbb{R} for each $r \in [0,1]$ and $[u]^1 \neq \emptyset$ [26]. Thus, if u is a fuzzy number, then $[u]^r = [\underline{u}(r), \bar{u}(r)]$, where $\underline{u}(r) = \min\{s: s \in [u]^r\}$ and $\bar{u}(r) = \max\{s: s \in [u]^r\}$ for each $r \in [0,1]$. The symbol $[u]^r$ is called the r -cut representation or parametric form of a fuzzy number u . We will let \mathbb{R}_F denote the set of fuzzy numbers on \mathbb{R} .

The question arises here is, if we have an interval-valued function $[\underline{z}(r), \bar{z}(r)]$ defined on $[0,1]$, then is there a fuzzy number u such that $[u(r)]^r = [\underline{z}(r), \bar{z}(r)]$. The next theorem characterizes fuzzy numbers through their r -cut representations.

Theorem 2.1. [26] Suppose that $\underline{u}: [0,1] \rightarrow \mathbb{R}$ and $\bar{u}: [0,1] \rightarrow \mathbb{R}$ satisfy the following conditions; first, \underline{u} is a bounded increasing function and \bar{u} is a bounded decreasing function with $\underline{u}(1) \leq \bar{u}(1)$; second, for each $k \in (0,1]$ \underline{u} and \bar{u} are left-hand continuous functions at $r = k$; third, \underline{u} and \bar{u} are right-hand continuous functions at $r = 0$. Then $u: \mathbb{R} \rightarrow [0,1]$ defined by

$$u(s) = \sup\{r: \underline{u}(r) \leq s \leq \bar{u}(r)\}, \quad (2.1)$$

is a fuzzy number with parameterization $[\underline{u}(r), \bar{u}(r)]$. Furthermore, if $u: \mathbb{R} \rightarrow [0,1]$ is a fuzzy number with parameterization $[\underline{u}(r), \bar{u}(r)]$, then the functions \underline{u} and \bar{u} satisfy the aforementioned conditions.

In general, we can represent an arbitrary fuzzy number u by an order pair of functions (\underline{u}, \bar{u}) which satisfy the requirements of Theorem 2.1. Frequently, we will write simply \underline{u}_r and \bar{u}_r instead of $\underline{u}(r)$ and $\bar{u}(r)$, respectively.

The metric structure on \mathbb{R}_F is given by $d_\infty: \mathbb{R}_F \times \mathbb{R}_F \rightarrow \mathbb{R}^+ \cup \{0\}$ such that $d_\infty(u, v) = \sup_{r \in [0,1]} d_H([u]^r, [v]^r)$ for arbitrary fuzzy numbers u and v , where d_H is the Hausdorff metric between $[u]^r$ and $[v]^r$. This metric is defined as $d_H([u]^r, [v]^r) = \inf\{\varepsilon: [u]^r \subset N([v]^r, \varepsilon), [v]^r \subset N([u]^r, \varepsilon)\} = \max\{|\underline{u}_r - \underline{v}_r|, |\bar{u}_r - \bar{v}_r|\}$, where the two set $N([u]^r, \varepsilon)$ and $N([v]^r, \varepsilon)$ are the ε -neighborhoods of $[u]^r$ and $[v]^r$, respectively. It is shown in [27] that (\mathbb{R}_F, d_∞) is a complete metric space.

Lemma 2.1. [27] For each $A, B, C, D \in \mathbb{R}_F$ with $\lambda \in \mathbb{R}$ the metric function d_∞ satisfies the following properties:

- i. $d_\infty(A + C, B + C) = d_\infty(A, B)$,
- ii. $d_\infty(A, B) \leq d_\infty(A, C) + d_\infty(C, B)$,
- iii. $d_\infty(A + C, B + D) \leq d_\infty(A, B) + d_\infty(C, D)$,
- iv. $d_\infty(\lambda A, \lambda B) = |\lambda| d_\infty(A, B)$.

For arithmetic operations on fuzzy numbers, the following results are well-known and follow from the theory of interval analysis. If u and v are two fuzzy number, then for each $r \in [0,1]$, we have; firstly, $[u + v]^r = [u]^r + [v]^r = [\underline{u}_r + \underline{v}_r, \bar{u}_r + \bar{v}_r]$; secondly, $[\lambda u]^r = \lambda[u]^r = [\min\{\lambda \underline{u}_r, \lambda \bar{u}_r\}, \max\{\lambda \underline{u}_r, \lambda \bar{u}_r\}]$; thirdly, $[uv]^r = [u]^r [v]^r = [\min\{\underline{u}_r \underline{v}_r, \underline{u}_r \bar{v}_r, \bar{u}_r \underline{v}_r, \bar{u}_r \bar{v}_r\}, \max\{\underline{u}_r \underline{v}_r, \underline{u}_r \bar{v}_r, \bar{u}_r \underline{v}_r, \bar{u}_r \bar{v}_r\}]$; fourthly, $u = v$ if and only if $[u]^r = [v]^r$ if and only if $\underline{u}_r = \underline{v}_r$ and $\bar{u}_r = \bar{v}_r$. In fact, the collection of all fuzzy number with aforementioned addition and scalar multiplication is a convex cone [28].

Let $u, v \in \mathbb{R}_F$. If there exists a $w \in \mathbb{R}_F$ such that $u = v + w$, then w is called the H-difference of u and v , denoted by $u \ominus v$. Here, the sign " \ominus " stands always for H-difference and let us remark that $u \ominus v \neq u + (-1)v$. Usually we denote $u + (-1)v$ by $u - v$, while $u \ominus v$ stands for the H-difference. It follows that Hukuhara differentiable function has increasing length of support [25]. To avoid this difficulty, we consider the following definition.

Definition 2.2. [8] Let $x: [0,1] \rightarrow \mathbb{R}_F$ and $t^* \in [0,1]$. We say that x is strongly generalized differentiable at t^* , if there exists an element $x'(t^*) \in \mathbb{R}_F$ such that either

- i. for all $h > 0$ sufficiently close to 0, the H-differences $x(t^* + h) \ominus x(t^*)$, $x(t^*) \ominus x(t^* - h)$ exist and
$$\lim_{h \rightarrow 0^+} \frac{x(t^* + h) \ominus x(t^*)}{h} = \lim_{h \rightarrow 0^+} \frac{x(t^*) \ominus x(t^* - h)}{h} = x'(t^*),$$
- ii. for all $h > 0$ sufficiently close to 0, the H-differences $x(t^*) \ominus x(t^* + h)$, $x(t^* - h) \ominus x(t^*)$ exist and
$$\lim_{h \rightarrow 0^+} \frac{x(t^*) \ominus x(t^* + h)}{-h} = \lim_{h \rightarrow 0^+} \frac{x(t^* - h) \ominus x(t^*)}{-h} = x'(t^*).$$

Here, the limit is taken in the metric space (\mathbb{R}_F, d_∞) and at the endpoints of $[0,1]$, we consider only one-sided derivatives. For customizing, in Definition 2.2, the first case corresponds to the H-derivative introduced in [28], so this differentiability concept is a generalization of the Hukuhara derivative.

Definition 2.3. [10] Let $x: [0,1] \rightarrow \mathbb{R}_F$. We say that x is (1)-differentiable on $[0,1]$ if x is differentiable in the sense (i) of Definition 2.2 and its derivative is denoted $D_1^1 x$. Similarly, we say that x is (2)-differentiable on $[0,1]$ if x is differentiable in the sense (ii) of Definition 2.2 and its derivative is denoted $D_2^1 x$.

The subsequent theorems show us a way to translate a fuzzy VIDE into a system of crisp VIDEs without the need to consider the fuzzy setting approach. Anyhow, these theorems have many uses in the applied mathematics and the numerical analysis fields.

Theorem 2.2. [10] Let $x: [0,1] \rightarrow \mathbb{R}_F$ and put $[x(t)]^r = [\underline{x}_r(t), \bar{x}_r(t)]$ for each $r \in [0,1]$.

- i. if x is (1)-differentiable, then \underline{x}_r and \bar{x}_r are differentiable functions on $[0,1]$ and $[D_1^1 x(t)]^r = [\underline{x}'_r(t), \bar{x}'_r(t)]$,
- ii. if x is (2)-differentiable, then \underline{x}_r and \bar{x}_r are differentiable functions on $[0,1]$ and $[D_2^1 x(t)]^r = [\bar{x}'_r(t), \underline{x}'_r(t)]$.

Next, we introduce the definitions for second fuzzy derivatives based on the selection of derivative type in each step of differentiation. For a given fuzzy-valued function x , we have two possibilities according to Definition 2.3 in order to obtain the derivative of x as follows: $D_1^1 x(t)$ and $D_2^1 x(t)$. Anyhow, for each of these two derivative, we have again two possibilities of derivatives: $D_1^1(D_1^1 x(t))$, $D_2^1(D_1^1 x(t))$ and $D_1^1(D_2^1 x(t))$, $D_2^1(D_2^1 x(t))$, respectively.

Definition 2.4. [29] Let $x: [0,1] \rightarrow \mathbb{R}_F$ and $n, m \in \{1,2\}$, we say that x is (n, m) -differentiable on $[0,1]$ if $D_2^1 x$ exist and its (m) -differentiable. The second derivatives of x are denoted by $D_{n,m}^2 x$.

Theorem 2.3. [29] Let $D_1^1 x: [0,1] \rightarrow \mathbb{R}_F$ or $D_2^1 x: [0,1] \rightarrow \mathbb{R}_F$, where $[x(t)]^r = [\underline{x}_r(t), \bar{x}_r(t)]$ for each $r \in [0,1]$:

- i. if $D_1^1 x$ is (1)-differentiable, then \underline{x}'_r and \bar{x}'_r are differentiable functions on $[0,1]$ and $[D_{1,1}^2 x(t)]^r = [\underline{x}''_r(t), \bar{x}''_r(t)]$,
- ii. if $D_1^1 x$ is (2)-differentiable, then \underline{x}'_r and \bar{x}'_r are differentiable functions on $[0,1]$ and $[D_{1,2}^2 x(t)]^r = [\bar{x}''_r(t), \underline{x}''_r(t)]$,
- iii. if $D_2^1 x$ is (1)-differentiable, then \underline{x}'_r and \bar{x}'_r are differentiable functions on $[0,1]$ and $[D_{2,1}^2 x(t)]^r = [\bar{x}''_r(t), \underline{x}''_r(t)]$,
- iv. if $D_2^1 x$ is (2)-differentiable, then \underline{x}'_r and \bar{x}'_r are differentiable functions on $[0,1]$ and $[D_{2,2}^2 x(t)]^r = [\underline{x}''_r(t), \bar{x}''_r(t)]$.

A fuzzy-valued function $x: [0,1] \rightarrow \mathbb{R}_F$ is called continuous at a point $t^* \in [0,1]$ provided for arbitrary fixed $\varepsilon > 0$, there exists an $\delta > 0$ such that $d_\infty(x(t), x(t^*)) < \varepsilon$ whenever $|t^* - t| < \delta$ for each $t \in [0,1]$. We say that x is continuous on $[0,1]$ if x is continuous at each $t^* \in [0,1]$ such that the continuity is one-sided at endpoints 0 and 1.

In order to complete the expert results about the fuzzy calculus theory we finalize the present section by some preliminary information about the fuzzy integral. Following [26], we define the integral of a fuzzy-valued function using the Riemann integral concept.

Definition 2.5. [26] Suppose that $x: [0,1] \rightarrow \mathbb{R}_F$, for each partition $\wp = \{t_0^*, t_1^*, \dots, t_n^*\}$ of $[0,1]$ and for arbitrary points $\xi_i \in [t_{i-1}^*, t_i^*]$, $1 \leq i \leq n$, let $\mathfrak{R}_\wp = \sum_{i=1}^n x(\xi_i)(t_i^* - t_{i-1}^*)$ and $\Delta = \max_{1 \leq i \leq n} |t_i^* - t_{i-1}^*|$. Then the definite integral of $x(t)$ over $[t_0, t_0 + a]$ is defined by $\int_0^1 x(t)dt = \lim_{\Delta \rightarrow 0} \mathfrak{R}_\wp$ provided the limit exists in the metric space (\mathbb{R}_F, d_∞) .

Theorem 2.4. [26] Let $x: [0,1] \rightarrow \mathbb{R}_F$ be continuous fuzzy-valued function and put $[x(t)]^r = [\underline{x}_r(t), \bar{x}_r(t)]$ for each $r \in [0,1]$. Then $\int_0^1 x(t)dt$ exist, belong to \mathbb{R}_F , \underline{x}_r and \bar{x}_r are integrable functions on $[0,1]$, and $\left[\int_0^1 x(t)dt\right]^r = \left[\int_0^1 \underline{x}_r(t)dt, \int_0^1 \bar{x}_r(t)dt\right]$.

Lemma 2.2. [30] Let $x, y: [0,1] \rightarrow \mathbb{R}_F$ be integrable fuzzy-valued functions and $\lambda \in \mathbb{R}$. Then the following are hold:

- i. $d_\infty(x(t), y(t))$ is integrable,
- ii. $d_\infty\left(\int_0^1 x(t)dt, \int_0^1 y(t)dt\right) \leq \int_0^1 d_\infty(x(t), y(t))dt$,
- iii. $\int_0^1 \lambda x(t)dt = \lambda \int_0^1 x(t)dt$,
- iv. $\int_0^1 (x(t) + y(t))dt = \int_0^1 x(t)dt + \int_0^1 y(t)dt$.

It should be noted that the fuzzy integral can be also defined using the Lebesgue-type approach [25] or the Henstock-type approach [31]. However, if x is continuous function, then all approaches yield the same value and results. Moreover, the representation of the fuzzy integral using Definition 2.5 is more convenient for numerical calculations and computational mathematics. The reader is kindly requested to go through [25,26,30-32] in order to know more details about the fuzzy integral, including its history and kinds, its properties and modification for use, its applications and characteristics, its justification and conditions for use, and its mathematical and geometric properties.

3. Algorithm of solving fuzzy VIDEs

The topic of fuzzy VIDEs are one of the most important modern mathematical fields that result from modeling of uncertain physical, engineering, and economical problems. In this section, we study fuzzy VIDEs using the concept of strongly generalized differentiability in which fuzzy equation is converted into equivalent system of crisp equations for each type of differentiability. Furthermore, we present an algorithm to solve the new system which consists of four crisp VIDEs.

Problem formulation is normally the most important part of the process. It is the determination of r -cut representation form of nonlinear terms f, g , the selection of the differentiability type, and the separation of fuzzy initial conditions. Next, fuzzy VIDE (1.1) and (1.2) is first formulated as an crisp set of VIDEs subject to crisp set of initial conditions, after that, a new discretized form of fuzzy VIDE (1.1) and (1.2) is presented. Anyhow, by considering the parametric form for both sides of fuzzy VIDE (1.1) and (1.2), one can write

$$[D_{n,m}^2 x(t)]^r = [f(t, x(t), D_n^1(t))]^r + \int_0^t [g(t, \tau, x(\tau), D_n^1(\tau))]^r d\tau, \quad (3.1)$$

subject to the crisp initial conditions

$$[x(0)]^r = [\alpha]^r, [D_n^1(0)]^r = [\beta]^r, \quad (3.2)$$

in which the endpoints functions of $[f(t, x(t), D_n^1(t))]^r$ and $[g(t, \tau, x(\tau), D_n^1(\tau))]^r$ are given, respectively, as follows:

$$\begin{aligned} [f(t, x(t), D_n^1(t))]^r &= f(t, [x(t)]^r, [D_n^1(t)]^r) = [\underline{f}_r(t, [x(t)]^r, [D_n^1(t)]^r), \bar{f}_r(t, [x(t)]^r, [D_n^1(t)]^r)] \\ &= [f_{1,r}(t, \underline{x}_r(t), \bar{x}_r(t), \underline{x}'_r(t), \bar{x}'_r(t)), f_{2,r}(t, \underline{x}_r(t), \bar{x}_r(t), \underline{x}'_r(t), \bar{x}'_r(t))], \end{aligned} \quad (3.3)$$

$$\begin{aligned} [g(t, \tau, x(\tau), D_n^1(\tau))]^r &= g(t, \tau, [x(\tau)]^r, [D_n^1(\tau)]^r) = [\underline{g}_r(t, \tau, [x(\tau)]^r, [D_n^1(\tau)]^r), \bar{g}_r(t, \tau, [x(\tau)]^r, [D_n^1(\tau)]^r)] \\ &= [g_{1,r}(t, \tau, \underline{x}_r(\tau), \bar{x}_r(\tau), \underline{x}'_r(\tau), \bar{x}'_r(\tau)), g_{2,r}(t, \tau, \underline{x}_r(\tau), \bar{x}_r(\tau), \underline{x}'_r(\tau), \bar{x}'_r(\tau))]. \end{aligned} \quad (3.4)$$

Definition 3.1. Let $x: [0,1] \rightarrow \mathbb{R}_F$ and $(n, m) \in \{1,2\}$, we say that x is a (n, m) -solution for fuzzy VIDE (1.1) and (1.2) on $[0,1]$, if $D_n^1 x$ and $D_{n,m}^2 x$ exist on $[0,1]$ and $D_{n,m}^2 x(t) = f(t, x(t), D_n^1 x(t)) + \int_0^t g(t, \tau, x(\tau), D_n^1 x(\tau)) d\tau$ with $x(0) = \alpha, x'(0) = \beta$.

The object of the next algorithm is to implement a procedure to solve fuzzy VIDE in parametric form in term of its r -cut representation. To do so, let x be a (n, m) -solution, utilizing Theorems 2.2 and 2.3, and considering fuzzy VIDE (1.1) and (1.2), we can thus translate it into system of crisp VIDEs, hereafter, called corresponding (n, m) -system. Anyhow, four IDEs systems are possible as given in the follow algorithm.

Algorithm 3.1: To find (n, m) -solution of fuzzy VIDE (1.1) and (1.2), we discuss the following four cases:

Input: The independent interval $[0,1]$, the unit truth interval $[0,1]$, and the fuzzy numbers α, β .

Output: The (n, m) -differentiable solution of VIDE (1.1) and (1.2) on $[0,1]$.

Step 1: Set $[f(t, x(t), D_n^1(t))]^r = [\underline{f}_r(t, [x(t)]^r, [D_n^1(t)]^r), \bar{f}_r(t, [x(t)]^r, [D_n^1(t)]^r)]$,

$$\text{Set } [g(t, \tau, x(\tau), D_n^1(\tau))]^r = [\underline{g}_r(t, \tau, [x(\tau)]^r, [D_n^1(\tau)]^r), \bar{g}_r(t, \tau, [x(\tau)]^r, [D_n^1(\tau)]^r)],$$

$$\text{Set } [\alpha]^r = [\underline{\alpha}_r, \bar{\alpha}_r] \text{ and } [\beta]^r = [\underline{\beta}_r, \bar{\beta}_r].$$

Case I. If $x(t)$ is (1,1)-differentiable, then use $[D_1^1 x(t)]^r$ and $[D_{1,1}^2 x(t)]^r$, and solving fuzzy VIDE (1.1) and (1.2) translates into the following (1,1)-system:

$$\begin{aligned} \underline{x}_r''(t) &= \underline{f}_r(t, [x(t)]^r, [D_1^1(t)]^r) + \int_0^t \underline{g}_r(t, \tau, [x(\tau)]^r, [D_1^1(\tau)]^r) d\tau, \\ \bar{x}_r''(t) &= \bar{f}_r(t, [x(t)]^r, [D_1^1(t)]^r) + \int_0^t \bar{g}_r(t, \tau, [x(\tau)]^r, [D_1^1(\tau)]^r) d\tau, \end{aligned} \quad (3.5)$$

subject to the crisp initial conditions

$$\underline{x}_r(0) = \underline{\alpha}_r, \bar{x}_r(0) = \bar{\alpha}_r, \underline{x}'_r(0) = \underline{\beta}_r, \bar{x}'_r(0) = \bar{\beta}_r. \quad (3.6)$$

Case II. If $x(t)$ is (1,2)-differentiable, then use $[D_1^1 x(t)]^r$ and $[D_{1,2}^2 x(t)]^r$, and solving fuzzy VIDE (1.1) and (1.2) translates into the following (1,2)-system:

$$\begin{aligned} \underline{x}_r''(t) &= \bar{f}_r(t, [x(t)]^r, [D_1^1(t)]^r) + \int_0^t \bar{g}_r(t, \tau, [x(\tau)]^r, [D_1^1(\tau)]^r) d\tau, \\ \bar{x}_r''(t) &= \underline{f}_r(t, [x(t)]^r, [D_1^1(t)]^r) + \int_0^t \underline{g}_r(t, \tau, [x(\tau)]^r, [D_1^1(\tau)]^r) d\tau, \end{aligned} \quad (3.7)$$

subject to the crisp initial conditions

$$\underline{x}_r(0) = \underline{\sigma}_r, \bar{x}_r(0) = \bar{\sigma}_r, \underline{x}'_r(0) = \underline{\beta}_r, \bar{x}'_r(0) = \bar{\beta}_r. \quad (3.8)$$

Case III. If $x(t)$ is (2,1)-differentiable, then use $[D_2^1 x(t)]^r$ and $[D_{2,1}^2 x(t)]^r$, and solving fuzzy VIDE (1.1) and (1.2) translates into the following (2,1)-system:

$$\begin{aligned} \underline{x}''_r(t) &= \underline{f}_r(t, [x(t)]^r, [D_2^1(t)]^r) + \int_0^t \underline{g}_r(t, \tau, [x(\tau)]^r, [D_2^1(\tau)]^r) d\tau, \\ \bar{x}''_r(t) &= \bar{f}_r(t, [x(t)]^r, [D_2^1(t)]^r) + \int_0^t \bar{g}_r(t, \tau, [x(\tau)]^r, [D_2^1(\tau)]^r) d\tau, \end{aligned} \quad (3.9)$$

subject to the crisp initial conditions

$$\underline{x}_r(0) = \underline{\sigma}_r, \bar{x}_r(0) = \bar{\sigma}_r, \underline{x}'_r(0) = \underline{\beta}_r, \bar{x}'_r(0) = \bar{\beta}_r. \quad (3.10)$$

Case IV. If $x(t)$ is (2,2)-differentiable, then use $[D_2^1 x(t)]^r$ and $[D_{2,2}^2 x(t)]^r$, and solving fuzzy VIDE (1.1) and (1.2) translates into the following (2,2)-system:

$$\begin{aligned} \underline{x}''_r(t) &= \underline{f}_r(t, [x(t)]^r, [D_2^1(t)]^r) + \int_0^t \underline{g}_r(t, \tau, [x(\tau)]^r, [D_2^1(\tau)]^r) d\tau, \\ \bar{x}''_r(t) &= \bar{f}_r(t, [x(t)]^r, [D_2^1(t)]^r) + \int_0^t \bar{g}_r(t, \tau, [x(\tau)]^r, [D_2^1(\tau)]^r) d\tau, \end{aligned} \quad (3.11)$$

subject to the crisp initial conditions

$$\underline{x}_r(0) = \underline{\sigma}_r, \bar{x}_r(0) = \bar{\sigma}_r, \underline{x}'_r(0) = \underline{\beta}_r, \bar{x}'_r(0) = \bar{\beta}_r. \quad (3.12)$$

Step 2: Solve the obtained (n, m) -system of crisp VIDEs for $\underline{x}_r(t)$ and $\bar{x}_r(t)$.

Step 3: Ensure that $x(t)$ is (n, m) -solution on the interval $[0, 1]$.

Step 4: Construct a (n, m) -differentiable solution such that $x(t) = [\underline{x}_r(t), \bar{x}_r(t)]$.

Step 5: Stop.

Sometimes, we can't decompose the membership function of the fuzzy solution $x(t)$ as a function defined on \mathbb{R} for each $t \in [0, 1]$. Then, using identity (2.1) we can leave a (n, m) -solution in term of its r -cut representation form. To summarize the evolution process; our strategy for solving fuzzy VIDE (1.1) and (1.2) is based on the selection of derivatives type in the given fuzzy VIDE. The first step is to choose the type of solution and translate fuzzy VIDE into the corresponding system of equations with coupled crisp VIDE for each type of differentiability. The second step is to solve the obtained VIDEs system, while aim of the third step is to use the representation Theorem 2.1 in order to construct the fuzzy solution.

Next, we construct a procedure based on Algorithm 3.1 to obtain the solutions of fuzzy VIDE (1.1) and (1.2). Here, we discussing and considering the (1,1)-differentiability in Case I of Algorithm 3.1 only; since the same procedure can be applied directly for the remaining cases. Anyhow, without the loss of generality and for simplicity, we assume that the function g takes the form $g(t, \tau, x(\tau), x'(\tau)) = k(t, \tau)G(x(\tau), x'(\tau))$. So, based on this, fuzzy VIDE (1.1) can be written in a new discretized form as $x''(t) = f(t, x(t), x'(t)) + \int_0^t k(t, \tau)G(x(\tau), x'(\tau))d\tau$, in which the r -cut representation form of $G(x(\tau), x'(\tau))$ should be of the form

$$[G(x(\tau), x'(\tau))]^r = [\underline{G}_r([x(\tau)]^r, [D_1^1(\tau)]^r), \bar{G}_r([x(\tau)]^r, [D_1^1(\tau)]^r)]. \quad (3.13)$$

In order to design a scheme for solving fuzzy VIDE (1.1) and (1.2), we first replace it by the following equivalent crisp system of VIDEs:

$$\begin{aligned} \underline{x}''_r(t) &= \underline{f}_r(t, [x(t)]^r, [D_1^1(t)]^r) + \int_0^t K_1(t, \tau, [x(\tau)]^r, [D_1^1(\tau)]^r) d\tau, \\ \bar{x}''_r(t) &= \bar{f}_r(t, [x(t)]^r, [D_1^1(t)]^r) + \int_0^t K_2(t, \tau, [x(\tau)]^r, [D_1^1(\tau)]^r) d\tau, \end{aligned} \quad (3.14)$$

subject to the crisp initial conditions

$$\underline{x}_r(0) = \underline{\sigma}_r, \bar{x}_r(0) = \bar{\sigma}_r, \underline{x}'_r(0) = \underline{\beta}_r, \bar{x}'_r(0) = \bar{\beta}_r, \quad (3.15)$$

where the new functions K_1, K_2 are given, respectively, as

$$\begin{aligned}
K_1(t, \tau, [x(\tau)]^r, [D_1^1(\tau)]^r) &= \begin{cases} k(t, \tau) \underline{G}_r([x(\tau)]^r, [D_1^1(\tau)]^r), & k(t, \tau) \geq 0, \\ k(t, \tau) \overline{G}_r([x(\tau)]^r, [D_1^1(\tau)]^r), & k(t, \tau) < 0, \end{cases} \\
K_2(t, \tau, [x(\tau)]^r, [D_1^1(\tau)]^r) &= \begin{cases} k(t, \tau) \overline{G}_r([x(\tau)]^r, [D_1^1(\tau)]^r), & k(t, \tau) \geq 0, \\ k(t, \tau) \underline{G}_r([x(\tau)]^r, [D_1^1(\tau)]^r), & k(t, \tau) < 0. \end{cases}
\end{aligned} \quad (3.16)$$

Prior to applying the analytic or the numerical methods for solving system of crisp VIDEs (3.14) and (3.15), we suppose that the kernel function $k(t, \tau)$ is nonnegative for $0 \leq \tau \leq c$ and nonpositive for $c \leq \tau \leq t$. Therefore, system of crisp VIDEs (3.14) can be translated again into the following form:

$$\begin{aligned}
\underline{x}'(t) &= \underline{f}_r(t, [x(t)]^r, [D_1^1(t)]^r) + \int_0^c k(t, \tau) \underline{G}_r([x(\tau)]^r, [D_1^1(\tau)]^r) d\tau + \int_c^t k(t, \tau) \overline{G}_r([x(\tau)]^r, [D_1^1(\tau)]^r) d\tau, \\
\overline{x}'_r(t) &= \overline{f}_r(t, [x(t)]^r, [D_1^1(t)]^r) + \int_0^c k(t, \tau) \overline{G}_r([x(\tau)]^r, [D_1^1(\tau)]^r) d\tau + \int_c^t k(t, \tau) \underline{G}_r([x(\tau)]^r, [D_1^1(\tau)]^r) d\tau.
\end{aligned} \quad (3.17)$$

4. Existence and uniqueness of four fuzzy solutions

It is worth stating that in many cases, since fuzzy VIDEs are often derived from problems in physical world, existence and uniqueness are often obvious for physical reasons. Notwithstanding this, a mathematical statement about existence and uniqueness is worthwhile. Uniqueness would be of importance if, for instance, we wished to approximate the solutions. If two solutions passed through a point, then successive approximations could very well jump from one solution to the other with misleading consequences.

Denote by $C([0,1], \mathbb{R}_{\mathcal{F}})$ the set of all continuous mapping from $[0,1]$ to $\mathbb{R}_{\mathcal{F}}$. The supremum metric on $C([0,1], \mathbb{R}_{\mathcal{F}})$ is defined by $d: C([0,1], \mathbb{R}_{\mathcal{F}}) \times C([0,1], \mathbb{R}_{\mathcal{F}}) \rightarrow \mathbb{R}^+ \cup \{0\}$ such that $d(x, y) = \sup_{t \in [0,1]} (d_{\infty}(x(t), y(t)) e^{-\eta t})$ for each $x, y \in C([0,1], \mathbb{R}_{\mathcal{F}})$, where $\eta \in \mathbb{R}$ is fixed. It is shown in [33] that $(C([0,1], \mathbb{R}_{\mathcal{F}}), d)$ is a complete metric space. On the other aspect as well, by $C^1([0,1], \mathbb{R}_{\mathcal{F}})$, we denote the set of all continuous mapping from $[0,1]$ to $\mathbb{R}_{\mathcal{F}}$ such that $x': [0,1] \rightarrow \mathbb{R}_{\mathcal{F}}$ exists as a continuous function. Anyhow, for $C^1([0,1], \mathbb{R}_{\mathcal{F}})$, we define the distance function $D: C^1([0,1], \mathbb{R}_{\mathcal{F}}) \times C^1([0,1], \mathbb{R}_{\mathcal{F}}) \rightarrow \mathbb{R}^+ \cup \{0\}$ such that $D(x, y) = d(x, y) + d(x', y')$. Indeed, it is shown in [33] that $(C^1([0,1], \mathbb{R}_{\mathcal{F}}), d)$ is also a complete metric space.

The following lemma transforms a fuzzy VIDE into four fuzzy Volterra integral equations. Here the equivalence between equations means that any solution of an equation is a solution too for the other one with respect to the differentiability type used.

Lemma 4.1. The fuzzy VIDE (1.1) and (1.2), where $f: [0,1] \times \mathbb{R}_{\mathcal{F}}^2 \rightarrow \mathbb{R}_{\mathcal{F}}$ and $g: [0,1]^2 \times \mathbb{R}_{\mathcal{F}}^2 \rightarrow \mathbb{R}_{\mathcal{F}}$ are supposed to be continuous is equivalent to one of the following fuzzy Volterra integral equations:

- $x(t) = \alpha + \beta t + \int_0^t (\int_0^z f(s, x(s), x'(s)) ds) dz + \int_0^t (\int_0^z (\int_0^s g(s, \tau, x(\tau), x'(\tau)) d\tau) ds) dz$, when x is (1,1)-differentiable,
- $x(t) = \alpha + \beta t \ominus (-1) \int_0^t (\int_0^z f(s, x(s), x'(s)) ds) dz \ominus (-1) \int_0^t (\int_0^z (\int_0^s g(s, \tau, x(\tau), x'(\tau)) d\tau) ds) dz$, when x is (1,2)-differentiable,
- $x(t) = \alpha \ominus (-1) (\beta t + \int_0^t (\int_0^z f(s, x(s), x'(s)) ds) dz + \int_0^t (\int_0^z (\int_0^s g(s, \tau, x(\tau), x'(\tau)) d\tau) ds) dz)$, when x is (2,1)-differentiable,
- $x(t) = \alpha \ominus (-1) (\beta t \ominus (-1) \int_0^t (\int_0^z f(s, x(s), x'(s)) ds) dz \ominus (-1) \int_0^t (\int_0^z (\int_0^s g(s, \tau, x(\tau), x'(\tau)) d\tau) ds) dz)$, when x is (2,2)-differentiable.

Proof. Since f and g are continuous functions; so they are integrable. Now, we determine the equivalent integral forms of fuzzy VIDE (1.1) and (1.2) under each type of strongly generalized differentiability as follows. Firstly, let us consider x is (1,1)-differentiable, then the equivalent integral form of fuzzy VIDE (1.1) and (1.2) can be written by implementation of fuzzy integration on both sides of the original equation two times as follows:

$$x'(z) = x'(0) + \int_0^z f(s, x(s), x'(s)) ds + \int_0^z \left(\int_0^s g(s, \tau, x(\tau), x'(\tau)) d\tau \right) ds, \quad (4.1)$$

for $z \in [0,1]$ and again for $t \in [0,1]$, one can write

$$x(t) = x(0) + x'(0)t + \int_0^t \left(\int_0^z f(s, x(s), x'(s)) ds \right) dz + \int_0^t \left(\int_0^z \left(\int_0^s g(s, \tau, x(\tau), x'(\tau)) d\tau \right) ds \right) dz. \quad (4.2)$$

Secondly, let us consider x is (1,2)-differentiable, then the equivalent integral form of fuzzy VIDE (1.1) and (1.2) can be written by implementation of fuzzy integration on both sides of the original equation two times as

$$x'(z) = x'(0) \ominus (-1) \int_0^z f(s, x(s), x'(s)) ds \ominus (-1) \int_0^z \left(\int_0^s g(s, \tau, \xi(\tau), x'(\tau)) d\tau \right) ds, \quad (4.3)$$

for $z \in [0,1]$, again for $t \in [0,1]$, we must have

$$x(t) = x(0) + \left(x'(0)t \ominus (-1) \int_0^t \left(\int_0^z f(s, x(s), x'(s)) ds \right) dz \right. \\ \left. \ominus (-1) \int_0^t \left(\int_0^z \left(\int_0^s g(s, \tau, \xi(\tau), x'(\tau)) d\tau \right) ds \right) dz \right). \quad (4.4)$$

Thirdly, if x is (2,1)-differentiable, then the equivalent form of fuzzy VIDE (1.1) and (1.2) can be written as

$$x'(z) = x'(0) + \int_0^z f(s, x(s), x'(s)) ds + \int_0^z \left(\int_0^s g(s, \tau, \xi(\tau), x'(\tau)) d\tau \right) ds, \quad (4.5)$$

for $z \in [0,1]$, which is equivalent for $t \in [0,1]$ to the integral equation of the form

$$x(t) = x(0) \ominus (-1) \left(x'(0)t + \int_0^t \left(\int_0^z f(s, x(s), x'(s)) ds \right) dz \right. \\ \left. + \int_0^t \left(\int_0^z \left(\int_0^s g(s, \tau, \xi(\tau), x'(\tau)) d\tau \right) ds \right) dz \right). \quad (4.6)$$

Fourthly, since x is (2,2)-differentiable, then one can write

$$x'(z) = x'(0) \ominus (-1) \int_0^z f(s, x(s), x'(s)) ds \ominus (-1) \int_0^z \left(\int_0^s g(s, \tau, \xi(\tau), x'(\tau)) d\tau \right) ds, \quad (4.7)$$

for $z \in [0,1]$ and for $t \in [0,1]$, we can also write

$$x(t) = x(0) \ominus (-1) \left(x'(0)t \ominus (-1) \int_0^t \left(\int_0^z f(s, x(s), x'(s)) ds \right) dz \right. \\ \left. \ominus (-1) \int_0^t \left(\int_0^z \left(\int_0^s g(s, \tau, \xi(\tau), x'(\tau)) d\tau \right) ds \right) dz \right), \quad (4.8)$$

which is equivalent to the form of part (iv).

In mathematics, the Banach fixed-point theorem; also known as the contraction mapping theorem; is an important tool in the theory of metric spaces; it guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces, and provides a constructive method to find those fixed points. The following results (Definition 4.1 and Theorem 4.1) were collected from [34].

Definition 4.1. Let (X, d_X) be a metric space. A mapping $G: X \rightarrow X$ is said to be a contraction mapping, if there exist a positive real number ρ with $\rho < 1$ such that $d_X(G(x), G(y)) \leq \rho d_X(x, y)$ for each $x, y \in X$.

We observe that, applying G to each of the two points of the space contracts the distance between them; obviously G is continuous. Anyhow, a point $x \in X$ is called a fixed point of the mapping $G: X \rightarrow X$ if $G(x) = x$. Next, we present the Banach fixed-point theorem.

Theorem 4.1. Any contraction mapping G of a nonempty complete metric space (X, d_X) into itself has a unique fixed point.

Lemma 4.2. The real-valued functions $\nu, \omega, \mu: [0,1] \rightarrow \mathbb{R}$ with $\eta \in \mathbb{R}$ represented by

$$\nu(t) = \frac{1}{\eta} (1 - e^{-\eta t}), \\ \omega(t) = \frac{1}{\eta^2} (1 - e^{-\eta t} - \eta t e^{-\eta t}), \\ \mu(t) = \frac{1}{\eta^3} \left(1 - e^{-\eta t} - \eta t e^{-\eta t} - \frac{\eta^2}{2} t^2 e^{-\eta t} \right), \quad (4.9)$$

are continuous nondecreasing functions on $[0,1]$. Furthermore, $\nu(1) = \sup_{t \in [0,1]} \nu(t)$, $\omega(1) = \sup_{t \in [0,1]} \omega(t)$, $\mu(1) = \sup_{t \in [0,1]} \mu(t)$, and $\lim_{\eta \rightarrow +\infty} (\nu(1) + \omega(1) + \mu(1)) = 0$.

Proof. Clearly ν, ω, μ are continuous functions on $[0,1]$ for each $\eta \in \mathbb{R}$. Since $\nu'(t) = e^{-\eta t} > 0$, $\omega'(t) = te^{-\eta t} > 0$, and $\mu'(t) = \frac{1}{2}t^2e^{-\eta t} > 0$ for each $t \in [0,1]$ and $\eta \in \mathbb{R}$; thus, ν, ω, μ are nondecreasing functions. As a result one can conclude that $\nu(1) = \sup_{t \in [0,1]} \nu(t)$, $\omega(1) = \sup_{t \in [0,1]} \omega(t)$, and $\mu(1) = \sup_{t \in [0,1]} \mu(t)$. On the other aspect as well, using the limit functions techniques it yields that

$$\begin{aligned} & \lim_{\eta \rightarrow +\infty} (\nu(1) + \omega(1) + \mu(1)) \\ &= \lim_{\eta \rightarrow +\infty} \left(\frac{1}{\eta} (1 - e^{-\eta t}) + \frac{1}{\eta^2} (1 - e^{-\eta t} - \eta t e^{-\eta t}) + \frac{1}{\eta^3} \left(1 - e^{-\eta t} - \eta t e^{-\eta t} - \frac{\eta^2}{2} t^2 e^{-\eta t} \right) \right) \\ &= 0. \end{aligned} \quad (4.10)$$

It should be mention here that Lemma 4.2 guarantees the existence of a unique fixed point for the next theorem. In other word, an existence of a unique solution for fuzzy VIDE (1.1) and (1.2) for each type of differentiability.

Throughout this paper, we will try to give the results of the all theorems; however, in some cases we will switch between the results obtained for the four type of differentiability in order not to increase the length of the paper without the loss of generality for the remaining results. Actually, in the same manner, we can employ the same technique to construct the proof for the omitted cases.

Theorem 4.2. Let $f: [0,1] \times \mathbb{R}_{\mathcal{F}}^2 \rightarrow \mathbb{R}_{\mathcal{F}}$ and $g: [0,1]^2 \times \mathbb{R}_{\mathcal{F}}^2 \rightarrow \mathbb{R}_{\mathcal{F}}$ are continuous fuzzy-valued functions. If there exists $K_1, K_2, L_1, L_2 > 0$ such that

$$\begin{aligned} d_{\infty}(f(t, \xi_1(t), \xi_2(t)), f(t, \zeta_1(t), \zeta_2(t))) &\leq K_1 d_{\infty}(\xi_1(t), \zeta_1(t)) + K_2 d_{\infty}(\xi_2(t), \zeta_2(t)), \\ d_{\infty}(g(t, \tau, \xi_1(\tau), \xi_2(\tau)), g(s, \tau, \zeta_1(\tau), \zeta_2(\tau))) &\leq L_1 d_{\infty}(\xi_1(\tau), \zeta_1(\tau)) + L_2 d_{\infty}(\xi_2(\tau), \zeta_2(\tau)), \end{aligned} \quad (4.11)$$

for each $t, \tau \in [0,1]$ and $\xi_1(\tau), \xi_2(\tau), \zeta_1(t), \zeta_2(t) \in \mathbb{R}_{\mathcal{F}}$. Then, the fuzzy VIDE (1.1) and (1.2) has four unique solutions on $[0,1]$ for each type of differentiability.

Proof. Without the loss of generality, we consider the (1,1)-differentiability only; actually, in the same manner, we can employ the same technique for the remaining types. For each $\xi(t) \in \mathbb{R}_{\mathcal{F}}$ and $t \in [0,1]$ define the operator $G\xi$ and $(G\xi)'$, respectively, as follows:

$$\begin{aligned} (G\xi)(t) &= \alpha + \beta t + \int_0^t \left(\int_0^z f(s, \xi(s), \xi'(s)) ds \right) dz + \int_0^t \left(\int_0^z \left(\int_0^s g(s, \tau, \xi(\tau), \xi'(\tau)) d\tau \right) ds \right) dz, \\ (G\xi)'(t) &= \beta + \int_0^t f(s, \xi(s), \xi'(s)) ds + \int_0^t \left(\int_0^s g(s, \tau, \xi(\tau), \xi'(\tau)) d\tau \right) ds. \end{aligned} \quad (4.12)$$

Thus, $G\xi: [0,1] \rightarrow \mathbb{R}_{\mathcal{F}}$ is continuous and $G: C^1([t_0, t_0 + a], \mathbb{R}_{\mathcal{F}}) \rightarrow C^1([t_0, t_0 + a], \mathbb{R}_{\mathcal{F}})$. Now, we are going to show that the operator $G\xi$ satisfies the hypothesis of the Banach-fixed point theorem. For each $\xi, \zeta \in C^1([t_0, t_0 + a], \mathbb{R}_{\mathcal{F}})$, we have

$$\begin{aligned} D(G\xi, G\zeta) &= d(G\xi, G\zeta) + d((G\xi)', (G\zeta)') \\ &= \sup_{t \in [0,1]} (d_{\infty}((G\xi)(t), (G\zeta)(t))e^{-\eta t}) + \sup_{t \in [0,1]} (d_{\infty}((G\xi)'(t), (G\zeta)'(t))e^{-\eta t}) \\ &= \sup_{t \in [0,1]} \left\{ d_{\infty} \left(\alpha + \beta t + \int_0^t \left(\int_0^z f(s, \xi(s), \xi'(s)) ds \right) dz + \int_0^t \left(\int_0^z \left(\int_0^s g(s, \tau, \xi(\tau), \xi'(\tau)) d\tau \right) ds \right) dz, \alpha \right. \right. \\ &\quad \left. \left. + \beta t + \int_0^t \left(\int_0^z f(s, \zeta(s), \zeta'(s)) ds \right) dz \right. \right. \\ &\quad \left. \left. + \int_0^t \left(\int_0^z \left(\int_0^s g(s, \tau, \zeta(\tau), \zeta'(\tau)) d\tau \right) ds \right) dz \right) e^{-\eta t} \right\} \\ &\quad + \sup_{t \in [0,1]} \left\{ d_{\infty} \left(\beta + \int_0^t f(s, \xi(s), \xi'(s)) ds + \int_0^t \left(\int_0^s g(s, \tau, \xi(\tau), \xi'(\tau)) d\tau \right) ds, \beta \right. \right. \\ &\quad \left. \left. + \int_0^t f(s, \zeta(s), \zeta'(s)) ds + \int_0^t \left(\int_0^s g(s, \tau, \zeta(\tau), \zeta'(\tau)) d\tau \right) ds \right) e^{-\eta t} \right\} \end{aligned} \quad (4.13)$$

$$\begin{aligned}
&= \sup_{t \in [0,1]} \left\{ d_{\infty} \left(\int_0^t \left(\int_0^z f(s, \xi(s), \xi'(s)) ds \right) dz \right. \right. \\
&\quad \left. \left. + \int_0^t \left(\int_0^z \left(\int_0^s g(s, \tau, \xi(\tau), \xi'(\tau)) d\tau \right) ds \right) dz, \int_0^t \left(\int_0^z f(s, \zeta(s), \zeta'(s)) ds \right) dz \right. \right. \\
&\quad \left. \left. + \int_0^t \left(\int_0^z \left(\int_0^s g(s, \tau, \zeta(\tau), \zeta'(\tau)) d\tau \right) ds \right) dz \right) e^{-\eta t} \right\} \\
&\quad + \sup_{t \in [0,1]} \left\{ d_{\infty} \left(\int_0^t f(s, \xi(s), \xi'(s)) ds + \int_0^t \left(\int_0^s g(s, \tau, \xi(\tau), \xi'(\tau)) d\tau \right) ds, \int_0^t f(s, \zeta(s), \zeta'(s)) ds \right. \right. \\
&\quad \left. \left. + \int_0^t \left(\int_0^s g(s, \tau, \zeta(\tau), \zeta'(\tau)) d\tau \right) ds \right) e^{-\eta t} \right\} \\
&\leq \sup_{t \in [0,1]} \left\{ d_{\infty} \left(\int_0^t \left(\int_0^z f(s, \xi(s), \xi'(s)) ds \right) dz, \int_0^t \left(\int_0^z f(s, \zeta(s), \zeta'(s)) ds \right) dz \right) e^{-\eta t} \right. \\
&\quad \left. + d_{\infty} \left(\int_0^t \left(\int_0^z \left(\int_0^s g(s, \tau, \xi(\tau), \xi'(\tau)) d\tau \right) ds \right) dz, \int_0^t \left(\int_0^z \left(\int_0^s g(s, \tau, \zeta(\tau), \zeta'(\tau)) d\tau \right) ds \right) dz \right) e^{-\eta t} \right\} \\
&\quad + \sup_{t \in [0,1]} \left\{ d_{\infty} \left(\int_0^t f(s, \xi(s), \xi'(s)) ds, \int_0^t f(s, \zeta(s), \zeta'(s)) ds \right) e^{-\eta t} \right. \\
&\quad \left. + d_{\infty} \left(\int_0^t \left(\int_0^s g(s, \tau, \xi(\tau), \xi'(\tau)) d\tau \right) ds, \int_0^t \left(\int_0^s g(s, \tau, \zeta(\tau), \zeta'(\tau)) d\tau \right) ds \right) e^{-\eta t} \right\} \\
&\leq \sup_{t \in [0,1]} \left\{ \int_0^t \int_0^z d_{\infty} (f(s, \xi(s), \xi'(s)), f(s, \zeta(s), \zeta'(s))) ds dz e^{-\eta t} \right. \\
&\quad \left. + \int_0^t \int_0^z \int_0^s d_{\infty} (g(s, \tau, \xi(\tau), \xi'(\tau)), g(s, \tau, \zeta(\tau), \zeta'(\tau))) d\tau ds dz e^{-\eta t} \right\} \\
&\quad + \sup_{t \in [0,1]} \left\{ \int_0^t d_{\infty} (f(s, \xi(s), \xi'(s)), f(s, \zeta(s), \zeta'(s))) ds e^{-\eta t} \right. \\
&\quad \left. + \int_0^t \int_0^s d_{\infty} (g(s, \tau, \xi(\tau), \xi'(\tau)), g(s, \tau, \zeta(\tau), \zeta'(\tau))) d\tau ds e^{-\eta t} \right\} \\
&\leq \sup_{t \in [0,1]} \left\{ \int_0^t \int_0^z (K_1 d_{\infty}(\xi(s), \zeta(s)) + K_2 d_{\infty}(\xi'(s), \zeta'(s))) ds dz e^{-\eta t} \right. \\
&\quad \left. + \int_0^t \int_0^z \int_0^s (L_1 d_{\infty}(\xi(\tau), \zeta(\tau)) + L_2 d_{\infty}(\xi'(\tau), \zeta'(\tau))) d\tau ds dz e^{-\eta t} \right\} \\
&\quad + \sup_{t \in [0,1]} \left\{ \int_0^t (K_1 d_{\infty}(\xi(s), \zeta(s)) + K_2 d_{\infty}(\xi'(s), \zeta'(s))) ds e^{-\eta t} \right. \\
&\quad \left. + \int_0^t \int_0^s (L_1 d_{\infty}(\xi(\tau), \zeta(\tau)) + L_2 d_{\infty}(\xi'(\tau), \zeta'(\tau))) d\tau ds e^{-\eta t} \right\} \\
&\leq \sup_{t \in [0,1]} \left\{ \int_0^t \int_0^z (K_1 d(\xi, \zeta) + K_2 d(\xi', \zeta')) e^{\eta s} ds dz e^{-\eta t} \right. \\
&\quad \left. + \int_0^t \int_0^z \int_0^s (L_1 d(\xi, \zeta) + L_2 d(\xi', \zeta')) e^{\eta \tau} d\tau ds dz e^{-\eta t} \right\} \\
&\quad + \sup_{t \in [0,1]} \left\{ \int_0^t (K_1 d(\xi, \zeta) + K_2 d(\xi', \zeta')) e^{\eta s} ds e^{-\eta t} + \int_0^t \int_0^s (L_1 d(\xi, \zeta) + L_2 d(\xi', \zeta')) e^{\eta \tau} d\tau ds e^{-\eta t} \right\} \\
&\leq (K_1 d(\xi, \zeta) + K_2 d(\xi', \zeta')) \left(\sup_{t \in [0,1]} \int_0^t \int_0^z e^{\eta s} ds dz e^{-\eta t} + \sup_{t \in [0,1]} \int_0^t e^{\eta s} ds e^{-\eta t} \right) \\
&\quad + (L_1 d(\xi, \zeta) + L_2 d(\xi', \zeta')) \left(\sup_{t \in [0,1]} \int_0^t \int_0^z \int_0^s e^{\eta \tau} d\tau ds dz e^{-\eta t} + \sup_{t \in [0,1]} \int_0^t \int_0^s e^{\eta \tau} d\tau ds e^{-\eta t} \right) \\
&= (K_1 d(\xi, \zeta) + K_2 d(\xi', \zeta')) \left(\sup_{t \in [0,1]} e^{-\eta t} \int_0^t \int_0^z e^{\eta s} ds dz + \sup_{t \in [0,1]} e^{-\eta t} \int_0^t e^{\eta s} ds \right) \\
&\quad + (L_1 d(\xi, \zeta) + L_2 d(\xi', \zeta')) \left(\sup_{t \in [0,1]} e^{-\eta t} \int_0^t \int_0^z \int_0^s e^{\eta \tau} d\tau ds dz + \sup_{t \in [0,1]} e^{-\eta t} \int_0^t \int_0^s e^{\eta \tau} d\tau ds \right)
\end{aligned}$$

$$\begin{aligned}
&\leq \max\{K_1, K_2\} (d(\xi, \zeta) + d(\xi', \zeta')) \left(\sup_{t \in [0,1]} e^{-\eta t} \int_0^t \int_0^z e^{\eta s} ds dz + \sup_{t \in [0,1]} e^{-\eta t} \int_0^t e^{\eta s} ds \right) \\
&\quad + \max\{L_1, L_2\} (d(\xi, \zeta) + d(\xi', \zeta')) \left(\sup_{t \in [0,1]} e^{-\eta t} \int_0^t \int_0^z \int_0^s e^{\eta \tau} d\tau ds dz + \sup_{t \in [0,1]} e^{-\eta t} \int_0^t \int_0^s e^{\eta \tau} d\tau ds \right) \\
&\leq \max\{K_1, K_2\} D(\xi, \zeta) \left(\sup_{t \in [0,1]} e^{-\eta t} \left(\frac{1}{\eta^2} (e^{\eta t} - 1 - \eta t) \right) + \sup_{t \in [0,1]} e^{-\eta t} \left(\frac{1}{\eta} (e^{\eta t} - 1) \right) \right) \\
&\quad + \max\{L_1, L_2\} D(\xi, \zeta) \left(\sup_{t \in [0,1]} e^{-\eta t} \left(\frac{1}{\eta^3} \left(e^{\eta t} - 1 - \eta t - \frac{\eta^2}{2} t^2 \right) \right) + \sup_{t \in [0,1]} e^{-\eta t} \left(\frac{1}{\eta^2} (e^{\eta t} - 1 - \eta t) \right) \right) \\
&\leq \max\{K_1, K_2\} D(\xi, \zeta) \left(e^{-\eta} \left(\frac{1}{\eta^2} (e^{\eta} - 1 - \eta) \right) + e^{-\eta} \left(\frac{1}{\eta} (e^{\eta} - 1) \right) \right) \\
&\quad + \max\{L_1, L_2\} D(\xi, \zeta) \left(e^{-\eta} \left(\frac{1}{\eta^3} \left(e^{\eta} - 1 - \eta - \frac{\eta^2}{2} \right) \right) + e^{-\eta} \left(\frac{1}{\eta^2} (e^{\eta} - 1 - \eta) \right) \right) \\
&\leq \max\{K_1, K_2, L_1, L_2\} \psi(\eta) D(\xi, \zeta),
\end{aligned}$$

where $\psi(\eta) = e^{-\eta} \left(\frac{1}{\eta^3} (e^{\eta} - 1 - \eta - \frac{\eta^2}{2}) + \frac{2}{\eta^2} (e^{\eta} - 1 - \eta) + \frac{1}{\eta} (e^{\eta} - 1) \right)$. But since $\lim_{\eta \rightarrow +\infty} \psi(\eta) = 0$ from Lemma 4.2, So, we can choose $\eta > 0$ such that

$$\max\{K_1, K_2, L_1, L_2\} \psi(\eta) < 1. \quad (4.14)$$

Anyhow, G is a contractive mapping; whilst the unique fixed point of G is in the space $C^1([0,1], \mathbb{R}_{\mathcal{F}})$. Using that $G\xi$ is the integral of a continuous function, we conclude that it is actually in the space $C^2([0,1], \mathbb{R}_{\mathcal{F}})$. Hence, by the Banach fixed-point theorem, fuzzy VIDE (1.1) and (1.2) has a unique fixed point $x \in C^1([0,1], \mathbb{R}_{\mathcal{F}})$. That is, a continuous function x on $[0,1]$ satisfying $Gx = x$. As a result, writing $(Gx)(t) = x(t)$ out, we have by Eq. (4.12)

$$x(t) = \alpha + \beta t + \int_0^t \left(\int_0^z f(s, x(s), x'(s)) ds \right) dz + \int_0^t \left(\int_0^z \left(\int_0^s g(s, \tau, x(\tau), x'(\tau)) d\tau \right) ds \right) dz. \quad (4.15)$$

On the other aspect as well, differentiate both sides Eq. (4.15) and substitute $t = 0$ to obtain fuzzy VIDE (1.1) and (1.2). Hence, every solution of fuzzy VIDE (1.1) and (1.2) must satisfy Eq. (4.15), and conversely. So, the proof of the theorem is complete.

Remark 4.1: The continuous nonlinear terms $f: [0,1] \times \mathbb{R}_{\mathcal{F}}^2 \rightarrow \mathbb{R}_{\mathcal{F}}$ and $g: [0,1]^2 \times \mathbb{R}_{\mathcal{F}}^2 \rightarrow \mathbb{R}_{\mathcal{F}}$ are said to satisfy a generalized Lipchitz condition relative to their last argument in fuzzy sense with respect to the metric space $(\mathbb{R}_{\mathcal{F}}, d_{\infty})$ if the conditions of Eq. (4.11) of Theorem 4.2 are hold.

5. Generalized characterization theorem

The characterization theorem shows us the following general hint on how to deal with the analytical or the numerical solutions of fuzzy VIDEs. We can translate the original fuzzy VIDE equivalently into a system of crisp VIDEs. The solutions techniques of the system of crisp VIDEs are extremely well studied in the literature, so any method we can consider for the system of crisp VIDEs, since the solution will be as well solution of the fuzzy VIDE under study. As a conclusion one does not need to rewrite the methods of solution for system of crisp VIDEs in fuzzy setting, but instead, we can use the methods directly on the obtained crisp system.

A function $f: [0,1] \times \mathbb{R}^4 \rightarrow \mathbb{R}$ is said to be equicontinuous if for any $\epsilon > 0$ and any $(t, x, y, z, w) \in [0,1] \times \mathbb{R}^4$, we have $|f(t, x, y, z, w) - f(t, x_1, y_1, z_1, w_1)| < \epsilon$, whenever $\|(t, x_1, y_1, z_1, w_1) - (t, x, y, z, w)\| < \delta$, and uniformly bounded on any bounded set. Similarly, for a function defined on $[0,1]^2 \times \mathbb{R}^4$ with the need for attention to change the metric used on $[0,1]^2 \times \mathbb{R}^4$.

Theorem 5.1. Consider the fuzzy VIDE (1.1) and (1.2), where $f: [0,1] \times \mathbb{R}_{\mathcal{F}}^2 \rightarrow \mathbb{R}_{\mathcal{F}}$ and $g: [0,1]^2 \times \mathbb{R}_{\mathcal{F}}^2 \rightarrow \mathbb{R}_{\mathcal{F}}$ are such that

$$\begin{aligned}
\text{i. } [f(t, x(t), D_n^1(t))]^r &= [f_{1,r}(t, \underline{x}_r(t), \bar{x}_r(t), \underline{x}'_r(t), \bar{x}'_r(t)), f_{2,r}(t, \underline{x}_r(t), \bar{x}_r(t), \underline{x}'_r(t), \bar{x}'_r(t))] \\
[g(t, \tau, x(\tau), D_n^1(\tau))]^r &= [g_{1,r}(t, \tau, \underline{x}_r(\tau), \bar{x}_r(\tau), \underline{x}'_r(\tau), \bar{x}'_r(\tau)), g_{2,r}(t, \tau, \underline{x}_r(\tau), \bar{x}_r(\tau), \underline{x}'_r(\tau), \bar{x}'_r(\tau))]
\end{aligned}$$

ii. $f_{1,r}, f_{2,r}$ and $g_{1,r}, g_{2,r}$ are equicontinuous functions and uniformly bounded on any bounded set,

iii. there exists real-finite constants $L, K, M, N > 0$ such that

$$\begin{aligned} |f_{1,2,r}(t, x_1, y_1, z_1, w_1) - f_{1,2,r}(t, x_2, y_2, z_2, w_2)| &\leq L \max\{|x_1 - x_2|, |y_1 - y_2|\} + K \max\{|z_1 - z_2|, |w_1 - w_2|\}, \\ |g_{1,2,r}(t, \tau, x_1, y_1, z_1, w_1) - g_{1,2,r}(t, \tau, x_2, y_2, z_2, w_2)| &\leq M \max\{|x_1 - x_2|, |y_1 - y_2|\} + N \max\{|z_1 - z_2|, |w_1 - w_2|\}, \end{aligned}$$

for each $t, \tau \in [t_0, t_0 + a]$, $r \in [0, 1]$, and $x_{1,2}, y_{1,2}, z_{1,2}, w_{1,2} \in \mathbb{R}$. Then, for (n, m) -differentiability, the fuzzy VIDE (1.1) and (1.2) and the corresponding (n, m) -system are equivalent.

Proof. Since the proof procedure is similar for each type of differentiability with respect to the corresponding (n, m) -system. Anyhow, we assume that x is $(1, 1)$ -differentiable (Case I of Algorithm 3.1) without the loss of generality. The equicontinuity of $f_{1,r}, f_{2,r}$ and $g_{1,r}, g_{2,r}$ implies the continuity of f and g , respectively. Furthermore, the Lipchitz property of condition (iii) ensures that f and g are satisfies a Lipchitz property in the metric space (\mathbb{R}_F, d_∞) as follows:

$$\begin{aligned} d_\infty(f(t, x(t), x'(t)), f(t, y(t), y'(t))) &= \sup_{r \in [0, 1]} d_H([f(t, x(t), x'(t))]^r, [f(t, y(t), y'(t))]^r) \\ &= \sup_{r \in [0, 1]} \max\left\{\left|\underline{f}_r(t, x(t), x'(t)) - \underline{f}_r(t, y(t), y'(t))\right|, \left|\bar{f}_r(t, x(t), x'(t)) - \bar{f}_r(t, y(t), y'(t))\right|\right\} \\ &= \sup_{r \in [0, 1]} \max\left\{\left|f_{1,r}(t, \underline{x}_r(t), \bar{x}_r(t), \underline{x}'_r(t), \bar{x}'_r(t))\right.\right. \\ &\quad \left.- f_{1,r}(t, \underline{y}_r(t), \bar{y}_r(t), \underline{y}'_r(t), \bar{y}'_r(t))\right|, \left|f_{2,r}(t, \underline{x}_r(t), \bar{x}_r(t), \underline{x}'_r(t), \bar{x}'_r(t))\right. \\ &\quad \left.- f_{2,r}(t, \underline{y}_r(t), \bar{y}_r(t), \underline{y}'_r(t), \bar{y}'_r(t))\right|\} \\ &\leq L \sup_{r \in [0, 1]} \max\left\{\left|\underline{x}_r(t) - \underline{y}_r(t)\right|, \left|\bar{x}_r(t) - \bar{y}_r(t)\right|\right\} + K \sup_{r \in [0, 1]} \max\left\{\left|\underline{x}'_r(t) - \underline{y}'_r(t)\right|, \left|\bar{x}'_r(t) - \bar{y}'_r(t)\right|\right\} \\ &= L \sup_{r \in [0, 1]} d_H([x(t)]^r, [y(t)]^r) + K \sup_{r \in [0, 1]} d_H([x'(t)]^r, [y'(t)]^r) \\ &= L d_\infty(x(t), y(t)) + K d_\infty(x'(t), y'(t)). \end{aligned} \quad (5.1)$$

Whilst on the other aspect as well, by similar fashion, it is easy to conclude that

$$d_\infty(g(t, \tau, x(\tau), x'(\tau)), g(t, \tau, y(\tau), y'(\tau))) \leq M d_\infty(x(\tau), y(\tau)) + N d_\infty(x'(\tau), y'(\tau)). \quad (5.2)$$

By the continuity of f and g , from this last Lipchitz conditions of Eqs. (5.1) and (5.2), and the boundedness property of condition (ii), it follows that fuzzy VIDE (1.1) and (1.2) has a unique solution on $[0, 1]$. Whilst, the solution of fuzzy VIDE (1.1) and (1.2) is $(1, 1)$ -differentiable and so, by Theorems 2.2 and 2.3, the functions $\underline{x}_r, \bar{x}_r$ and $\underline{x}'_r, \bar{x}'_r$ are differentiable on $[0, 1]$. As a conclusion one can obtained that $(\underline{x}_r(t), \bar{x}_r(t))$ is a solution of crisp VIDEs (3.5) and (3.6).

Conversely, suppose that we have a solution $(\underline{x}_r(t), \bar{x}_r(t))$ with $r \in [0, 1]$ is fixed, of fuzzy VIDE (1.1) and (1.2) (note that this solution exists by property of condition (iii)). Whilst, the Lipchitz conditions of Eqs. (5.1) and (5.2) implies the existence and uniqueness of fuzzy solution $\tilde{x}(t)$. Indeed, since \tilde{x} is $(1, 1)$ -differentiable, then $\underline{\tilde{x}}_r(t)$ and $\bar{\tilde{x}}_r(t)$ the endpoints of $[\tilde{x}(t)]^r$ are a solution of crisp VIDEs (3.5) and (3.6) (note that $[\tilde{x}]^r$ and $[D_1^1 \tilde{x}]^r$ are obviously valid level sets of fuzzy-valued functions). But since the solution of crisp VIDEs (3.5) and (3.6) is unique, we have $[\tilde{x}(t)]^r = [\underline{\tilde{x}}_r(t), \bar{\tilde{x}}_r(t)]^r = [\underline{x}_r(t), \bar{x}_r(t)]^r = [x(t)]^r$. That is the fuzzy VIDE (1.1) and (1.2) and the system of crisp VIDEs (3.5) and (3.6) are equivalent. This completes the proof of the theorem.

The purpose of the next corollary is not to make an essential improvement of Theorem 5.1, but rather to give alternate conditions under which fuzzy VIDE (1.1) and (1.2) and the corresponding system of crisp VIDEs are equivalent.

Corollary 5.1. Suppose that $f: [0, 1] \times \mathbb{R}_F^2 \rightarrow \mathbb{R}_F$ and $g: [0, 1]^2 \times \mathbb{R}_F^2 \rightarrow \mathbb{R}_F$ are such that the condition (i) of Theorem 5.1 hold. If there exists real-finite constants $L, K, M, N > 0$ such that

$$\begin{aligned} |f_{1,2,r}(t_1, x_1, y_1, z_1, w_1) - f_{1,2,r}(t_2, x_2, y_2, z_2, w_2)| \\ \leq L \max\{|t_1 - t_2|, |x_1 - x_2|, |y_1 - y_2|\} + K \max\{|t_1 - t_2|, |z_1 - z_2|, |w_1 - w_2|\}, \\ |g_{1,2,r}(t_1, \tau_1, x_1, y_1, z_1, w_1) - g_{1,2,r}(t_2, \tau_2, x_2, y_2, z_2, w_2)| \\ \leq M \max\{|t_1 - t_2|, |\tau_1 - \tau_2|, |x_1 - x_2|, |y_1 - y_2|\} + N \max\{|t_1 - t_2|, |\tau_1 - \tau_2|, |z_1 - z_2|, |w_1 - w_2|\}, \end{aligned} \quad (5.3)$$

for each $t_{1,2}, \tau_{1,2} \in [0, 1]$, $r \in [0, 1]$, and $x_{1,2}, y_{1,2}, z_{1,2}, w_{1,2} \in \mathbb{R}$. Then, for (n, m) -differentiability, the fuzzy VIDE (1.1) and (1.2) and the corresponding (n, m) -system are equivalent.

Proof. Here, we consider the (1,1)-differentiability only; actually, in the same manner, we can employ the same technique for the remaining types of (n, m) -differentiability. To this end, assume the hypothesis of Corollary 5.1, then the conditions (i) and (iii) of Theorem 5.1 are clearly hold. To establish condition (ii), apply the following: fix $\epsilon > 0$, choose $\delta_1 = \epsilon/(2L)$ and $\delta_2 = \epsilon/(2K)$, and suppose $\|(t, x, y) - (t_1, x_1, y_1)\| < \delta_1$ and $\|(t, z, w) - (t_1, z_1, w_1)\| < \delta_2$. Then, for each $r \in [0, 1]$, one can write

$$\begin{aligned} |f_{1,2,r}(t, x, y, z, w) - f_{1,2,r}(t_1, x_1, y_1, z_1, w_1)| \\ \leq L \max\{|t - t_1|, |x - x_1|, |y - y_1|\} + K \max\{|t - t_1|, |z - z_1|, |w - w_1|\} \\ \leq L\|(t, x, y) - (t_1, x_1, y_1)\| + K\|(t, z, w) - (t_1, z_1, w_1)\| \\ \leq L\delta_1 + K\delta_2 = \epsilon. \end{aligned} \quad (5.4)$$

Next, we want to show that $f_{1,r}, f_{2,r}$ are uniformly bounded on any bounded set. To do so, let S be any bounded subset of $[0, 1] \times \mathbb{R}^4$. Then there exist constants $x_1, y_1, z_1, w_1, x_2, y_2, z_2, w_2 \in \mathbb{R}$ such that if $w = (t, x, y, z) \in S$, then $t \in [0, 1]$, $x \in [x_1, x_2]$, $y \in [y_1, y_2]$, $z \in [z_1, z_2]$, and $w \in [w_1, w_2]$. For the conduct of proceedings in the proof, fix $r^* \in [0, 1]$ and $w^* \in S$, further, let $L^* = \max\{1, |x_1 - x_2|, |y_1 - y_2|\}$, $K^* = \max\{1, |z_1 - z_2|, |w_1 - w_2|\}$, and $C = LL^* + KK^* + \text{supp } f(w^*)$, where $\text{supp } f(w^*)$ is the support of $f(w^*)$. Suppose that $r \in [0, 1]$ and $w \in S$. Then one can write

$$|f_{1,r}(w) - f_{1,r^*}(w^*)| \leq L \max\{1, |x_1 - x_2|, |y_1 - y_2|\} + K \max\{1, |z_1 - z_2|, |w_1 - w_2|\} = LL^* + KK^*, \quad (5.5)$$

while on the other aspect as well, the triangle inequality will gives

$$\begin{aligned} |f_{1,r}(w) - f_{1,r^*}(w^*)| &= |f_{1,r}(w) - f_{1,r}(w^*) + f_{1,r}(w^*) - f_{1,r^*}(w^*)| \\ &\leq |f_{1,r}(w) - f_{1,r}(w^*)| + |f_{1,r}(w^*) - f_{1,r^*}(w^*)| \\ &= LL^* + KK^* + \text{supp } f(w^*) = C. \end{aligned} \quad (5.6)$$

But since $|f_{1,r}(w)| - |f_{1,r^*}(w^*)| \leq |f_{1,r}(w) - f_{1,r^*}(w^*)| \leq C$ or $|f_{1,r}(w)| \leq C + |f_{1,r^*}(w^*)|$, therefore $f_{1,r}$ is uniformly bounded on S . Similarly, $f_{2,r}$ is uniformly bounded on any bounded set. The same procedure can be applied directly for $g_{1,r}, g_{2,r}$. Hence, fuzzy VIDE (1.1) and (1.2) and the corresponding (1,1)-system are equivalent by Theorem 5.1.

Remark 5.1. The following requirement conditions on f and g :

$$\begin{aligned} [f(t, x(t), x'(t))]^r &= [f_{1,r}(t, \underline{x}_r(t), \bar{x}_r(t), \underline{x}'_r(t), \bar{x}'_r(t)), f_{2,r}(t, \underline{x}_r(t), \bar{x}_r(t), \underline{x}'_r(t), \bar{x}'_r(t))] \\ [g(t, \tau, x(\tau), x'(\tau))]^r &= [g_{1,r}(t, \tau, \underline{x}_r(\tau), \bar{x}_r(\tau), \underline{x}'_r(\tau), \bar{x}'_r(\tau)), g_{2,r}(t, \tau, \underline{x}_r(\tau), \bar{x}_r(\tau), \underline{x}'_r(\tau), \bar{x}'_r(\tau))] \end{aligned} \quad (5.7)$$

are fulfilled by any fuzzy-valued functions obtained from continuous real-valued functions by Zadeh's extension principle and Nguyen theorem [35-37]. So these conditions are not too restrictive.

6. Conclusion

Existence and uniqueness theorem is the tool which makes it possible for us to conclude that there exists only one solution to a given problem which satisfies a constraint condition. How does it work? Why is it the case? We believe it but it would be interesting to see the main ideas behind. To this end, in this paper we investigated and proved the existence, uniqueness, and other properties of solutions of a certain nonlinear second-order fuzzy VIDE under strongly generalized differentiability by considered four cases of differentiability. We make use of the standard tools of the fixed point theorem and a certain integral inequality with explicit estimate to establish the main results. In addition to that, some results for characterizing solution by an equivalent system of crisp VIDEs are presented and proved.

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$\alpha\beta$ -statistical convergence and strong $\alpha\beta$ -convergence of order γ for a sequence of fuzzy numbers[†]

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Abstract The purpose of this paper is to introduce the concepts of $\alpha\beta$ -statistical convergence of order γ and strong $\alpha\beta$ -convergence of order γ for a sequence of fuzzy numbers. At the same time, some connections between $\alpha\beta$ -statistical convergence of order γ and strong $\alpha\beta$ -convergence of order γ for a sequence of fuzzy numbers are established. It also shows that if a sequence of fuzzy numbers is strongly $\alpha\beta$ -convergent of order γ then it is $\alpha\beta$ -statistically convergent of order γ .

Keywords: Fuzzy numbers; sequence of fuzzy numbers; statistical convergence.

1. Introduction

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [1] and subsequently several authors have discussed various aspects of theory and applications of fuzzy sets. Recently Matloka [2] introduced bounded and convergent sequences of fuzzy numbers, studied some of their properties, and showed that every convergent sequence of fuzzy numbers is bounded. In addition, sequences of fuzzy numbers have been discussed by Aytar and Pehlivan [3], Basarir and Mursaleen [4,5] and many others. The notion of statistical convergence was introduced by Fast [6] which is a very useful functional tool for studying the convergence problems of numerical sequences. Some applications of statistical convergence in number theory and mathematical analysis can be found in [7, 8]. The idea is based on the notion of natural density of subsets of N , and the natural density of a subset A of N is denoted by $\delta(A)$ and defined by

$$\delta(A) = \lim_{n \rightarrow \infty} \frac{1}{n} |\{k < n : k \in A\}|.$$

In 2014, Hüseyin Aktuğlu [9] introduced the concepts of $\alpha\beta$ -statistically convergence and $\alpha\beta$ -statistically convergence of order γ for a sequence, which shows that $\alpha\beta$ -statistically convergence is a non-trivial extension of ordinary and statistical convergences.

In this paper, we define the sequence spaces of $\alpha\beta$ -statistical convergence of order γ and strong $\alpha\beta$ -convergence of order γ , and testify some properties of these spaces. At the same time, some connections between $\alpha\beta$ -statistical convergence of order γ and strong $\alpha\beta$ -convergence for a sequence of order γ of fuzzy numbers are established. In Section 2 we will give a brief overview about fuzzy numbers, statistical convergence, and present $\delta^{\alpha,\beta}(k, \gamma)$. In Section 3 we show that $\alpha\beta$ -statistical convergence for a sequence of fuzzy numbers can reduce to statistical convergence, λ -statistical convergence, and lacunary statistical convergence. Meanwhile, strong $\alpha\beta$ -convergence for a sequence of fuzzy numbers can reduce to strong convergence, strong λ -convergence and strongly lacunary convergence.

2. Definitions and preliminaries

Let $\tilde{A} \in \tilde{F}(R)$ be a fuzzy subset on R . If \tilde{A} is convex, normal, upper semi-continuous and has compact support, we say that \tilde{A} is a fuzzy number. Let \tilde{R}^c denote the set of all fuzzy numbers [10,11,12].

For $\tilde{A} \in \tilde{R}^c$, we write the level set of \tilde{A} as $A_\lambda = \{x : A(x) \geq \lambda\}$ and $A_\lambda = [A_\lambda^-, A_\lambda^+]$. Let $\tilde{A}, \tilde{B} \in \tilde{R}^c$, we define $\tilde{A} + \tilde{B} = \tilde{C}$ iff $A_\lambda + B_\lambda = C_\lambda$, $\lambda \in [0, 1]$ iff $A_\lambda^- + B_\lambda^- = C_\lambda^-$ and $A_\lambda^+ + B_\lambda^+ = C_\lambda^+$ for any $\lambda \in [0, 1]$.

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Define

$$D(\tilde{A}, \tilde{B}) = \sup_{\lambda \in [0,1]} d(A_\lambda, B_\lambda) = \sup_{\lambda \in [0,1]} \max\{|A_\lambda^- - B_\lambda^-|, |A_\lambda^+ - B_\lambda^+|\},$$

where d is the Hausdorff metric. $D(\tilde{A}, \tilde{B})$ is called the distance between \tilde{A} and \tilde{B} [11,13,14].

Using the results of [10,11], we see that

- (1) (\tilde{R}^c, D) is a complete metric space,
- (2) $D(u + w, v + w) = D(u, v)$,
- (3) $D(ku, kv) = |k|D(u, v)$, $k \in R$,
- (4) $D(u + v, w + e) \leq D(u, w) + D(v, e)$,
- (5) $D(u + v, \bar{0}) \leq D(u, \bar{0}) + D(v, \bar{0})$,
- (6) $D(u + v, w) \leq D(u, w) + D(v + \bar{0})$,

where $u, v, w, e \in \tilde{R}^c$, $\tilde{0}(t) = \begin{cases} 1, & t = (0, 0, 0, \dots, 0), \\ 0, & \text{otherwise.} \end{cases}$

Definitions 2.1.[15] A sequence $\{x_n\}$ of fuzzy numbers is said to be statistically convergent to a fuzzy number x_0 if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in N : D(x_n, x_0) \geq \varepsilon\}$ has natural density zero. The fuzzy number x_0 is called the statistical limit of the sequence $\{x_n\}$ and we write $\lim_{n \rightarrow \infty} x_n = x_0$.

Now let $\alpha(n)$ and $\beta(n)$ be two sequences of positive numbers satisfying the following conditions:

- (1) $\alpha(n)$ and $\beta(n)$ are both non-decreasing,
- (2) $\beta(n) \geq \alpha(n)$,
- (3) $\beta(n) - \alpha(n) \rightarrow \infty$, as $n \rightarrow \infty$,

and let Λ denote the set of pairs (α, β) satisfying (1), (2) and (3).

For each pair $(\alpha, \beta) \in \Lambda$, $0 < \gamma \leq 1$ and $K \subset N$, we define $\delta^{\alpha, \beta}(K, \gamma)$ in the following way:

$$\delta^{\alpha, \beta}(K, \gamma) = \lim_n \frac{|K \cap P_n^{\alpha, \beta}|}{(\beta(n) - \alpha(n) + 1)^\gamma}$$

where $P_n^{\alpha, \beta}$ is the closed interval $[\alpha(n), \beta(n)]$ and $|S|$ represents the cardinality of S .

Lemma 2.1. Let K and M be two subsets of N and $0 < \gamma \leq \delta \leq 1$. Then for all $(\alpha, \beta) \in \Lambda$, we have

- (1) $\delta^{\alpha, \beta}(\emptyset, \gamma) = 0$,
- (2) $\delta^{\alpha, \beta}(N, 1) = 1$,
- (3) if K is a finite set, the $\delta^{\alpha, \beta}(K, \gamma) = 0$,
- (4) $K \subset M \Rightarrow \delta^{\alpha, \beta}(K, \gamma) \leq \delta^{\alpha, \beta}(M, \gamma)$,
- (5) $\delta^{\alpha, \beta}(K, \delta) \leq \delta^{\alpha, \beta}(K, \gamma)$.

3. Main results

Definition 3.1. A sequence of fuzzy numbers is said to be $\alpha\beta$ -statistically convergent of order γ to x_0 , if for every $\varepsilon > 0$,

$$\delta^{\alpha, \beta}(\{k : D(x_k, x_0) \geq \varepsilon\}, \gamma) = \lim_n \frac{|\{k \in P_n^{\alpha, \beta} : D(x_k, x_0) \geq \varepsilon\}|}{(\beta(n) - \alpha(n) + 1)^\gamma} = 0.$$

In this case, we write $\tilde{S}_\gamma^{\alpha, \beta} - \lim x_k = x_0$. The set of all $\alpha\beta$ -statistically convergent of order γ will be denoted simply by $\tilde{S}_\gamma^{\alpha, \beta}$.

For $\gamma = 1$, we say that x is $\alpha\beta$ -statistically convergent to x_0 and this is denoted by $\tilde{S}^{\alpha, \beta} - \lim x_k = x_0$.

The following example shows that Definition 3.1 is non-trivial generalization of both ordinary and statistical convergence.

Example 3.1. Taking $\alpha(n) = 1$ and $\beta(n) = n^{\frac{1}{\gamma}}$, where $0 < \gamma < 1$ is fixed, then

$$\delta^{\alpha, \beta}(\{k : D(x_k, x_0) \geq \varepsilon\}, \gamma) = \lim_n \frac{|\{k \in [1, n^{\frac{1}{\gamma}}] : D(x_k, x_0) \geq \varepsilon\}|}{n}$$

and, in particular, for $\gamma = \frac{1}{2}$ we have

$$\delta^{\alpha, \beta}(\{k : D(x_k, x_0) \geq \varepsilon\}, \frac{1}{2}) = \lim_n \frac{|\{k \in [1, n^2] : D(x_k, x_0) \geq \varepsilon\}|}{n}.$$

Consider the sequence of fuzzy numbers

$$x_k(t) = \begin{cases} t+1, & -1 \leq t \leq 0, k \neq n^2, \\ -t+1, & 0 < t \leq 1, k \neq n^2, \\ t, & 0 \leq t \leq 1, k = n^2, \\ 2-t, & 1 < t \leq 2, k = n^2, \\ 0, & \text{others;} \end{cases} \quad x_0(t) = \begin{cases} t+1, & -1 \leq t \leq 0, \\ -t+1, & 0 < t \leq 1, \\ 0, & \text{others;} \end{cases}$$

Obviously $st - \lim_n x_k = x_0$, however

$$\delta^{\alpha,\beta}(\{k : D(x_k, x_0) \geq \varepsilon\}, \frac{1}{2}) = \lim_n \frac{|\{k \in [1, n^2] : D(x_k, x_0) \geq \varepsilon\}|}{n} \neq 0.$$

for all $\varepsilon > 0$, $\tilde{S}_\gamma^{\alpha,\beta} - \lim x_k \neq x_0$.

Definition 3.2. Based on strongly $\alpha\beta$ -convergence of order γ , for every $\varepsilon > 0$, we define the following sets

$$\begin{aligned} \tilde{W}_\gamma^{\alpha,\beta} &= \{x = \{x_k\} : \lim_n \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}} D(x_k, x_0) = 0\}, \\ \tilde{W}_{\gamma 0}^{\alpha,\beta} &= \{x = \{x_k\} : \lim_n \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}} D(x_k, \bar{0}) = 0\}, \\ \tilde{W}_{\gamma\infty}^{\alpha,\beta} &= \{x = \{x_k\} : \sup_n \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}} D(x_k, \bar{0}) < \infty\}, \end{aligned}$$

where

$$\tilde{0}(t) = \begin{cases} 1, & t = (0, 0, 0, \dots, 0), \\ 0, & \text{otherwise.} \end{cases}$$

If $x \in \tilde{W}_\gamma^{\alpha,\beta}$, we say that x is strongly $\alpha\beta$ -convergent of order γ to x_0 and we write $\tilde{W}_\gamma^{\alpha,\beta} - \lim x_k = x_0$. For $\gamma = 1$, we say that x is strongly $\alpha\beta$ -convergent to x_0 and this is denoted by $\tilde{W}^{\alpha,\beta} - \lim x_k = x_0$.

Remark 3.1. Take $\alpha(n) = 1$, $\beta(n) = n$ and $\gamma = 1$, then $P_n^{\alpha,\beta} = [1, n]$ and

$$\delta^{\alpha,\beta}(\{k : D(x_k, x_0) \geq \varepsilon\}, \gamma) = \lim_n \frac{|\{k \leq n : D(x_k, x_0) \geq \varepsilon\}|}{n} = 0.$$

This shows that in this case, $\alpha\beta$ -statistical convergence of order γ reduces to statistical convergence which we denoted by \tilde{S} . Meanwhile, the sequences space $\tilde{W}_\gamma^{\alpha,\beta}$ reduces to \tilde{W} , $\tilde{W}_{\gamma 0}^{\alpha,\beta}$ reduces to \tilde{W}_0 and $\tilde{W}_{\gamma\infty}^{\alpha,\beta}$ reduces to \tilde{W}_∞ . Where \tilde{W} , \tilde{W}_0 and \tilde{W}_∞ are defined by Mursaleen and Basarir [16].

$$\begin{aligned} \tilde{W} &= \{x = \{x_k\} : \lim_n \frac{1}{n} \sum_{k=1}^n D(x_k, x_0) = 0\}, \\ \tilde{W}_0 &= \{x = \{x_k\} : \lim_n \frac{1}{n} \sum_{k=1}^n D(x_k, \bar{0}) = 0\}, \\ \tilde{W}_\infty &= \{x = \{x_k\} : \sup_n \frac{1}{n} \sum_{k=1}^n D(x_k, \bar{0}) < \infty\}. \end{aligned}$$

Remark 3.2. Let λ_n be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$ and $I_n = [n - \lambda_n + 1, n]$. We choose $\alpha(n) = n - \lambda_n + 1$, $\beta(n) = n$ and $\gamma = 1$, then $P_n^{\alpha,\beta} = [n - \lambda_n + 1, n]$. Moreover,

$$\delta^{\alpha,\beta}(\{k : D(x_k, x_0) \geq \varepsilon\}, \gamma) = \lim_n \frac{|\{k \in I_n : D(x_k, x_0) \geq \varepsilon\}|}{\lambda_n} = 0.$$

This shows that in this case, $\alpha\beta$ -statistical convergence of order γ reduces to λ -statistical convergence which we denoted by $\tilde{S}(\lambda)$. Meanwhile, the sequences space $\tilde{W}_\gamma^{\alpha,\beta}$ reduces to $\tilde{W}(\lambda)$, $\tilde{W}_{\gamma_0}^{\alpha,\beta}$ reduces to $\tilde{W}_0(\lambda)$ and $\tilde{W}_{\gamma\infty}^{\alpha,\beta}$ reduces to $\tilde{W}_\infty(\lambda)$. Where $\tilde{W}(\lambda)$, $\tilde{W}_0(\lambda)$ and $\tilde{W}_\infty(\lambda)$ are defined by Savas [17].

$$\tilde{W}(\lambda) = \{x = \{x_k\} : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} D(x_k, x_0) = 0\},$$

$$\tilde{W}_0(\lambda) = \{x = \{x_k\} : \lim_n \frac{1}{\lambda_n} \sum_{k \in I_n} D(x_k, \bar{0}) = 0\},$$

$$\tilde{W}_\infty(\lambda) = \{x = \{x_k\} : \sup_n \frac{1}{\lambda_n} \sum_{k \in I_n} D(x_k, \bar{0}) < \infty\}.$$

Remark 3.3. A lacunary sequence $\theta = \{k_r\}$ is an increasing sequence such that $k_0 = 0$, $h_r = k_r - k_{r-1} \rightarrow \infty$, $r \rightarrow \infty$ and $I_r = (k_{r-1}, k_r]$. Take $\alpha(r) = k_{r-1} + 1$, $\beta(r) = k_r$ and $\gamma = 1$, then $P_r^{\alpha,\beta} = [k_{r-1} + 1, k_r]$. However $(k_{r-1}, k_r] \cap N = [k_{r-1} + 1, k_r] \cap N$, we have

$$\delta^{\alpha,\beta}(\{k : D(x_k, x_0) \geq \varepsilon\}, \gamma) = \lim_r \frac{|\{k \in I_r : D(x_k, x_0) \geq \varepsilon\}|}{h_r} = 0.$$

This shows that in this case, $\alpha\beta$ -statistical convergence of order γ coincides with lacunary statistical convergence which we denoted by $\tilde{S}(\theta)$. Meanwhile, the sequences space $\tilde{W}_\gamma^{\alpha,\beta}$ reduces to $\tilde{W}(\theta)$, $\tilde{W}_{\gamma_0}^{\alpha,\beta}$ reduces to $\tilde{W}_0(\theta)$ and $\tilde{W}_{\gamma\infty}^{\alpha,\beta}$ reduces to $\tilde{W}_\infty(\theta)$.

Where

$$\tilde{W}(\theta) = \{x = \{x_k\} : \lim_r \frac{1}{h_r} \sum_{k \in I_r} D(x_k, x_0) = 0\},$$

$$\tilde{W}_0(\theta) = \{x = \{x_k\} : \lim_r \frac{1}{h_r} \sum_{k \in I_r} D(x_k, \bar{0}) = 0\},$$

$$\tilde{W}_\infty(\theta) = \{x = \{x_k\} : \sup_r \frac{1}{h_r} \sum_{k \in I_r} D(x_k, \bar{0}) < \infty\}.$$

Theorem 3.1. Let $x = \{x_k\}$, $y = \{y_k\}$ be two sequences of fuzzy numbers. We have

(1) If $\tilde{S}_\gamma^{\alpha,\beta} - \lim x_k = x_0$ and $c \in R$, then $\tilde{S}_\gamma^{\alpha,\beta} - \lim cx_k = cx_0$;

(2) If $\tilde{S}_\gamma^{\alpha,\beta} - \lim x_k = x_0$, $\tilde{S}_\gamma^{\alpha,\beta} - \lim y_k = y_0$, then $\tilde{S}_\gamma^{\alpha,\beta} - \lim(x_k + y_k) = x_0 + y_0$.

Proof. (1) The proof is obvious when $c=0$. Suppose that $c \neq 0$, then the proof of (1) follows from the following inequality,

$$\frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(cx_k, cx_0) \geq \varepsilon\}| \leq \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_0) \geq \frac{\varepsilon}{|c|}\}|.$$

(2) Suppose that $\tilde{S}_\gamma^{\alpha,\beta} - \lim x_k = x_0$, $\tilde{S}_\gamma^{\alpha,\beta} - \lim y_k = y_0$, then

$$\lim_n \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_0) \geq \frac{\varepsilon}{2}\}| = 0,$$

$$\lim_n \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(y_k, y_0) \geq \frac{\varepsilon}{2}\}| = 0.$$

Since

$$D(x_k + y_k, x_0 + y_0) \leq D(x_k + y_k, x_0 + y_k) + D(x_0 + y_k, x_0 + y_0) = D(x_k, x_0) + D(y_k, y_0).$$

For $\varepsilon > 0$, we have

$$\begin{aligned} & \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(x_k + y_k, x_0 + y_0) \geq \varepsilon\}| \\ & \leq \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_0) \geq \frac{\varepsilon}{2}\}| + \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(y_k, y_0) \geq \frac{\varepsilon}{2}\}| \end{aligned}$$

$\rightarrow 0, n \rightarrow \infty$. Hence $\tilde{S}_\gamma^{\alpha,\beta} - \lim(x_k + y_k) = x_0 + y_0$.

Definition 3.3. The sequence of fuzzy numbers $x = \{x_k\}$ is a $\alpha\beta$ -statistically Cauchy sequence of order γ , if for every $\varepsilon > 0$ there exists a number $N(=N(\varepsilon))$ such that

$$\lim_n \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_N) \geq \varepsilon\}| = 0.$$

Theorem 3.2. Let $x = \{x_k\}$ be a sequence of fuzzy numbers. It is a $\alpha\beta$ -statistically convergent sequence of order γ if and only if x is a $\alpha\beta$ -statistical Cauchy sequence of order γ .

Proof. Suppose that $\tilde{S}_\gamma^{\alpha,\beta} - \lim x_k = x_0$ and let $\varepsilon > 0$, then

$$\lim_n \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_0) \geq \varepsilon\}| = 0,$$

and N is chosen such that $\lim_n \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(x_N, x_0) \geq \varepsilon\}| = 0$, then we have

$$\begin{aligned} & \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_N) \geq \varepsilon\}| \\ & \leq \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_0) \geq \varepsilon\}| + \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(x_N, x_0) \geq \varepsilon\}|. \end{aligned}$$

Hence $x = \{x_k\}$ is a $\alpha\beta$ -statistically Cauchy sequence of order γ .

Next, assume that $x = \{x_k\}$ be $\alpha\beta$ -statistical Cauchy sequence of order γ , then there exists a strictly increasing sequence N_p of positive integers such that $\lim_n \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_{N_p}) \geq \varepsilon_p\}| = 0$, where $\varepsilon_p : p = 1, 2, 3, \dots$ is a strictly decreasing sequence of numbers converging to zero for each p and q pair ($p \neq q$) of positive integers, we can select K_{pq} such $D(x_{K_{pq}}, x_{N_p}) < \varepsilon_p$ and $D(x_{K_{pq}}, x_{N_q}) < \varepsilon_q$. It follows that

$$D(x_{N_p}, x_{N_q}) \leq D(x_{K_{pq}}, x_{N_p}) + D(x_{K_{pq}}, x_{N_q}) < \varepsilon_p + \varepsilon_q \rightarrow 0, p, q \rightarrow \infty.$$

Hence, $\{x_{N_p} : p = 1, 2, \dots\}$ is a Cauchy sequence and satisfies the Cauchy convergence criterion. Let $\{x_{N_p}\}$ converge to x_0 . Since $\varepsilon_p : p = 1, 2, \dots \rightarrow 0$, so for $\varepsilon > 0$, there exists p_0 such that $\varepsilon_{p_0} < \frac{\varepsilon}{2}$ and $D(x_{N_{p_0}}, x_0) < \frac{\varepsilon}{2}$, $p \geq p_0$, then

$$D(x_k, x_0) \leq D(x_k, x_{N_{p_0}}) + D(x_{N_{p_0}}, x_0) \leq D(x_k, x_{N_{p_0}}) + \frac{\varepsilon}{2},$$

we have

$$\begin{aligned} & \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_0) \geq \varepsilon\}| \leq \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_{N_{p_0}}) \geq \frac{\varepsilon}{2}\}| \\ & \leq \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_{N_{p_0}}) \geq \varepsilon_{p_0}\}| \rightarrow 0, n \rightarrow \infty. \end{aligned}$$

This shows that $x = \{x_k\}$ is $\alpha\beta$ -statistically convergent of order γ .

Theorem 3.3. Let $x = \{x_k\}$ is a sequence of fuzzy numbers. There exists a $\alpha\beta$ -statistically convergent of order γ sequence $y = \{y_k\}$ such that $x_k = y_k$ for almost all k according to γ , then $x = \{x_k\}$ is a $\alpha\beta$ -statistically convergent sequence of order γ .

Proof. Let $x_k = y_k$ for almost all k according to γ and $\tilde{S}_\gamma^{\alpha,\beta} - \lim y_k = x_0$. Suppose $\varepsilon > 0$. Then for each n ,

$$\{k \in P_n^{\alpha,\beta} : D(x_k, x_0) \geq \varepsilon\} \subseteq \{k \in P_n^{\alpha,\beta} : D(y_k, x_0) \geq \varepsilon\} \cup \{k \in P_n^{\alpha,\beta} : x_k \neq y_k\}.$$

Since $x_k = y_k$ for almost all k according to γ , the latter set contains a fixed number of integers, say $S = S(\varepsilon)$. Then

$$\begin{aligned} & \lim_n \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_0) \geq \varepsilon\}| \\ & \leq \lim_n \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(y_k, x_0) \geq \varepsilon\}| + \lim_n \frac{S}{(\beta(n) - \alpha(n) + 1)^\gamma}, \end{aligned}$$

Hence $\tilde{S}_\gamma^{\alpha,\beta} - \lim x_k = x_0$, i.e. $x = \{x_k\}$ is a $\alpha\beta$ -statistically convergent sequence of order γ .

Theorem 3.4. Let $0 < \gamma_1 \leq \gamma_2 \leq 1$, then $\tilde{S}_{\gamma_1}^{\alpha,\beta} \subseteq \tilde{S}_{\gamma_2}^{\alpha,\beta}$.

Proof. Let $0 < \gamma_1 \leq \gamma_2 \leq 1$. Then we have

$$\frac{1}{(\beta(n) - \alpha(n) + 1)^{\gamma_2}} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_0) \geq \varepsilon\}| \leq \frac{1}{(\beta(n) - \alpha(n) + 1)^{\gamma_1}} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_0) \geq \varepsilon\}|,$$

for every $\varepsilon > 0$ and so we get $\tilde{S}_{\gamma_1}^{\alpha,\beta} \subseteq \tilde{S}_{\gamma_2}^{\alpha,\beta}$.

Corollary 3.1. If a sequence $x = \{x_k\}$ of fuzzy numbers is $\alpha\beta$ -statistically convergent of order γ , then it is $\alpha\beta$ -statistically convergent, for each $\gamma \in (0, 1]$, i.e. $\tilde{S}_\gamma^{\alpha,\beta} \subseteq \tilde{S}^{\alpha,\beta}$.

Theorem 3.5. The sequence spaces of fuzzy numbers $\tilde{W}_{\gamma_0}^{\alpha,\beta}$, $\tilde{W}_\gamma^{\alpha,\beta}$ and $\tilde{W}_{\gamma_\infty}^{\alpha,\beta}$ satisfy the relationship: $\tilde{W}_{\gamma_0}^{\alpha,\beta} \subset \tilde{W}_\gamma^{\alpha,\beta} \subset \tilde{W}_{\gamma_\infty}^{\alpha,\beta}$.

Proof. Let $x = \{x_k\} \in \tilde{W}_\gamma^{\alpha,\beta}$. Note that

$$\begin{aligned} & \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}} D(x_k, \bar{0}) \\ & \leq \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}} D(x_k, x_0) + \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}} D(x_0, \bar{0}) \\ & \leq \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}} D(x_k, x_0) + \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} D(x_0, \bar{0}), \end{aligned}$$

according to the above inequality, we have $\sup_n \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}} D(x_k, \bar{0}) < \infty$, thus we get $x \in \tilde{W}_{\gamma_\infty}^{\alpha,\beta}$.

The proof of $\tilde{W}_{\gamma_0}^{\alpha,\beta} \subset \tilde{W}_\gamma^{\alpha,\beta}$ is obvious.

Theorem 3.6. The sequence spaces of fuzzy numbers $\tilde{W}_{\gamma_0}^{\alpha,\beta}$, $\tilde{W}_\gamma^{\alpha,\beta}$ and $\tilde{W}_{\gamma_\infty}^{\alpha,\beta}$ are linear spaces over the set of real numbers.

Proof. Let $x = \{x_k\}$, $y = \{y_k\} \in \tilde{W}_{\gamma_0}^{\alpha,\beta}$, $\alpha, \beta \in R$. In order to get result we need to prove the following

$$\lim_n \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}} D(\alpha x_k + \beta y_k, \bar{0}) = 0.$$

Since $x = \{x_k\}$, $y = \{y_k\} \in \tilde{W}_{\gamma_0}^{\alpha,\beta}$, we have

$$\begin{aligned} \lim_n \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}} D(x_k, \bar{0}) &= 0, \\ \lim_n \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}} D(y_k, \bar{0}) &= 0. \end{aligned}$$

And $D(\alpha x_k + \beta y_k, \bar{0}) \leq D(\alpha x_k, \bar{0}) + D(\beta y_k, \bar{0}) = |\alpha| D(x_k, \bar{0}) + |\beta| D(y_k, \bar{0})$, we get

$$\begin{aligned} & \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}} D(\alpha x_k + \beta y_k, \bar{0}) \leq \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}} [D(\alpha x_k, \bar{0}) + D(\beta y_k, \bar{0})] \\ & \leq \frac{|\alpha|}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}} D(x_k, x_0) + \frac{|\beta|}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}} D(y_k, x_0) \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

Thus $\alpha x + \beta y \in \tilde{W}_{\gamma_0}^{\alpha,\beta}$. Similarly it can be shown that the other spaces are also linear spaces.

Theorem 3.7. Let $0 < \gamma \leq 1$. If a sequence $x = \{x_k\}$ of fuzzy number is strongly $\alpha\beta$ -convergent of order γ , then it is $\alpha\beta$ -statistically convergent of order γ , i.e. $\tilde{W}_\gamma^{\alpha,\beta} \subset \tilde{S}_\gamma^{\alpha,\beta}$.

Proof. Given $\varepsilon > 0$ and any sequence $x = \{x_k\}$ of fuzzy numbers, we write

$$\begin{aligned} \sum_{k \in P_n^{\alpha, \beta}} D(x_k, x_0) &= \sum_{k \in P_n^{\alpha, \beta}, D(x_k, x_0) < \varepsilon} D(x_k, x_0) + \sum_{k \in P_n^{\alpha, \beta}, D(x_k, x_0) \geq \varepsilon} D(x_k, x_0) \\ &\geq \sum_{k \in P_n^{\alpha, \beta}, D(x_k, x_0) \geq \varepsilon} D(x_k, x_0) \geq |\{k \in P_n^{\alpha, \beta} : D(x_k, x_0) \geq \varepsilon\}| \cdot \varepsilon \end{aligned}$$

and hence

$$\frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k \in P_n^{\alpha, \beta}} D(x_k, x_0) \geq \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} |\{k \in P_n^{\alpha, \beta} : D(x_k, x_0) \geq \varepsilon\}| \cdot \varepsilon.$$

Here, it can be easily to see that if a sequence $x = \{x_k\}$ of fuzzy number is strongly $\alpha\beta$ -convergent of order γ , then it is $\alpha\beta$ -statistically convergent of order γ .

Corollary 3.2. Let $0 < \gamma \leq \eta \leq 1$. If a sequence $x = \{x_k\}$ of fuzzy number is strongly $\alpha\beta$ -convergent of order γ , then it is $\alpha\beta$ -statistically convergent of order η , i.e. $\tilde{W}_\gamma^{\alpha, \beta} \subset \tilde{S}_\eta^{\alpha, \beta}$.

Definition 3.4. Let $p = \{p_k\}$ be any sequence of strictly positive real numbers. A sequence $x = \{x_k\}$ of fuzzy numbers is said to be strongly $\alpha\beta(p)$ -convergent of order γ , if for $\gamma \in (0, 1]$, there is a fuzzy number x_0 such that

$$\lim_n \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k \in P_n^{\alpha, \beta}} [D(x_k, x_0)]^{p_k} = 0,$$

we denote the set of all strongly $\alpha\beta(p)$ -convergent of order γ for fuzzy sequences by $\tilde{W}_\gamma^{\alpha, \beta}(p)$. Where

$$\tilde{W}_\gamma^{\alpha, \beta}(p) = \{x = \{x_k\} : \lim_n \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k \in P_n^{\alpha, \beta}} [D(x_k, x_0)]^{p_k} = 0\},$$

$$\tilde{W}_{\gamma 0}^{\alpha, \beta}(p) = \{x = \{x_k\} : \lim_n \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k \in P_n^{\alpha, \beta}} [D(x_k, \bar{0})]^{p_k} = 0\},$$

$$\tilde{W}_{\gamma \infty}^{\alpha, \beta}(p) = \{x = \{x_k\} : \sup_n \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k \in P_n^{\alpha, \beta}} [D(x_k, \bar{0})]^{p_k} < \infty\}.$$

It similar to the proofs of Theorem 3.5, 3.6, for strongly $\alpha\beta(p)$ -convergent of order γ we have the following results.

Theorem 3.8. The sequence spaces of fuzzy numbers $\tilde{W}_{\gamma 0}^{\alpha, \beta}(p)$, $\tilde{W}_\gamma^{\alpha, \beta}(p)$ and $\tilde{W}_{\gamma \infty}^{\alpha, \beta}(p)$ satisfy the relationship: $\tilde{W}_{\gamma 0}^{\alpha, \beta}(p) \subset \tilde{W}_\gamma^{\alpha, \beta}(p) \subset \tilde{W}_{\gamma \infty}^{\alpha, \beta}(p)$.

Theorem 3.9. The sequence spaces of fuzzy numbers $\tilde{W}_{\gamma 0}^{\alpha, \beta}(p)$, $\tilde{W}_\gamma^{\alpha, \beta}(p)$ and $\tilde{W}_{\gamma \infty}^{\alpha, \beta}(p)$ are linear spaces over the set of real numbers.

Theorem 3.10. Let $0 < p_k \leq q_k$, and $\{\frac{q_k}{p_k}\}$ be bounded. Then $\tilde{W}_\gamma^{\alpha, \beta}(q) \subset \tilde{W}_\gamma^{\alpha, \beta}(p)$.

Proof. Let $x = \{x_k\} \in \tilde{W}_\gamma^{\alpha, \beta}(q)$, and $t_k = [D(x_k, x_0)]^{q_k}$, $\lambda_k = \frac{p_k}{q_k}$, $0 < \lambda_k \leq 1$. Let $0 < \lambda < \lambda_k$,

and define $u_k = \begin{cases} t_k, & t_k \geq 1, \\ 0, & t_k < 1, \end{cases}$ $v_k = \begin{cases} 0, & t_k \geq 1, \\ t_k, & t_k < 1, \end{cases}$ then $t_k = u_k + v_k$, $t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$, and

$u_k^{\lambda_k} \leq u_k \leq t_k$, $v_k^{\lambda_k} \leq v_k$. We have

$$\begin{aligned} &\frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k \in P_n^{\alpha, \beta}} [D(x_k, x_0)]^{p_k} = \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k \in P_n^{\alpha, \beta}} t_k^{\lambda_k} \\ &= \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k \in P_n^{\alpha, \beta}} (u_k^{\lambda_k} + v_k^{\lambda_k}) \leq \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k \in P_n^{\alpha, \beta}} t_k \\ &+ \frac{1}{(\beta(n) - \alpha(n) + 1)^\gamma} \sum_{k \in P_n^{\alpha, \beta}} v_k^\lambda \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

Since $x = \{x_k\} \in \tilde{W}_\gamma^{\alpha,\beta}(q)$, we have $\lim_n \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}} t_k = 0$. And since $v_k < 1$, $\lambda < 1$, we get $\lim_n \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}} v_k^\lambda = 0$. Hence, $\tilde{W}_\gamma^{\alpha,\beta}(q) \subset \tilde{W}_\gamma^{\alpha,\beta}(p)$.

In the following theorem, we shall discuss the relationship between the space $\tilde{W}_\gamma^{\alpha,\beta}(p)$ and $\tilde{S}_\gamma^{\alpha,\beta}$.

Theorem 3.11. Let $0 < h = \inf_k p_k \leq \sup_k p_k = H < \infty$, l_∞ be a set of all bounded sequence of fuzzy numbers. Then

- (1) $\tilde{W}_\gamma^{\alpha,\beta}(p) \subset \tilde{S}_\gamma^{\alpha,\beta}$;
- (2) If $x = \{x_k\} \in l_\infty \cap \tilde{S}_\gamma^{\alpha,\beta}$, then $x = \{x_k\} \in \tilde{W}_\gamma^{\alpha,\beta}(p)$;
- (3) $l_\infty \cap \tilde{S}_\gamma^{\alpha,\beta} = l_\infty \cap \tilde{W}_\gamma^{\alpha,\beta}(p)$.

Proof. (1) Let $x = \{x_k\} \in \tilde{W}_\gamma^{\alpha,\beta}(p)$, Note that

$$\begin{aligned} & \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}} [D(x_k, x_0)]^{p_k} \geq \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}, D(x_k, x_0) \geq \varepsilon} [D(x_k, x_0)]^{p_k} \\ & \geq \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}, D(x_k, x_0) \geq \varepsilon} \min\{\varepsilon^h, \varepsilon^H\} \\ & = \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_0) \geq \varepsilon\}| \cdot \min\{\varepsilon^h, \varepsilon^H\}, \end{aligned}$$

follow from the above inequality, we have $\lim_n \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_0) \geq \varepsilon\}| = 0$. Thus we get $x = \{x_k\} \in \tilde{S}_\gamma^{\alpha,\beta}$.

(2) Let $x = \{x_k\} \in l_\infty \cap \tilde{S}_\gamma^{\alpha,\beta}$, then there is a constant $T > 0$, such that $D(x_k, x_0) \leq T$. Therefore

$$\begin{aligned} & \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}} [D(x_k, x_0)]^{p_k} \\ & = \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}, D(x_k, x_0) \geq \varepsilon} [D(x_k, x_0)]^{p_k} + \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}, D(x_k, x_0) < \varepsilon} [D(x_k, x_0)]^{p_k} \\ & \leq \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}, D(x_k, x_0) \geq \varepsilon} \max\{T^h, T^H\} + \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}, D(x_k, x_0) < \varepsilon} \varepsilon^{p_k} \\ & \leq \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} |\{k \in P_n^{\alpha,\beta} : D(x_k, x_0) \geq \varepsilon\}| \cdot \max\{T^h, T^H\} \\ & \quad + \max\{\varepsilon^h, \varepsilon^H\}, \end{aligned}$$

follow from the above inequality, we have $\lim_n \frac{1}{(\beta(n)-\alpha(n)+1)^\gamma} \sum_{k \in P_n^{\alpha,\beta}} [D(x_k, x_0)]^{p_k} = 0$. Thus we get $x = \{x_k\} \in \tilde{W}_\gamma^{\alpha,\beta}(p)$.

(3) From (1) and (2), (3) is obvious.

4. Conclusion

In this article, we introduced some classes of sequences of fuzzy numbers defined by $\alpha\beta$ -statistically convergence of order γ , strong $\alpha\beta$ -convergence of order γ , and strong $\alpha\beta(p)$ -convergence of order γ . We have proved some properties and relationships of these spaces. At the same time, it also shows that if a sequence of fuzzy numbers is strongly $\alpha\beta$ -convergent of order γ then it is $\alpha\beta$ -statistically convergent of order γ .

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IF rough approximations based on lattices*

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Abstract: An IF rough set, which is the result of approximation of an IF set with respect to an IF approximation space, is an extension of fuzzy rough sets. This paper studies rough set theory within the context of lattices. First, we introduce the concepts of IF rough sets and IF rough approximation operators based on lattices. Then, we give some properties on IF rough approximations of IF sublattices such as IF ideals and IF filters.

Keywords: Lattice; IF set; Full congruence relation; IF approximate space; IF rough set; IF sublattice; IF rough approximation.

1 Introduction

Rough set theory was originally proposed by Pawlak [11, 12] as a mathematical approach to handle imprecision and uncertainty in data analysis. Usefulness and versatility of this theory have amply been demonstrated by successful applications in a variety of problems [15, 16].

The basic structure of rough set theory is an approximation space. Based on it, lower and upper approximations can be induced. Using these approximations, knowledge hidden in information systems may be revealed and expressed in the form of decision rules [11].

Intuitionistic fuzzy (IF, for short) sets were originated by Atanassov [1, 2]. It is an intuitively straightforward extension of Zadeh's fuzzy sets [19]. IF sets have played an useful role in the research of uncertainty theories. Unlike a fuzzy set, which gives a degree of which element belongs to a set, an IF set gives both

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a membership degree and a nonmembership degree. Thus, an IF set is more objective than a fuzzy set to describe the vagueness of data or information.

Recently, rough set approximation was introduced into IF sets [14, 20, 21, 22]. For example, Zhou et al. [20, 21, 22] proposed a general framework for the study of IF rough sets, Zhang et al. [24] gave a general frame for IF rough sets on two universes.

The purpose of this paper is to investigate IF rough approximations based on lattices.

2 Preliminaries

Throughout this paper, “Intuitionistic fuzzy” is briefly written “IF”, U denotes a universe, I denotes $[0, 1]$, L denotes a lattice with the least element 0_L and the greatest element 1_L . $J = \{(a, b) \in I \times I : a + b \leq 1\}$.

In this section, we recall some basic notions and properties.

2.1 IF sets

Definition 2.1 ([8]). Let $(a, b), (c, d) \in I \times I$. Define

- (1) $(a, b) = (c, d) \iff a = c, b = d$.
- (2) $(a, b) \sqcup (c, d) = (a \vee c, b \wedge d)$, $(a, b) \sqcap (c, d) = (a \wedge c, b \vee d)$.
- (3) $(a, b)^c = (b, a)$.

Moreover, for $\{(a_\alpha, b_\alpha) : \alpha \in \Gamma\} \subseteq I \times I$,

$$\bigsqcup_{\alpha \in \Gamma} (a_\alpha, b_\alpha) = (\bigvee_{\alpha \in \Gamma} a_\alpha, \bigwedge_{\alpha \in \Gamma} b_\alpha), \quad \bigsqcap_{\alpha \in \Gamma} (a_\alpha, b_\alpha) = (\bigwedge_{\alpha \in \Gamma} a_\alpha, \bigvee_{\alpha \in \Gamma} b_\alpha).$$

Definition 2.2 ([8]). Let $(a, b), (c, d) \in J$ and let $S \subseteq J \times J$. $(a, b)S(c, d)$, if $a \leq c$ and $b \geq d$. We denote S by \leq .

Remark 2.3. (1) Let (J, \leq) be a poset with $0_J = (0, 1)$ and $1_J = (1, 0)$.

- (2) $(a, b)^{cc} = (a, b)$.
- (3) $((a, b) \sqcup (c, d)) \sqcup (e, f) = (a, b) \sqcup ((c, d) \sqcup (e, f))$,
 $((a, b) \sqcap (c, d)) \sqcap (e, f) = (a, b) \sqcap ((c, d) \sqcap (e, f))$.
- (4) $(a, b) \sqcup (c, d) = (c, d) \sqcup (a, b)$, $(a, b) \sqcap (c, d) = (c, d) \sqcap (a, b)$.
- (5) $((a, b) \sqcup (c, d)) \sqcap (e, f) = ((a, b) \sqcap (e, f)) \sqcup ((c, d) \sqcap (e, f))$.
 $((a, b) \sqcap (c, d)) \sqcup (e, f) = ((a, b) \sqcup (e, f)) \sqcap ((c, d) \sqcup (e, f))$.
- (6) $(\bigsqcup_{\alpha \in \Gamma} (a_\alpha, b_\alpha))^c = \bigsqcap_{\alpha \in \Gamma} (a_\alpha, b_\alpha)^c$, $(\bigsqcap_{\alpha \in \Gamma} (a_\alpha, b_\alpha))^c = \bigsqcup_{\alpha \in \Gamma} (a_\alpha, b_\alpha)^c$.

Definition 2.4 ([1]). An IF set A in U is an object having the form

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle : x \in U \},$$

where $\mu_A, \nu_A \in F(U)$ satisfying $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in U$, and $\mu_A(x), \nu_A(x)$ are used to define the degree of membership and the degree of non-membership of the element x to A , respectively.

$IF(U)$ denotes the family of all IF sets in U .

For the sake of simplicity, we give the following definition.

Definition 2.5. A is called an IF set in U , if $A = (A^*, A_*) \in F(U) \times F(U)$ and for each $x \in U$, $A(x) = (A^*(x), A_*(x)) \in J$, where $A^*(x), A_*(x)$ are used to define the degree of membership and the degree of non-membership of the element x to A , respectively.

For each $\mathcal{A} \subseteq IF(U)$, we denote

$$\mathcal{A}^c = \{A^c : A \in \mathcal{A}\},$$

$$\mathcal{A}^* = \{A^* : A \in \mathcal{A}\} \text{ and } \mathcal{A}_* = \{A_* : A \in \mathcal{A}\}.$$

For each $\lambda \in J$, $\hat{\lambda}$ represents a constant IF set which satisfies $\hat{\lambda}(x) = \lambda$ for each $x \in U$.

$A \in IF(U)$ is called proper if $A \neq \hat{\lambda}$ for any $\lambda \in J$.

In this paper, if we concern IF sets in U without special statements, we always refer to the proper IF subset.

Some IF relations and IF operations are defined as follows ([19]): for any $A, B \in IF(U)$ and $\{A_\alpha : \alpha \in \Gamma\} \subseteq IF(U)$,

$$(1) A = B \iff A(x) = B(x) \text{ for each } x \in U.$$

$$(2) A \subseteq B \iff A(x) \leq B(x) \text{ for each } x \in U.$$

$$(3) (\bigcup_{\alpha \in \Gamma} A_\alpha)(x) = \bigcup_{\alpha \in \Gamma} A_\alpha(x) \text{ for each } x \in U.$$

$$(4) (\bigcap_{\alpha \in \Gamma} A_\alpha)(x) = \bigcap_{\alpha \in \Gamma} A_\alpha(x) \text{ for each } x \in U.$$

$$(5) A^c(x) = A(x)^c \text{ for each } x \in U.$$

$$(6) (\lambda A)(x) = \lambda \sqcap (A^*(x), A_*(x)) \text{ for any } x \in U \text{ and } \lambda \in J.$$

Obviously, $A = B \iff A^* = B^*$ and $A_* = B_* \iff A \subseteq B$ and $B \subseteq A$.

We define a special IF sets $1_y = ((1_y)^*, (1_y)_*)$ for some $y \in U$ as follows:

$$(1_y)^*(x) = \begin{cases} 1, & x = y, \\ 0, & x \neq y. \end{cases} \quad (1_y)_*(x) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

Remark 2.6. For each $A \in IF(U)$,

$$A = \bigcup_{y \in U} (A(y)1_y).$$

Let $\mu \in IF(U)$ and $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$, the (α, β) -level cut set of μ , denoted by μ_α^β , is defined as follows:

$$\mu_\alpha^\beta = \{x \in U : \mu^*(x) \geq \alpha, \mu_*(x) \leq \beta\}.$$

We respectively call the sets

$$\mu_\alpha = \{x \in U : \mu^*(x) \geq \alpha\}, \quad \mu^\beta = \{x \in U : \mu_*(x) \leq \beta\}$$

the α -level cut set, the β -level set of membership generated by A .

For $x \in U$ and $(a, b) \in J - \{(0, 1)\}$, $x^{(a,b)} \in IF(U)$ is called an IF point if

$$x^{(a,b)}(y) = \begin{cases} (0, 1), & y \neq x, \\ (a, b), & y = x. \end{cases}$$

It is said that the IF point $x^{(a,b)}$ belongs to $\mu \in IF(U)$, which is written $x^{(a,b)} \in \mu$. Obviously,

$$x^{(a,b)} \in \mu \iff \mu(x) \geq (a, b).$$

$IFP(U)$ denotes the set of all IF point of U .

For $\mu, \lambda \in IF(U)$,

$$\mu \subseteq \lambda \iff \forall x^{(a,b)} \in IFP(U), x^{(a,b)} \in \mu \text{ implies } x^{(a,b)} \in \lambda.$$

2.2 Lattices

Definition 2.7. Let L be a set and let \leq be a binary relation on L . Then \leq is called a partial order on L , if

- (i) $a \leq a$ for any $a \in L$, (ii) $a \leq b$ and $b \leq a$ imply $a = b$ for any $a, b \in L$,
- (iii) $a \leq b$ and $b \leq c$ imply $a \leq c$ for any $a, b, c \in L$.

Moreover, the pair (L, \leq) is called a partial order set (briefly, a poset).

Definition 2.8. Let (L, \leq) be a poset and $a, b \in L$.

- (1) a is called a top (or maximal) element of L , if $x \leq a$ for any $x \in L$.
- (2) b is called a bottom (or minimal) element of L , if $b \leq x$ for any $x \in L$.

If a poset L has top elements a_1, a_2 (resp. bottom elements b_1, b_2), then $a_1 = a_2$ (resp. $b_1 = b_2$). We denote this sole top element (resp. this sole bottom element) by 1_L (resp. 0_L).

Definition 2.9. Let (L, \leq) be a poset, $S \subseteq L$ and $a, b \in L$.

- (1) a is called a above boundary in S , if $x \leq a$ for any $x \in S$.
- (2) b is called a under boundary in S , if $b \leq x$ for any $x \in S$.
- (3) $a = \sup S$ or $\vee S$, if a is a minimal above boundary in S .
- (4) $b = \inf S$ or $\wedge S$, if b is a maximal under boundary in S .

Let (L, \leq) be a poset and $S \subseteq L$. If $S = \{a, b\}$, then we denote $\vee S = a \vee b$ and $\wedge S = a \wedge b$.

Obviously, if (L, \leq) is a poset and $a, b \in L$, then

$$a = a \wedge b \iff a \leq b \iff b = a \vee b.$$

A poset L is called a lattice, if for any $a, b \in L$, $a \vee b \in L$ and $a \wedge b \in L$.

Let L be a lattice. For $X \subseteq L$, we denote

- (1) $\downarrow X = \{y \in L : y \leq x \text{ for some } x \in X\}$,
- (2) $\uparrow X = \{y \in L : y \geq x \text{ for some } x \in X\}$.

Especially, $\downarrow x = \downarrow \{x\}$, $\uparrow x = \uparrow \{x\}$.

$F(L)$ (resp. $IF(L)$) denotes the family of all fuzzy (resp. IF) sets in L .
 $\mu \in F(L)$ is called a fuzzy sublattice of L , if $\mu(x \wedge y) \wedge \mu(x \vee y) \geq \mu(x) \wedge \mu(y)$ for any $x, y \in L$.

Let μ be a fuzzy sublattice of L .

(1) μ is a fuzzy ideal of L , if $\mu(x \vee y) = \mu(x) \wedge \mu(y)$ for any $x, y \in L$.

(2) μ is a fuzzy filter of L , if $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$ for any $x, y \in L$.

2.3 Fuzzy rough approximation operators based on lattices

Definition 2.10 ([3]). Let θ be an equivalence relation on L . The pair (L, θ) is called Pawlak approximation space. For each $\mu \in F(L)$, the fuzzy lower and the fuzzy upper approximation of μ with respect to (L, θ) , denoted by $\underline{\theta}(\mu)$ and $\bar{\theta}(\mu)$, are defined as follows: for each $x \in L$,

$$\underline{\theta}(\mu)(x) = \bigwedge_{a \in [x]_{\theta}} \mu(a), \quad \bar{\theta}(\mu)(x) = \bigvee_{a \in [x]_{\theta}} \mu(a).$$

The pair $(\underline{\theta}(\mu), \bar{\theta}(\mu))$ is called the fuzzy rough set of μ with respect to (L, θ) .

$\underline{\theta} : F(L) \rightarrow F(L)$ and $\bar{\theta} : F(L) \rightarrow F(L)$ are called the fuzzy lower approximation operator and the fuzzy upper approximation operator, respectively. In general, we refer to $\underline{\theta}$ and $\bar{\theta}$ as the fuzzy rough approximation operators.

Proposition 2.11 ([3]). Let θ be an equivalence relation on L . Then for $\mu, \lambda \in F(L)$,

- (1) $\underline{\theta}(\mu) \subseteq \mu \subseteq \bar{\theta}(\mu)$.
- (2) If $\mu \subseteq \lambda$, then $\underline{\theta}(\mu) \subseteq \underline{\theta}(\lambda)$ and $\bar{\theta}(\mu) \subseteq \bar{\theta}(\lambda)$.
- (3) $\underline{\theta}(\mu^c) = (\bar{\theta}(\mu))^c$ and $\bar{\theta}(\mu^c) = (\underline{\theta}(\mu))^c$.
- (4) $\underline{\theta}\bar{\theta}(\mu) = \bar{\theta}(\mu)$ and $\bar{\theta}\underline{\theta}(\mu) = \underline{\theta}(\mu)$.
- (5) $\underline{\theta}(\mu)(x) = \underline{\theta}(\mu)(a)$ and $\bar{\theta}(\mu)(x) = \bar{\theta}(\mu)(a)$ for any $x \in L$ and $a \in [x]_{\theta}$.
- (6) $\underline{\theta}\bar{\theta}(\mu) = \bar{\theta}(\mu)$ and $\bar{\theta}\underline{\theta}(\mu) = \underline{\theta}(\mu)$.

Definition 2.12 ([3]). Let θ be an equivalence relation on L . Then θ is called a full congruence relation, if $(a, b) \in \theta$ implies that $(a \vee x, b \vee x) \in \theta$ and $(a \wedge x, b \wedge x) \in \theta$ for any $x \in L$.

For $a \in L$, denote

$$[a]_{\theta} = \{x \in L : (a, x) \in \theta\}, \quad L/\theta = \{[a]_{\theta} : a \in L\}.$$

Lemma 2.13 ([3]). Let θ be a full congruence relation on L . Then for any $a, b, c, d \in L$,

- (1) If $(a, b), (c, d) \in \theta$, then $(a \vee c, b \vee d), (a \wedge c, b \wedge d) \in \theta$.
- (2) If $x \in [a]_{\theta}, y \in [b]_{\theta}$, then $x \vee y \in [a \vee b]_{\theta}$.
- (3) If $x \in [a]_{\theta}, y \in [b]_{\theta}$, then $x \wedge y \in [a \wedge b]_{\theta}$.

Proposition 2.14 ([3]). Let θ be a full congruence relation on L .

(1) If μ is a fuzzy ideal, then for $x, y \in L$,

$$\underline{\theta}(\mu)(x \wedge y) = \bigwedge_{a \in [x]_{\theta}, b \in [y]_{\theta}} \mu(a \wedge b), \quad \bar{\theta}(\mu)(x \vee y) = \bigvee_{a \in [x]_{\theta}, b \in [y]_{\theta}} \mu(a \vee b).$$

(2) If μ is a fuzzy filter, then for $x, y \in L$,

$$\underline{\theta}(\mu)(x \vee y) = \bigwedge_{a \in [x]_{\theta}, b \in [y]_{\theta}} \mu(a \vee b), \quad \bar{\theta}(\mu)(x \wedge y) = \bigvee_{a \in [x]_{\theta}, b \in [y]_{\theta}} \mu(a \wedge b).$$

3 IF rough sets and IF rough approximation operators based on lattices

Definition 3.1. Let θ be an equivalence relation on L . The pair (L, θ) is called Pawlak approximation space. For each $\mu \in IF(L)$, the IF lower and the IF upper approximation of μ with respect to (L, θ) , denoted by $\underline{\theta}(\mu)$ and $\bar{\theta}(\mu)$, are defined as follows:

$$\begin{aligned} \underline{\theta}(\mu) &= ((\underline{\theta}(\mu))^*, (\underline{\theta}(\mu))_*), \\ \bar{\theta}(\mu) &= ((\bar{\theta}(\mu))^*, (\bar{\theta}(\mu))_*), \end{aligned}$$

where for each $x \in L$,

$$\begin{aligned} (\underline{\theta}(\mu))^*(x) &= \bigwedge_{a \in [x]_{\theta}} \mu^*(a), & (\underline{\theta}(\mu))_*(x) &= \bigvee_{a \in [x]_{\theta}} \mu_*(a), \\ (\bar{\theta}(\mu))^*(x) &= \bigvee_{a \in [x]_{\theta}} \mu^*(a), & (\bar{\theta}(\mu))_*(x) &= \bigwedge_{a \in [x]_{\theta}} \mu_*(a). \end{aligned}$$

The pair $(\underline{\theta}(\mu), \bar{\theta}(\mu))$ is called the IF rough set of μ with respect to (L, θ) .

$\underline{\theta} : IF(L) \rightarrow IF(L)$ and $\bar{\theta} : IF(L) \rightarrow IF(L)$ are called the IF lower approximation operator and the IF upper approximation operator, respectively. In general, we refer to $\underline{\theta}$ and $\bar{\theta}$ as the IF rough approximation operators.

Remark 3.2. (1) $(\underline{\theta}(\mu))^* = \underline{\theta}(\mu^*)$ $(\underline{\theta}(\mu))_* = \bar{\theta}(\mu_*)$
 (2) $(\bar{\theta}(\mu))^* = \bar{\theta}(\mu^*)$ $(\bar{\theta}(\mu))_* = \underline{\theta}(\mu_*)$

Proposition 3.3. For any $x \in L$,

$$\underline{\theta}(\mu)(x) = \bigcap_{a \in [x]_{\theta}} \mu(a), \quad \bar{\theta}(\mu)(x) = \bigcup_{a \in [x]_{\theta}} \mu(a).$$

Proof.

$$\begin{aligned} \underline{\theta}(\mu)(x) &= \left(\bigwedge_{a \in [x]_{\theta}} \mu^*(a), \bigvee_{a \in [x]_{\theta}} \mu_*(a) \right) \\ &= \bigcap_{a \in [x]_{\theta}} (\mu^*(a), \mu_*(a)) \\ &= \bigcap_{a \in [x]_{\theta}} \mu(a). \end{aligned}$$

$$\begin{aligned}
\bar{\theta}(\mu)(x) &= (\bigvee_{a \in [x]_{\theta}} \mu^*(a), \bigwedge_{a \in [x]_{\theta}} \mu_*(a)) \\
&= \bigvee_{a \in [x]_{\theta}} (\mu^*(a), \mu_*(a)) \\
&= \bigvee_{a \in [x]_{\theta}} \mu(a).
\end{aligned}$$

□

Proposition 3.4. Let θ be an equivalence relation on L . Then for any $\mu, \lambda \in IF(L)$,

- (1) $\underline{\theta}(\mu) \subseteq \mu \subseteq \bar{\theta}(\mu)$.
- (2) If $\mu \subseteq \lambda$, then $\bar{\theta}(\mu) \subseteq \bar{\theta}(\lambda)$ and $\underline{\theta}(\mu) \subseteq \underline{\theta}(\lambda)$.
- (3) $\bar{\theta}\bar{\theta}(\mu) = \bar{\theta}(\mu)$ and $\underline{\theta}\underline{\theta}(\mu) = \underline{\theta}(\mu)$.
- (4) $\underline{\theta}(\mu)(x) = \underline{\theta}(\mu)(a)$ and $\bar{\theta}(\mu)(x) = \bar{\theta}(\mu)(a)$ for any $x \in L$ and $a \in [x]_{\theta}$.
- (5) $\bar{\theta}\bar{\theta}(\mu) = \bar{\theta}(\mu)$ and $\underline{\theta}\underline{\theta}(\mu) = \underline{\theta}(\mu)$.

Proof. It is straightforward. □

Proposition 3.5. Let θ be an equivalence relation on L . Then for any $\{\mu_i : i \in I\} \subseteq IF(L)$,

- (1) $\underline{\theta}(\bigvee_{i \in I} \mu_i) \supseteq \bigvee_{i \in I} \underline{\theta}(\mu_i)$, $\underline{\theta}(\bigwedge_{i \in I} \mu_i) = \bigwedge_{i \in I} \underline{\theta}(\mu_i)$.
- (2) $\bar{\theta}(\bigvee_{i \in I} \mu_i) = \bigvee_{i \in I} \bar{\theta}(\mu_i)$, $\bar{\theta}(\bigwedge_{i \in I} \mu_i) \subseteq \bigwedge_{i \in I} \bar{\theta}(\mu_i)$.

Proof. (1) For any $x \in L$,

$$\begin{aligned}
\underline{\theta}(\bigvee_{i \in I} \mu_i)(x) &= \bigwedge_{a \in [x]_{\theta}} \bigvee_{i \in I} \mu_i(a) \supseteq \bigvee_{i \in I} \bigwedge_{a \in [x]_{\theta}} \mu_i(a) = \bigvee_{i \in I} \underline{\theta}(\mu_i)(x), \\
\underline{\theta}(\bigwedge_{i \in I} \mu_i)(x) &= \bigwedge_{a \in [x]_{\theta}} \bigwedge_{i \in I} \mu_i(a) = \bigwedge_{i \in I} \bigwedge_{a \in [x]_{\theta}} \mu_i(a) = \bigwedge_{i \in I} \underline{\theta}(\mu_i)(x).
\end{aligned}$$

Thus, $\underline{\theta}(\bigvee_{i \in I} \mu_i) \supseteq \bigvee_{i \in I} \underline{\theta}(\mu_i)$, $\underline{\theta}(\bigwedge_{i \in I} \mu_i) = \bigwedge_{i \in I} \underline{\theta}(\mu_i)$.

- (2) The proof is similar to (1). □

4 IF sublattices and IF rough approximations based on lattices

4.1 IF sublattices

Definition 4.1. $\mu \in IF(L)$ is called an IF sublattice of L , if $\mu(x \wedge y) \sqcap \mu(x \vee y) \geq \mu(x) \sqcap \mu(y)$ for any $x, y \in L$.

Definition 4.2. Let μ be an IF sublattice of L . Then

- (1) μ is an IF ideal of L , if $\mu(x \vee y) = \mu(x) \sqcap \mu(y)$ for any $x, y \in L$.
- (2) μ is an IF filter of L , if $\mu(x \wedge y) = \mu(x) \sqcap \mu(y)$ for any $x, y \in L$.

Denote the set of all IF ideals of L by $IFI(L)$.

Proposition 4.3. *Let μ be an IF sublattice of L . Then*

- (1) μ is an IF ideal of $L \iff x \leq y$ implies that $\mu(x) \geq \mu(y)$ for any $x, y \in L$.
 (2) μ is an IF filter of $L \iff x \leq y$ implies that $\mu(x) \leq \mu(y)$ for any $x, y \in L$.

Proof. It is straightforward. \square

Let μ be a proper IF ideal of L . Then

- (1) μ is called an IF prime ideal of L , if $\mu(x \wedge y) \leq \mu(x) \sqcup \mu(y)$ for any $x, y \in L$.
 (2) μ is called an IF prime filter of L , if $\mu(x \vee y) \leq \mu(x) \sqcup \mu(y)$ for any $x, y \in L$.

4.2 IF rough approximations of some IF sublattices

Lemma 4.4. *Let θ be a full congruence relation on L .*

- (1) *If μ is an IF ideal of L , then for any $x, y \in L$,*

$$\underline{\theta}(\mu)(x \wedge y) = \bigcap_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \wedge b), \quad \bar{\theta}(\mu)(x \vee y) = \bigcup_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \vee b).$$

- (2) *If μ is an IF filter of L , then for any $x, y \in L$,*

$$\underline{\theta}(\mu)(x \vee y) = \bigcap_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \vee b), \quad \bar{\theta}(\mu)(x \wedge y) = \bigcup_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \wedge b).$$

Proof. (1) By Lemma 2.12,

$$\bigwedge_{z \in [x \wedge y]_\theta} \mu^*(z) \leq \bigwedge_{a \in [x]_\theta, b \in [y]_\theta} \mu^*(a \wedge b), \quad \bigvee_{z \in [x \wedge y]_\theta} \mu_*(z) \geq \bigvee_{a \in [x]_\theta, b \in [y]_\theta} \mu_*(a \wedge b).$$

Then

$$\underline{\theta}(\mu)(x \wedge y) = \bigcap_{z \in [x \wedge y]_\theta} \mu(z) \leq \bigcap_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \wedge b).$$

Now assume that $z \in [x \wedge y]_\theta$. Then $z \vee x \in [x]_\theta$, $z \vee y \in [y]_\theta$.

Since $z \leq (z \vee x) \wedge (z \vee y)$, by Proposition 2.14, we have

$$\mu(z) \geq \mu(z \vee x) \wedge \mu(z \vee y).$$

Then

$$\mu^*(z) \geq \mu^*((z \vee x) \wedge (z \vee y)), \quad \mu_*(z) \leq \mu_*((z \vee x) \wedge (z \vee y)).$$

Note that

$$\bigwedge_{z \in [x \wedge y]_\theta} \mu^*(z) \geq \bigwedge_{a \in [x]_\theta, b \in [y]_\theta} \mu^*(a \wedge b), \quad \bigvee_{z \in [x \wedge y]_\theta} \mu_*(z) \leq \bigvee_{a \in [x]_\theta, b \in [y]_\theta} \mu_*(a \wedge b).$$

Then $\bigcap_{z \in [x \wedge y]_\theta} \mu(z) \geq \bigcap_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \wedge b)$.

Thus

$$\underline{\theta}(\mu)(x \wedge y) = \bigcap_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \wedge b).$$

Note that

$$\bigvee_{z \in [x \wedge y]_\theta} \mu^*(z) \geq \bigvee_{a \in [x]_\theta, b \in [y]_\theta} \mu^*(a \vee b), \quad \bigwedge_{z \in [x \vee y]_\theta} \mu_*(z) \geq \bigwedge_{a \in [x]_\theta, b \in [y]_\theta} \mu_*(a \vee b).$$

Then

$$\bar{\theta}(\mu)(x \vee y) = \bigcup_{z \in [x \vee y]_\theta} \mu(z) \geq \bigcup_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \vee b).$$

Now assume that $z \in [x \vee y]_\theta$. Then

$$z \wedge x \in [x]_\theta, \quad z \wedge y \in [y]_\theta.$$

Since $z \geq (z \wedge x) \vee (z \wedge y)$, by Proposition 2.14, we have

$$\mu(z) \leq \mu(z \wedge x) \vee \mu(z \wedge y).$$

Then $\mu^*(z) \leq \mu^*((z \wedge x) \vee (z \wedge y))$, $\mu_*(z) \geq \mu_*((z \wedge x) \vee (z \wedge y))$.

Since

$$\bigvee_{z \in [x \vee y]_\theta} \mu^*(z) \leq \bigvee_{a \in [x]_\theta, b \in [y]_\theta} \mu^*(a \vee b), \quad \bigwedge_{z \in [x \vee y]_\theta} \mu_*(z) \geq \bigwedge_{a \in [x]_\theta, b \in [y]_\theta} \mu_*(a \vee b).$$

we have

$$\bigcup_{z \in [x \vee y]_\theta} \mu(z) \leq \bigcup_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \vee b).$$

Thus

$$\bar{\theta}(\mu)(x \vee y) = \bigcup_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \vee b).$$

(2) The proof is similar to (1). \square

Proposition 4.5. *Let θ be a full congruence relation on L . Let $\mu \in IF(L)$ and let $\underline{\theta}(\mu)$ be an IF sublattice of L . Then*

- (1) *If μ is an IF ideal of L , then $\underline{\theta}(\mu)$ is an IF ideal of L .*
- (2) *If μ is an IF filter of L , then $\underline{\theta}(\mu)$ is an IF filter of L .*

Proof. (1) Since $\underline{\theta}(\mu)$ is an IF sublattice of L , we conclude that for any $x, y \in L$

$$\underline{\theta}(\mu)(x \vee y) \geq \underline{\theta}(\mu)(x \wedge y) \sqcap \underline{\theta}(\mu)(x \vee y) \geq \underline{\theta}(\mu)(x) \sqcap \underline{\theta}(\mu)(y).$$

By Lemma 4.4, for any $x, y \in L$,

$$\begin{aligned}
\underline{\theta}(\mu)(x \vee y) &= \bigcap_{z \in [x \vee y]_\theta} \mu(z) \\
&\leq \bigcap_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \vee b) \\
&= \bigcap_{a \in [x]_\theta, b \in [y]_\theta} (\mu(a) \sqcap \mu(b)) \\
&= \bigcap_{a \in [x]_\theta, b \in [y]_\theta} (\mu^*(a) \wedge \mu^*(b), \mu_*(a) \vee \mu_*(b)) \\
&= (\bigwedge_{a \in [x]_\theta, b \in [y]_\theta} (\mu^*(a) \wedge \mu^*(b)), \bigvee_{a \in [x]_\theta, b \in [y]_\theta} (\mu_*(a) \vee \mu_*(b))) \\
&= ((\bigwedge_{a \in [x]_\theta} \mu(a)) \wedge (\bigwedge_{b \in [y]_\theta} \mu(b)), (\bigvee_{a \in [x]_\theta} \mu(a)) \vee (\bigvee_{b \in [y]_\theta} \mu(b))) \\
&= (\bigwedge_{a \in [x]_\theta} \mu(a), \bigvee_{a \in [x]_\theta} \mu(a)) \sqcap (\bigwedge_{b \in [y]_\theta} \mu(b), \bigvee_{b \in [y]_\theta} \mu(b)) \\
&= (\bigcap_{a \in [x]_\theta} \mu(a)) \sqcap (\bigcap_{b \in [y]_\theta} \mu(b)) \\
&= \underline{\theta}(\mu)(x) \sqcap \underline{\theta}(\mu)(y).
\end{aligned}$$

(2) The proof is similar to (1). \square

Proposition 4.6. Let θ be a full congruence relation on L . Then for $\mu \in IF(L)$,

- (1) If μ is an IF sublattice of L , then $\bar{\theta}(\mu)$ is an IF sublattice of L .
- (2) If μ is an IF ideal of L , then $\bar{\theta}(\mu)$ is an IF ideal of L .
- (3) If μ is an IF filter of L , then $\bar{\theta}(\mu)$ is an IF filter of L .

Proof. (1) Suppose that μ is an IF sublattice of L . Then for any $x, y \in L$,

$$\begin{aligned}
&\bar{\theta}(\mu)(x \wedge y) \sqcap \bar{\theta}(\mu)(x \vee y) \\
&= (\bigvee_{a \in [x \wedge y]_\theta} \mu^*(a), \bigwedge_{a \in [x \wedge y]_\theta} \mu_*(a)) \sqcap (\bigvee_{b \in [x \vee y]_\theta} \mu^*(b), \bigwedge_{b \in [x \vee y]_\theta} \mu_*(b)) \\
&= (\bigvee_{a \in [x \wedge y]_\theta} \mu^*(a) \wedge \bigvee_{b \in [x \vee y]_\theta} \mu^*(b), \bigwedge_{a \in [x \wedge y]_\theta} \mu^*(a) \vee \bigwedge_{b \in [x \vee y]_\theta} \mu^*(b)) \\
&\geq (\bigvee_{a \in [x]_\theta, c \in [y]_\theta} \mu^*(a \wedge c) \wedge \bigvee_{b \in [x]_\theta, d \in [y]_\theta} \mu^*(b \vee d), \\
&\quad \bigwedge_{a \in [x]_\theta, c \in [y]_\theta} \mu_*(a \wedge c) \vee \bigwedge_{b \in [x]_\theta, d \in [y]_\theta} \mu_*(b \vee d)) \\
&\geq (\bigvee_{a, b \in [x]_\theta, c, d \in [y]_\theta} \mu^*(a \wedge c) \wedge \mu^*(b \vee d), \bigwedge_{a, b \in [x]_\theta, c, d \in [y]_\theta} \mu_*(a \wedge c) \vee \mu_*(b \vee d)) \\
&\geq (\bigvee_{a \in [x]_\theta, c \in [y]_\theta} \mu^*(a \wedge c) \wedge \mu^*(a \vee c), \bigwedge_{a \in [x]_\theta, c \in [y]_\theta} \mu_*(a \wedge c) \vee \mu_*(a \vee c)) \\
&\geq (\bigvee_{a \in [x]_\theta, c \in [y]_\theta} \mu^*(a) \wedge \mu^*(c), \bigwedge_{a \in [x]_\theta, c \in [y]_\theta} \mu_*(a) \vee \mu_*(c))
\end{aligned}$$

$$\begin{aligned}
&= (\bigvee_{a \in [x]_\theta} \mu^*(a) \wedge \bigvee_{c \in [y]_\theta} \mu^*(c), \bigwedge_{a \in [x]_\theta} \mu_*(a) \vee \bigwedge_{c \in [y]_\theta} \mu_*(c)) \\
&= (\bigvee_{a \in [x]_\theta} \mu^*(a), \bigwedge_{a \in [x]_\theta} \mu_*(a)) \sqcap (\bigvee_{b \in [y]_\theta} \mu^*(b), \bigwedge_{b \in [y]_\theta} \mu_*(b)) \\
&= \bar{\theta}(\mu)(x) \sqcap \bar{\theta}(\mu)(y)
\end{aligned}$$

Thus $\bar{\theta}(\mu)$ is an IF sublattice of L .

(2) Suppose that μ is an IF ideal of L . Then μ is an IF sublattice of L .

By (1), $\bar{\theta}(\mu)$ is an IF sublattice of L .

For any $x, y \in L$,

$$\begin{aligned}
\bar{\theta}(\mu)(x \vee y) &= \bigsqcup_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \vee b) \\
&= \bigsqcup_{a \in [x]_\theta, b \in [y]_\theta} (\mu(a) \sqcap \mu(b)) \\
&= \bigsqcup_{a \in [x]_\theta, b \in [y]_\theta} (\mu^*(a) \wedge \mu^*(b), \mu_*(a) \vee \mu_*(b)) \\
&= (\bigvee_{a \in [x]_\theta, b \in [y]_\theta} (\mu^*(a) \wedge \mu^*(b)), \bigwedge_{a \in [x]_\theta, b \in [y]_\theta} (\mu_*(a) \vee \mu_*(b))) \\
&= ((\bigvee_{a \in [x]_\theta} \mu^*(a)) \wedge (\bigvee_{b \in [y]_\theta} \mu^*(b)), (\bigwedge_{a \in [x]_\theta} \mu_*(a)) \vee (\bigwedge_{b \in [y]_\theta} \mu_*(b))) \\
&= (\bigvee_{a \in [x]_\theta} \mu^*(a), \bigwedge_{a \in [x]_\theta} \mu_*(a)) \sqcap (\bigvee_{b \in [y]_\theta} \mu^*(b), \bigwedge_{b \in [y]_\theta} \mu_*(b)) \\
&= \bigsqcup_{a \in [x]_\theta} \mu(a) \sqcap \bigsqcup_{b \in [y]_\theta} \mu(b) \\
&= \bar{\theta}(\mu)(x) \sqcap \bar{\theta}(\mu)(y)
\end{aligned}$$

Thus $\bar{\theta}(\mu)$ is an IF ideal of L .

(3) The proof is similar to (2). \square

Proposition 4.7. *Let θ be a full congruence relation on L and let $\underline{\theta}(\mu)$ is a proper IF sublattice of L .*

(1) *If $\mu \in IF(L)$ is an IF prime ideal of L , then $\underline{\theta}(\mu)$ is an IF prime ideal of L .*

(2) *If $\mu \in IF(L)$ is an IF prime filter of L , then $\underline{\theta}(\mu)$ is an IF prime filter of L .*

Proof. (1) Suppose that μ is an IF prime ideal of L .

By Proposition 4.5, $\underline{\theta}(\mu)$ is an IF ideal of L .

By Proposition 4.3 and Lemma 4.4, for any $x, y \in L$.

$$\begin{aligned}
\underline{\theta}(\mu)(x \wedge y) &= \bigcap_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \wedge b) \\
&= \bigcap_{a \in [x]_\theta, b \in [y]_\theta} (\mu^*(a \wedge b), \mu_*(a \wedge b)) \\
&= \left(\bigwedge_{a \in [x]_\theta, b \in [y]_\theta} \mu^*(a \wedge b), \bigvee_{a \in [x]_\theta, b \in [y]_\theta} \mu_*(a \wedge b) \right) \\
&\leq \left(\bigwedge_{a \in [x]_\theta, b \in [y]_\theta} (\mu^*(a) \wedge \mu^*(b)), \bigvee_{a \in [x]_\theta, b \in [y]_\theta} (\mu_*(a) \wedge \mu_*(b)) \right) \\
&= \left(\bigwedge_{a \in [x]_\theta} \mu^*(a) \vee \bigwedge_{b \in [y]_\theta} \mu^*(b), \bigvee_{a \in [x]_\theta} \mu_*(a) \wedge \bigvee_{b \in [y]_\theta} \mu_*(b) \right) \\
&= \left(\bigwedge_{a \in [x]_\theta} \mu^*(a), \bigvee_{a \in [x]_\theta} \mu_*(a) \right) \sqcup \left(\bigwedge_{b \in [y]_\theta} \mu^*(b), \bigvee_{b \in [y]_\theta} \mu_*(b) \right) \\
&= \bigcap_{a \in [x]_\theta} \mu(a) \sqcup \bigcap_{b \in [y]_\theta} \mu(b) \\
&= \underline{\theta}(\mu)(x) \sqcup \underline{\theta}(\mu)(y).
\end{aligned}$$

Thus $\underline{\theta}(\mu)$ is an IF prime ideal of L .

(2) The proof is similar to (1). □

Definition 4.8. Let θ be a full congruence relation on L . Then

(1) θ is called \vee -complete, if $\{x \vee y : x \in [a]_\theta, y \in [b]_\theta\} = [a \vee b]_\theta$ for any $a, b \in L$.

(2) θ is called \wedge -complete, if $\{x \wedge y : x \in [a]_\theta, y \in [b]_\theta\} = [a \wedge b]_\theta$ for any $a, b \in L$.

(3) θ is called complete, if θ is both \vee -complete and \wedge -complete.

Proposition 4.9. Let θ be a full congruence relation on L .

(1) Let μ be an IF prime ideal of L and let θ be \wedge -complete. If $\bar{\theta}(\mu)$ is proper, then $\bar{\theta}(\mu)$ is an IF prime ideal of L .

(2) Let μ be an IF prime filter of L and let θ be \vee -complete. If $\bar{\theta}(\mu)$ is proper, then $\bar{\theta}(\mu)$ is an IF filter ideal.

Proof. (1) By Proposition 4.6, $\bar{\theta}(\mu)$ is an IF ideal of L .

Since θ is \wedge -complete, for any $x, y \in L$, we have

$$\begin{aligned}
\bar{\theta}(\mu)(x \wedge y) &= \bigsqcup_{a \in [x]_\theta, b \in [y]_\theta} \mu(a \wedge b) \\
&\leq \bigsqcup_{a \in [x]_\theta, b \in [y]_\theta} \mu(a) \sqcup \mu(b) \\
&= \bigsqcup_{a \in [x]_\theta, b \in [y]_\theta} (\mu^*(a) \vee \mu^*(b), \mu_*(a) \wedge \mu_*(b)) \\
&= (\bigsqcup_{a \in [x]_\theta, b \in [y]_\theta} (\mu^*(a) \vee \mu^*(b)), \bigwedge_{a \in [x]_\theta, b \in [y]_\theta} (\mu_*(a) \wedge \mu_*(b))) \\
&= ((\bigsqcup_{a \in [x]_\theta} \mu^*(a)) \vee (\bigsqcup_{b \in [y]_\theta} \mu^*(b)), (\bigwedge_{a \in [x]_\theta} \mu_*(a)) \wedge (\bigwedge_{b \in [y]_\theta} \mu_*(b))) \\
&= (\bigsqcup_{a \in [x]_\theta} \mu^*(a), \bigwedge_{a \in [x]_\theta} \mu_*(a)) \sqcup (\bigsqcup_{b \in [y]_\theta} \mu^*(b), \bigwedge_{b \in [y]_\theta} \mu_*(b)) \\
&= \bigsqcup_{a \in [x]_\theta} \mu(a) \sqcup \bigsqcup_{b \in [y]_\theta} \mu(b) \\
&= \bar{\theta}(\mu)(x) \sqcup \bar{\theta}(\mu)(y)
\end{aligned}$$

Thus $\bar{\theta}(\mu)$ is an IF prime ideal of L .

(2) The proof is similar to (1) □

Definition 4.10. Let $\mu \in IF(L)$. The least IF ideal of L containing μ is called an IF ideal of L induced by μ . We denote it by $\langle \mu \rangle$.

For any $\mu \in IF(L)$, we denote

$$\mu^\diamond(x) = \bigsqcup \{(\alpha, \beta) \in J : x \in I(\mu_\alpha^\beta)\} \quad (x \in L).$$

Proposition 4.11. Let $\mu \in IF(L)$. Then

- (1) $\mu \subseteq \mu^\diamond$.
- (2) $\mu^\diamond = \bigcap \{\nu \in IFI(L) : \mu \subseteq \nu, \nu(0_L) = 1_J\}$.

Proof. (1) Consider that $\mu_\alpha^\beta = \{x \in U : \mu(x) \geq (\alpha, \beta)\}$. Then

$$\mu(x) = \sqcup \{(\alpha, \beta) : x \in \mu_\alpha^\beta\} \leq \sqcup \{(\alpha, \beta) : x \in I(\mu_\alpha^\beta)\} = \mu^\diamond.$$

(2) Firstly, we can prove that $\mu^\diamond \in IFI(L)$.

For any $x, y \in L$,

$$\begin{aligned}
\mu^\diamond(x) &= \bigsqcup \{(\alpha, \beta) \in J : x \in I(\mu_\alpha^\beta)\}, \\
\mu^\diamond(y) &= \bigsqcup \{(\alpha, \beta) \in J : y \in I(\mu_\alpha^\beta)\}.
\end{aligned}$$

Put

$$A = \{(\alpha, \beta) \in J : x \in I(\mu_\alpha^\beta)\}, \quad B = \{(\alpha, \beta) \in J : y \in I(\mu_\alpha^\beta)\}.$$

Suppose $x \leq y$. Then $A \subseteq B$. So $\sqcup A \leq \sqcup B$. This implies that

$$\mu^\diamond(y) \leq \mu^\diamond(x).$$

Secondly, since $0_L \in I(\mu_1^0)$, we have $1_J \leq \mu^\diamond(0_L)$. Then $\mu^\diamond(0_L) = 1_J$. Combined with (1), we have

$$\mu^\diamond \in \{\nu \in IFI(L) : \mu \subseteq \nu, \nu(0_L) = 1_J\}.$$

Then

$$\mu^\diamond \supseteq \cap\{\nu \in IFI(L) : \mu \subseteq \nu, \nu(0_L) = 1_J\}.$$

Now, we need to prove that

$$\mu^\diamond \subseteq \cap\{\nu \in IFI(L) : \mu \subseteq \nu, \nu(0_L) = 1_J\}.$$

For any $\nu \in \{\nu \in IFI(L) : \mu \subseteq \nu, \nu(0_L) = 1_J\}$, we have $\mu \subseteq \nu$. This implies

$$\mu_\alpha^\beta \subseteq \nu_\alpha^\beta.$$

Then $I(\mu_\alpha^\beta) \subseteq I(\nu_\alpha^\beta)$.

Denote

$$C = \{(\alpha, \beta) \in J : x \in I(\nu_\alpha^\beta)\}.$$

Then $A \subseteq C$ and so $\mu^\diamond(x) \leq \nu^\diamond(x)$.

Note that $\nu \in IFI(L)$. Then $\nu_\alpha^\beta \in IFI(L)$. So $I(\nu_\alpha^\beta) = \nu_\alpha^\beta$.

This implies that

$$\nu^\diamond(x) = \sqcup C = \sqcup\{(\alpha, \beta) \in J : x \in \nu_\alpha^\beta\} = \nu(x).$$

Thus $\mu^\diamond(x) \leq \nu(x)$.

Hence

$$\mu^\diamond \subseteq \cap\{\nu \in IFI(L) : \mu \subseteq \nu, \nu(0_L) = 1_J\}.$$

□

Proposition 4.12. *Let θ be a full congruence relation on L . Then for any $\mu \in IF(L)$,*

$$(1) \bar{\theta}(< \mu >) = \bar{\theta}(< \bar{\theta}(\mu) >).$$

$$(2) \bar{\theta}(\mu^\diamond) = \bar{\theta}((\bar{\theta}(\mu))^\diamond).$$

Proof. (1) Since $\mu \subseteq < \mu >$, we conclude from Proposition 3.4 that

$$\bar{\theta}(\mu) \subseteq \bar{\theta}(< \mu >).$$

By Proposition 4.6 and Proposition 4.11,

$$< \bar{\theta}(\mu) > \subseteq \bar{\theta}(< \mu >).$$

By Proposition 3.4,

$$\bar{\theta}(< \bar{\theta}(\mu) >) \subseteq \bar{\theta}(< \mu >).$$

Note that $\mu \subseteq \bar{\theta}(\mu)$. Then $\langle \mu \rangle \subseteq \langle \bar{\theta}(\mu) \rangle$.
By Proposition 3.4,

$$\bar{\theta}(\langle \mu \rangle) \subseteq \bar{\theta}(\langle \bar{\theta}(\mu) \rangle).$$

Thus

$$\bar{\theta}(\langle \mu \rangle) = \bar{\theta}(\langle \bar{\theta}(\mu) \rangle).$$

(2) Since $\langle \mu \rangle \subseteq \mu^\diamond$, by Proposition 3.4, we have $\bar{\theta}(\langle \mu \rangle) \subseteq \bar{\theta}(\mu^\diamond)$.
It is clear that

$$\bar{\theta}(\mu^\diamond)(0_L) = 1_J, \quad \langle \bar{\theta}(\mu) \rangle \subseteq \langle \bar{\theta}(\mu^\diamond) \rangle = \bar{\theta}(\mu^\diamond).$$

Then $(\bar{\theta}(\mu))^\diamond \subseteq \bar{\theta}(\mu^\diamond)$

By Proposition 3.4,

$$\bar{\theta}((\bar{\theta}(\mu))^\diamond) \subseteq \bar{\theta}(\mu^\diamond).$$

Since $\mu^\diamond \subseteq (\bar{\theta}(\mu))^\diamond$ we conclude $\bar{\theta}(\mu^\diamond) \subseteq \bar{\theta}((\bar{\theta}(\mu))^\diamond)$.

Thus

$$\bar{\theta}(\mu^\diamond) = \bar{\theta}((\bar{\theta}(\mu))^\diamond).$$

□

Proposition 4.13. Let $a^{(r,s)}, b^{(p,q)} \in IFP(L)$ and $\mu \in IF(L)$. Then

$$(1) \bar{\theta}(a^{(r,s)}) = \chi_{[a]_\theta}^{(r,s)}.$$

$$(2) \langle a^{(r,s)} \rangle(x) = \chi_{\downarrow a}^{(r,s)} \text{ and } (a^{(r,s)})^\diamond(x) = \begin{cases} (1, 0) & x = 0_L, \\ (r, s) & 0_L \neq x, \\ (0, 1) & \text{otherwise.} \end{cases}$$

$$(3) \bar{\theta}(\langle a^{(r,s)} \rangle)(x) = \begin{cases} (r, s) & \downarrow a \cap [x]_\theta \neq \emptyset, \\ (0, 1) & \text{otherwise.} \end{cases}$$

$$\bar{\theta}((a^{(r,s)})^\diamond)(x) = \begin{cases} (1, 0) & 0_L \in [x]_\theta \\ (r, s) & \downarrow a \cap [x]_\theta \neq \emptyset, \\ (0, 1) & \text{otherwise.} \end{cases}$$

$$(4) \langle a^{(r,s)} \rangle \wedge \langle b^{(p,q)} \rangle = \langle (a \wedge b)^{(r,s) \wedge (p,q)} \rangle$$

Proof. It is straightforward. □

Let θ be an equivalence relation on L . $\mu \in F(L)$ is called a fixed-point of θ -upper (resp. θ -lower) rough approximation, if $\bar{\theta}(\mu) = \mu$ (resp. $\underline{\theta}(\mu) = \mu$).

Denote

$$Fix(\bar{\theta}) = \{\mu \in F(L) \mid \bar{\theta}(\mu) = \mu\}, \quad Fix(\underline{\theta}) = \{\mu \in F(L) \mid \underline{\theta}(\mu) = \mu\}.$$

Proposition 4.14. *Let θ_1 and θ_2 be two equivalence relations on L . Then the following are equivalent:*

- (1) *For each $\mu \in F(L)$, $\bar{\theta}_1(\mu) \leq \bar{\theta}_2(\mu)$;*
- (2) *For each $\mu \in F(L)$, $\underline{\theta}_1(\mu) \geq \underline{\theta}_2(\mu)$;*
- (3) *$Fix(\bar{\theta}_2) \subseteq Fix(\bar{\theta}_1)$;*
- (4) *$Fix(\underline{\theta}_2) \subseteq Fix(\underline{\theta}_1)$.*

Proof. (1) \implies (2). This holds by Proposition 2.10.

(2) \implies (3) Let $\mu \in F(L)$ and $\underline{\theta}_1(\mu^c) \geq \underline{\theta}_2(\mu^c)$.

By Proposition 2.10, $(\bar{\theta}_1(\mu))^c \geq (\bar{\theta}_2(\mu))^c$.

Thus $\bar{\theta}_1(\mu) \leq \bar{\theta}_2(\mu)$.

Note that $\bar{\theta}_2(\mu) = \mu$. Then $\mu \leq \bar{\theta}_1(\mu) \leq \bar{\theta}_2(\mu) = \mu$.

It follows that $\bar{\theta}_1(\mu) = \mu$.

(3) \implies (1) Let $\mu \in F(L)$. Since $\bar{\theta}_2(\mu) \in Fix(\bar{\theta}_2)$, we have $\bar{\theta}_2(\mu) \in Fix(\bar{\theta}_1)$.

Thus $\bar{\theta}_1(\mu) \leq \bar{\theta}_1(\bar{\theta}_2(\mu)) = \bar{\theta}_2(\mu)$.

(2) \implies (4) Let $\mu \in F(L)$ and $\underline{\theta}_2(\mu) = \mu$. Then $\mu = \underline{\theta}_2(\mu) \leq \underline{\theta}_1(\mu) = \mu$.

It follows that $\underline{\theta}_1(\mu) = \mu$.

(4) \implies (2) Let $\mu \in F(L)$. By Proposition 3.4, $\underline{\theta}_2(\mu) \in Fix(\theta_2)$.

Then $\underline{\theta}_2(\mu) \in Fix(\theta_1)$. Thus

$$\underline{\theta}_2(\mu) = \underline{\theta}_1(\underline{\theta}_2(\mu)) \leq \underline{\theta}_1(\mu).$$

□

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Some results on approximating spaces ^{*}

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Abstract: Topology and rough set theory are widely used in research field of computer science. In this paper, we study properties of topologies induced by binary relations, investigate a particular type of topological spaces which associate with some equivalence relation (i.e., approximating spaces) and obtain some characteristic conditions of approximating spaces.

Keywords: Binary relation; Rough set; Topology; Approximating space

1 Introduction

Rough set theory, proposed by Pawlak [8], is a new mathematical tool for data reasoning. It may be seen as an extension of classical set theory and has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert systems and many other fields [9, 10, 11, 12].

The basic structure of rough set theory is an approximation space. Based on it, lower and upper approximations can be induced. Using these approximations, knowledge hidden in information systems may be revealed and expressed in the form of decision rules. A key notion in Pawlak rough set model is equivalence relations. The equivalence classes are the building blocks for the construction of these approximations. In the real world, the equivalence relation is, however, too restrictive for many practical applications. To address this issue, many interesting and meaningful extensions of Pawlak rough sets have been presented. Equivalence relations can be replaced by tolerance relations [15], binary relations [20] and so on.

Topological structure is an important base for knowledge extraction and processing. Then, an interesting research topic in rough set theory is to study relationships between rough sets and topologies. Many authors studied topological properties of rough sets [3, 4, 7, 18, 22]. It is known that the pair of lower and upper approximation operators induced by a reflexive and transitive relation is exactly the pair of interior and closure operators of a topology [21].

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The purpose of this paper is to investigate further approximating spaces.

2 Preliminaries

Throughout this paper, I denotes $[0, 1]$, N is the set of natural number. U denotes a non-empty set, 2^U denotes the set of all subsets of U , $|X|$ denotes the cardinality of X .

2.1 Binary relations

Recall that R is called a binary relation on U if $R \in 2^{U \times U}$.

Let R be a binary relation on U . R is called preorder if R is reflexive and transitive. R is called tolerance if R is both reflexive and symmetric. R is called equivalence if R is reflexive, symmetric and transitive.

Let R be a binary relation on U . For $u, v, w \in U$, we define

$$R^{uvw} = R \cup S^{uvw} \quad \text{and} \quad R^{uv} = \bigcup_{w \in U} R^{uvw},$$

$$\text{where } S^{uvw} = \begin{cases} \{(u, v)\}, & (u, w) \in R \text{ and } (w, v) \in R \\ \emptyset, & (u, w) \notin R \text{ or } (w, v) \notin R \end{cases}.$$

If $S^{uvw} \neq \emptyset$, then

$$S^{uvw}(x) = \begin{cases} \{v\}, & x = u \\ \emptyset, & x \neq u \end{cases}.$$

Definition 2.1 ([4]). Let R and R_s be two binary relations on U . If for all $x, y \in U$, xR_sy if and only if xRy or there exists $\{v_1, v_2, \dots, v_n\} \subseteq U$ such that $xRv_1, v_1Rv_2, \dots, v_nRy$, then R_s is called the transmitting expression of R .

Theorem 2.2 ([4]). Let R be a binary relation on U and R_s the transmitting expression of R . Then R_s is a transitive relation on U . Moreover,

- (1) If R is reflexive, then R_s is also reflexive;
- (2) If R is transitive, then $R_s = R$;
- (3) If R is symmetric, then R_s is also symmetric.

2.2 Rough sets

Let R be an equivalence relation on U . Then the pair (U, R) is called a Pawlak approximation space. Based on (U, R) , one can define the following two rough approximations:

$$R_*(X) = \{x \in U : [x]_R \subseteq X\},$$

$$R^*(X) = \{x \in U : [x]_R \cap X \neq \emptyset\}.$$

$R_*(X)$ and $R^*(X)$ are called the Pawlak lower approximation and the Pawlak upper approximation of X , respectively.

Definition 2.3 ([19]). Let R be a binary relation on U . $\forall x \in U$, denote

$$R(x) = \{y \in U : (x, y) \in R\}.$$

Then $R(x)$ is called the successor neighborhood of x , the pair (U, R) is called an approximation space. The lower and upper approximations of $X \in 2^U$ with regard to (U, R) , denoted by $\underline{R}(X)$ and $\overline{R}(X)$ are respectively, defined as follows:

$$\underline{R}(X) = \{x \in U : R(x) \subseteq X\} \text{ and } \overline{R}(X) = \{x \in U : R(x) \cap X \neq \emptyset\}.$$

Proposition 2.4. Let $\{R_\alpha : \alpha \in \Gamma\}$ be a family of binary relations on U . Then $\forall X \in 2^U$,

$$\bigcap_{\alpha \in \Gamma} \underline{R}_\alpha(X) = \underline{\bigcup_{\alpha \in \Gamma} R_\alpha(X)}.$$

Proof. Put $R = \bigcup_{\alpha \in \Gamma} R_\alpha$. By $R_\beta \subseteq R$ for each $\beta \in \Gamma$, $\underline{R}_\beta(X) \supseteq \underline{R}(X)$. Then $\bigcap_{\alpha \in \Gamma} \underline{R}_\alpha(X) \supseteq \underline{R}(X)$.

Let $x \in \bigcap_{\alpha \in \Gamma} \underline{R}_\alpha(X)$. Then $x \in \underline{R}_\alpha(X)$ and so $R_\alpha(x) \subseteq X$ for each $\alpha \in \Gamma$. Thus $(\bigcup_{\alpha \in \Gamma} R_\alpha)(x) = \bigcup_{\alpha \in \Gamma} (R_\alpha(x)) \subseteq X$. So $x \in \underline{\bigcup_{\alpha \in \Gamma} R_\alpha(X)}$. Hence $\underline{R}_\beta(X) \subseteq \underline{\bigcup_{\alpha \in \Gamma} R_\alpha(X)}$.

Therefore, $\bigcap_{\alpha \in \Gamma} \underline{R}_\alpha(X) = \underline{\bigcup_{\alpha \in \Gamma} R_\alpha(X)}$. □

Proposition 2.5. Let R be a binary relation on U . Then $\forall u, v, w \in U$,

$$\underline{R}^{uvw}(X) - \{u\} = \underline{R}(X) - \{u\}.$$

Proof. (1) If $R^{uvw} = R$, then $\underline{R}^{uvw}(X) - \{u\} = \underline{R}(X) - \{u\}$.

(2) If $R^{uvw} \neq R$, then $(u, w), (w, v) \in R$ and $(u, v) \notin R$.

Obviously, $\underline{R}^{uvw}(X) - \{u\} \subseteq \underline{R}(X) - \{u\}$.

For $x \in \underline{R}(X) - \{u\}$, note that $S^{uvw}(x) = \emptyset$ ($x \in U - \{u\}$), then

$$R^{uvw}(x) = (R \cup S^{uvw})(x) = R(x) \cup S^{uvw}(x) = R(x) \subseteq X \text{ } (x \in U - \{u\}).$$

So $x \in \underline{R}^{uvw}(X) - \{u\}$. It follows $\underline{R}^{uvw}(X) - \{u\} \supseteq \underline{R}(X) - \{u\}$.

Hence

$$\underline{R}^{uvw}(X) - \{u\} = \underline{R}(X) - \{u\}.$$

□

Theorem 2.6. Let R be a binary relation on U and τ a topology on U . If one of the following conditions is satisfied, then R is preorder.

- (1) \overline{R} is the closure operator of τ .
- (2) \underline{R} is the interior operator of τ .

Proof. (1) Let $x, y, z \in U$. Denote $cl_\tau(z_1)(y) = \lambda$.

Note that \underline{R} is the interior operator of τ and $x \in cl_\tau(\{x\}) = \overline{R}(\{x\})$. Then $(x, x) \in R$. So R is reflexive.

Let $(x, y), (y, z) \in R$. Then $x \in \overline{R}(\{y\}), y \in \overline{R}(\{z\})$.

Note that \overline{R} is the closure operator of τ . Then $x \in cl(\{y\}), y \in cl(\{z\})$. So

$$x \in cl(\{x\}) \subseteq cl(cl(\{y\})) = cl(\{y\}) \subseteq cl(cl(\{z\})) = cl(\{z\}) = \overline{R}(\{z\}).$$

This implies $(x, z) \in R$. So R is transitive.

Hence R is preorder.

(2) This proof is similar to (1). □

3 Topologies induced by binary relations

3.1 Topologies induced by reflexive relations

Let R be a reflexive relation on U . Denote

$$\tau_R = \{X \in 2^U : \underline{R}(X) = X\},$$

$$\sigma_R = \{\underline{R}(X) : X \in 2^U\}.$$

Kondo [2] proved that if R is a reflexive relation on X , then τ_R is a topology on X , which may be called the topology induced by R on X .

Remark 3.1. (1) If R is preorder, then $\tau_R = \sigma_R$.

(2) If R is equivalence, then $\tau_R = \{ \bigcup_{x \in X} [x]_R : X \in 2^U \}$.

Theorem 3.2 ([7]). Let R be a preorder relation on U . Then

- (1) σ_R is a topology on U .
- (2) \underline{R} is an interior operator of σ_R .
- (3) \overline{R} is a closure operator of σ_R .

Proposition 3.3. Let ρ and R be two reflexive relations on U . Then

- (1) $\rho \subseteq R \implies \tau_\rho \supseteq \tau_R$.
- (2) If ρ and R are preorder, then $\tau_\rho = \tau_R \iff \rho = R$.

Proof. (1) $\forall X \in \tau_R, \underline{R}(X) = X$. By $\rho \subseteq R$ and the reflexivity of ρ ,

$$X = \underline{R}(X) \subseteq \underline{\rho}(X) \subseteq X.$$

Then $\underline{\rho}(X) = X$ and so $X \in \tau_\rho$. Thus $\tau_\rho \supseteq \tau_R$.

(2) Necessity. Suppose $\tau_\rho = \tau_R$. Note that ρ and R are preorder. Then $\tau_\rho = \sigma_\rho = \sigma_R = \tau_R$.

By Theorem 3.2(3),

$$\begin{aligned} (x, y) \in \rho &\iff x \in \overline{\rho}(\{y\}) \iff x \in cl_{\sigma_\rho}(\{y\}) \\ &\iff x \in cl_{\sigma_R}(\{y\}) \iff x \in \overline{R}(\{y\}) \iff (x, y) \in R. \end{aligned}$$

Then $\rho = R$.

Sufficiency. Obviously. □

Proposition 3.4. Let $\{R_\alpha : \alpha \in \Gamma\}$ be a family of reflexive relations on U . Then

$$\tau_{\bigcup_{\alpha \in \Gamma} R_\alpha} = \bigcap_{\alpha \in \Gamma} \tau_{R_\alpha}.$$

Proof. By Proposition 3.3(1), $\tau_{\bigcup_{\alpha \in \Gamma} R_\alpha} \subseteq \bigcap_{\alpha \in \Gamma} \tau_{R_\alpha}$.

Let $X \in \bigcap_{\alpha \in \Gamma} \sigma_{R_\alpha}$. Then $\forall \alpha \in \Gamma$, $\underline{R}_\alpha(X) = X$. By Proposition 2.4,

$$X = \bigcap_{\alpha \in \Gamma} \underline{R}_\alpha(X) = \bigcup_{\alpha \in \Gamma} R_\alpha(X).$$

So $X \in \tau_{\bigcup_{\alpha \in \Gamma} R_\alpha}$. This implies $\tau_{\bigcup_{\alpha \in \Gamma} R_\alpha} \supseteq \bigcap_{\alpha \in \Gamma} \tau_{R_\alpha}$.

Hence $\tau_{\bigcup_{\alpha \in \Gamma} R_\alpha} = \bigcap_{\alpha \in \Gamma} \tau_{R_\alpha}$. \square

3.2 The topologies induced by some binary relations

Theorem 3.5. Let ρ, λ, R be three reflexive relations on U . If $\tau_\rho = \tau_R = \tau_\lambda$ and $\rho \subseteq \delta \subseteq \lambda$, then $\tau_\delta = \tau_R$.

Proof. By $\rho \subseteq \delta \subseteq \lambda$ and Proposition 3.3(1),

$$\tau_R = \tau_\lambda \subseteq \tau_\delta \subseteq \tau_\rho = \tau_R.$$

Then $\tau_\delta = \tau_R$. \square

Theorem 3.6. Let R be a reflexive relation on U . Then $\forall u, v, w \in U$, $\tau_{R^{uvw}} = \tau_R = \tau_{R^{uv}}$.

Proof. Obviously, R^{uvw} and R^{uv} both are reflexive.

(1) 1) If $u = v$, then $R^{uvw} = R$ and so $\tau_{R^{uvw}} = \tau_R$.

2) If $u \neq v$, $R^{uvw} = R$, we have $\tau_{R^{uvw}} = \tau_R$.

3) If $u \neq v$, $R^{uvw} \neq R$, we have $(u, w) \in R$, $(w, v) \in R$, $(u, v) \notin R$ and $S^{uvw} = \{(u, v)\}$.

Let $X \in \sigma_R$. Then $X \subseteq \underline{R}(X)$. By Proposition 3.3(1), $\sigma_R \supseteq \sigma_{R^{uvw}}$. By Proposition 2.5, $X - \{u\} \subseteq \underline{R}(X) - \{u\} = \underline{R}^{uvw}(X) - \{u\}$.

i) If $u \notin X$, then $X \subseteq \underline{R}^{uvw}(X)$.

ii) If $u \in X$, then $u \in \underline{R}(X)$ and so

$$w \in R(u) \subseteq X \subseteq \underline{R}(X).$$

We can obtain $R(w) \subseteq \underline{R}(X)$. Note that $v \in R(w)$. Then $v \in \underline{R}(X)$. We have

$$R^{uvw}(u) = (R \cup S^{uvw})(u) = R(u) \cup S^{uvw}(u) = R(u) \cup \{v\} \subseteq X.$$

Then $u \in \underline{R}^{uvw}(X)$. Thus $X \subseteq \underline{R}^{uvw}(X)$. By the reflexivity of ρ , $X \supseteq \underline{R}^{uvw}(X)$. Then $\underline{R}^{uvw}(X) = X$ and So $X \in \sigma_{R^{uvw}}$.

By i) and ii), $\tau_R \subseteq \tau_{R^{uvw}}$.

Thus

$$\tau_{R^{uvw}} = \tau_R (w \in U).$$

(2) By (1) and Proposition 3.4,

$$\tau_{R^{uv}} = \tau \bigcup_{w \in U} R^{uvw} = \bigcap_{w \in U} \tau_{R^{uvw}} = \tau_R.$$

□

Denote $R_0 = R$. R_n ($n \in \omega$) are defined as follows:

$$R_{n+1} = \bigcup_{u,v \in X} (R_n)^{uv}.$$

Put

$$R^* = \lim_{n \rightarrow \infty} R_n.$$

Obviously, $R^* = \bigcup_{n=0}^{\infty} R_n$.

Corollary 3.7. *Let R be a reflexive relation on U . Then $\tau_{R_n} = \tau_R = \tau_{R^*}$.*

Proof. This holds by Proposition 3.4 and Theorem 3.6. □

Theorem 3.8. *Let R be a binary relation on U . Then*

$$R \text{ is transitive} \iff R = R_1.$$

Proof. Necessity. Obviously.

Sufficiency. Suppose that R is not transitive. Then there exist x, y, z such that $(x, z), (z, y) \in R$, $(x, y) \notin R$. So $(x, y) \in R^{xy}$. This implies

$$(x, y) \in R_1 = \bigcup_{u,v \in X} R^{uv}.$$

We have $R_1 \neq R$. This is a contradiction.

Thus R is transitive. □

Corollary 3.9. *If R is a preorder relation on U , then $\forall n \in \omega$, $R_n = R$.*

Proof. This holds by Theorem 4.6. □

Denote $R_0 = R$. R_n ($n \in \omega$) are defined as follows: $R_{n+1} = \bigcup_{u,v \in X} (R_n)^{uv}$

Denote

$$R^* = \lim_{n \rightarrow \infty} R_n.$$

Theorem 3.10. *If R is a reflexive relation on U , then R^* is transitive.*

Proof. Let $(u, w), (w, v) \in R^*$. Then there exist $n_1, n_2 \in N$ such that $(u, w) \in R_{n_1}, (w, v) \in R_{n_2}$. Pick $n_0 = n_1 + n_2$. Then $(u, w) \in R_{n_0}, (w, v) \in R_{n_0}$. So

$$(u, v) \in (R_{n_0})^{uvw} \subseteq (R_{n_0})^{uv} \subseteq R_{n_0+1} \subseteq R^*.$$

So R^* is transitive. □

Theorem 3.11. *Let R be a reflexive relation on U . Then $R_s = R^*$.*

Proof. Note that

$$\begin{aligned} (x, y) &\in R^* & (R^* &= \bigcup_{n=0}^{\infty} R_n) \\ \iff \exists n \in N, (x, y) &\in R_n & (R_n &= \bigcup_{u, v \in X} (R_{n-1})^{uv}) \\ \iff (x, y) &\in (R_{n-1})^{xy} & ((R_{n-1})^{xy} &= \bigcup_{w \in U} (R_{n-1})^{xyw}) \\ \iff \exists w_{2^n} \in U, (x, y) &\in (R_{n-1})^{xyw_{2^n}} & \\ \iff \exists w_{2^n} \in U, (x, w_{2^n}), (w_{2^n}, y) &\in R_{n-1} & \\ \iff \exists w_{2^n-1}, w_{2^n}, w_{2^n-2} \in U, & & \\ & (x, w_{2^n-1}), (w_{2^n-1}, w_{2^n}), (w_{2^n}, w_{2^n-2}), (w_{2^n-2}, y) \in R_{n-2}. \\ & \dots \dots \dots \\ \iff \exists w_2, w_3, \dots, w_{2^n} \in U, & & \\ & (x, w_3), \dots, (w_{2^n-1}, w_{2^n}), (w_{2^n}, w_{2^n-2}), \dots, (w_2, y) \in R_0 = R. \\ \iff (x, y) &\in R_s. \end{aligned}$$

Then $R_s = R^*$. □

Corollary 3.12. *Let R be a tolerance relation on U . Then*

- (1) R_s is equivalence.
- (2) $\tau_{R_s} = \tau_R$.
- (3) $\underline{R_s}$ is an interior operator of τ_R .
- (4) $\overline{R_s}$ is a closure operator of τ_R .

Proof. (1) This holds by Theorem 3.11.

(2) This holds by Corollary 3.7 and Theorem 3.11

(3) This holds by (2) and Theorem 3.2.

(4) This holds by (2) and Theorem 3.2. □

4 Some characteristic conditions of approximating spaces

Definition 4.1 ([4]). *Let (U, μ) be a topological space. If there exists an equivalence relation R on U such that $\tau_R = \mu$, then (U, τ) is called a approximating space.*

Definition 4.2. Let μ be a topology on U . Define a binary relation R_μ on U by

$$(x, y) \in R_\mu \iff x \in cl_\mu(\{y\}).$$

Then R_μ is called the binary relation induced by μ on U .

Theorem 4.3. Let (U, μ) be a topological space. Then the following are equivalent:

- (1) (U, μ) is an approximating space;
- (2) There exists a tolerance relation R on U such that $\tau_R = \mu$;
- (3) There exists a tolerance relation R on U such that \underline{R} is an interior operator of μ ;
- (4) There exists a tolerance relation R on U such that \overline{R} is a closure operator of μ ;
- (5) There exists an equivalence relation R on U such that

$$\mu = \{ \bigcup_{x \in X} [x]_R : X \in 2^U \}.$$

Proof. (1) \implies (2) is obvious.

(1) \implies (3) and (1) \implies (4) hold by Theorem 3.2.

(1) \implies (5) holds by Remark 3.2.

(2) \implies (1) Suppose that there exists a tolerance relation R on U such that $\tau_R = \mu$.

By Theorem 2.2 and Corollary 3.12, R_s is equivalence and $\tau_{R_s} = \tau_R$.

Then $\tau_{R_s} = \mu$.

Thus (U, μ) is an approximating space.

(3) \implies (1) Suppose that there exists a tolerance relation R on U such that \underline{R} is an interior operator of μ . Then

$$X \in \tau_R \iff \underline{R}(X) = X \iff int_\mu(X) = X \iff X \in \mu.$$

Then $\tau_R = \mu$.

By Theorem 2.6(2), R is preorder. So R is equivalence.

Thus (U, μ) is an approximating space.

(4) \implies (1) The proof is similar to (3) \implies (1).

(5) \implies (1) holds by Remark 3.2. □

Corollary 4.4. If (U, μ) is an approximating space, then R_μ is an equivalence relation.

Proof. Obviously, R_μ is reflexive.

By Theorem 4.3, there exists an equivalence relation R on U such that

$$\mu = \{ \bigcup_{x \in X} [x]_R : X \in 2^U \}.$$

By Remark 2.4, we have

$$(x, y) \in R_\mu \Rightarrow x \in cl_\mu(\{y\}) = [y]_R \Rightarrow y \in [y]_R = [x]_R = cl_\mu(\{x\}) \Rightarrow (y, x) \in R_\mu.$$

$$\begin{aligned}
(x, y), (y, z) \in R_\mu &\implies x \in cl_\mu(\{y\}) = [y]_R, y \in cl_\mu(\{z\}) = [z]_R \\
&\implies x \in [y]_R = [z]_R = cl_\mu(\{z\}) \implies (x, z) \in R_\mu.
\end{aligned}$$

Thus R_μ is equivalence. \square

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Divisible and strong fuzzy filters of residuated lattices

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Abstract. In a residuated lattice, divisible fuzzy filters and strong fuzzy filters are introduced, and their properties are investigated. Characterizations of a divisible and strong fuzzy filter are discussed. Conditions for a fuzzy filter to be divisible are established. Relations between a divisible fuzzy filter and a strong fuzzy filter are considered.

1. Introduction

In order to deal with fuzzy and uncertain informations, non-classical logic has become a formal and useful tool. As the semantical systems of non-classical logic systems, various logical algebras have been proposed. Residuated lattices are important algebraic structures which are basic of *MTL*-algebras, *BL*-algebras, *MV*-algebras, Gödel algebras, *R*₀-algebras, lattice implication algebras, etc. The filter theory plays an important role in studying logical systems and the related algebraic structures, and various filters have been proposed in the literature. Zhang et al. [8] introduced the notions of IMTL-filters (NM-filters, MV-filters) of residuated lattices, and presented their characterizations. Ma and Hu [4] introduced divisible filters, strong filters and *n*-contractive filters in residuated lattices.

In this paper, we consider the fuzzification of divisible filters and strong filters in residuated lattices. We define divisible fuzzy filters and strong fuzzy filters, and investigate related properties. We discussed characterizations of a divisible and strong fuzzy filter, and provided conditions for a fuzzy filter to be divisible. We establish relations between a divisible fuzzy filter and a strong fuzzy filter.

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2. Preliminaries

Definition 2.1 ([1, 2, 3]). A *residuated lattice* is an algebra $\mathcal{L} := (L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that

- (1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice.
- (2) $(L, \odot, 1)$ is a commutative monoid.
- (3) \odot and \rightarrow form an adjoint pair, that is,

$$(\forall x, y, z \in L) (x \leq y \rightarrow z \Leftrightarrow x \odot y \leq z).$$

In a residuated lattice \mathcal{L} , the ordering \leq and negation \neg are defined as follows:

$$(\forall x, y \in L) (x \leq y \Leftrightarrow x \wedge y = x \Leftrightarrow x \vee y = y \Leftrightarrow x \rightarrow y = 1)$$

and $\neg x = x \rightarrow 0$ for all $x \in L$.

Proposition 2.2 ([1, 2, 3, 4, 6, 7]). *In a residuated lattice \mathcal{L} , the following properties are valid.*

$$1 \rightarrow x = x, x \rightarrow 1 = 1, x \rightarrow x = 1, 0 \rightarrow x = 1, x \rightarrow (y \rightarrow x) = 1. \quad (2.1)$$

$$x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z). \quad (2.2)$$

$$x \leq y \Rightarrow z \rightarrow x \leq z \rightarrow y, y \rightarrow z \leq x \rightarrow z. \quad (2.3)$$

$$z \rightarrow y \leq (x \rightarrow z) \rightarrow (x \rightarrow y), z \rightarrow y \leq (y \rightarrow x) \rightarrow (z \rightarrow x). \quad (2.4)$$

$$(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z. \quad (2.5)$$

$$\neg x = \neg \neg \neg x, x \leq \neg \neg x, \neg 1 = 0, \neg 0 = 1. \quad (2.6)$$

$$x \odot y \leq x \odot (x \rightarrow y) \leq x \wedge y \leq x \wedge (x \rightarrow y) \leq x. \quad (2.7)$$

$$x \leq y \Rightarrow x \odot z \leq y \odot z. \quad (2.8)$$

$$x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z), (x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z). \quad (2.9)$$

$$x \rightarrow y \leq (x \odot z) \rightarrow (y \odot z). \quad (2.10)$$

$$\neg \neg (x \rightarrow y) \leq \neg \neg x \rightarrow \neg \neg y. \quad (2.11)$$

$$x \rightarrow (x \wedge y) = x \rightarrow y. \quad (2.12)$$

Definition 2.3 ([5]). A nonempty subset F of a residuated lattice \mathcal{L} is called a *filter* of \mathcal{L} if it satisfies the conditions:

$$(\forall x, y \in L) (x, y \in F \Rightarrow x \odot y \in F). \quad (2.13)$$

$$(\forall x, y \in L) (x \in F, x \leq y \Rightarrow y \in F). \quad (2.14)$$

Divisible and strong fuzzy filters of residuated lattices

Proposition 2.4 ([5]). *A nonempty subset F of a residuated lattice \mathcal{L} is a filter of \mathcal{L} if and only if it satisfies:*

$$1 \in F. \quad (2.15)$$

$$(\forall x \in F) (\forall y \in L) (x \rightarrow y \in F \Rightarrow y \in F). \quad (2.16)$$

Definition 2.5 ([9]). A fuzzy set μ in a residuated lattice \mathcal{L} is called a *fuzzy filter* of \mathcal{L} if it satisfies:

$$(\forall x, y \in L) (\mu(x \odot y) \geq \min\{\mu(x), \mu(y)\}). \quad (2.17)$$

$$(\forall x, y \in L) (x \leq y \Rightarrow \mu(x) \leq \mu(y)). \quad (2.18)$$

Theorem 2.6 ([9]). *A fuzzy set μ in a residuated lattice \mathcal{L} is a fuzzy filter of \mathcal{L} if and only if the following assertions are valid:*

$$(\forall x \in L) (\mu(1) \geq \mu(x)). \quad (2.19)$$

$$(\forall x, y \in L) (\mu(y) \geq \min\{\mu(x \rightarrow y), \mu(x)\}). \quad (2.20)$$

3. Divisible and strong fuzzy filters

In what follows let \mathcal{L} denote a residuated lattice unless otherwise specified.

Definition 3.1 ([4]). A filter F of \mathcal{L} is said to be *divisible* if it satisfies:

$$(\forall x, y \in L) ((x \wedge y) \rightarrow [x \odot (x \rightarrow y)] \in F). \quad (3.1)$$

Definition 3.2. A fuzzy filter μ of \mathcal{L} is said to be *divisible* if it satisfies:

$$(\forall x, y \in L) (\mu((x \wedge y) \rightarrow [x \odot (x \rightarrow y)]) = \mu(1)). \quad (3.2)$$

Example 3.3. Let $L = \{0, a, b, 1\}$ be a chain with Cayley tables which are given in Tables 1 and 2.

TABLE 1. Cayley table for the “ \odot ”-operation

\odot	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

Then $\mathcal{L} := (L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a residuated lattice. Define a fuzzy set μ in \mathcal{L} by $\mu(1) = 0.7$ and $\mu(x) = 0.2$ for all $x(\neq 1) \in L$. It is routine to verify that μ is a divisible fuzzy filter of \mathcal{L} .

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TABLE 2. Cayley table for the “ \rightarrow ”-operation

\rightarrow	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Example 3.4. Consider a residuated lattice $L = [0, 1]$ in which two operations “ \odot ” and “ \rightarrow ” are defined as follows:

$$x \odot y = \begin{cases} 0 & \text{if } x + y \leq \frac{1}{2}, \\ x \wedge y & \text{otherwise.} \end{cases}$$

$$x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ (\frac{1}{2} - x) \vee y & \text{otherwise.} \end{cases}$$

The fuzzy set μ of \mathcal{L} given by $\mu(1) = 0.9$ and $\mu(x) = 0.2$ for all $x(\neq 1) \in L$ is a fuzzy filter of \mathcal{L} . But it is not divisible since

$$\mu((0.3 \wedge 0.2) \rightarrow (0.3 \odot (0.3 \rightarrow 0.2))) = \mu(0.3) \neq \mu(1).$$

Proposition 3.5. Every divisible fuzzy filter μ of \mathcal{L} satisfies the following identity.

$$(\forall x, y, z \in L) (\mu(((x \odot y) \wedge (x \odot z)) \rightarrow (x \odot (y \wedge z))) = \mu(1)). \quad (3.3)$$

Proof. Let $x, y, z \in L$. If we let $x := x \odot y$ and $y := x \odot z$ in (3.2), then

$$\mu(((x \odot y) \wedge (x \odot z)) \rightarrow ((x \odot y) \odot ((x \odot y) \rightarrow (x \odot z)))) = \mu(1). \quad (3.4)$$

Using (2.2) and (2.7), we have

$$\begin{aligned} (x \odot y) \odot ((x \odot y) \rightarrow (x \odot z)) &= x \odot y \odot (y \rightarrow (x \rightarrow (x \odot z))) \\ &\leq x \odot (y \wedge (x \rightarrow (x \odot z))), \end{aligned}$$

and so

$$\begin{aligned} ((x \odot y) \wedge (x \odot z)) &\rightarrow ((x \odot y) \odot ((x \odot y) \rightarrow (x \odot z))) \\ &\leq ((x \odot y) \wedge (x \odot z)) \rightarrow (x \odot (y \wedge (x \rightarrow (x \odot z)))) \end{aligned}$$

by (2.3). It follows from (3.4) and (2.18) that

$$\begin{aligned} \mu(1) &= \mu(((x \odot y) \wedge (x \odot z)) \rightarrow ((x \odot y) \odot ((x \odot y) \rightarrow (x \odot z)))) \\ &\leq \mu(((x \odot y) \wedge (x \odot z)) \rightarrow (x \odot (y \wedge (x \rightarrow (x \odot z))))) \end{aligned}$$

and so that

$$\mu(((x \odot y) \wedge (x \odot z)) \rightarrow (x \odot (y \wedge (x \rightarrow (x \odot z))))) = \mu(1) \quad (3.5)$$

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since $\mu(1) \geq \mu(x)$ for all $x \in L$. On the other hand, if we take $x := x \rightarrow (x \odot z)$ in (3.2) then

$$\begin{aligned}\mu(1) &= \mu((y \wedge (x \rightarrow (x \odot z))) \rightarrow ((x \rightarrow (x \odot z)) \odot ((x \rightarrow (x \odot z)) \rightarrow y))) \\ &\leq \mu((x \odot (y \wedge (x \rightarrow (x \odot z)))) \rightarrow (x \odot ((x \rightarrow (x \odot z)) \odot ((x \rightarrow (x \odot z)) \rightarrow y)))) \\ &= \mu((x \odot (y \wedge (x \rightarrow (x \odot z)))) \rightarrow (x \odot (x \rightarrow (x \odot z)) \odot ((x \rightarrow (x \odot z)) \rightarrow y)))\end{aligned}$$

by using (2.10), (2.18) and the commutativity and associativity of \odot . Hence

$$\mu((x \odot (y \wedge (x \rightarrow (x \odot z)))) \rightarrow (x \odot (x \rightarrow (x \odot z)) \odot ((x \rightarrow (x \odot z)) \rightarrow y))) = \mu(1). \quad (3.6)$$

Using (2.5), we get

$$\begin{aligned}&(((x \odot y) \wedge (x \odot z)) \rightarrow (x \odot (y \wedge (x \rightarrow (x \odot z))))) \odot \\ &((x \odot (y \wedge (x \rightarrow (x \odot z)))) \rightarrow (x \odot (x \rightarrow (x \odot z)) \odot ((x \rightarrow (x \odot z)) \rightarrow y))) \\ &\leq ((x \odot y) \wedge (x \odot z)) \rightarrow (x \odot (x \rightarrow (x \odot z)) \odot ((x \rightarrow (x \odot z)) \rightarrow y)).\end{aligned}$$

It follows from (2.18), (2.17), (3.5) and (3.6) that

$$\begin{aligned}&\mu(((x \odot y) \wedge (x \odot z)) \rightarrow (x \odot (x \rightarrow (x \odot z)) \odot ((x \rightarrow (x \odot z)) \rightarrow y))) \\ &\geq \mu((((x \odot y) \wedge (x \odot z)) \rightarrow (x \odot (y \wedge (x \rightarrow (x \odot z))))) \odot \\ &((x \odot (y \wedge (x \rightarrow (x \odot z)))) \rightarrow (x \odot (x \rightarrow (x \odot z)) \odot ((x \rightarrow (x \odot z)) \rightarrow y)))) \\ &\geq \min\{\mu((((x \odot y) \wedge (x \odot z)) \rightarrow (x \odot (y \wedge (x \rightarrow (x \odot z))))) \odot \\ &\mu(((x \odot (y \wedge (x \rightarrow (x \odot z)))) \rightarrow (x \odot (x \rightarrow (x \odot z)) \odot ((x \rightarrow (x \odot z)) \rightarrow y))))\} \\ &= \mu(1)\end{aligned}$$

Thus

$$\mu(((x \odot y) \wedge (x \odot z)) \rightarrow (x \odot (x \rightarrow (x \odot z)) \odot ((x \rightarrow (x \odot z)) \rightarrow y))) = \mu(1). \quad (3.7)$$

Since $x \odot (x \rightarrow (x \odot z)) \odot ((x \rightarrow (x \odot z)) \rightarrow y) \leq x \odot z \odot (z \rightarrow y) \leq x \odot (y \wedge z)$, we obtain

$$\begin{aligned}&((x \odot y) \wedge (x \odot z)) \rightarrow (x \odot (x \rightarrow (x \odot z)) \odot ((x \rightarrow (x \odot z)) \rightarrow y)) \\ &\leq ((x \odot y) \wedge (x \odot z)) \rightarrow (x \odot (y \wedge z)).\end{aligned}$$

It follows that

$$\begin{aligned}&\mu(((x \odot y) \wedge (x \odot z)) \rightarrow (x \odot (y \wedge z))) \\ &\geq \mu(((x \odot y) \wedge (x \odot z)) \rightarrow (x \odot (x \rightarrow (x \odot z)) \odot ((x \rightarrow (x \odot z)) \rightarrow y))) \\ &= \mu(1)\end{aligned}$$

and that $\mu(((x \odot y) \wedge (x \odot z)) \rightarrow (x \odot (y \wedge z))) = \mu(1)$. \square

We consider characterizations of a divisible fuzzy filter.

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Theorem 3.6. *A fuzzy filter μ of \mathcal{L} is divisible if and only if the following assertion is valid:*

$$(\forall x, y, z \in L) (\mu([x \rightarrow (y \wedge z)] \rightarrow [(x \rightarrow y) \odot ((x \wedge y) \rightarrow z)]) = \mu(1)). \quad (3.8)$$

Proof. Assume that μ is a divisible fuzzy filter of \mathcal{L} . If we take $x := x \rightarrow y$ and $y := x \rightarrow z$ in (3.2) and use (2.9) and (2.2), then

$$\begin{aligned} \mu(1) &= \mu([(x \rightarrow y) \wedge (x \rightarrow z)] \rightarrow [(x \rightarrow y) \odot ((x \rightarrow y) \rightarrow (x \rightarrow z))]) \\ &= \mu([x \rightarrow (y \wedge z)] \rightarrow [(x \rightarrow y) \odot ((x \odot (x \rightarrow y)) \rightarrow z)]) . \end{aligned}$$

Using (2.4) and (2.10), we have

$$\begin{aligned} (x \wedge y) \rightarrow [x \odot (x \rightarrow y)] &\leq [(x \odot (x \rightarrow y)) \rightarrow z] \rightarrow [(x \wedge y) \rightarrow z] \\ &\leq [(x \rightarrow y) \odot ((x \odot (x \rightarrow y)) \rightarrow z)] \rightarrow [(x \rightarrow y) \odot ((x \wedge y) \rightarrow z)] \end{aligned}$$

for all $x, y, z \in L$. Since μ is a divisible fuzzy filter of \mathcal{L} , it follows from (3.2) and (2.18) that

$$\begin{aligned} \mu(1) &= \mu((x \wedge y) \rightarrow [x \odot (x \rightarrow y)]) \\ &\leq \mu([(x \rightarrow y) \odot ((x \odot (x \rightarrow y)) \rightarrow z)] \rightarrow [(x \rightarrow y) \odot ((x \wedge y) \rightarrow z)]) \end{aligned}$$

and so from (2.19) that

$$\mu([(x \rightarrow y) \odot ((x \odot (x \rightarrow y)) \rightarrow z)] \rightarrow [(x \rightarrow y) \odot ((x \wedge y) \rightarrow z)]) = \mu(1)$$

for all $x, y, z \in L$. Using (2.5), we get

$$\begin{aligned} &([x \rightarrow (y \wedge z)] \rightarrow [(x \rightarrow y) \odot ((x \odot (x \rightarrow y)) \rightarrow z)]) \odot \\ &\quad \left([(x \rightarrow y) \odot ((x \odot (x \rightarrow y)) \rightarrow z)] \rightarrow [(x \rightarrow y) \odot ((x \wedge y) \rightarrow z)] \right) \\ &\leq [x \rightarrow (y \wedge z)] \rightarrow [(x \rightarrow y) \odot ((x \wedge y) \rightarrow z)], \end{aligned}$$

and so

$$\begin{aligned} &\mu([x \rightarrow (y \wedge z)] \rightarrow [(x \rightarrow y) \odot ((x \wedge y) \rightarrow z)]) \\ &\geq \mu([x \rightarrow (y \wedge z)] \rightarrow [(x \rightarrow y) \odot ((x \odot (x \rightarrow y)) \rightarrow z)]) \odot \\ &\quad \left([(x \rightarrow y) \odot ((x \odot (x \rightarrow y)) \rightarrow z)] \rightarrow [(x \rightarrow y) \odot ((x \wedge y) \rightarrow z)] \right) \\ &\geq \min \left\{ \mu([x \rightarrow (y \wedge z)] \rightarrow [(x \rightarrow y) \odot ((x \odot (x \rightarrow y)) \rightarrow z)]), \right. \\ &\quad \left. \mu([(x \rightarrow y) \odot ((x \odot (x \rightarrow y)) \rightarrow z)] \rightarrow [(x \rightarrow y) \odot ((x \wedge y) \rightarrow z)]) \right\} \\ &= \mu(1). \end{aligned}$$

Therefore $\mu([x \rightarrow (y \wedge z)] \rightarrow [(x \rightarrow y) \odot ((x \wedge y) \rightarrow z)]) = \mu(1)$ for all $x, y, z \in L$.

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Conversely, let μ be a fuzzy filter that satisfies the condition (3.8). if we take $x := 1$ in (3.8) and use (2.1), then we obtain (3.2). \square

Theorem 3.7. *A fuzzy filter μ of \mathcal{L} is divisible if and only if it satisfies:*

$$(\forall x, y \in L) (\mu([y \odot (y \rightarrow x)] \rightarrow [x \odot (x \rightarrow y)]) = \mu(1)). \quad (3.9)$$

Proof. Suppose that μ is a divisible fuzzy filter of \mathcal{L} . Note that

$$(x \wedge y) \rightarrow [x \odot (x \rightarrow y)] \leq [y \odot (y \rightarrow x)] \rightarrow [x \odot (x \rightarrow y)]$$

for all $x, y \in L$. It follows from (3.2) and (2.18) that

$$\begin{aligned} \mu(1) &= \mu((x \wedge y) \rightarrow [x \odot (x \rightarrow y)]) \\ &\leq \mu([y \odot (y \rightarrow x)] \rightarrow [x \odot (x \rightarrow y)]) \end{aligned}$$

and that $\mu([y \odot (y \rightarrow x)] \rightarrow [x \odot (x \rightarrow y)]) = \mu(1)$.

Conversely, let μ be a fuzzy filter of \mathcal{L} that satisfies the condition (3.9). Since

$$y \rightarrow x = y \rightarrow (y \wedge x) \text{ for all } x, y \in L,$$

the condition (3.9) implies that

$$\mu([y \odot (y \rightarrow (x \wedge y))] \rightarrow [x \odot (x \rightarrow (x \wedge y))]) = \mu(1). \quad (3.10)$$

If we take $y := x \wedge z$ in (3.10), then

$$\begin{aligned} \mu(1) &= \mu([(x \wedge z) \odot ((x \wedge z) \rightarrow (x \wedge (x \wedge z)))] \rightarrow [x \odot (x \rightarrow (x \wedge (x \wedge z)))]]) \\ &= \mu((x \wedge z) \rightarrow [x \odot (x \rightarrow z)]). \end{aligned}$$

Therefore μ is a divisible fuzzy filter of \mathcal{L} . \square

We discuss conditions for a fuzzy filter to be divisible.

Theorem 3.8. *If a fuzzy filter μ of \mathcal{L} satisfies the following assertion:*

$$(\forall x, y \in L) (\mu((x \wedge y) \rightarrow (x \odot y)) = \mu(1)), \quad (3.11)$$

then μ is divisible.

Proof. Note that $x \odot y \leq x \odot (x \rightarrow y)$ for all $x, y \in L$. It follows from (2.3) that

$$(x \wedge y) \rightarrow (x \odot y) \leq (x \wedge y) \rightarrow (x \odot (x \rightarrow y)).$$

Hence, by (3.11) and (2.18), we have

$$\mu(1) = \mu((x \wedge y) \rightarrow (x \odot y)) \leq \mu((x \wedge y) \rightarrow (x \odot (x \rightarrow y))),$$

and so $\mu((x \wedge y) \rightarrow (x \odot (x \rightarrow y))) = \mu(1)$ for all $x, y \in L$. Therefore μ is a divisible fuzzy filter of \mathcal{L} . \square

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Theorem 3.9. *If a fuzzy filter μ of \mathcal{L} satisfies the following assertion:*

$$(\forall x, y \in L) (\mu((x \wedge (x \rightarrow y)) \rightarrow y) = \mu(1)), \quad (3.12)$$

then μ is divisible.

Proof. Taking $y := x \odot y$ in (3.12) implies that

$$\begin{aligned} \mu(1) &= \mu((x \wedge (x \rightarrow (x \odot y))) \rightarrow (x \odot y)) \\ &\leq \mu((x \wedge y) \rightarrow (x \odot y)) \end{aligned}$$

and so $\mu((x \wedge y) \rightarrow (x \odot y)) = \mu(1)$ for all $x, y \in L$. It follows from Theorem 3.8 that μ is a divisible fuzzy filter of \mathcal{L} . \square

Theorem 3.10. *If a fuzzy filter μ of \mathcal{L} satisfies the following assertion:*

$$(\forall x, y, z \in L) (\mu(x \rightarrow z) \geq \min\{\mu((x \odot y) \rightarrow z), \mu(x \rightarrow y)\}), \quad (3.13)$$

then μ is divisible.

Proof. If we take $x := x \wedge (x \rightarrow y)$, $y := x$ and $z := y$ in (3.13), then

$$\begin{aligned} \mu((x \wedge (x \rightarrow y)) \rightarrow y) &\geq \min\{\mu(((x \wedge (x \rightarrow y)) \odot x) \rightarrow y), \mu((x \wedge (x \rightarrow y)) \rightarrow x)\} \\ &= \mu(1) \end{aligned}$$

Thus $\mu((x \wedge (x \rightarrow y)) \rightarrow y) = \mu(1)$ for all $x, y \in L$, and so μ is a divisible fuzzy filter of \mathcal{L} by Theorem 3.9. \square

Theorem 3.11. *If a fuzzy filter μ of \mathcal{L} satisfies the following assertion:*

$$(\forall x \in L) (\mu(x \rightarrow (x \odot x)) = \mu(1)), \quad (3.14)$$

then μ is divisible.

Proof. Let μ be a fuzzy filter of \mathcal{L} that satisfies the condition (3.14). Using (2.10) and the commutativity of \odot , we have $x \rightarrow y \leq (x \odot x) \rightarrow (x \odot y)$, and so

$$(x \rightarrow (x \odot x)) \odot (x \rightarrow y) \leq (x \rightarrow (x \odot x)) \odot ((x \odot x) \rightarrow (x \odot y))$$

for all $x, y \in L$ by (2.8) and the commutativity of \odot . It follows from (2.5), (2.8) and the commutativity of \odot that

$$\begin{aligned} &((x \rightarrow (x \odot x)) \odot (x \rightarrow y)) \odot ((x \odot y) \rightarrow z) \\ &\leq ((x \rightarrow (x \odot x)) \odot ((x \odot x) \rightarrow (x \odot y))) \odot ((x \odot y) \rightarrow z) \\ &\leq (x \rightarrow (x \odot y)) \odot ((x \odot y) \rightarrow z) \\ &\leq x \rightarrow z \end{aligned}$$

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and so from (2.17), (2.18), (2.19) and (3.14) that

$$\begin{aligned}
 \mu(x \rightarrow z) &\geq \mu((x \rightarrow (x \odot x)) \odot (x \rightarrow y)) \odot ((x \odot y) \rightarrow z)) \\
 &\geq \min\{\mu((x \rightarrow (x \odot x)) \odot (x \rightarrow y)), \mu((x \odot y) \rightarrow z)\} \\
 &\geq \min\{\mu(x \rightarrow (x \odot x)), \mu(x \rightarrow y), \mu((x \odot y) \rightarrow z)\} \\
 &= \min\{\mu(1), \mu(x \rightarrow y), \mu((x \odot y) \rightarrow z)\} \\
 &= \min\{\mu((x \odot y) \rightarrow z), \mu(x \rightarrow y)\}
 \end{aligned}$$

for all $x, y, z \in L$. Therefore μ is a divisible fuzzy filter of \mathcal{L} by Theorem 3.10. \square

Definition 3.12 ([4]). A filter F of \mathcal{L} is said to be *strong* if it satisfies:

$$(\forall x \in L) (\neg\neg(\neg\neg x \rightarrow x) \in F). \quad (3.15)$$

Definition 3.13. A fuzzy filter μ of \mathcal{L} is said to be *strong* if it satisfies:

$$(\forall x \in L) (\mu(\neg\neg(\neg\neg x \rightarrow x)) = \mu(1)). \quad (3.16)$$

Example 3.14. Consider a residuated lattice $L := \{0, a, b, c, d, 1\}$ with the following Hasse diagram (Figure 3.1) and Cayley tables (see Table 3 and Table 4).

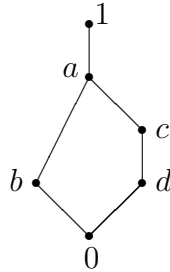


Figure 3.1

TABLE 3. Cayley table for the “ \odot ”-operation

\odot	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	b	d	d	a
b	c	b	b	0	0	b
c	b	d	0	d	d	c
d	b	d	0	d	d	d
1	0	a	b	c	d	1

Define a fuzzy set μ in \mathcal{L} by $\mu(1) = 0.6$ and $\mu(x) = 0.5$ for all $x(\neq 1) \in L$. It is routine to check that μ is a strong fuzzy filter of \mathcal{L} .

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TABLE 4. Cayley table for the “ \rightarrow ”-operation

\rightarrow	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	c	c	1
b	c	a	1	c	c	1
c	b	a	b	1	a	1
d	b	a	b	a	1	1
1	0	a	b	c	d	1

We provide characterizations of a strong fuzzy filter.

Theorem 3.15. *Given a fuzzy set μ of \mathcal{L} , the following assertions are equivalent.*

- (1) μ is a strong fuzzy filter of \mathcal{L} .
- (2) μ is a fuzzy filter of \mathcal{L} that satisfies

$$(\forall x, y \in L) (\mu((y \rightarrow \neg\neg x) \rightarrow \neg\neg(y \rightarrow x)) = \mu(1)). \quad (3.17)$$

- (3) μ is a fuzzy filter of \mathcal{L} that satisfies

$$(\forall x, y \in L) (\mu((\neg x \rightarrow y) \rightarrow \neg\neg(\neg y \rightarrow x)) = \mu(1)). \quad (3.18)$$

Proof. Assume that μ is a strong fuzzy filter of \mathcal{L} . Then μ is a fuzzy filter of \mathcal{L} . Note that

$$\begin{aligned} \neg\neg(\neg\neg x \rightarrow x) &\leq \neg\neg((y \rightarrow \neg\neg x) \rightarrow (y \rightarrow x)) \\ &\leq \neg\neg((y \rightarrow \neg\neg x) \rightarrow \neg\neg(y \rightarrow x)) \\ &= (y \rightarrow \neg\neg x) \rightarrow \neg\neg(y \rightarrow x) \end{aligned}$$

and

$$\begin{aligned} \neg\neg(\neg\neg x \rightarrow x) &\leq \neg\neg(((\neg x \rightarrow y) \odot \neg y) \rightarrow x) \\ &= \neg\neg((\neg x \rightarrow y) \rightarrow (\neg y \rightarrow x)) \\ &\leq \neg\neg((\neg x \rightarrow y) \rightarrow \neg\neg(\neg y \rightarrow x)) \\ &= (\neg x \rightarrow y) \rightarrow \neg\neg(\neg y \rightarrow x) \end{aligned}$$

for all $x, y \in L$. It follows from (3.16) and (2.18) that

$$\mu(1) = \mu(\neg\neg(\neg\neg x \rightarrow x)) \leq \mu((y \rightarrow \neg\neg x) \rightarrow \neg\neg(y \rightarrow x)) \quad (3.19)$$

and

$$\mu(1) = \mu(\neg\neg(\neg\neg x \rightarrow x)) \leq \mu((\neg x \rightarrow y) \rightarrow \neg\neg(\neg y \rightarrow x)). \quad (3.20)$$

Combining (2.19), (3.19) and (3.20), we have $\mu((y \rightarrow \neg\neg x) \rightarrow \neg\neg(y \rightarrow x)) = \mu(1)$ and $\mu((\neg x \rightarrow y) \rightarrow \neg\neg(\neg y \rightarrow x)) = \mu(1)$ for all $x, y \in L$. Therefore (2) and (3) are valid. Let μ be a fuzzy

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filter of \mathcal{L} that satisfies the condition (3.17). If we take $y := \neg\neg x$ in (3.17) and use (2.1), then we can induce the condition (3.16) and so μ is a strong fuzzy filter of \mathcal{L} . Let μ be a fuzzy filter of \mathcal{L} that satisfies the condition (3.18). Taking $y := \neg x$ in (3.18) and using (2.1) induces the condition (3.16). Hence μ is a strong fuzzy filter of \mathcal{L} . \square

We investigate relationship between a divisible fuzzy filter and a strong fuzzy filter.

Theorem 3.16. *Every divisible fuzzy filter is a strong fuzzy filter.*

Proof. Let μ be a divisible fuzzy filter of \mathcal{L} . If we put $x := \neg\neg x$ and $y := x$ in (3.2), then we have

$$\mu((\neg\neg x \wedge x) \rightarrow (\neg\neg x \odot (\neg\neg x \rightarrow x))) = \mu(1). \quad (3.21)$$

Using (2.4) and (2.8), we get

$$\begin{aligned} (\neg\neg x \wedge x) \rightarrow (\neg\neg x \odot (\neg\neg x \rightarrow x)) &\leq \neg(\neg\neg x \odot (\neg\neg x \rightarrow x)) \rightarrow \neg(\neg\neg x \wedge x) \\ &\leq (\neg\neg x \odot \neg(\neg\neg x \odot (\neg\neg x \rightarrow x))) \rightarrow (\neg\neg x \odot \neg(\neg\neg x \wedge x)) \\ &\leq \neg(\neg\neg x \odot \neg(\neg\neg x \wedge x)) \rightarrow \neg(\neg\neg x \odot \neg(\neg\neg x \odot (\neg\neg x \rightarrow x))) \end{aligned}$$

for all $x \in L$. It follows from (3.21) and (2.18) that

$$\begin{aligned} \mu(1) &= \mu((\neg\neg x \wedge x) \rightarrow (\neg\neg x \odot (\neg\neg x \rightarrow x))) \\ &\leq \mu(\neg(\neg\neg x \odot \neg(\neg\neg x \wedge x)) \rightarrow \neg(\neg\neg x \odot \neg(\neg\neg x \odot (\neg\neg x \rightarrow x)))). \end{aligned} \quad (3.22)$$

Combining (3.22) with (2.19), we have

$$\mu(\neg(\neg\neg x \odot \neg(\neg\neg x \wedge x)) \rightarrow \neg(\neg\neg x \odot \neg(\neg\neg x \odot (\neg\neg x \rightarrow x)))) = \mu(1) \quad (3.23)$$

for all $x \in L$. Using (2.2), (2.11), (2.12) and (2.6), we get

$$\begin{aligned} \neg(\neg\neg x \odot \neg(\neg\neg x \wedge x)) &= \neg\neg x \rightarrow \neg\neg(\neg\neg x \wedge x) \\ &\geq \neg\neg(x \rightarrow (\neg\neg x \wedge x)) \\ &= \neg\neg(x \rightarrow (x \wedge \neg\neg x)) \\ &= \neg\neg(x \rightarrow \neg\neg x) = \neg\neg 1 = 1 \end{aligned}$$

and so $\neg(\neg\neg x \odot \neg(\neg\neg x \wedge x)) = 1$ for all $x \in L$. It follows from (3.23) and (2.20) that

$$\begin{aligned} &\mu(\neg(\neg\neg x \odot \neg(\neg\neg x \odot (\neg\neg x \rightarrow x)))) \\ &\geq \min\{\mu(\neg(\neg\neg x \odot \neg(\neg\neg x \wedge x)) \rightarrow \neg(\neg\neg x \odot \neg(\neg\neg x \odot (\neg\neg x \rightarrow x)))), \\ &\quad \mu(\neg(\neg\neg x \odot \neg(\neg\neg x \wedge x)))\} \\ &= \mu(1) \end{aligned}$$

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and so that

$$\begin{aligned}\mu(1) &= \mu(\neg(\neg x \odot \neg(\neg x \odot (\neg x \rightarrow x)))) \\ &= \mu(\neg(\neg x \odot (\neg x \rightarrow \neg(\neg x \rightarrow x)))).\end{aligned}\tag{3.24}$$

Taking $x := \neg x$ and $y := \neg(\neg x \rightarrow x)$ in (3.2) induces

$$\begin{aligned}\mu(1) &= \mu((\neg x \wedge \neg(\neg x \rightarrow x)) \rightarrow (\neg x \odot (\neg x \rightarrow \neg(\neg x \rightarrow x)))) \\ &\leq \mu(\neg(\neg x \odot (\neg x \rightarrow \neg(\neg x \rightarrow x))) \rightarrow \neg(\neg x \wedge \neg(\neg x \rightarrow x)))\end{aligned}$$

by using (2.3) and (2.18). Thus

$$\mu(\neg(\neg x \odot (\neg x \rightarrow \neg(\neg x \rightarrow x))) \rightarrow \neg(\neg x \wedge \neg(\neg x \rightarrow x))) = \mu(1).\tag{3.25}$$

Since $\neg(\neg x \rightarrow x) \leq \neg x$ for all $x \in L$, it follows from (2.19), (2.20), (3.24) and (3.25) that

$$\mu(1) = \mu(\neg(\neg x \wedge \neg(\neg x \rightarrow x))) = \mu(\neg(\neg x \rightarrow x))$$

for all $x \in L$. Therefore μ is a strong fuzzy filter of \mathcal{L} . \square

Corollary 3.17. *If a fuzzy filter μ of \mathcal{L} satisfies one of conditions (3.8), (3.9), (3.11), (3.12), (3.13) and (3.14), then μ is a strong fuzzy filter of \mathcal{L} .*

The following example shows that the converse of Theorem 3.16 may not be true in general.

Example 3.18. The strong fuzzy filter μ of \mathcal{L} which is given in Example 3.14 is not a divisible fuzzy filter of \mathcal{L} since $\mu((a \wedge c) \rightarrow (a \odot (a \rightarrow c))) = \mu(a) \neq \mu(1)$.

4. Conclusions

The filter theory plays an important role in studying logical systems and the related algebraic structures, and various filters have been proposed in the literature. Zhang et al. [8] introduced the notions of IMTL-filters (NM-filters, MV-filters) of residuated lattices, and presented their characterizations. Ma and Hu [4] introduced divisible filters, strong filters and n -contractive filters in residuated lattices.

In this paper, we have considered the fuzzification of divisible filters and strong filters in residuated lattices. We have defined divisible fuzzy filters and strong fuzzy filters, and have investigated related properties. We have discussed characterizations of a divisible and strong fuzzy filter, and have provided conditions for a fuzzy filter to be divisible. We have establish relations between a divisible fuzzy filter and a strong fuzzy filter. In a forthcoming paper, we will study the fuzzification of n -contractive filters in residuated lattices, and apply the results to other algebraic structures.

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FREQUENT HYPERCYCLICITY OF WEIGHTED COMPOSITION OPERATORS ON CLASSICAL BANACH SPACES

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ABSTRACT. In this paper we characterize the frequent hypercyclicity of weighted composition operators on some classical Banach spaces, such as the weighted Dirichlet space S_v . Besides, we also discuss the frequent hypercyclicity of the weighted composition operators on the weighted Bergman space A_α^p .

1. INTRODUCTION AND TERMINOLOGY

Let $H(\mathbb{D})$ be the space of all holomorphic functions on \mathbb{D} , where \mathbb{D} is the open unit disk of the complex plane \mathbb{C} . The collection of all holomorphic self-maps of \mathbb{D} will be denoted by $S(\mathbb{D})$, and let $Aut(\mathbb{D})$ denote the set of all automorphisms on \mathbb{D} . The disk algebra, denoted by $A(\mathbb{D})$, consists of all functions in $H(\mathbb{D})$ that are continuous up to the boundary $\partial\mathbb{D}$ of the unit disk \mathbb{D} . Let dA denote the normalized Lebesgue measure on \mathbb{D} . The space of bounded analytic functions on \mathbb{D} will be denoted by H^∞ , with the sup norm $\|\cdot\|_\infty$.

For $\alpha > -1$ and $1 < p < \infty$, the weighted Bergman space A_α^p consists of analytic functions f such that

$$\|f\|^p = \int_{\mathbb{D}} |f(z)|^p d\nu_\alpha(z) < \infty,$$

where $d\nu_\alpha$ on \mathbb{D} is defined by

$$d\nu_\alpha = (\alpha + 1) (1 - |z|^2)^\alpha d\nu(z)$$

and $\nu_\alpha(\mathbb{D}) = 1$. Under the norm $\|\cdot\|$, A_α^p is a separable infinite dimensional Banach space, since the set of polynomials is dense in A_α^p .

For each real number v , the weighted Dirichlet space S_v is the space of holomorphic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$ such that the following norm

$$\|f\|_v^2 = \sum_{n=0}^{\infty} |a_n|^2 (n+1)^{2v}$$

is finite. Observe that the space S_v is Hilbert space, where the inner product is defined by

$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n} (n+1)^{2v},$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$. For instance, if $v = 0, -1/2, 1/2$, then S_v is, respectively, the classical Hardy space H^2 , the Bergman space A^2 , and the Dirichlet space \mathcal{D} .

By Lemma 1.2 in [5], we know the following expression

$$\|f\|^2 = \sum_{i=0}^l |f^{(i)}(0)|^2 + \int_{\mathbb{D}} |f^{(l+1)}(z)|^2 (1 - |z|^2)^{2l+1-2v} dA(z)$$

defines an equivalent norm on S_v , where $l \geq -1$ is an integer such that $v < l + 1$, and when $l = -1$, the first term in the right hand side above does not appear.

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A bounded linear operator T acting on a separable Banach space X is said to be hypercyclic if there is an $f \in X$ such that orbit $\{T^n f\}_{n \geq 0}$ is dense in X . One bounded operator T is called similar to another bounded operator S on X if there exists a bounded and invertible operator V on H such that $TV = VS$. And the similarity preserve hypercyclicity. A continuous linear operator T acting on a separable Banach space X is said to be mixing, if for any pair U, V of nonempty open subsets of X , there exists some $N \geq 0$ such that

$$T^n(U) \cap (V) \neq \emptyset, \text{ for all } n \geq N.$$

The lower density of a subset A of \mathbb{N} is defined as

$$\underline{\text{dens}}(A) = \liminf_{N \rightarrow \infty} \frac{\text{card}\{0 \leq n \leq N; n \in A\}}{N+1}.$$

A vector $x \in X$ is called frequently hypercyclic for T , if for every non-empty open subset U of X ,

$$\underline{\text{dens}}\{n \in \mathbb{N}, T^n x \in U\} > 0.$$

The operator T is called frequently hypercyclic if it possesses a frequently hypercyclic vector. It is obvious that if the operator T is frequently hypercyclic, then T is hypercyclic. More related details can be founded in chapter 9 in the book [6].

Let $u \in H(\mathbb{D})$ and $\varphi \in S(\mathbb{D})$, the weighted composition operator uC_φ is defined as

$$(uC_\varphi f)(z) = u(z)f(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

And when $u \equiv 1$, we just have the composition operator C_φ and when $\varphi(z) = z$, we get the multiplication operator M_u .

For $\varphi \in LFT(\mathbb{D})$, we define φ as following:

$$\varphi(z) = \frac{az + b}{cz + d},$$

where $ad - bc \neq 0$.

Note that the linear fractional self-maps of \mathbb{D} fall into distinct classes determined by their fixed point properties (see [1]). There are:

- (a) Maps with interior fixed point. By the Schwarz Lemma the interior fixed point is either attractive, or the map is an elliptic automorphism.
- (b) Parabolic maps. Its fixed point is on $\partial\mathbb{D}$, and the derivative = 1 at the fixed point.
- (c) Hyperbolic maps with attractive fixed point on $\partial\mathbb{D}$ and their repulsive fixed point outside of \mathbb{D} . Both fixed points are on $\partial\mathbb{D}$ if and only if the map is the automorphism of \mathbb{D} . In this case, the derivative < 1 at the attractive fixed point.

According to a result by P.R. Hurst [8], the composition operator $C_\varphi : S_v \rightarrow S_v$ is bounded for any $v \in \mathbb{R}$ and any $\varphi \in LFT(\mathbb{D})$. In [4], the authors partially characterized the frequent hypercyclicity of scalar multiples of composition operators, whose symbols are linear fractional maps, acting on the weighted Dirichlet space S_v . E. Gallado and A. Montes [5] have furnished a complete characterization of the hypercyclicity of λC_φ on S_v in terms of λ, v, φ . Readers interested in related topics can refer to [3, 7, 9, 12, 13].

In this note, we will discuss the conditions of the frequent hypercyclicity of weighted composition operators on some classical Banach spaces, such as the weighted Dirichlet space S_v and the weighted Bergman space A_α^p .

2. FREQUENT HYPERCYCLICITY OF uC_φ ON S_v

In this section, we begin to discuss the frequent hypercyclicity of the weighted composition operator uC_φ on S_v .

Theorem 2.1. *If uC_φ is frequently hypercyclic on S_v , then φ is univalent and has no fixed point in \mathbb{D} , and $u(z) \neq 0$ for every $z \in \mathbb{D}$.*

Proof. It is well known that uC_φ is hypercyclic on S_v , so by Theorem 1 in [11], we obtain it. \square

The following result can be found in [11, Theorem 2].

Theorem 2.2. *Let $v > 1/2$. Then*

- (a) *No weighted composition operator on S_v is hypercyclic.*
- (b) *If φ has two fixed points α, β in $\overline{\mathbb{D}}$, and $u(\alpha) = u(\beta)$, then uC_φ is not cyclic on S_v .*

Combining with the comparison principle, to discuss frequent hypercyclicity of the weighted composition operator uC_φ on S_v , we may assume without loss of generality that $0 \leq v \leq \frac{1}{2}$.

2.1. The case for $v = 0$. In general, composition operators are bounded on H^2 (see [2, Chapter 3]). M_u is also a bounded operator on S_v if $u \in H^\infty$. So when $v = 0$, $\varphi \in S(\mathbb{D})$ and $u \in H^\infty$, $uC_\varphi = M_u C_\varphi$.

According to the definition of [9], for any $w \in \partial\mathbb{D}$ and any positive number α , $Lip_\alpha(w)$ corresponds to the class of holomorphic functions φ such that there is some neighborhood G of w in $\partial\mathbb{D}$ and a positive constant M with

$$|\varphi(z) - \varphi(w)| \leq M |z - w|^\alpha, \quad \text{for } z \in G.$$

For example, if an analytic function φ on \mathbb{D} is also analytic at $w \in \partial\mathbb{D}$, then $\varphi \in Lip_\alpha(w)$ whenever $0 \leq \alpha \leq 1$. Moreover, if $\varphi'(w) = 0$, then $\varphi \in Lip_\alpha(w)$ whenever $0 \leq \alpha \leq 2$.

We have the following proposition.

Proposition 2.3. *Let $\varphi \in LFT(\mathbb{D})$, $w \in \partial\mathbb{D}$ be the Denjoy-Wolff point of φ , $u \in Lip_\alpha(w) \cap A(\mathbb{D})$, $\|u\|_\infty = |u(w)| \neq 0$. Then $u(w)$ is an eigenvalue for uC_φ , whenever φ is hyperbolic and $\alpha > 0$ or φ is parabolic automorphism and $\alpha > 1$. Moreover, if u never vanishes on $\overline{\mathbb{D}}$, then the eigenfunction also never vanishes.*

Proof. According to the proof of Proposition 2.4 of [9], we have that the function $g(z) = \prod_{n=0}^{\infty} \frac{u(\varphi_n(z))}{u(w)}$ is a nonzero holomorphic function on \mathbb{D} . Since $\|u\|_\infty = |u(w)| \neq 0$, then for every $j \geq 0$ and $z \in \mathbb{D}$, $\left| \frac{u(\varphi_j(z))}{u(w)} \right| \leq 1$. And note that for fixed $z \in \mathbb{D}$, $\prod_{j=0}^n \left| \frac{u(\varphi_j(z))}{u(w)} \right|$ is

decreasing with respect to n . Therefore, $\|g\|_\infty = \sup_{z \in \mathbb{D}} \left| \prod_{n=0}^{\infty} \frac{u(\varphi_n(z))}{u(w)} \right| \leq \sup_{z \in \mathbb{D}} \left| \frac{u(\varphi(z))}{u(w)} \right| \leq 1$.

That is, $g \in H^\infty \subset S_v$ and $u(z)g(\varphi(z)) = u(w)g(z)$. Thus $u(w)$ is an eigenvalue for uC_φ . Since $u(z) \neq 0$ for every $z \in \overline{\mathbb{D}}$, $g(z) \neq 0$ for $z \in \overline{\mathbb{D}}$. \square

Next, we obtain the following result.

Theorem 2.4. *Let $\varphi \in LFT(\mathbb{D})$, $w \in \partial\mathbb{D}$ be the Denjoy-Wolff point of φ , $u \in Lip_\alpha(w) \cap A(\mathbb{D})$, $\|u\|_\infty = |u(w)| \neq 0$ and $u(z) \neq 0$ for every $z \in \overline{\mathbb{D}}$, then*

- (a) *If φ is hyperbolic automorphism, $\alpha > 0$ and $\varphi'(w)^{1/2} < |u(w)| < \varphi'(w)^{-1/2}$, then uC_φ is frequently hypercyclic on $H^2(\mathbb{D})$.*
- (b) *If φ is parabolic automorphism, $\alpha > 1$ and $|u(w)| = 1$, then uC_φ is frequently hypercyclic on $H^2(\mathbb{D})$.*
- (c) *If φ is hyperbolic non-automorphism, $\alpha > 0$ and $|u(w)| > \varphi'(w)^{1/2}$, then uC_φ is frequently hypercyclic on $H^2(\mathbb{D})$.*

Proof. By the proof of Proposition 2.4, $g(z) = \prod_{n=0}^{\infty} \frac{u(\varphi_n(z))}{u(w)} \neq 0$ for $z \in \overline{\mathbb{D}}$, it is easy to see that M_g is a bounded operator on $H^2(\mathbb{D})$ and $uC_\varphi M_g = u(w) M_g C_\varphi$. Combining with the comparison principle, we obtain this theorem. \square

2.2. The case for $0 < v < 1/2$. For $v \in (0, 1/2)$, using the equivalent norm in S_v , we define the Banach space Q_c as follows:

$$Q_c = \{f \in S_v : \|f\|_{Q_c} = |f(0)| + \sup_{w \in \mathbb{D}} \|f \circ \varphi_w - f\| < \infty\},$$

where $c = 1 - 2v$ and $\varphi_w(z) = (w - z)/(1 - \bar{w}z)$. For different $p \in (0, 1)$, $Q_{p_1} \subset Q_{p_2}$ when $0 < p_1 < p_2 \leq 1$. In particular, $Q_1 = BMOA$, the bounded mean oscillation space of analytic functions and when $p > 1$, $Q_p = \mathcal{B}$, the Bloch space on \mathbb{D} .

Let $g \in Q_{1-2v}$, by Corollary 2 in [10], we know that if

$$\sup_{\zeta \in \partial \mathbb{D}} \int_{D(\zeta, r)} |g(z)|^2 (1 - |z|)^{1-2v} dA(z) = O(r^{3-2v}), \quad (2.1)$$

then M_g is bounded on S_v .

Thus we get the following theorems.

Theorem 2.5. *Let $0 < v < 1/2$ and $\alpha > 0$. And let $\varphi \in LFM(\mathbb{D})$ and φ be a hyperbolic automorphism of the unit disc with Denjoy-Wolff point $w \in \partial \mathbb{D}$, $u \in Lip_\alpha(w)$ and $u(w) \neq 0$, the function $g = \prod_{i=0}^{\infty} \frac{1}{u(w)} u(\varphi_i(w)) \in Q_{1-2v}$, $\|I - M_g\|_{S_v \rightarrow S_v} < 1$ and (2.1) holds, then the following are equivalent:*

- (a) uC_φ is frequently hypercyclic.
- (b) uC_φ is hypercyclic.
- (c) $\varphi'(w)^{(1-2v)/2} < |u(w)| < \varphi'(w)^{(2v-1)/2}$.

Proof. The implication (a) \Rightarrow (b) is trivial. If $\varphi \in LFM(\mathbb{D})$ with Denjoy-Wolff point $w \in \partial \mathbb{D}$ and $u \in Lip_\alpha(w)$, $u(w) \neq 0$, as we saw in the proof of Proposition 2.4 in [9], the map $g(z) = \prod_{i=0}^{\infty} \frac{1}{u(w)} u(\varphi_i(w))$ is a nonzero holomorphic function satisfying $uC_\varphi g = u(w)g$.

Since $g \in Q_{1-2v}$ and (2.1) holds, we have M_g is bounded operator on S_v , so $g \in S_v$, thus the function g is an eigenfunction of uC_φ corresponding to $u(w)$ on S_v , and $uC_\varphi M_g = u(w)M_g C_\varphi$.

Note that $\|I - M_g\|_{S_v \rightarrow S_v} \leq 1 + \|M_g\|_{S_v \rightarrow S_v}$. So $I - M_g$ is also a bounded operator on S_v . Because $\|I - M_g\|_{S_v \rightarrow S_v} < 1$, then M_g is a invertible operator. It is obvious that (b) \Leftrightarrow (c). Besides, suppose that the condition (c) holds, by the proof of Theorem 2.6 in [4], $u(w)C_\varphi$ satisfies the Frequent Hypercyclicity Criterion. The implication (c) \Rightarrow (a) is obvious. \square

2.3. The case for $v = 1/2$. If so, we know that S_v is the Dirichlet space \mathcal{D} .

Theorem 2.6. *Let $\varphi \in LFT(\mathbb{D})$, $\alpha > 1$, $w \in \partial \mathbb{D}$ be the Denjoy-Wolff point of φ , $u \in Lip_\alpha(w) \cap A(\mathbb{D})$, $\|u\|_\infty = |u(w)| > 1$ and $u(z) \neq 0$ for every $z \in \mathbb{D}$. If φ is hyperbolic non-automorphism, then uC_φ is frequently hypercyclic on the Dirichlet space \mathcal{D} .*

Proof. By the proof of Proposition 2.4, $g(z) = \prod_{n=0}^{\infty} \frac{u(\varphi_n(z))}{u(w)} \neq 0$ for $z \in \overline{\mathbb{D}}$. Since $\|u\|_\infty = |u(w)| > 1$, so $g \in H^\infty \subset \mathcal{D}$ and $u(z)g(\varphi(z)) = u(w)g(z)$. It is easy to see that M_g is a bounded operator on the Dirichlet space \mathcal{D} and $uC_\varphi M_g = u(w)M_g C_\varphi$. By Theorem 1.8 in [5] and the comparison principle, we complete the proof. \square

3. FREQUENT HYPERCYCLICITY OF uC_φ ON A_α^p

In this section, we study in detail frequent hypercyclicity of uC_φ on the weighted Bergman space A_α^p and we suppose that the weighted composition operator uC_φ is bounded on A_α^p .

Proposition 3.1. *Let $\alpha > -1$, $1 < p < \infty$ and $\varphi \in LFT(\mathbb{D})$. If uC_φ is frequently hypercyclic on A_α^p , then*

- (i) φ has no fixed point in \mathbb{D} and φ is univalent.
- (ii) $u(z) \neq 0$ for every $z \in \mathbb{D}$.

Proof. The proof is obvious, so we omit it. \square

Next, we obtain the following results.

Theorem 3.2. *Let $\alpha > -1$, $\beta > 0$, $1 < p < \infty$, $\varphi \in LFT(\mathbb{D})$ and φ be a hyperbolic automorphism and $w \in \partial \mathbb{D}$ be the Denjoy-Wolff point of φ , $u \in Lip_\beta(w) \cap A(\mathbb{D})$, $\|u\|_\infty = |u(w)| \neq 0$ and $u(z) \neq 0$ for every $z \in \mathbb{D}$. If $\varphi'(w)^{(2+\alpha)/p} < |u(w)| < \varphi'(w)^{(-2-\alpha)/p}$, then uC_φ is frequently hypercyclic on A_α^p .*

Proof. First, since this space under consideration is unitarily invariant, we may assume that 1 and -1 are fixed points of φ and 1 is the attractive fixed point. The change of variables

$$\sigma(z) = \frac{i(1-z)}{1+z}$$

takes the unit disk onto the upper half plane, 1 and -1 to 0 and ∞ . We obtain that φ is conjugate to the translation map

$$\psi(z) = \rho z,$$

where $0 < \rho < 1$. By using the equation $\sigma \circ \varphi = \psi \circ \sigma$, we can get

$$\varphi(z) = \frac{(1+\rho)z + 1 - \rho}{(1-\rho)z + 1 + \rho},$$

where $\varphi'(1) = \rho$.

Let X_0 denote the subspace of polynomials vanishing m at 1, where $m > \frac{2(\alpha+2)}{p}$. It is obvious that X_0 is dense on A_α^p . Fix $f \in X_0$. It is similarly proved as in Theorem 3.5 in [5] that

$$\|\lambda^n C_\varphi^n f\|^p \leq C |\lambda|^{np} \rho^{(\alpha+2)n}, n \in \mathbb{N},$$

where C is a constant independent of n . If $\varphi'(1)^{(2+\alpha)/p} < |\lambda| < \varphi'(1)^{(-2-\alpha)/p}$, we obtain that

$$\sum_{n=1}^{\infty} \|(\lambda C_\varphi)^n f\| < \infty, \text{ for all } f \in X_0. \quad (3.1)$$

Similarly, let Y_0 denote the subspace of polynomials vanishing m at -1 and Y_0 is dense on A_α^p . We take $S = (\lambda C_\varphi)^{-1}$. Observe that -1 is the attractive fixed point of φ^{-1} with $(\varphi^{-1})'(-1) = \frac{1}{\varphi'(-1)} = \rho$ and $\varphi'(1)^{(2+\alpha)/p} < |\lambda| < \varphi'(1)^{(-2-\alpha)/p}$. Therefore, a similar argument leads to

$$\sum_{n=1}^{\infty} \|S^n f\| < \infty, \text{ for all } f \in Y_0. \quad (3.2)$$

If we set $X := X_0 \cap Y_0$, then we obtain that X is dense in A_α^p . Clearly (3.1) and (3.2) hold for all $f \in X$. It is obvious that $\lambda C_\varphi S$ is the identity on X . Consequently, λC_φ satisfies the Frequent Hypercyclicity Criterion. By Proposition 2.4, then $u C_\varphi$ is frequently hypercyclic on A_α^p . \square

Theorem 3.3. Let $\alpha > -1$, $\beta > 0$, $1 < p < \infty$, $\varphi \in LFT(\mathbb{D})$ and φ is a hyperbolic non-automorphism, $w \in \partial\mathbb{D}$ be the Denjoy-Wolff point of φ , $u \in Lip_\beta(w) \cap A(\mathbb{D})$, $\|u\|_\infty = |u(w)| \neq 0$ and $u(z) \neq 0$ for every $z \in \mathbb{D}$. If $|u(w)| > \varphi'(\zeta)^{(2+\alpha)/p}$, then $u C_\varphi$ is frequently hypercyclic on A_α^p .

Proof. First, we prove that if $|\lambda| > \varphi'(w)^{(2+\alpha)/p}$, then λC_φ is frequently hypercyclic on A_α^p .

Now, we assume that $w = 1$ is the boundary fixed point and β is a exterior fixed point. Upon conjugating with an appropriate map, φ is conjugate to

$$\rho z + 1 - \rho,$$

where $0 < \rho < 1$. Hence we may assume that $\varphi(z) = \rho z + 1 - \rho$, where $\varphi'(1) = \rho$. For any $n \in \mathbb{N}$, we have

$$\varphi_n(z) = \rho^n z + 1 - \rho^n. \quad (3.3)$$

Let X_0 denote the subspace of polynomials vanishing m at 1, where m is to be determined later on. Obviously, X_0 is dense on A_α^p . Fix $f \in X_0$. It is similarly proved as in Theorem 2.11 in [5] that

$$\|\lambda^n C_\varphi^n f\|^p \leq C |\lambda|^{np} \rho^{mnp}, n \in \mathbb{N},$$

where C is a constant independent of n . Since $0 < \rho < 1$, we can choose m large enough to have $|\lambda \rho^m| < 1$. By the assumption, we obtain that

$$\sum_{n=1}^{\infty} \|(\lambda C_\varphi)^n f\| < \infty, \text{ for all } f \in X_0. \quad (3.4)$$

Define $T = \lambda C_\varphi$ and the inverse $S = \lambda^{-1} C_{\varphi^{-1}}$. Let Y be the set of all polynomials that vanish m times at β where m will be suitable number. The set Y_0 will be

$$Y_0 = \bigcup_{n=0}^{\infty} \lambda^{-n} C_{\varphi^{-1}}^n(Y) = \bigcup_{n=0}^{\infty} \lambda^{-n} C_{\varphi^{-n}}(Y).$$

Similarly, we obtain that for n large enough

$$\|\lambda^{-n}C_{\varphi-n}f\|^p \leq C|\lambda|^{-np}\rho^{n(\alpha+2)},$$

where C is a constant independent of n . By the assumption, we have

$$\sum_{n=1}^{\infty} \|S^n f\| < \infty, \text{ for all } f \in Y_0. \quad (3.5)$$

If we set $X := \cup_{n=0}^{\infty} S^n(X \cap Y)$, then we obtain that X is dense in A_{α}^p . Clearly (3.4) and (3.5) hold for all $f \in X$. It is obvious that $\lambda C_{\varphi}S$ is the identity on X . Consequently, λC_{φ} satisfies the Frequent Hypercyclicity Criterion. By Proposition 2.4, then uC_{φ} is frequently hypercyclic on A_{α}^p . \square

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ON THE SPECIAL TWISTED q -POLYNOMIALS

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ABSTRACT. In this paper, we found some interesting identities of q -extension of special twisted polynomials which are derive from the bosonic q -integral and fermionic q -integral on \mathbb{Z}_p .

1. INTRODUCTION

Let p be a given odd prime number. Throughout this paper, we assume that \mathbb{Z}_p , \mathbb{Q}_p and \mathbb{C}_p will, respectively, denote the rings of p -adic integers, the fields of p -adic rational numbers and the completion of algebraic closure of \mathbb{Q}_p . The p -adic norm $|\cdot|_p$ is normalized by $|p|_p = \frac{1}{p}$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on \mathbb{Z}_p . For $f \in UD(\mathbb{Z}_p)$, the *bosonic p -adic q -integral on \mathbb{Z}_p* is defined by T. Kim to be

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \quad (\text{see [9, 10]}), \quad (1.1)$$

and the *fermionic p -adic q -integral on \mathbb{Z}_p* is also defined by Kim to be

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x) (-q)^x, \quad (\text{see [9, 11]}). \quad (1.2)$$

Let $f_1(x) = f(x+1)$. Then, by (1.1) and (1.2), we get

$$qI_q(f_1) - I_q(f) = (q-1)f(0) + \frac{q-1}{\log q} f'(0), \quad (1.3)$$

and

$$qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \quad (1.4)$$

where $f'(0) = \frac{d}{dx} f(x)|_{x=0}$ (see [9, 10, 11]).

It is well known that the *q -Bernoulli polynomials* are defined by the generating function to be

$$\frac{q-1 + \frac{(q-1)t}{\log q}}{qe^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,q}(x) \frac{t^n}{n!}, \quad (1.5)$$

and the *q -Euler polynomials* are given by

$$\frac{[2]_q}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}. \quad (1.6)$$

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When $x = 0$, $B_{n,q} = B_{n,q}(0)$ ($E_{n,q} = E_{n,q}(0)$) are called the n th q -Bernoulli numbers (n th q -Euler numbers, respectively) (see [7, 8, 14, 16]).

The *Stirling numbers of the first kind* are defined by

$$(x)_n = x(x-1) \cdots (x-n+1) = \sum_{l=0}^n S_1(n, l) x^l, \quad (n \geq 0),$$

and the *Stirling numbers of the second kind* are defined by

$$(e^t - 1)^n = n! \sum_{l=n}^{\infty} S_2(l, n) \frac{t^l}{l!}, \quad (\text{see [1, 12]}).$$

The *Daehee polynomials of the first kind* are defined by the generating function to be

$$\left(\frac{\log(1+t)}{t} \right) (1+t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!} \quad (\text{see [4, 5]}).$$

Recently, the q -Daehee polynomials are defined by the generating function to be

$$\left(\frac{1-q + \frac{1-q}{\log q}}{1-q-qt} \right) (1+t)^x = \sum_{n=0}^{\infty} D_{n,q}(x) \frac{t^n}{n!}, \quad (\text{see [2, 13]}), \quad (1.7)$$

and the q -Changhee polynomials are defined by the generating function to be

$$\frac{[2]_q}{[2]_q + qt} (1+t)^x = \sum_{n=0}^{\infty} Ch_{n,q}(x) \frac{t^n}{n!}, \quad (\text{see [3]}) \quad (1.8)$$

where $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$. When $x = 0$, $D_{n,q} = D_{n,q}(0)$ ($Ch_{n,q} = Ch_{n,q}(0)$) are called the n th q -Daehee numbers (n th q -Changhee numbers, respectively).

The Daehee polynomials and Changhee polynomials are introduced by T. Kim et. al. in [4, 6], and found interesting identities in [2, 4, 5, 6, 13, 15, 16]. In this paper, we found some interesting identities of q -extension of special twisted polynomials which are derive from the bosonic q -integral and fermionic q -integral on \mathbb{Z}_p .

2. TWISTED q -DAEHEE NUMBERS AND POLYNOMIALS OF HIGHER-ORDER

For $n \in \mathbb{N}$, let T_p be the p -adic locally constant space defined by

$$T_p = \bigcup_{n \geq 1} C_{p^n} = \lim_{n \rightarrow \infty} C_{p^n},$$

where $C_{p^n} = \{\omega | \omega^{p^n} = 1\}$ is the cyclic group of order p^n .

In this section, we assume that $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$. We define the *higher order q -Bernoulli polynomials* as follows:

$$\left(\frac{q-1 + \frac{q-1}{\log q} t}{qe^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(r)}(x) \frac{t^n}{n!}. \quad (2.1)$$

When $x = 0$, $B_{n,q}^{(r)}(0) = B_{n,q}^{(r)}$ are called the *higher order q -Bernoulli numbers*.

For $\varepsilon \in T_p$, we consider the *twisted q -Daehee polynomials of order r* as follows:

$$\left(\frac{q-1 + \frac{q-1}{\log q} \log(1+\varepsilon t)}{q\varepsilon t + q - 1} \right)^r (1+\varepsilon t)^x = \sum_{n=0}^{\infty} D_{n,\varepsilon,q}^{(r)}(x) \frac{t^n}{n!}. \quad (2.2)$$

When $x = 0$, $D_{n,\varepsilon,q}^{(r)}(0) = D_{n,\varepsilon,q}^{(r)}$ are called *twisted q -Daehee numbers of order r* .

From (1.1), we can obtain the equation:

$$\begin{aligned} & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + \epsilon t)^{x_1 + \cdots + x_r + x} d\mu_q(x_1) \cdots d\mu_q(x_r) \\ &= \left(\frac{q - 1 + \frac{q-1}{\log q} \log(1 + \epsilon t)}{q\epsilon t + q - 1} \right)^r (1 + \epsilon t)^x \\ &= \sum_{n=0}^{\infty} D_{n,\epsilon,q}^{(r)}(x) \frac{t^n}{n!}. \end{aligned} \quad (2.3)$$

By (2.3), we get

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \epsilon^n \binom{x_1 + \cdots + x_r + x}{n} d\mu_q(x_1) \cdots d\mu_q(x_r) = \frac{D_{n,\epsilon,q}^{(r)}(x)}{n!} \quad (n \geq 0). \quad (2.4)$$

By replacing t by $\frac{1}{\epsilon}(e^t - 1)$ in (2.3), we have

$$\sum_{n=0}^{\infty} D_{n,\epsilon,q}^{(r)}(x) \frac{\left(\frac{1}{\epsilon}(e^t - 1)\right)^n}{n!} = \left(\frac{q - 1 + \frac{q-1}{\log q} t}{qe^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(r)}(x) \frac{t^n}{n!} \quad (2.5)$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} D_{n,\epsilon,q}^{(r)}(x) \frac{1}{\epsilon^n n!} (e^t - 1)^n &= \sum_{n=0}^{\infty} D_{n,\epsilon,q}^{(r)}(x) \frac{1}{\epsilon^n n!} \sum_{m=n}^{\infty} S_2(m, n) \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \frac{D_{m,\epsilon,q}^{(r)}(x) S_2(n, m)}{\epsilon^m} \right) \frac{t^n}{n!}. \end{aligned} \quad (2.6)$$

Thus, by (2.5) and (2.6), we have

$$B_{n,q}^{(r)}(x) = \sum_{m=0}^n \frac{D_{m,\epsilon,q}^{(r)}(x) S_2(n, m)}{\epsilon^m}. \quad (2.7)$$

Therefore, by (2.4) and (2.7), we obtain the following theorem.

Theorem 2.1. *For $n \geq 0$, we have*

$$B_{n,q}^{(r)}(x) = \sum_{m=0}^n \frac{D_{m,\epsilon,q}^{(r)}(x) S_2(n, m)}{\epsilon^m}$$

and

$$\frac{D_{n,\epsilon,q}^{(r)}(x)}{n!} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \epsilon^n \binom{x_1 + \cdots + x_r + x}{n} d\mu_q(x_1) \cdots d\mu_q(x_r)$$

where $S_2(m, n)$ is the Stirling number of the second kind.

From (2.1), by replacing t by $\log(1 + \epsilon t)$, we have

$$\begin{aligned}
 & \left(\frac{q - 1 + \frac{q-1}{\log q} \log(1 + \epsilon t)}{q\epsilon t + q - 1} \right)^r (1 + \epsilon t)^x \\
 &= \sum_{n=0}^{\infty} B_{n,q}^{(r)}(x) \frac{1}{n!} (\log(1 + \epsilon t))^n \\
 &= \sum_{n=0}^{\infty} B_{n,q}^{(r)}(x) \frac{1}{n!} n! \sum_{m=n}^{\infty} S_1(m, n) \frac{(\epsilon t)^m}{m!} \\
 &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^m \epsilon^m S_1(m, n) B_{n,q}^{(r)}(x) \right) \frac{t^m}{m!},
 \end{aligned} \tag{2.8}$$

where $S_1(m, n)$ is the Stirling number of the first kind. Thus, by (2.2) and (2.8), we obtain the following theorem.

Theorem 2.2. *For $n \geq 0$, we have*

$$D_{n,\epsilon,q}^{(r)}(x) = \sum_{n=0}^m \epsilon^m S_1(m, n) B_{n,q}^{(r)}(x).$$

Now, we consider the q -Changhee polynomials of order r which are defined by the generating function as follows:

$$\frac{[2]_q}{q\epsilon t + [2]_q} (1 + \epsilon t)^x = \sum_{n=0}^{\infty} Ch_{n,\epsilon,q}^{(r)}(x) \frac{t^n}{n!}. \tag{2.9}$$

In the special case $x = 0$, $Ch_{n,\epsilon,q}^{(r)}(0) = Ch_{n,\epsilon,q}^{(r)}$ are called the q -Changhee numbers of order r .

From (1.2), we note that

$$\begin{aligned}
 & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1 + \epsilon t)^{x_1 + \cdots + x_r + x} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) \\
 &= \left(\frac{[2]_q}{q\epsilon t + [2]_q} \right)^r (1 + \epsilon t)^x.
 \end{aligned} \tag{2.10}$$

By (2.10), we have

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \epsilon^n \binom{x_1 + \cdots + x_r + x}{n} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) = \frac{Ch_{n,\epsilon,q}^{(r)}(x)}{n!}. \tag{2.11}$$

In view of (1.6), we define the higher order q -Euler polynomials by generating function to be

$$\left(\frac{[2]_q}{qe^t + 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x) \frac{t^n}{n!}. \tag{2.12}$$

From (2.10), we note that

$$\begin{aligned}
 & \int_{\mathbb{Z}_p} (1 + \epsilon t)^{x_1 + \cdots + x_r + x} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) \\
 &= \left(\frac{[2]_q}{qe^{\log(1+\epsilon t)} + 1} \right)^r e^{x \log(1+\epsilon t)} \\
 &= \sum_{n=0}^{\infty} E_{n,q}^{(r)} \frac{1}{n!} (\log(1 + \epsilon t))^n \\
 &= \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x) \frac{1}{n!} \sum_{m=n}^{\infty} S_1(m, n) \frac{(\epsilon t)^m}{m!} \\
 &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^m \epsilon^m E_{n,q}^{(r)}(x) S_1(m, n) \right) \frac{t^m}{m!}.
 \end{aligned} \tag{2.13}$$

Hence, by (2.11) and (2.13), we obtain the following theorem.

Theorem 2.3. *For $n \geq 0$, we have*

$$\begin{aligned}
 & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \epsilon^n \binom{x_1 + \cdots + x_r + x}{n} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) \\
 &= \frac{Ch_{n,\epsilon,q}^{(r)}(x)}{n!} = \frac{1}{n!} \sum_{m=0}^n \epsilon^n E_{m,q}^{(r)}(x) S_1(n, m).
 \end{aligned}$$

By replacing t by $\frac{1}{\epsilon}(e^t - 1)$ in (2.9), we have

$$\sum_{n=0}^{\infty} Ch_{n,\epsilon,q}^{(r)}(x) \frac{(e^t - 1)^n}{\epsilon^n n!} = \left(\frac{[2]_q}{qe^t + 1} \right)^r e^{xt} \tag{2.14}$$

and

$$\begin{aligned}
 \sum_{n=0}^{\infty} \epsilon^{-n} Ch_{n,\epsilon,q}^{(r)}(x) \frac{1}{n!} (e^t - 1)^n &= \sum_{n=0}^{\infty} \epsilon^{-n} Ch_{n,\epsilon,q}^{(r)}(x) \sum_{m=n}^{\infty} S_2(m, n) \frac{t^m}{m!} \\
 &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^m \epsilon^{-n} Ch_{n,\epsilon,q}^{(r)}(x) S_2(m, n) \right) \frac{t^m}{m!}
 \end{aligned} \tag{2.15}$$

By (2.12), (2.14) and (2.15), we obtain the following theorem.

Theorem 2.4. *For $n \geq 0$, we have*

$$E_{n,q}^{(r)}(x) = \sum_{m=0}^n \epsilon^{-m} Ch_{m,\epsilon,q}^{(r)}(x) S_2(n, m).$$

From now on, we consider the q -analogue of the *twisted Cauchy polynomials of order r* , which are defined by the generating function to be

$$\left(\frac{q(1 + \epsilon t) - 1}{(q - 1) + \frac{q-1}{\log q} \log(1 + \epsilon t)} \right)^r (1 + \epsilon t)^x = \sum_{n=0}^{\infty} C_{n,\epsilon,q}^{(r)}(x) \frac{t^n}{n!}. \tag{2.16}$$

In the special case $x = 0$, $C_{n,\epsilon,q}^{(r)}(0) = C_{n,\epsilon,q}^{(r)}$ are called the *twisted Cauchy numbers of order r* . Note that

$$\begin{aligned} & \lim_{q \rightarrow 1} \left(\frac{q(1+\epsilon t) - 1}{(q-1) + \frac{q-1}{\log q} \log(1+\epsilon t)} \right)^r (1+r)^x \\ &= \left(\frac{\epsilon t}{\log(1+\epsilon t)} \right)^r (1+t)^x = \sum_{n=0}^{\infty} C_{n,\epsilon}^{(r)}(x) \frac{t^n}{n!}, \end{aligned} \quad (2.17)$$

where $C_{n,\epsilon}^{(r)}$ are called the *Cauchy polynomials of order r* .

By (2.2), we can derive the followings:

$$\begin{aligned} (1+\epsilon t)^x &= \left(\frac{q(1+\epsilon t) - 1}{(q-1) + \frac{q-1}{\log q} \log(1+\epsilon t)} \right)^r (1+\epsilon t)^x \left(\frac{(q-1) + \frac{q-1}{\log q} \log(1+\epsilon t)}{q(1+\epsilon t) - 1} \right)^r \\ &= \left(\sum_{k=0}^{\infty} C_{k,\epsilon,q}^{(r)} \right) \left(\sum_{m=0}^{\infty} D_{m,\epsilon,q}^{(r)} \frac{t^m}{m!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} C_{l,\epsilon,q}^{(r)}(x) D_{n-l,\epsilon,q}^{(r)} \right) \frac{t^n}{n!} \end{aligned} \quad (2.18)$$

and

$$(1+\epsilon t)^x = \sum_{n=0}^{\infty} \epsilon^n(x)_n \frac{t^n}{n!}. \quad (2.19)$$

By (2.18) and (2.19), we obtain the following theorem.

Theorem 2.5. For $n \geq 0$, we have

$$\binom{x}{n} = \frac{1}{\epsilon^n n!} \sum_{l=0}^n \binom{n}{l} C_{l,\epsilon,q}^{(r)}(x) D_{n-l,\epsilon,q}^{(r)}.$$

Let n be a given nonnegative integer. In [2], authors defined q -analogue of the Bernoulli-Euler mixed-type polynomials of order (r, s) $BE_{n,q}^{(r,s)}(x)$, and derived the following equation.

$$\sum_{n=0}^{\infty} BE_{n,q}^{(r,s)}(x) \frac{t^n}{n!} = \left(\frac{[2]_q}{qe^t + 1} \right)^s \left(\frac{q-1 + \frac{q-1}{\log q} t}{qe^t - 1} \right)^r e^{xt}. \quad (2.20)$$

By replacing t by $\log(1+\epsilon t)$, we get

$$\begin{aligned} & \sum_{n=0}^{\infty} BE_{n,q}^{(r,s)}(x) \frac{(\log(1+\epsilon t))^n}{n!} \\ &= \left(\frac{[2]_q}{q(1+\epsilon t) + 1} \right)^s \left(\frac{q-1 + \frac{q-1}{\log q} \log(1+\epsilon t)}{q(1+\epsilon t) - 1} \right)^r (1+\epsilon t)^x \\ &= \left(\sum_{m=0}^{\infty} Ch_{m,\epsilon,q}^{(s)}(x) \frac{t^m}{m!} \right) \left(\sum_{l=0}^{\infty} D_{l,\epsilon,q}^{(r)} \frac{t^l}{l!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \binom{n}{l} Ch_{l,\epsilon,q}^{(s)}(x) D_{n-l,\epsilon,q}^{(r)} \right) \frac{t^n}{n!}, \end{aligned} \quad (2.21)$$

and

$$\sum_{n=0}^{\infty} BE_{n,q}^{(r,s)}(x) \frac{(\log(1+\epsilon t))^n}{n!} = \sum_{n=0}^{\infty} \left(\epsilon^n \sum_{m=0}^n BE_{m,q}^{(r,s)}(x) S_1(n, m) \right) \frac{t^m}{m!}. \quad (2.22)$$

Thus, by (2.21) and (2.22), we obtain the following theorem.

Theorem 2.6. For $n \geq 0$, we have

$$\sum_{l=0}^n \binom{n}{l} Ch_{l,\epsilon,q}^{(s)}(x) D_{n-l,\epsilon,q}^{(r)} = \epsilon^n \sum_{m=0}^n BE_{m,q}^{(r,s)}(x) S_1(n, m).$$

From now on, we consider the q -analogue of the *twisted Daehee-Changhee mixed-type polynomials of order (r, s)* as follows:

$$DC_{n,\epsilon,q}^{(r,s)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} D_{n,\epsilon,q}^{(r)}(x + y_1 + \cdots + y_s) d\mu_{-q}(y_1) \cdots d\mu_{-q}(y_s) \quad (2.23)$$

where n is a given nonnegative integer.

By (2.23), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} DC_{n,\epsilon,q}^{(r,s)}(x) \frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} D_{n,\epsilon,q}^{(r)}(x + y_1 + \cdots + y_s) \frac{t^n}{n!} d\mu_{-q}(y_1) \cdots d\mu_{-q}(y_s) \\ &= \left(\frac{q-1 + \frac{q-1}{\log q} \log(1+\epsilon t)}{q\epsilon t + q-1} \right)^r \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+\epsilon t)^{x+y_1+\cdots+y_s} d\mu_{-q}(y_1) \cdots d\mu_{-q}(y_s) \\ &= \left(\frac{q-1 + \frac{q-1}{\log q} \log(1+\epsilon t)}{q\epsilon t + q-1} \right)^r \left(\frac{[2]_q}{q\epsilon t + [2]_q} \right)^s (1+\epsilon t)^x \\ &= \left(\sum_{n=0}^{\infty} D_{n,\epsilon,q}^{(r)}(x) \frac{t^n}{n!} \right) \left(\sum_{m=0}^{\infty} Ch_{m,\epsilon,q}^{(s)} \frac{t^m}{m!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \binom{n}{m} D_{m,\epsilon,q}^{(r)}(x) Ch_{m-n,\epsilon,q}^{(s)} \right) \frac{t^n}{n!} \end{aligned} \quad (2.24)$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} DC_{n,\epsilon,q}^{(r,s)}(x) \frac{(\frac{1}{\epsilon}(e^t-1))^n}{n!} &= \left(\frac{q-1 + \frac{q-1}{\log q} t}{qe^t-1} \right)^r \left(\frac{[2]_q}{qe^t+1} \right)^s e^{xt} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \binom{n}{m} B_{m,q}^{(r)}(x) E_{n-m,q}^{(s)} \right) \frac{t^n}{n!}. \end{aligned} \quad (2.25)$$

Note that

$$\begin{aligned} \sum_{n=0}^{\infty} DC_{m,\epsilon,q}^{(r,s)}(x) \frac{(\frac{1}{\epsilon}(e^t-1))^n}{n!} &= \sum_{n=0}^{\infty} DC_{n,\epsilon,q}^{(r,s)}(x) \frac{1}{\epsilon^n n!} \sum_{m=n}^{\infty} S_2(m, n) \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \epsilon^{-m} DC_{m,\epsilon,q}^{(r,s)}(x) S_2(n, m) \right) \frac{t^n}{n!}. \end{aligned} \quad (2.26)$$

Hence, by (2.24), (2.25) and (2.26), we obtain the following theorem.

Theorem 2.7. For $n \geq 0$, we get

$$DC_{n,\epsilon,q}^{(r,s)}(x) = \sum_{m=0}^{\infty} \binom{n}{m} D_{m,\epsilon,q}^{(r)}(x) Ch_{m-n,\epsilon,q}^{(s)}$$

and

$$\sum_{m=0}^n \binom{n}{m} B_{m,q}^{(r)}(x) E_{n-m,q}^{(s)} = \sum_{m=0}^n \epsilon^{-m} DC_{m,\epsilon,q}^{(r,s)}(x) S_2(n, m).$$

Now, we consider the q -analogue of the *twisted Cauchy-Changhee mixed-type polynomials of order (r, s)* as follows:

$$CC_{n,\epsilon,q}^{(r,s)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} C_{n,\epsilon,q}^{(r)}(x + y_1 + \cdots + y_s) d\mu_{-q}(y_1) \cdots d\mu_{-q}(y_s) \quad (2.27)$$

where n is a given nonnegative integer.

By (2.27), we have

$$\begin{aligned} & \sum_{n=0}^{\infty} CC_{n,\epsilon,q}^{(r,s)}(x) \frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} C_{n,\epsilon,q}^{(r)}(x + y_1 + \cdots + y_s) \frac{t^n}{n!} d\mu_{-q}(y_1) \cdots d\mu_{-q}(y_s) \\ &= \left(\frac{q(1 + \epsilon t) - 1}{(q - 1) + \frac{q-1}{\log q} \log(1 + \epsilon t)} \right)^r \left(\frac{[2]_q}{q\epsilon t + [2]_q} \right)^s (1 + \epsilon t)^x \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} \binom{n}{m} C_{m,\epsilon,q}^{(r)}(x) Ch_{m-n,\epsilon,q}^{(s)} \right) \frac{t^n}{n!} \end{aligned} \quad (2.28)$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} CC_{n,\epsilon,q}^{(r,s)}(x) \frac{\left(\frac{1}{\epsilon}(e^t - 1)\right)^n}{n!} &= \left(\frac{qe^t - 1}{q - 1 + \frac{q-1}{\log q} t} \right)^r \left(\frac{[2]_q}{qe^t + 1} \right)^s e^{xt} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \binom{n}{m} B_{m,q}^{(-r)}(x) E_{n-m,q}^{(s)} \right) \frac{t^n}{n!}. \end{aligned} \quad (2.29)$$

Now, we observe that

$$\begin{aligned} \sum_{n=0}^{\infty} CC_{n,\epsilon,q}^{(r,s)}(x) \frac{\left(\frac{1}{\epsilon}(e^t - 1)\right)^n}{n!} &= \sum_{n=0}^{\infty} CC_{n,\epsilon,q}^{(r,s)}(x) \frac{1}{\epsilon^n n!} \sum_{m=n}^{\infty} S_2(m, n) \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \epsilon^{-m} CC_{m,\epsilon,q}^{(r,s)}(x) S_2(n, m) \right) \frac{t^n}{n!}. \end{aligned} \quad (2.30)$$

Therefore, by (2.28), (2.29) and (2.30), we obtain the following theorem.

Theorem 2.8. For $n \geq 0$, we have

$$CC_{n,\epsilon,q}^{(r,s)}(x) = \sum_{m=0}^{\infty} \binom{n}{m} C_{m,\epsilon,q}^{(r)}(x) Ch_{m-n,\epsilon,q}^{(s)}$$

and

$$\sum_{m=0}^n \binom{n}{m} B_{m,q}^{(-r)}(x) E_{n-m,q}^{(s)} = \sum_{m=0}^n \epsilon^{-m} CC_{m,\epsilon,q}^{(r,s)}(x) S_2(n, m).$$

From now on, we consider the q -analogue of *twisted Cauchy-Daehee mixed-type polynomials of order (r, s)* as follows:

$$CD_{n,\epsilon,q}^{(r,s)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} C_{n,\epsilon,q}^{(r)}(x + y_1 + \cdots + y_s) d\mu_q(y_1) \cdots d\mu_q(y_s) \quad (2.31)$$

where n is a given nonnegative integer.

By (2.31), we have

$$\begin{aligned} & \sum_{n=0}^{\infty} CD_{n,\epsilon,q}^{(r,s)}(x) \frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} C_{n,\epsilon,q}^{(r)}(x + y_1 + \cdots + y_s) \frac{t^n}{n!} d\mu_q(y_1) \cdots d\mu_q(y_s) \\ &= \left(\frac{q(1+\epsilon t) - 1}{(q-1) + \frac{q-1}{\log q} \log(1+\epsilon t)} \right)^r \left(\frac{q-1 + \frac{q-1}{\log q} \log(1+\epsilon t)}{q\epsilon t + q - 1} \right)^s (1+\epsilon t)^x \quad (2.32) \\ &= \begin{cases} \sum_{n=0}^{\infty} C_{n,\epsilon,q}^{(r-s)}(x) \frac{t^n}{n!} & \text{if } r > s, \\ \sum_{n=0}^{\infty} D_{n,\epsilon,q}^{(s-r)}(x) \frac{t^n}{n!} & \text{if } r < s, \\ \sum_{n=0}^{\infty} (x)_n \frac{t^n}{n!} & \text{if } r = s. \end{cases} \end{aligned}$$

Thus, by (2.32), we obtain the following theorem.

Theorem 2.9. For $n \geq 0$, we have

$$CD_{n,\epsilon,q}^{(r,s)} = \begin{cases} C_{n,\epsilon,q}^{(r-s)}(x) & \text{if } r > s, \\ D_{n,\epsilon,q}^{(s-r)}(x) & \text{if } r < s, \\ (x)_n & \text{if } r = s. \end{cases}$$

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Equicontinuity of Maps on $[0, 1)$

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Abstract: We mainly study the equicontinuity of maps on $[0, 1)$. Let $f : X \rightarrow X$ be a continuous map on $X = [0, 1)$. We show that if f is an equicontinuous map with $F(f)$ nonempty, then one of the following two conditions holds: (1) $F(f)$ consists of a single point and $F(f^2) = \bigcap_{n=1}^{\infty} f^n(X)$; (2) $F(f) = \bigcap_{n=1}^{\infty} f^n(X)$. Last we construct two examples to show that the converse result doesn't hold.

Keywords: *Interval, Equicontinuous, Periodic point.*

Mathematics Subject Classification (2000): Primary: 54B20, 54E40

1 Introduction

Let (X, d) be a metric space with the metric d (not necessary compact) and $f : X \rightarrow X$ be a continuous map. For every nonnegative integer n define f^n inductively by $f^n = f \circ f^{n-1}$, where f^0 is the identity map on X . A point x of X is said to be a *periodic point* of f if there is a positive integer n such that $f^n(x) = x$. The least such n is called the *period* of x . A point of period one is called a *fixed point*. Let $F(f)$ denote the fixed point set of f and $P(f)$ the set of periodic points of f .

If $x \in X$ then the *trajectory* (or *orbit*) of x is the sequence $orb(x, f) = \{f^n(x) : n \geq 0\}$ and the ω -*limit set* of x is

$$\omega(x, f) = \bigcap_{m \geq 0} \overline{\bigcup_{n \geq m} f^n(x)}.$$

Equivalently, $y \in \omega(x, f)$ if and only if $y \in X$ is a limit point of the trajectory $orb(x, f)$, i.e., $f^{n_k}(x) \rightarrow y$ for some sequence of integers $n_k \rightarrow \infty$.

The map f is said to be *equicontinuous* (in some terminology also *Lyapunov stable*) if given $\epsilon > 0$ there exists a $\delta > 0$ such that $d(f^i(x), f^i(y)) < \epsilon$ whenever $d(x, y) < \delta$ for all $x, y \in X$ and all $i \geq 1$.

In 1982, J. Cano [4] proved the following theorem on equicontinuous map for the closed interval I .

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Theorem 1.1 *Let $f : I \rightarrow I$ be an equicontinuous map. Then $F(f)$ is connected and if it is non-degenerate then $F(f) = P(f)$.*

The next theorem was due to Bruckner and Hu [3]. This result was also proved by Blokh in [2].

Theorem 1.2 *Let $f : I \rightarrow I$ be a continuous map. Then f is equicontinuous if and only if $\bigcap_{i=1}^{\infty} f^n(I) = F(f^2)$.*

In [9], Valaristos described the characters of equicontinuous circle maps: A continuous map f of the unit circle S^1 to itself is equicontinuous if and only if one of the following four statements holds: (1) f is topologically conjugate to a rotation; (2) $F(f)$ contains exactly two points and $F(f^2) = S^1$; (3) $F(f)$ contains exactly one point and $F(f^2) = \bigcap_{n=1}^{\infty} f^n(S^1)$; (4) $F(f) = \bigcap_{n=1}^{\infty} f^n(S^1)$. In 2000, Sun [8] obtained some necessary and sufficient conditions of equicontinuous σ -maps. Later, Mai [6] studied the structure of equicontinuous maps of general metric spaces, and given some still simpler necessary and sufficient conditions of equicontinuous graph maps.

In [5], Gu showed that a map on Warsaw circle W is equicontinuous if and only if $F(f)$ consists of a single point and $F(f^2) = \bigcap_{n=1}^{\infty} f^n(X)$ or $F(f) = \bigcap_{n=1}^{\infty} f^n(X)$.

Warsaw circle W is simple connected but not locally connected, and it often appears as an example of circle-like and non arc-like in the theory of continuum (see [7]). In addition, Warsaw circle is not a continuous image of the closed interval. So it is not a Peano continuum. However, it is easily to see that there is a continuous bijective map $\phi : [0, 1) \rightarrow X$. Moreover, if f is a continuous self-map of Warsaw circle W , then there is unique continuous map $\tilde{f} : [0, 1) \rightarrow [0, 1)$ such that $\phi \circ \tilde{f} = f \circ \phi$ (see [10]). Note that ϕ is not a homeomorphism since $[0, 1)$ is not compact but Warsaw circle W is compact. It follows that f and \tilde{f} are not topologically conjugate. So, it may be that there are some different dynamical properties between maps on $[0, 1)$ and on Warsaw circle.

In this paper we shall deal with the problem of equicontinuity of maps on $[0, 1)$. Our main results are the following theorems.

Theorem 1.3 *Let $X = [0, 1)$ and $f : X \rightarrow X$ be an equicontinuous map. If $F(f) \neq \emptyset$, then every periodic point of f has periodic 1 or 2, both $F(f^2)$ and $F(f)$ are connected. Furthermore, if $F(f)$ is non-degenerate then $F(f) = P(f)$.*

Theorem 1.4 *Let $X = [0, 1)$ and $f : X \rightarrow X$ be a continuous map with $F(f) \neq \emptyset$. If f is equicontinuous, then one of the following two conditions holds:*

- (1) $F(f)$ consists of a single point and $F(f^2) = \bigcap_{n=1}^{\infty} f^n(X)$;
- (2) $F(f) = \bigcap_{n=1}^{\infty} f^n(X)$.

Moreover, it is equivalent whenever f is uniformly continuous.

In Section 3, we will construct two examples to show that the converse result of Theorem 1.4 doesn't hold.

2 Proof of Theorem 1.3 and 1.4

In this section, we mainly prove Theorem 1.3 and 1.4.

2.1 Some lemmas

In this section, we give some lemmas which are needed in proof of Theorem 1.3 and 1.4.

Lemma 2.1 *Let $f : X \rightarrow X$ be an continuous map of $X = [0, 1)$. If $F(f^2) = X$, then f is the identity map on X .*

Proof It is not hard to see that $f(X) = X$. Assume there exist $x \in X$ such that $f(x) \neq x$. Then we can choose $0 \leq p_1 < p_2 < 1$ such that $f(p_1) = p_2$ and $f(p_2) = p_1$. Let $m = \max_{x \in [0, p_2]} f(x)$. It is clear that $p_2 \leq m < 1$. For each $x \in (p_2, 1)$, we have $f(x) < p_2$ or there exists $q \in (p_2, 1)$ such that $f(q) = p_2$ by the continuity of f , which contradicts to $q \in F(f^2)$. Hence $f(X) = [0, m]$. This also contradicts to $f(X) = X$. Therefore, f is the identity map on X .

Lemma 2.2 *Let $X = [0, 1)$ and $f : X \rightarrow X$ be an equicontinuous map with a fixed point p . Suppose J is a component of $F(f^4)$ containing p . Then $\omega(x, f) \subset J$ for every $x \in X$.*

Proof Without loss of generality, we may assume that J is a proper subset of X (note that it is clearly hold whenever $J = X$). Firstly, we prove that there is a connected open subset $K \supset J$ such that $\omega(x, f) \subset J$ for every $x \in K$.

Case 1 $J = \{p\}$. Let $\epsilon = (1/2) \min(p, 1 - p)$. By the equicontinuity of f , there is an open interval K of p such that $|f^n(x) - p| < \epsilon$ for every x in K and every positive integer n . Let $L = \bigcup_{j \geq 0} f^j(K)$. Then L is a closed proper invariant interval of X . It follows from Theorem 1.1 and 1.2 that the fixed point set of $f|_L$ and $f^2|_L$ is connected, and therefore it is $\{p\}$. Moreover, all periodic points of $f|_L$ have period 1 or 2. But the fixed point p is the only periodic point of f in L . Therefore $P(f|_L) = F(f|_L)$ and by Proposition 15 in [1, p. 78] the ω -limit points coincide with the fixed points. Hence p is the only ω -limit point of f in L . Thus $\omega(x, f) = \{p\} = J$ for every $x \in L$. Since $K \subset L$, we have $\omega(x, f) = J$ for every $x \in K$.

Case 2 $J = [q_1, q_2]$ is a closed interval of X . For every $i = 1, 2$ we consider the orbit $\{q_i, f(q_i), f^2(q_i), f^3(q_i)\}$ of q_i . Let $\epsilon = (1/2) \min(f^j(q_i), 1 - f^j(q_i))$. By the equicontinuity of f , there is an open interval K_{ij} containing $f^j(q_i)$ such that $|f^{4n}(x) - f^j(q_i)| < \epsilon$ for every $x \in K_{ij}$ and every positive integer n . Let $K_i = \bigcap_{j=0}^3 f^{-j}(K_{ij})$, define $L = \bigcup_{j=0}^{\infty} f^j(K_1 \cup J \cup K_2)$. Then L is a closed proper invariant interval of X . We know from Theorem 1.1 and 1.2 that fixed point set of $f|_L$ and $f^2|_L$ is connected and therefore, it is contained in J . Moreover, all periodic points of $f|_L$ have period 1 and 2. Since $P(f|_L)$ is closed, by Proposition 15 in [1, p. 78], it coincides with the set of ω -limits points. Therefore, $\omega(x, f) \subset J$ for each $x \in L$. Let $K = K_1 \cup J \cup K_2$. Then $K \supset J$ and $\omega(x, f) \subset J$ for each $x \in K$.

Case 3 $J = [q, 1)$, where $0 < q < 1$. Obviously, $f(J) \subset J$. By Lemma 2.1, we have $J \subset F(f)$. Then $\lim_{x \rightarrow 1} f(x) = 1$. Let $g : [0, 1] \rightarrow [0, 1]$ such that $g|_X = f$ and $g(1) = 1$. So g is a equicontinuous map on $[0, 1]$. It follows from Theorem 1.1 and 1.2 that the fixed point set of g^2 is connected and $P(g) = F(g^2)$. Hence $P(g) = [q, 1]$. By Proposition 15 in [1, p. 78], $\omega(x, g) \subset [q, 1]$ for each $x \in [0, 1]$. Let $K = X$, then $\omega(x, f) \subset J$ for each $x \in K$.

Secondly, we show that $\omega(x, f) \subset J$ for each $x \in X$. Let

$$S = \{x \in X : \omega(x, f) \subset J\}.$$

Note that S is a nonempty set since $K \subset S$. Let $y \in S$. Then there is a positive integer m such that $f^m(y) \in K$. By the continuity of f^m , there exists an open subset U containing y such that $f^m(U) \subset K$. Hence $U \subset S$ and S is an open set. Let T be the component of S containing J and therefore K as well. Then T is open and connected. It is sufficient to show that $T = X$. Suppose that $T \neq X$. Let $\epsilon = (1/2) \min\{|x - y| : x \in J, y \in X - T\}$. Then $\epsilon > 0$. Assume that z is an endpoint of $X - T$. Then we have $f^n(z) \notin T$ for each positive integer n . On the other hand, for any $\delta > 0$ we can choose $x \in T$ such that $|x - z| < \delta$. Since $\omega(x, f) \subset J$, there is a positive integer m such that $f^m(x) \in B(J, \epsilon/2)$. Hence $|f^m(x) - f^m(z)| > \epsilon/2$. This is a contradiction. Therefore, $T = X$ and the proof is completed.

The following two lemmas are obviously facts on any compact metric space.

Lemma 2.3 *Let $f : X \rightarrow X$ be a continuous map, where X is a compact metric space. Let k be a positive integer and $g = f^k$. Then f is equicontinuous if and only if g is equicontinuous.*

Lemma 2.4 *Let $f : X \rightarrow X$ be a continuous map, where X is a compact metric space. If $f|_{f(X)}$ is equicontinuous then f is equicontinuous.*

2.2 Proof of Theorem 1.3

Let $X = [0, 1)$ and $f : X \rightarrow X$ be an equicontinuous map. If p is a fixed point of f and J is a component of $F(f^4)$ containing p , then we consider the following three case.

Case 1 $J = \{p\}$. By Lemma 2.2, $\omega(x, f) \subset \{p\}$ for each $x \in X$. This shows that p is a unique periodic point of f . Hence $F(f) = F(f^2) = \{p\}$ is connected.

Case 2 $J = [q_1, q_2]$. By Lemma 2.2, $\omega(x, f) \subset J$ for each $x \in X$. This shows that $P(f) \subset J$. Hence $P(f) = F(f^4) = J$ and $F(f^4)$ is connected. Applying Theorem 1.1 to $f|_J$, we know that all periodic points of f have period 1 or 2, both $F(f)$ and $F(f^2)$ are connected. Furthermore, if $F(f)$ is non-degenerate then $F(f) = P(f)$.

Case 3 $J = [q, 1)$. By Lemma 2.2, $\omega(x, f) \subset J$ for each $x \in X$. This shows that $P(f) \subset J$. Hence $P(f) = F(f^4) = J$ and $F(f^4)$ is connected. Applying Lemma 2.1 to $f|_J$, we have $P(f) = F(f) = J$ is connected.

This complete the proof of Theorem 1.3.

2.3 Proof of Theorem 1.4

Let $X = [0, 1)$ and $f : X \rightarrow X$ be a continuous map. We suppose that f is equicontinuous. By Theorem 1.3, both $F(f^2)$ and $F(f)$ are connected.

(1) If $F(f)$ consists a single point p then $F(f^2) = P(f)$. Moreover by Lemma 2.2, we have $\omega(x, f) \subset F(f^2)$ for every $x \in X$.

Case 1 If $F(f^2) = [q_1, q_2]$, where $0 \leq q_1 \leq q_2 < 1$. Similar the proof of Lemma 2.2, there exists an open, connected subset K containing $F(f^2)$ such that $L = \bigcup_{j=0}^{\infty} f^j(K) \subset X$ is a closed and invariant interval. Fixed $\epsilon > 0$, there is $\delta > 0$ such that $|x - y| < \delta$ implies $|f^n(x) - f^n(y)| < \epsilon$ for all $n \geq 0$. For $x \in X$, since $\omega(x, f) \subset F(f^2) \subset K \subset L$, there exists a positive integer N_x such that $f^{N_x}(x) \in K$. Then $f^m(x) \in L$ for each $m > N_x$. By the continuity of f^{N_x} , there is an open neighborhood V_x of x such that $f^{N_x}(V_x) \subset K$ and hence $f^m(V_x) \subset L$ for every $m \geq N_x$. Note that the collection $\{V_x\}_{x \in I_\delta}$ forms an open cover of $I_\delta = [0, 1 - \delta]$. By the compactness of I_δ , there is a finite subcover $\{V_{x_1}, \dots, V_{x_s}\}$. Set $N = \max\{N_{x_1}, \dots, N_{x_s}\}$. Then $f^m(V_{x_i}) \subset L$ for every $m \geq N$ and any $1 \leq i \leq s$. Thus, $f^m(I_\delta) \subset L$ for every $m \geq N$, and hence $f^m(X) \subset B(L, \epsilon)$ for all $m \geq N$, where $B(L, \epsilon) = \{y \in X : d(x, y) < \epsilon \text{ for some } x \in L\}$. By the arbitrary of ϵ , we can get $\bigcap_{n=1}^{\infty} f^n(X) \subset L$. Using Theorem 1.2, we have $\bigcap_{n=1}^{\infty} f^n(L) = F(f^2)$. It follows that $F(f^2) = \bigcap_{n=1}^{\infty} f^n(L) = \bigcap_{n=1}^{\infty} f^n(X)$, i.e., (1) holds.

Case 2 If $F(f^2) = [q, 1]$. Applying Lemma 2.1 to $f|_{[q, 1]}$, we have $f(x) = x$ for all $x \in [q, 1]$, i.e., $[q, 1] \subset F(f)$. This contradicts to $F(f)$ consists a single point.

(2) If $F(f)$ is non-degenerate, then $F(f) = P(f)$ by Theorem 1.3. Similar to the above argument we can get $F(f) = \bigcap_{n=1}^{\infty} f^n(X)$ whenever $F(f) = [q_1, q_2]$. Now we assume $F(f) = [q, 1]$ for some $0 \leq q < 1$. Define $g : [0, 1] \rightarrow [0, 1]$ as $g|_X = f$ and $g(1) = 1$. So g is a equicontinuous map on $[0, 1]$. It follows from Theorem 1.1 and 1.2 that $\bigcap_{n=1}^{\infty} g^n([0, 1]) = F(g^2) = F(g)$. Thus, $\bigcap_{n=1}^{\infty} f^n(X) = F(f)$.

3 Examples

In this section, we will construct two examples to show that the converse result of Theorem 1.4 doesn't hold.

Example 3.1 Let $I = [0, 1]$ and let $a_n = 1 - 1/2^n$ for every $n = 1, 2, \dots$. Now we define a piecewise linear continuous map $f : I \rightarrow I$ as follows (See Figure 1):

- (1) $f(x) = 1 - x$ for each $x \in [0, 1/2]$;
- (2) $f(a_{2n}) = 1/2$ and $f(a_{2n-1}) = 0$ for all $n = 1, 2, \dots$.

It is easily to see that $F(f)$ consists of a single point and $F(f^2) = \bigcap_{n=1}^{\infty} f^n(I) = [0, 1/2]$. However, f is not equicontinuous since $|a_{n+1} - a_n| = \frac{1}{2^{n+1}} \rightarrow 0$ but $|f(a_{n+1}) - f(a_n)| = 1/2$ for all $n \geq 1$.

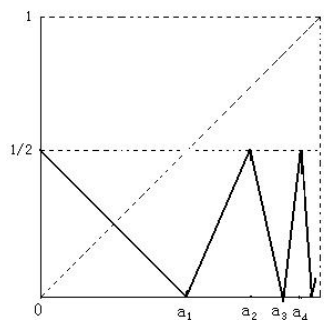


Figure 1

Example 3.2 Let $I = [0, 1)$ and let $a_n = 1 - 1/2^n$ for every $n = 1, 2, \dots$. Now we define a piecewise linear continuous map $f : I \rightarrow I$ as follows (See Figure 2):

- (1) $f(x) = x$ for each $x \in [0, 1/2]$;
- (2) $f(a_{2n}) = 0$ and $f(a_{2n-1}) = 1/2$ for all $n = 1, 2, \dots$.

It is easily to see that $F(f)$ is non-degenerate and $F(f) = \bigcap_{n=1}^{\infty} f^n(I) = [0, 1/2]$. However, f is not equicontinuous since $|a_{n+1} - a_n| = \frac{1}{2^{n+1}} \rightarrow 0$ but $|f(a_{n+1}) - f(a_n)| = 1/2$ for all $n \geq 1$.

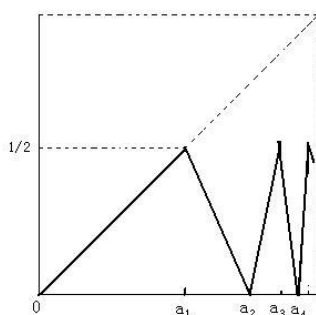


Figure 2

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On mixed type Riemann-Liouville and Hadamard fractional integral inequalities

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Abstract

In this paper, some new mixed type Riemann-Liouville and Hadamard fractional integral inequalities are established, in the case where the functions are bounded by integrable functions. Moreover, mixed type Riemann-Liouville and Hadamard fractional integral inequalities of Chebyshev type are presented.

Key words and phrases: Fractional integral; fractional integral inequalities; Riemann-Liouville fractional integral; Hadamard fractional integral; Chebyshev inequalities.

AMS (MOS) Subject Classifications: 26D10; 26A33.

1 Introduction

The study of mathematical inequalities play very important role in classical differential and integral equations which has applications in many fields. Fractional inequalities are important in studying the existence, uniqueness and other properties of fractional differential equations. Recently many authors have studied integral inequalities on fractional calculus using Riemann-Liouville and Caputo derivative, see [1], [2], [3], [4], [5], [6] and the references therein.

Another kind of fractional derivative that appears in the literature is the fractional derivative due to Hadamard introduced in 1892 [7], which differs from the Riemann-Liouville and Caputo derivatives in the sense that the kernel of the integral contains logarithmic function of arbitrary exponent. Details and properties of Hadamard fractional derivative and integral can be found in [8, 9, 10, 11, 12, 13]. Recently in the literature, were appeared some results on fractional integral inequalities using Hadamard fractional integral; see [14, 15, 16].

Recently, we have been established some new Riemann-Liouville fractional integral inequalities in [17], and some fractional integral inequalities via Hadamard's fractional integral in [18]. In the present paper we combine the results of [17] and [18] and obtain some new mixed type Riemann-Liouville and Hadamard fractional integral inequalities. In Section 3, we consider the case where the functions are bounded by integrable functions and are not necessary increasing or decreasing as are the synchronous functions. In Section 4, we establish mixed type Riemann-Liouville and Hadamard fractional integral inequalities of Chebyshev type, concerning the integral of the product of two functions and the product of two integrals. As applications, in Section 5, we present a way for constructing the four bounding functions, and use them to give some estimates of Chebyshev type inequalities of Riemann-Liouville and Hadamard fractional integrals for two unknown functions.

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2 Preliminaries

In this section, we give some preliminaries and basic properties used in our subsequent discussion. The necessary background details are given in the book by Kilbas et al. [8].

Definition 2.1 The Riemann-Liouville fractional integral of order $\alpha > 0$ of a continuous function $f : (a, \infty) \rightarrow \mathbb{R}$ is defined by

$$I_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds,$$

provided the right-hand side is point-wise defined on (a, ∞) , where Γ is the gamma function.

Definition 2.2 The Hadamard fractional integral of order $\alpha \in \mathbb{R}^+$ of a function $f(t)$, for all $0 < a < t < \infty$, is defined as

$$J_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s} \right)^{\alpha-1} f(s) \frac{ds}{s},$$

provided the integral exists.

From Definitions (2.1) and (2.2), we derive the following properties:

$$\begin{aligned} I_a^\alpha I_a^\beta f(t) &= I_a^{\alpha+\beta} f(t) = I_a^\beta I_a^\alpha f(t), \\ J_a^\alpha J_a^\beta f(t) &= J_a^{\alpha+\beta} f(t) = J_a^\beta J_a^\alpha f(t), \end{aligned}$$

for $\alpha, \beta > 0$ and

$$\begin{aligned} I_a^\alpha (t^\gamma) &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} (t-a)^{\gamma+\alpha}, \\ J_a^\alpha (\log t)^\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} \left(\log \frac{t}{a} \right)^{\gamma+\alpha}, \end{aligned}$$

for $\alpha > 0, \gamma > -1, t > a > 0$.

3 Inequalities Involving Mixed Type of Riemann-Liouville and Hadamard Fractional Integral for Bounded Functions

In this section we obtain some new inequalities of mixed type for Riemann-Liouville and Hadamard fractional integral in the case where the functions are bounded by integrable functions and are not necessary increasing or decreasing as are the synchronous functions.

Theorem 3.1 Let f be an integrable function on $[a, \infty)$, $a > 0$. Assume that:

(H₁) There exist two integrable functions φ_1, φ_2 on $[a, \infty)$ such that

$$\varphi_1(t) \leq f(t) \leq \varphi_2(t), \quad \text{for all } t \in [a, \infty), \quad a > 0. \quad (1)$$

Then for $0 < a < t < \infty$ and $\alpha, \beta > 0$, the following two inequalities hold:

$$(A_1) \quad J_a^\alpha \varphi_2(t) I_a^\beta f(t) + J_a^\alpha f(t) I_a^\beta \varphi_1(t) \geq J_a^\alpha \varphi_2(t) I_a^\beta \varphi_1(t) + J_a^\alpha f(t) I_a^\beta f(t),$$

$$(B_1) \quad I_a^\alpha \varphi_2(t) J_a^\beta f(t) + I_a^\alpha f(t) J_a^\beta \varphi_1(t) \geq I_a^\alpha \varphi_2(t) J_a^\beta \varphi_1(t) + I_a^\alpha f(t) J_a^\beta f(t).$$

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Proof. From condition (H1), for all $\tau, \rho > a$, we have

$$(\varphi_2(\tau) - f(\tau))(f(\rho) - \varphi_1(\rho)) \geq 0,$$

which implies

$$\varphi_2(\tau)f(\rho) + \varphi_1(\rho)f(\tau) \geq \varphi_1(\rho)\varphi_2(\tau) + f(\tau)f(\rho). \quad (2)$$

Multiplying both sides of (2) by $(\log(t/\tau))^{\alpha-1}/\tau\Gamma(\alpha)$, $\tau \in (a, t)$, we get

$$\begin{aligned} f(\rho) \frac{(\log(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)} \varphi_2(\tau) + \varphi_1(\rho) \frac{(\log(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)} f(\tau) \\ \geq \varphi_1(\rho) \frac{(\log(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)} \varphi_2(\tau) + f(\rho) \frac{(\log(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)} f(\tau). \end{aligned} \quad (3)$$

Integrating both sides of (3) with respect to τ on (a, t) , we obtain

$$\begin{aligned} f(\rho) \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \varphi_2(\tau) \frac{d\tau}{\tau} + \varphi_1(\rho) \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} f(\tau) \frac{d\tau}{\tau} \\ \geq \varphi_1(\rho) \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \varphi_2(\tau) \frac{d\tau}{\tau} + f(\rho) \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} f(\tau) \frac{d\tau}{\tau}, \end{aligned}$$

which yields

$$f(\rho) J_a^\alpha \varphi_2(t) + \varphi_1(\rho) J_a^\alpha f(t) \geq \varphi_1(\rho) J_a^\alpha \varphi_2(t) + f(\rho) J_a^\alpha f(t). \quad (4)$$

Multiplying both sides of (4) by $(t-\rho)^{\beta-1}/\Gamma(\beta)$, $\rho \in (a, t)$, we have

$$\begin{aligned} J_a^\alpha \varphi_2(t) \frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)} f(\rho) + J_a^\alpha f(t) \frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)} \varphi_1(\rho) \\ \geq J_a^\alpha \varphi_2(t) \frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)} \varphi_1(\rho) + J_a^\alpha f(t) \frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)} f(\rho). \end{aligned} \quad (5)$$

Integrating both sides of (5) with respect to ρ on (a, t) , we get

$$\begin{aligned} J_a^\alpha \varphi_2(t) \frac{1}{\Gamma(\beta)} \int_a^t (t-\rho)^{\beta-1} f(\rho) d\rho + J_a^\alpha f(t) \frac{1}{\Gamma(\beta)} \int_a^t (t-\rho)^{\beta-1} \varphi_1(\rho) d\rho \\ \geq J_a^\alpha \varphi_2(t) \frac{1}{\Gamma(\beta)} \int_a^t (t-\rho)^{\beta-1} \varphi_1(\rho) d\rho + J_a^\alpha f(t) \frac{1}{\Gamma(\beta)} \int_a^t (t-\rho)^{\beta-1} f(\rho) d\rho. \end{aligned} \quad (6)$$

Hence, we get the desired inequality in (A_1) . The inequality (B_1) , is proved by similar arguments. \square

Corollary 3.2 Let f be an integrable function on $[a, \infty)$, $a > 0$ satisfying $m \leq f(t) \leq M$, for all $t \in [a, \infty)$ and $m, M \in \mathbb{R}$. Then for $0 < a < t < \infty$ and $\alpha, \beta > 0$, the following two inequalities hold:

$$(A_2) \quad M \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} J_a^\beta f(t) + m \frac{(t-a)^\beta}{\Gamma(\beta+1)} J_a^\alpha f(t) \geq mM \frac{(\log \frac{t}{a})^\alpha (t-a)^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} + J_a^\alpha f(t) I_a^\beta f(t),$$

$$(B_2) \quad M \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\beta f(t) + m \frac{(\log \frac{t}{a})^\beta}{\Gamma(\beta+1)} I_a^\alpha f(t) \geq mM \frac{(t-a)^\alpha (\log \frac{t}{a})^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} + I_a^\alpha f(t) J_a^\beta f(t).$$

Theorem 3.3 Let f be an integrable function on $[a, \infty)$, $a > 0$ and $\theta_1, \theta_2 > 0$ satisfying $1/\theta_1 + 1/\theta_2 = 1$. In addition, suppose that the condition (H_1) holds. Then for $0 < a < t < \infty$ and $\alpha, \beta > 0$, the following two inequalities hold:

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$$\begin{aligned}
(A_3) \quad & J_a^\alpha \varphi_2(t) I_a^\beta \varphi_1(t) + J_a^\alpha f(t) I_a^\beta f(t) + \frac{1}{\theta_1} \frac{(t-a)^\beta}{\Gamma(\beta+1)} J_a^\alpha (\varphi_2 - f)^{\theta_1}(t) + \frac{1}{\theta_2} \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\beta (f - \varphi_1)^{\theta_2}(t) \\
& \geq J_a^\alpha \varphi_2(t) I_a^\beta f(t) + J_a^\alpha f(t) I_a^\beta \varphi_1(t), \\
(B_3) \quad & I_a^\alpha \varphi_2(t) J_a^\beta \varphi_1(t) + I_a^\alpha f(t) J_a^\beta f(t) + \frac{1}{\theta_1} \frac{(\log \frac{t}{a})^\beta}{\Gamma(\beta+1)} I_a^\alpha (\varphi_2 - f)^{\theta_1}(t) + \frac{1}{\theta_2} \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\beta (f - \varphi_1)^{\theta_2}(t) \\
& \geq I_a^\alpha \varphi_2(t) J_a^\beta f(t) + I_a^\alpha f(t) J_a^\beta \varphi_1(t).
\end{aligned}$$

Proof. Firstly, we recall the well-known Young's inequality as

$$\frac{1}{\theta_1} x^{\theta_1} + \frac{1}{\theta_2} y^{\theta_2} \geq xy, \quad \forall x, y \geq 0, \quad \theta_1, \theta_2 > 0,$$

where $1/\theta_1 + 1/\theta_2 = 1$. By setting $x = \varphi_2(\tau) - f(\tau)$ and $y = f(\rho) - \varphi_1(\rho)$, $\tau, \rho > a$, we have

$$\frac{1}{\theta_1} (\varphi_2(\tau) - f(\tau))^{\theta_1} + \frac{1}{\theta_2} (f(\rho) - \varphi_1(\rho))^{\theta_2} \geq (\varphi_2(\tau) - f(\tau))(f(\rho) - \varphi_1(\rho)). \quad (7)$$

Multiplying both sides of (7) by $(\log(t/\tau))^{\alpha-1}(t-\rho)^{\beta-1}/\tau\Gamma(\alpha)\Gamma(\beta)$, $\tau, \rho \in (a, t)$, we get

$$\begin{aligned}
& \frac{1}{\theta_1} \frac{(\log t/\tau)^{\alpha-1}(t-\rho)^{\beta-1}}{\tau\Gamma(\alpha)\Gamma(\beta)} (\varphi_2(\tau) - f(\tau))^{\theta_1} + \frac{1}{\theta_2} \frac{(\log t/\tau)^{\alpha-1}(t-\rho)^{\beta-1}}{\tau\Gamma(\alpha)\Gamma(\beta)} (f(\rho) - \varphi_1(\rho))^{\theta_2} \\
& \geq \frac{(\log t/\tau)^{\alpha-1}}{\tau\Gamma(\alpha)} (\varphi_2(\tau) - f(\tau)) \frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)} (f(\rho) - \varphi_1(\rho)).
\end{aligned}$$

Double integrating the above inequality with respect to τ and ρ from a to t , we have

$$\frac{1}{\theta_1} J_a^\alpha (\varphi_2 - f)^{\theta_1}(t) I_a^\beta (1)(t) + \frac{1}{\theta_2} J_a^\alpha (1)(t) I_a^\beta (f - \varphi_1)^{\theta_2}(t) \geq J_a^\alpha (\varphi_2 - f)(t) I_a^\beta (f - \varphi_1)(t),$$

which implies the result in (A_3) . By using the similar method, we obtain the inequality in (B_3) . \square

Corollary 3.4 Let f be an integrable function on $[a, \infty)$, $a > 0$ satisfying $m \leq f(t) \leq M$, $\theta_1 = \theta_2 = 2$ for all $t \in [a, \infty)$ and $m, M \in \mathbb{R}$. Then for $0 < a < t < \infty$ and $\alpha, \beta > 0$, the following two inequalities hold:

$$\begin{aligned}
(A_4) \quad & (m+M)^2 \frac{(\log \frac{t}{a})^\alpha (t-a)^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{(t-a)^\beta}{\Gamma(\beta+1)} J_a^\alpha f^2(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\beta f^2(t) + 2J_a^\alpha f(t) I_a^\beta f(t) \\
& \geq 2(m+M) \left(\frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\beta f(t) + \frac{(t-a)^\beta}{\Gamma(\beta+1)} J_a^\alpha f(t) \right), \\
(B_4) \quad & (m+M)^2 \frac{(t-a)^\alpha (\log \frac{t}{a})^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)} + \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\beta f^2(t) + \frac{(\log \frac{t}{a})^\beta}{\Gamma(\beta+1)} I_a^\alpha f^2(t) + 2J_a^\beta f(t) I_a^\alpha f(t) \\
& \geq 2(m+M) \left(\frac{(\log \frac{t}{a})^\beta}{\Gamma(\beta+1)} I_a^\alpha f(t) + \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\beta f(t) \right).
\end{aligned}$$

Theorem 3.5 Let f be an integrable function on $[a, \infty)$, $a > 0$ and $\theta_1, \theta_2 > 0$ satisfying $\theta_1 + \theta_2 = 1$. In addition, suppose that the condition (H_1) holds. Then for $0 < a < t < \infty$, and $\alpha, \beta > 0$, the following two inequalities hold:

$$\begin{aligned}
(A_5) \quad & \theta_1 \frac{(t-a)^\beta}{\Gamma(\beta+1)} J_a^\alpha \varphi_2(t) + \theta_2 \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\beta f(t) \\
& \geq J_a^\alpha (\varphi_2 - f)^{\theta_1}(t) I_a^\beta (f - \varphi_1)^{\theta_2}(t) + \theta_1 \frac{(t-a)^\beta}{\Gamma(\beta+1)} J_a^\alpha f(t) + \theta_2 \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\beta \varphi_1(t),
\end{aligned}$$

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$$\begin{aligned}
(B_5) \quad & \theta_1 \frac{(\log \frac{t}{a})^\beta}{\Gamma(\beta+1)} I_a^\alpha \varphi_2(t) + \theta_2 \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\beta f(t) \\
& \geq I_a^\alpha (\varphi_2 - f)^{\theta_1}(t) J_a^\beta (f - \varphi_1)^{\theta_2}(t) + \theta_1 \frac{(\log \frac{t}{a})^\beta}{\Gamma(\beta+1)} I_a^\alpha f(t) + \theta_2 \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\beta \varphi_1(t).
\end{aligned}$$

Proof. From the well-known weighted AM-GM inequality

$$\theta_1 x + \theta_2 y \geq x^{\theta_1} y^{\theta_2}, \quad \forall x, y \geq 0, \quad \theta_1, \theta_2 > 0,$$

where $\theta_1 + \theta_2 = 1$, and setting $x = \varphi_2(\tau) - f(\tau)$ and $y = f(\rho) - \varphi_1(\rho)$, $\tau, \rho > a$, we have

$$\theta_1 (\varphi_2(\tau) - f(\tau)) + \theta_2 (f(\rho) - \varphi_1(\rho)) \geq (\varphi_2(\tau) - f(\tau))^{\theta_1} (f(\rho) - \varphi_1(\rho))^{\theta_2}. \quad (8)$$

Multiplying both sides of (8) by $(\log(t/\tau))^{\alpha-1} (t-\rho)^{\beta-1} / \tau \Gamma(\alpha) \Gamma(\beta)$, $\tau, \rho \in (a, t)$, we obtain

$$\begin{aligned}
& \theta_1 \frac{(\log(t/\tau))^{\alpha-1} (t-\rho)^{\beta-1}}{\tau \Gamma(\alpha) \Gamma(\beta)} (\varphi_2(\tau) - f(\tau)) + \theta_2 \frac{(\log(t/\tau))^{\alpha-1} (t-\rho)^{\beta-1}}{\tau \Gamma(\alpha) \Gamma(\beta)} (f(\rho) - \varphi_1(\rho)) \\
& \geq \frac{(\log t/\tau)^{\alpha-1}}{\tau \Gamma(\alpha)} (\varphi_2(\tau) - f(\tau))^{\theta_1} \frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)} (f(\rho) - \varphi_1(\rho))^{\theta_2}.
\end{aligned}$$

Double integration the above inequality with respect to τ and ρ from a to t , we have

$$\theta_1 J_a^\alpha (\varphi_2 - f)(t) I_a^\beta (1)(t) + \theta_2 J_a^\alpha (1)(t) I_a^\beta (f - \varphi_1)(t) \geq J_a^\alpha (\varphi_2 - f)^{\theta_1}(t) I_a^\beta (f - \varphi_1)^{\theta_2}(t).$$

Therefore, we deduce the inequality in (A_5) . By using the similar method, we obtain the desired bound in (B_5) . \square

Corollary 3.6 Let f be an integrable function on $[a, \infty)$, $a > 0$ satisfying $m \leq f(t) \leq M$, $\theta_1 = \theta_2 = 1/2$ for all $0 < a < t < \infty$ and $m, M \in \mathbb{R}$. Then for $0 < a < t < \infty$ and $\alpha, \beta > 0$, the following two inequalities hold:

$$\begin{aligned}
(A_6) \quad & M \frac{(\log \frac{t}{a})^\alpha (t-a)^\beta}{\Gamma(\alpha+1) \Gamma(\beta+1)} + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\beta f(t) \\
& \geq m \frac{(\log \frac{t}{a})^\alpha (t-a)^\beta}{\Gamma(\alpha+1) \Gamma(\beta+1)} + \frac{(t-a)^\beta}{\Gamma(\beta+1)} J_a^\alpha f(t) + 2 J_a^\alpha (M-f)^{1/2}(t) I_a^\beta (f-m)^{1/2}(t), \\
(B_6) \quad & M \frac{(t-a)^\alpha (\log \frac{t}{a})^\beta}{\Gamma(\alpha+1) \Gamma(\beta+1)} + \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\beta f(t) \\
& \geq m \frac{(t-a)^\alpha (\log \frac{t}{a})^\beta}{\Gamma(\alpha+1) \Gamma(\beta+1)} + \frac{(\log \frac{t}{a})^\beta}{\Gamma(\beta+1)} I_a^\alpha f(t) + 2 I_a^\alpha (M-f)^{1/2}(t) J_a^\beta (f-m)^{1/2}(t).
\end{aligned}$$

Lemma 3.7 [19] Assume that $a \geq 0$, $p \geq q \geq 0$, and $p \neq 0$. Then, we have

$$a^{q/p} \leq \left(\frac{q}{p} k^{(q-p)/p} a + \frac{p-q}{p} k^{q/p} \right).$$

Theorem 3.8 Let f be an integrable function on $[a, \infty)$, $a > 0$ and constants $p \geq q \geq 0$, $p \neq 0$. In addition, assume that the condition (H_1) holds. Then for any $k > 0$, $0 < a < t < \infty$, $\alpha > 0$, the following two inequalities hold:

$$\begin{aligned}
(A_7) \quad & J_a^\alpha (\varphi_2 - f)^{q/p}(t) I_a^\alpha (f - \varphi_1)^{q/p}(t) + \frac{q}{p} k^{(q-p)/p} (J_a^\alpha \varphi_2(t) I_a^\alpha \varphi_1(t) + J_a^\alpha f(t) I_a^\alpha f(t)) \\
& \leq \frac{q}{p} k^{(q-p)/p} (J_a^\alpha \varphi_2(t) I_a^\alpha f(t) + J_a^\alpha f(t) I_a^\alpha \varphi_1(t)) + \frac{p-q}{p} k^{q/p} \frac{(t-a)^\alpha (\log \frac{t}{a})^\alpha}{\Gamma^2(\alpha+1)},
\end{aligned}$$

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$$\begin{aligned}
(B_7) \quad & I_a^\alpha (\varphi_2 - f)^{q/p}(t) J_a^\alpha (f - \varphi_1)^{q/p}(t) + \frac{q}{p} k^{(q-p)/p} (I_a^\alpha \varphi_2(t) J_a^\alpha \varphi_1(t) + I_a^\alpha f(t) J_a^\alpha f(t)) \\
& \leq \frac{q}{p} k^{(q-p)/p} (I_a^\alpha \varphi_2(t) J_a^\alpha f(t) + I_a^\alpha f(t) J_a^\alpha \varphi_1(t)) + \frac{p-q}{p} k^{q/p} \frac{(t-a)^\alpha (\log \frac{t}{a})^\alpha}{\Gamma^2(\alpha+1)}.
\end{aligned}$$

Proof. From condition (H_1) and Lemma 3.7, for $p \geq q \geq 0$, $p \neq 0$, it follows that

$$((\varphi_2(\tau) - f(\tau))(f(\rho) - \varphi_1(\rho)))^{q/p} \leq \frac{q}{p} k^{(q-p)/p} (\varphi_2(\tau) - f(\tau))(f(\rho) - \varphi_1(\rho)) + \frac{p-q}{p} k^{q/p}, \quad (9)$$

for any $k > 0$. Multiplying both sides of (9) by $(\log(t/\tau))^{\alpha-1}/\tau\Gamma(\alpha)$, $\tau \in (a, t)$, and integrating the resulting identity with respect to τ from a to t , one has

$$\begin{aligned}
& (f(\rho) - \varphi_1(\rho))^{q/p} J_a^\alpha (\varphi_2 - f)^{q/p}(t) \\
& \leq \frac{q}{p} k^{(q-p)/p} (f(\rho) - \varphi_1(\rho)) J_a^\alpha (\varphi_2 - f)(t) + \frac{p-q}{p} k^{q/p} \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)}.
\end{aligned} \quad (10)$$

Multiplying both sides of (10) by $(t-\rho)^{\alpha-1}/\Gamma(\alpha)$, $\rho \in (a, t)$, and integrating the resulting identity with respect to ρ from a to t , we obtain

$$\begin{aligned}
& J_a^\alpha (\varphi_2 - f)^{q/p}(t) I_a^\alpha (f - \varphi_1)(t)^{q/p} \\
& \leq \frac{q}{p} k^{(q-p)/p} J_a^\alpha (\varphi_2 - f)(t) I_a^\alpha (f - \varphi_1)(t) + \frac{p-q}{p} k^{q/p} \frac{(t-a)^\alpha (\log \frac{t}{a})^\alpha}{\Gamma^2(\alpha+1)},
\end{aligned}$$

which leads to inequality in (A_6) . Using the similar arguments, we get the required inequality in (B_6) . \square

Corollary 3.9 Let f be an integrable function on $[a, \infty)$, $a > 0$ satisfying $m \leq f(t) \leq M$ for all $t \in [a, \infty)$, constants $q = 1$, $p = 2$, $k = 1$ and $m, M \in \mathbb{R}$. Then for $0 < a < t < \infty$ and $\alpha > 0$, the following two inequalities hold:

$$\begin{aligned}
(A_8) \quad & 2J_a^\alpha (M - f)^{1/2}(t) I_a^\alpha (f - m)^{1/2}(t) + J_a^\alpha f(t) I_a^\alpha f(t) \\
& \leq \frac{M(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha f(t) + \frac{m(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha f(t) + (1 - mM) \frac{(t-a)^\alpha (\log \frac{t}{a})^\alpha}{\Gamma^2(\alpha+1)},
\end{aligned}$$

$$\begin{aligned}
(B_8) \quad & 2I_a^\alpha (M - f)^{1/2}(t) J_a^\alpha (f - m)^{1/2}(t) + I_a^\alpha f(t) J_a^\alpha f(t) \\
& \leq \frac{M(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha f(t) + \frac{m(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha f(t) + (1 - mM) \frac{(t-a)^\alpha (\log \frac{t}{a})^\alpha}{\Gamma^2(\alpha+1)}.
\end{aligned}$$

4 Chebyshev Type Inequalities for Riemann-Liouville and Hadamard Fractional Integrals

In this section, we establish our main fractional integral inequalities of Chebyshev type, concerning the integral of the product of two functions and the product of two integrals, with the help of the following lemma.

Lemma 4.1 Let f be an integrable function on $[a, \infty)$, $a > 0$ and φ_1, φ_2 are two integrable functions on $[a, \infty)$. Assume that the condition (H_1) holds. Then for $0 < a < t < \infty$, and $\alpha, \beta > 0$, the following two equalities hold:

$$\begin{aligned}
(A_9) \quad & \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha f^2(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha f^2(t) - 2J_a^\alpha f(t) I_a^\alpha f(t) \\
& = J_a^\alpha (f - \varphi_1)(t) I_a^\alpha (\varphi_2 - f)(t) + J_a^\alpha (\varphi_2 - f)(t) I_a^\alpha (f - \varphi_1)(t)
\end{aligned}$$

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$$\begin{aligned}
& + \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} (J_a^\alpha(\varphi_1 f + \varphi_2 f - \varphi_1 \varphi_2)(t) - J_a^\alpha((\varphi_2 - f)(f - \varphi_1))(t)) \\
& + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} (I_a^\alpha(\varphi_1 f + \varphi_2 f - \varphi_1 \varphi_2)(t) - I_a^\alpha((\varphi_2 - f)(f - \varphi_1))(t)) \\
& + J_a^\alpha \varphi_1(t) I_a^\alpha(\varphi_2 - f)(t) + J_a^\alpha \varphi_2(t) I_a^\alpha(\varphi_1 - f)(t) - J_a^\alpha f(t) I_a^\alpha(\varphi_1 + \varphi_2)(t), \\
(B_9) \quad & \frac{(t-a)^\beta}{\Gamma(\beta+1)} J_a^\alpha f^2(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\beta f^2(t) - 2J_a^\alpha f(t) I_a^\beta f(t) \\
& = J_a^\alpha(f - \varphi_1)(t) I_a^\beta(\varphi_2 - f)(t) + J_a^\alpha(\varphi_2 - f)(t) I_a^\beta(f - \varphi_1)(t) \\
& + \frac{(t-a)^\beta}{\Gamma(\beta+1)} (J_a^\alpha(\varphi_1 f + \varphi_2 f - \varphi_1 \varphi_2)(t) - J_a^\alpha((\varphi_2 - f)(f - \varphi_1))(t)) \\
& + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} (I_a^\beta(\varphi_1 f + \varphi_2 f - \varphi_1 \varphi_2)(t) - I_a^\beta((\varphi_2 - f)(f - \varphi_1))(t)) \\
& + J_a^\alpha \varphi_1(t) I_a^\beta(\varphi_2 - f)(t) + J_a^\alpha \varphi_2(t) I_a^\beta(\varphi_1 - f)(t) - J_a^\alpha f(t) I_a^\beta(\varphi_1 + \varphi_2)(t).
\end{aligned}$$

Proof. For any $0 < a < \tau, \rho < t < \infty$, we have

$$\begin{aligned}
& (\varphi_2(\rho) - f(\rho))(f(\tau) - \varphi_1(\tau)) + (\varphi_2(\tau) - f(\tau))(f(\rho) - \varphi_1(\rho)) \\
& - (\varphi_2(\tau) - f(\tau))(f(\tau) - \varphi_1(\tau)) - (\varphi_2(\rho) - f(\rho))(f(\rho) - \varphi_1(\rho)) \\
& = f^2(\tau) + f^2(\rho) - 2f(\tau)f(\rho) + \varphi_2(\rho)f(\tau) + \varphi_1(\tau)f(\rho) - \varphi_1(\tau)\varphi_2(\rho) \\
& + \varphi_2(\tau)f(\rho) + \varphi_1(\rho)f(\tau) - \varphi_1(\rho)\varphi_2(\tau) - \varphi_2(\tau)f(\tau) + \varphi_1(\tau)\varphi_2(\tau) \\
& - \varphi_1(\tau)f(\tau) - \varphi_2(\rho)f(\rho) + \varphi_1(\rho)\varphi_2(\rho) - \varphi_1(\rho)f(\rho).
\end{aligned} \tag{11}$$

Multiplying (11) by $(\log(t/\tau))^{\alpha-1}/\tau\Gamma(\alpha)$, $\tau \in (a, t)$, $0 < a < t < \infty$, and integrating the resulting identity with respect to τ from a to t , we get

$$\begin{aligned}
& (\varphi_2(\rho) - f(\rho))(J_a^\alpha f(t) - J_a^\alpha \varphi_1(t)) + (J_a^\alpha \varphi_2(t) - J_a^\alpha f(t))(f(\rho) - \varphi_1(\rho)) \\
& - J_a^\alpha((\varphi_2 - f)(f - \varphi_1))(t) - (\varphi_2(\rho) - f(\rho))(f(\rho) - \varphi_1(\rho)) \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} \\
& = J_a^\alpha f^2(t) + f^2(\rho) \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} - 2f(\rho) {}_H I_a^\alpha f(t) + \varphi_2(\rho) {}_H I_a^\alpha f(t) + f(\rho) J_a^\alpha \varphi_1(t) \\
& - \varphi_2(\rho) J_a^\alpha \varphi_1(t) + f(\rho) J_a^\alpha \varphi_2(t) + \varphi_1(\rho) J_a^\alpha f(t) - \varphi_1(\rho) J_a^\alpha \varphi_2(t) \\
& - J_a^\alpha \varphi_2 f(t) + J_a^\alpha \varphi_1 \varphi_2(t) - J_a^\alpha \varphi_1 f(t) - \varphi_2(\rho) f(\rho) \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} \\
& + \varphi_1(\rho) \varphi_2(\rho) \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} - \varphi_1(\rho) f(\rho) \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)}.
\end{aligned} \tag{12}$$

Multiplying (12) by $(t-\rho)^{\alpha-1}/\Gamma(\alpha)$, $\rho \in (a, t)$, $0 < a < t < \infty$, and integrating the resulting identity with respect to ρ from a to t , we have

$$\begin{aligned}
& (J_a^\alpha f(t) - J_a^\alpha \varphi_1(t))(I_a^\alpha \varphi_2(t) - I_a^\alpha f(t)) + (J_a^\alpha \varphi_2(t) - J_a^\alpha f(t))(I_a^\alpha f(t) - I_a^\alpha \varphi_1(t)) \\
& - J_a^\alpha((\varphi_2 - f)(f - \varphi_1))(t) \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} - I_a^\alpha((\varphi_2 - f)(f - \varphi_1))(t) \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} \\
& = J_a^\alpha f^2(t) \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} + I_a^\alpha f^2(t) \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} - 2J_a^\alpha f(t) I_a^\alpha f(t) \\
& + J_a^\alpha f(t) I_a^\alpha \varphi_2(t) + J_a^\alpha \varphi_1(t) I_a^\alpha f(t) - J_a^\alpha \varphi_1(t) I_a^\alpha \varphi_2(t) \\
& + J_a^\alpha \varphi_2(t) I_a^\alpha f(t) + J_a^\alpha f(t) I_a^\alpha \varphi_1(t) - J_a^\alpha \varphi_2(t) I_a^\alpha \varphi_1(t) \\
& - J_a^\alpha \varphi_2 f(t) \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} + J_a^\alpha \varphi_1 \varphi_2(t) \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} - J_a^\alpha \varphi_1 f(t) \frac{(t-a)^\alpha}{\Gamma(\alpha+1)}
\end{aligned}$$

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$$- I_a^\alpha \varphi_2 f(t) \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} + I_a^\alpha \varphi_1 \varphi_2(t) \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} - I_a^\alpha \varphi_1 f(t) \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)}.$$

Therefore, the desired equality (A₉) is proved. The equality (B₉) is derived by using the similar arguments. \square

Let now g be an integrable function on $[a, \infty)$, $a > 0$ satisfying the assumption:

(H₂) There exist ψ_1 and ψ_2 integrable functions on $[a, \infty)$ such that

$$\psi_1(t) \leq g(t) \leq \psi_2(t) \quad \text{for } 0 < a < t < \infty.$$

Theorem 4.2 Let f and g be two integrable functions on $[a, \infty)$, $a > 0$ and $\varphi_1, \varphi_2, \psi_1$ and ψ_2 are four integrable functions on $[a, \infty)$ satisfying the conditions (H₁) and (H₂) on $[a, \infty)$. Then for all $0 < a < t < \infty$ and $\alpha > 0$, the following inequality holds:

$$\begin{aligned} & \left| \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha f g(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha f g(t) - J_a^\alpha f(t) I_a^\alpha g(t) - I_a^\alpha f(t) J_a^\alpha g(t) \right| \\ & \leq |K(f, \varphi_1, \varphi_2)|^{1/2} |K(g, \psi_1, \psi_2)|^{1/2}. \end{aligned} \quad (13)$$

where $K(u, v, w)$ is defined by

$$K(u, v, w) = \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha (uw + uv - vw)(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha (uw + uv - vw)(t) - 2J_a^\alpha u(t) I_a^\alpha v(t).$$

Proof. Let f and g be two integrable functions defined on $[a, \infty)$ satisfying (H₁) and (H₂), respectively. We define a function H for $0 < a < t < \infty$ as follows

$$H(\tau, \rho) := (f(\tau) - f(\rho))(g(\tau) - g(\rho)), \quad \tau, \rho \in (a, t). \quad (14)$$

Multiplying both sides of (14) by $(\log(t/\tau))^{\alpha-1}(t-\rho)^{\alpha-1}/\tau\Gamma^2(\alpha)$, $\tau, \rho \in (a, t)$, and double integrating the resulting identity with respect to τ and ρ from a to t , we have

$$\begin{aligned} & \frac{1}{\Gamma^2(\alpha)} \int_a^t \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} (t-\rho)^{\alpha-1} H(\tau, \rho) d\rho \frac{d\tau}{\tau} \\ & = \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha f g(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha f g(t) - J_a^\alpha f(t) I_a^\alpha g(t) - I_a^\alpha f(t) J_a^\alpha g(t). \end{aligned} \quad (15)$$

Applying the Cauchy-Schwarz inequality to (15), we have

$$\begin{aligned} & \left(\frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha f g(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha f g(t) - J_a^\alpha f(t) I_a^\alpha g(t) - I_a^\alpha f(t) J_a^\alpha g(t) \right)^2 \\ & \leq \left(\frac{1}{\Gamma^2(\alpha)} \int_a^t \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} (t-\rho)^{\alpha-1} (f(\tau) - f(\rho))^2 d\rho \frac{d\tau}{\tau} \right) \\ & \quad \times \left(\frac{1}{\Gamma^2(\alpha)} \int_a^t \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} (t-\rho)^{\alpha-1} (g(\tau) - g(\rho))^2 d\rho \frac{d\tau}{\tau} \right) \\ & = \left(\frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha f^2(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha f^2(t) - 2J_a^\alpha f(t) I_a^\alpha f(t) \right) \\ & \quad \times \left(\frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha g^2(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha g^2(t) - 2J_a^\alpha g(t) I_a^\alpha g(t) \right). \end{aligned} \quad (16)$$

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Since $(\varphi_2(t) - f(t))(f(t) - \varphi_1(t)) \geq 0$ and $(\psi_2(t) - f(t))(f(t) - \psi_1(t)) \geq 0$ for $t \in [a, \infty)$, we get

$$\begin{aligned}\frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha((\varphi_2 - f)(f - \varphi_1))(t) &\geq 0, \\ \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha((\varphi_2 - f)(f - \varphi_1))(t) &\geq 0, \\ \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha((\psi_2 - g)(g - \psi_1))(t) &\geq 0, \\ \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha((\psi_2 - g)(g - \psi_1))(t) &\geq 0.\end{aligned}$$

Thus, from Lemma 4.1, we obtain

$$\begin{aligned}& \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha f^2(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha f^2(t) - 2J_a^\alpha f(t) I_a^\alpha f(t) \\ & \leq J_a^\alpha(f - \varphi_1)(t) I_a^\alpha(\varphi_2 - f)(t) + J_a^\alpha(\varphi_2 - f)(t) I_a^\alpha(f - \varphi_1)(t) \\ & \quad + \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha(\varphi_2 f + \varphi_1 f - \varphi_1 \varphi_2)(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha(\varphi_2 f + \varphi_1 f - \varphi_1 \varphi_2)(t) \\ & \quad + J_a^\alpha \varphi_1(t) I_a^\alpha(\varphi_2 - f)(t) + J_a^\alpha \varphi_2(t) I_a^\alpha(\varphi_1 - f)(t) - J_a^\alpha f(t) I_a^\alpha(\varphi_1 + \varphi_2)(t) \\ & = \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha(\varphi_2 f + \varphi_1 f - \varphi_1 \varphi_2)(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha(\varphi_2 f + \varphi_1 f - \varphi_1 \varphi_2)(t) - 2J_a^\alpha f(t) I_a^\alpha f(t) \\ & = K(f, \varphi_1, \varphi_2),\end{aligned}\tag{17}$$

and

$$\begin{aligned}& \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha g^2(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha g^2(t) - 2J_a^\alpha g(t) I_a^\alpha g(t) \\ & \leq J_a^\alpha(g - \psi_1)(t) I_a^\alpha(\psi_2 - g)(t) + J_a^\alpha(\psi_2 - g)(t) I_a^\alpha(g - \psi_1)(t) \\ & \quad + \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha(\psi_2 g + \psi_1 g - \psi_1 \psi_2)(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha(\psi_2 g + \psi_1 g - \psi_1 \psi_2)(t) \\ & \quad + J_a^\alpha \psi_1(t) I_a^\alpha(\psi_2 - g)(t) + J_a^\alpha \psi_2(t) I_a^\alpha(\psi_1 - g)(t) - J_a^\alpha g(t) I_a^\alpha(\psi_1 + \psi_2)(t), \\ & = \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha(\psi_2 g + \psi_1 g - \psi_1 \psi_2)(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha(\psi_2 g + \psi_1 g - \psi_1 \psi_2)(t) - 2J_a^\alpha g(t) I_a^\alpha g(t) \\ & = K(g, \psi_1, \psi_2).\end{aligned}\tag{18}$$

From (16), (17) and (18), the required inequality in (13) is proved. \square

Corollary 4.3 If $K(f, \varphi_1, \varphi_2) = K(f, m, M)$ and $K(g, \psi_1, \psi_2) = K(g, p, P)$, $m, M, p, P \in \mathbb{R}$, then inequality (13) reduces to the following fractional integral inequality:

$$\begin{aligned}& \left| \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha f g(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha f g(t) - J_a^\alpha f(t) I_a^\alpha g(t) - I_a^\alpha f(t) J_a^\alpha g(t) \right| \\ & \leq \frac{1}{4} \left\{ \left[\left(J_a^\alpha f(t) - I_a^\alpha f(t) + M \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} - m \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} \right)^2 \right. \right. \\ & \quad \left. \left. + \left(I_a^\alpha f(t) - J_a^\alpha f(t) + M \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} - m \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} \right)^2 \right]^{1/2} \right. \\ & \quad \left. \times \left[\left(J_a^\alpha g(t) - I_a^\alpha g(t) + P \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} - p \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} \right)^2 \right]^{1/2} \right.\end{aligned}$$

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$$+ \left(J_a^\alpha g(t) - I_a^\alpha g(t) + p \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} - P \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} \right)^2 \Big]^{1/2} \Big\}.$$

Theorem 4.4 Let f and g be two integrable function on $[a, \infty)$, $a > 0$. Assume that there exist four integrable functions $\varphi_1, \varphi_2, \psi_1$ and ψ_2 satisfying the conditions (H_1) and (H_2) on $[a, \infty)$. Then for all $0 < a < t < \infty$ and $\alpha, \beta > 0$, the following inequality holds:

$$\begin{aligned} & \left| \frac{(t-a)^\beta}{\Gamma(\beta+1)} J_a^\alpha f g(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\beta f g(t) - J_a^\alpha f(t) I_a^\beta g(t) - I_a^\beta f(t) J_a^\alpha g(t) \right| \\ & \leq |K_1(f, \varphi_1, \varphi_2)|^{1/2} |K_1(g, \psi_1, \psi_2)|^{1/2}, \end{aligned} \quad (19)$$

where $K_1(u, v, w)$ is defined by

$$K_1(u, v, w) = \frac{(t-a)^\beta}{\Gamma(\beta+1)} J_a^\alpha (uw + uv - vw)(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\beta (uw + uv - vw)(t) - 2J_a^\alpha u(t) I_a^\beta u(t).$$

Proof. Multiplying both sides of (14) by $(\log(t/\tau))^{\alpha-1} (t-\rho)^{\beta-1} / \tau \Gamma(\alpha) \Gamma(\beta)$, $\tau, \rho \in (a, t)$, and double integrating with respect to τ and ρ from a to t we get

$$\begin{aligned} & \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_a^t \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} (t-\rho)^{\beta-1} H(\tau, \rho) d\rho \frac{d\tau}{\tau} \\ & = \frac{(t-a)^\beta}{\Gamma(\alpha+1)} J_a^\alpha f g(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\beta f g(t) - J_a^\alpha f(t) I_a^\beta g(t) - I_a^\beta f(t) J_a^\alpha g(t). \end{aligned} \quad (20)$$

By using the Cauchy-Schwarz inequality for double integrals, we have

$$\begin{aligned} & \left| \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha f g(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha f g(t) - J_a^\alpha f(t) I_a^\alpha g(t) - I_a^\alpha f(t) J_a^\alpha g(t) \right| \\ & \leq \left[\frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_a^t \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} (t-\rho)^{\beta-1} f^2(\tau) d\rho \frac{d\tau}{\tau} \right. \\ & \quad + \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_a^t \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} (t-\rho)^{\beta-1} f^2(\rho) d\rho \frac{d\tau}{\tau} \\ & \quad \left. - \frac{2}{\Gamma(\alpha) \Gamma(\beta)} \int_a^t \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} (t-\rho)^{\beta-1} f(\tau) f(\rho) d\rho \frac{d\tau}{\tau} \right]^{1/2} \\ & \quad \times \left[\frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_a^t \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} (t-\rho)^{\beta-1} g^2(\tau) d\rho \frac{d\tau}{\tau} \right. \\ & \quad + \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_a^t \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} (t-\rho)^{\beta-1} g^2(\rho) d\rho \frac{d\tau}{\tau} \\ & \quad \left. - \frac{2}{\Gamma(\alpha) \Gamma(\beta)} \int_a^t \int_a^t \left(\log \frac{t}{\tau} \right)^{\alpha-1} (t-\rho)^{\beta-1} g(\tau) g(\rho) d\rho \frac{d\tau}{\tau} \right]^{1/2}. \end{aligned}$$

Therefore, we get

$$\begin{aligned} & \left| \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha f g(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha f g(t) - J_a^\alpha f(t) I_a^\alpha g(t) - I_a^\alpha f(t) J_a^\alpha g(t) \right| \\ & \leq \left[\frac{(t-a)^\beta}{\Gamma(\beta+1)} J_a^\alpha f^2(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\beta f^2(t) - 2J_a^\alpha f(t) I_a^\beta f(t) \right]^{1/2} \\ & \quad \times \left[\frac{(t-a)^\beta}{\Gamma(\beta+1)} J_a^\alpha g^2(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\beta g^2(t) - 2J_a^\alpha g(t) I_a^\beta g(t) \right]^{1/2}. \end{aligned}$$

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Thus, from Lemma 4.1, we get

$$\begin{aligned}
& \frac{(t-a)^\beta}{\Gamma(\beta+1)} J_a^\alpha f^2(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\beta f^2(t) - 2J_a^\alpha f(t) I_a^\beta f(t) \\
& \leq J_a^\alpha (f - \varphi_1)(t) I_a^\beta (\varphi_2 - f)(t) + J_a^\alpha (\varphi_2 - f)(t) I_a^\beta (f - \varphi_1)(t) \\
& \quad + \frac{(t-a)^\beta}{\Gamma(\beta+1)} J_a^\alpha (\varphi_2 f + \varphi_1 f - \varphi_1 \varphi_2)(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\beta (\varphi_2 f + \varphi_1 f - \varphi_1 \varphi_2)(t) \\
& \quad + J_a^\alpha \varphi_1(t) I_a^\beta (\varphi_2 - f)(t) + J_a^\alpha \varphi_2(t) I_a^\beta (\varphi_1 - f)(t) - J_a^\alpha f(t) I_a^\beta (\varphi_1 + \varphi_2)(t) \\
& = \frac{(t-a)^\beta}{\Gamma(\beta+1)} J_a^\alpha (\varphi_2 f + \varphi_1 f - \varphi_1 \varphi_2)(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\beta (\varphi_2 f + \varphi_1 f - \varphi_1 \varphi_2)(t) - 2J_a^\alpha f(t) I_a^\beta f(t) \\
& = K_1(f, \varphi_1, \varphi_2),
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
& \frac{(t-a)^\beta}{\Gamma(\beta+1)} J_a^\alpha g^2(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\beta g^2(t) - 2J_a^\alpha g(t) I_a^\beta g(t) \\
& \leq J_a^\alpha (g - \psi_1)(t) I_a^\beta (\psi_2 - g)(t) + J_a^\alpha (\psi_2 - g)(t) I_a^\beta (g - \psi_1)(t) \\
& \quad + \frac{(t-a)^\beta}{\Gamma(\beta+1)} J_a^\alpha (\psi_2 g + \psi_1 g - \psi_1 \psi_2)(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\beta (\psi_2 g + \psi_1 g - \psi_1 \psi_2)(t) \\
& \quad + J_a^\alpha \psi_1(t) I_a^\beta (\psi_2 - g)(t) + J_a^\alpha \psi_2(t) I_a^\beta (\psi_1 - g)(t) - J_a^\alpha g(t) I_a^\beta (\psi_1 + \psi_2)(t), \\
& = \frac{(t-a)^\beta}{\Gamma(\beta+1)} J_a^\alpha (\psi_2 g + \psi_1 g - \psi_1 \psi_2)(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\beta (\psi_2 g + \psi_1 g - \psi_1 \psi_2)(t) - 2J_a^\alpha g(t) I_a^\beta g(t) \\
& = K_1(g, \psi_1, \psi_2).
\end{aligned} \tag{22}$$

From (15), (21) and (22), we obtain the desired bound in (19). \square

Corollary 4.5 *If $K(f, \varphi_1, \varphi_2) = K(f, m, M)$ and $K(g, \psi_1, \psi_2) = K(g, p, P)$, $m, M, p, P \in \mathbb{R}$, then inequality (13) reduces to the following fractional integral inequality:*

$$\begin{aligned}
& \left| \frac{(t-a)^\beta}{\Gamma(\beta+1)} J_a^\alpha f g(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\beta f g(t) - J_a^\alpha f(t) I_a^\beta g(t) - I_a^\beta f(t) J_a^\alpha g(t) \right| \\
& \leq \frac{1}{4} \left\{ \left[\left(J_a^\alpha f(t) - I_a^\beta f(t) + M \frac{(t-a)^\beta}{\Gamma(\beta+1)} - m \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} \right)^2 \right. \right. \\
& \quad \left. \left. + \left(I_a^\beta f(t) - J_a^\alpha f(t) + M \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} - m \frac{(t-a)^\beta}{\Gamma(\beta+1)} \right)^2 \right]^{1/2} \right. \\
& \quad \times \left[\left(J_a^\alpha g(t) - I_a^\beta g(t) + P \frac{(t-a)^\beta}{\Gamma(\beta+1)} - p \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} \right)^2 \right. \\
& \quad \left. \left. + \left(I_a^\beta g(t) - J_a^\alpha g(t) + p \frac{(t-a)^\beta}{\Gamma(\beta+1)} - P \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} \right)^2 \right]^{1/2} \right\}.
\end{aligned}$$

5 Applications

In this section we present a way for constructing the four bounding functions, and use them to give some estimates of Chebyshev type inequalities of Riemann-Liouville and Hadamard fractional integrals for two unknown functions.

From the Definitions 2.1 and 2.2, for $0 < a = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T$, we define two

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notations of sub-integrals for Riemann-Liouville and Hadamard fractional integrals as

$$I_{t_j, t_{j+1}}^\alpha f(T) = \frac{1}{\Gamma(\alpha)} \int_{t_j}^{t_{j+1}} (T - \tau)^{\alpha-1} f(\tau) d\tau, \quad j = 0, 1, \dots, p. \quad (23)$$

and

$$J_{t_j, t_{j+1}}^\alpha f(T) = \frac{1}{\Gamma(\alpha)} \int_{t_j}^{t_{j+1}} \left(\log \frac{T}{\tau} \right)^{\alpha-1} f(\tau) \frac{d\tau}{\tau}, \quad j = 0, 1, \dots, p. \quad (24)$$

Note that

$$\begin{aligned} I_a^\alpha f(T) &= \sum_{j=0}^p I_{t_j, t_{j+1}}^\alpha f(T) \\ &= \frac{1}{\Gamma(\alpha)} \int_a^{t_1} (T - \tau)^{\alpha-1} f(\tau) d\tau + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (T - \tau)^{\alpha-1} f(\tau) d\tau \\ &\quad + \dots + \frac{1}{\Gamma(\alpha)} \int_{t_p}^T (T - \tau)^{\alpha-1} f(\tau) d\tau, \end{aligned}$$

and

$$\begin{aligned} J_a^\alpha f(T) &= \sum_{j=0}^p J_{t_j, t_{j+1}}^\alpha f(T) \\ &= \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \left(\log \frac{T}{\tau} \right)^{\alpha-1} f(\tau) d\tau + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{T}{\tau} \right)^{\alpha-1} f(\tau) d\tau \\ &\quad + \dots + \frac{1}{\Gamma(\alpha)} \int_{t_p}^T \left(\log \frac{T}{\tau} \right)^{\alpha-1} f(\tau) d\tau. \end{aligned}$$

Let u be a unit step function defined by

$$u(t) = \begin{cases} 1, & t > 0, \\ 0, & t \leq 0, \end{cases} \quad (25)$$

and let $u_a(t)$ be the Heaviside unit step function defined by

$$u_a(t) = u(t - a) = \begin{cases} 1, & t > a, \\ 0, & t \leq a. \end{cases} \quad (26)$$

Let φ_1 be a piecewise continuous functions on $[0, T]$ defined by

$$\begin{aligned} \varphi_1(t) &= m_1(u_0(t) - u_{t_1}(t)) + m_2(u_{t_1}(t) - u_{t_2}(t)) + m_3(u_{t_2}(t) - u_{t_3}(t)) + \dots + m_{p+1}u_{t_p}(t) \\ &= m_1u_0(t) + (m_2 - m_1)u_{t_1}(t) + (m_3 - m_2)u_{t_2}(t) + \dots + (m_{p+1} - m_p)u_{t_p}(t) \\ &= \sum_{j=0}^p (m_{j+1} - m_j)u_{t_j}(t), \end{aligned} \quad (27)$$

where $m_0 = 0$ and $0 < a = t_0 < t_1 < t_2 < \dots < t_p < t_{p+1} = T$.

Analogously, we define the functions φ_2, ψ_1 and ψ_2 as

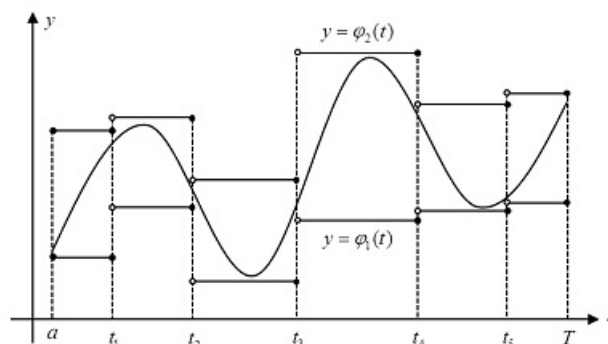
$$\varphi_2(t) = \sum_{j=0}^p (M_{j+1} - M_j)u_{t_j}(t), \quad (28)$$

$$\psi_1(t) = \sum_{j=0}^p (n_{j+1} - n_j)u_{t_j}(t), \quad (29)$$

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$$\psi_2(t) = \sum_{j=0}^p (N_{j+1} - N_j) u_{t_j}(t), \quad (30)$$

where the constants $n_0 = N_0 = M_0 = 0$. If there is an integrable function f on $[a, T]$ satisfying condition (H_1) then we get $m_{j+1} \leq f(t) \leq M_{j+1}$ for each $t \in (t_j, t_{j+1}]$, $j = 0, 1, 2, \dots, p$. In particular, $p = 4$, the time history of f can be shown as in figure 1.

Figure 1: Functions f , φ_1 and φ_2 .

Proposition 5.1 Let f and g be two integrable functions on $[a, T]$, $a > 0$. Assume that the functions $\varphi_1, \varphi_2, \psi_1$ and ψ_2 are defined by (27), (28), (29) and (30), respectively, satisfying (H_1) – (H_2) . Then for $\alpha > 0$, the following inequality holds:

$$\begin{aligned} & \left| \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} J_a^\alpha f g(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\alpha f g(t) - J_a^\alpha f(t) I_a^\alpha g(t) - I_a^\alpha f(t) J_a^\alpha g(t) \right| \\ & \leq |K^*(f, \varphi_1, \varphi_2)|^{1/2} |K^*(g, \psi_1, \psi_2)|^{1/2}, \end{aligned} \quad (31)$$

where

$$\begin{aligned} & K^*(u, v, w)(T) \\ & \leq \frac{(T-a)^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^p \left\{ w J_{t_i, t_{j+1}}^\alpha u(T) + v J_{t_i, t_{j+1}}^\alpha u(T) - vw \left[\left(\log \frac{T}{t_j} \right)^\alpha - \left(\log \frac{T}{t_{j+1}} \right)^\alpha \right] \right\} \\ & + \frac{(\log \frac{T}{a})^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^p \left\{ w I_{t_i, t_{j+1}}^\alpha u(T) + v I_{t_i, t_{j+1}}^\alpha u(T) - vw [(T-t_j)^\alpha - (T-t_{j+1})^\alpha] \right\} \\ & - 2 \left(\sum_{j=0}^p J_{t_i, t_{j+1}}^\alpha u(T) \right) \left(\sum_{j=0}^p I_{t_i, t_{j+1}}^\alpha u(T) \right). \end{aligned}$$

Proof. Since

$$\begin{aligned} I_{t_j, t_{j+1}}^\alpha(1)(T) &= \frac{1}{\Gamma(\alpha)} \int_{t_j}^{t_{j+1}} (T-\tau)^{\alpha-1} d\tau \\ &= \frac{1}{\Gamma(\alpha+1)} [(T-t_j)^\alpha - (T-t_{j+1})^\alpha], \\ J_{t_j, t_{j+1}}^\alpha(1)(T) &= \frac{1}{\Gamma(\alpha)} \int_{t_j}^{t_{j+1}} \left(\log \frac{T}{\tau} \right)^{\alpha-1} \frac{d\tau}{\tau} \\ &= \frac{1}{\Gamma(\alpha+1)} \left[\left(\log \frac{T}{t_j} \right)^\alpha - \left(\log \frac{T}{t_{j+1}} \right)^\alpha \right], \end{aligned}$$

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we have

$$I_a^\alpha(\varphi_1\varphi_2)(T) = \sum_{j=0}^p \frac{m_{j+1}M_{j+1}}{\Gamma(\alpha+1)} [(T-t_j)^\alpha - (T-t_{j+1})^\alpha],$$

$$J_{t_j, t_{j+1}}^\alpha(\psi_1\psi_2)(T) = \sum_{j=0}^p \frac{n_{j+1}N_{j+1}}{\Gamma(\alpha+1)} \left[\left(\log \frac{T}{t_j} \right)^\alpha - \left(\log \frac{T}{t_{j+1}} \right)^\alpha \right].$$

Therefore, two functional $K^*(f, \varphi_1, \varphi_2)(T)$ and $K^*(g, \psi_1, \psi_2)(T)$ can be expressed by

$$\begin{aligned} K^*(f, \varphi_1, \varphi_2)(T) &\leq \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^p \left\{ M_{j+1} J_{t_i, t_{j+1}}^\alpha f(T) + m_{j+1} J_{t_i, t_{j+1}}^\alpha f(T) \right. \\ &\quad \left. - m_{j+1} M_{j+1} \left[\left(\log \frac{T}{t_j} \right)^\alpha - \left(\log \frac{T}{t_{j+1}} \right)^\alpha \right] \right\} \\ &\quad + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^p \left\{ M_{j+1} I_{t_i, t_{j+1}}^\alpha f(T) + m_{j+1} I_{t_i, t_{j+1}}^\alpha f(T) \right. \\ &\quad \left. - m_{j+1} M_{j+1} [(T-t_j)^\alpha - (T-t_{j+1})^\alpha] \right\} \\ &\quad - 2 \left(\sum_{j=0}^p J_{t_i, t_{j+1}}^\alpha f(T) \right) \left(\sum_{j=0}^p I_{t_i, t_{j+1}}^\alpha f(T) \right), \end{aligned}$$

and

$$\begin{aligned} K^*(g, \psi_1, \psi_2)(T) &\leq \frac{(t-a)^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^p \left\{ N_{j+1} J_{t_i, t_{j+1}}^\alpha g(T) + n_{j+1} J_{t_i, t_{j+1}}^\alpha g(T) \right. \\ &\quad \left. - n_{j+1} M_{j+1} \left[\left(\log \frac{T}{t_j} \right)^\alpha - \left(\log \frac{T}{t_{j+1}} \right)^\alpha \right] \right\} \\ &\quad + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^p \left\{ N_{j+1} I_{t_i, t_{j+1}}^\alpha g(T) + n_{j+1} I_{t_i, t_{j+1}}^\alpha g(T) \right. \\ &\quad \left. - n_{j+1} N_{j+1} [(T-t_j)^\alpha - (T-t_{j+1})^\alpha] \right\} \\ &\quad - 2 \left(\sum_{j=0}^p J_{t_i, t_{j+1}}^\alpha g(T) \right) \left(\sum_{j=0}^p I_{t_i, t_{j+1}}^\alpha g(T) \right). \end{aligned}$$

By applying Theorem (4.2), the required inequality (31) is established. \square

Proposition 5.2 Let f and g be two integrable functions on $[a, T]$, $a > 0$. Assume that the functions $\varphi_1, \varphi_2, \psi_1$ and ψ_2 are defined by (27), (28), (29) and (30), respectively, satisfying (H_1) -(H_2). Then for $\alpha, \beta > 0$, the following inequality holds:

$$\begin{aligned} &\left| \frac{(t-a)^\beta}{\Gamma(\beta+1)} J_a^\alpha f g(t) + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} I_a^\beta f g(t) - J_a^\alpha f(t) I_a^\beta g(t) - I_a^\beta f(t) J_a^\alpha g(t) \right| \\ &\leq |K_1^*(f, \varphi_1, \varphi_2)|^{1/2} |K_1^*(g, \psi_1, \psi_2)|^{1/2}, \end{aligned} \quad (32)$$

where

$$K_1^*(u, v, w)(T) \leq \frac{(T-a)^\beta}{\Gamma(\beta+1)} \sum_{j=0}^p \left\{ w J_{t_i, t_{j+1}}^\alpha u(T) + v J_{t_i, t_{j+1}}^\alpha u(T) - vw \left[\left(\log \frac{T}{t_j} \right)^\alpha - \left(\log \frac{T}{t_{j+1}} \right)^\alpha \right] \right\}$$

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$$\begin{aligned}
& + \frac{(\log \frac{T}{a})^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^p \left\{ w I_{t_i, t_{j+1}}^\beta u(T) + v I_{t_i, t_{j+1}}^\beta u(T) - vw [(T-t_j)^\beta - (T-t_{j+1})^\beta] \right\} \\
& - 2 \left(\sum_{j=0}^p J_{t_i, t_{j+1}}^\alpha u(T) \right) \left(\sum_{j=0}^p I_{t_i, t_{j+1}}^\beta u(T) \right).
\end{aligned}$$

Proof. By direct computations, we have

$$\begin{aligned}
K_1^*(f, \varphi_1, \varphi_2)(T) & \leq \frac{(t-a)^\beta}{\Gamma(\beta+1)} \sum_{j=0}^p \left\{ M_{j+1} J_{t_i, t_{j+1}}^\alpha f(T) + m_{j+1} J_{t_i, t_{j+1}}^\alpha f(T) \right. \\
& \quad \left. - m_{j+1} M_{j+1} \left[\left(\log \frac{T}{t_j} \right)^\alpha - \left(\log \frac{T}{t_{j+1}} \right)^\alpha \right] \right\} \\
& \quad + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^p \left\{ M_{j+1} I_{t_i, t_{j+1}}^\alpha f(T) + m_{j+1} I_{t_i, t_{j+1}}^\beta f(T) \right. \\
& \quad \left. - m_{j+1} M_{j+1} [(T-t_j)^\beta - (T-t_{j+1})^\beta] \right\} \\
& \quad - 2 \left(\sum_{j=0}^p J_{t_i, t_{j+1}}^\alpha f(T) \right) \left(\sum_{j=0}^p I_{t_i, t_{j+1}}^\beta f(T) \right),
\end{aligned}$$

and

$$\begin{aligned}
K_1^*(g, \psi_1, \psi_2)(T) & \leq \frac{(t-a)^\beta}{\Gamma(\beta+1)} \sum_{j=0}^p \left\{ N_{j+1} J_{t_i, t_{j+1}}^\alpha g(T) + n_{j+1} J_{t_i, t_{j+1}}^\alpha g(T) \right. \\
& \quad \left. - n_{j+1} M_{j+1} \left[\left(\log \frac{T}{t_j} \right)^\alpha - \left(\log \frac{T}{t_{j+1}} \right)^\alpha \right] \right\} \\
& \quad + \frac{(\log \frac{t}{a})^\alpha}{\Gamma(\alpha+1)} \sum_{j=0}^p \left\{ N_{j+1} I_{t_i, t_{j+1}}^\beta g(T) + n_{j+1} I_{t_i, t_{j+1}}^\beta g(T) \right. \\
& \quad \left. - n_{j+1} N_{j+1} [(T-t_j)^\beta - (T-t_{j+1})^\beta] \right\} \\
& \quad - 2 \left(\sum_{j=0}^p J_{t_i, t_{j+1}}^\alpha g(T) \right) \left(\sum_{j=0}^p I_{t_i, t_{j+1}}^\beta g(T) \right),
\end{aligned}$$

By applying Theorem (4.4), the required inequality (32) is established. \square

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Weighted composition operators from $F(p, q, s)$ spaces to n th weighted-Orlicz spaces

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Abstract

The boundedness and compactness of the weighted composition operator from $F(p, q, s)$ spaces to n th weighted-Orlicz spaces are characterized in this paper.

Keywords: weighted composition operator, $F(p, q, s)$ spaces, n th weighted-Orlicz spaces.

1 Introduction

Let $\mathcal{H}(\mathbb{D})$ be the space of all holomorphic functions on the open unit disk \mathbb{D} in the complex plane \mathbb{C} , \mathbb{N}_0 the set of all nonnegative integers, \mathbb{N} the set of all positive integers, and dA the Lebesgue measure on \mathbb{D} normalized so that $A(\mathbb{D}) = 1$. Let $u \in \mathcal{H}(\mathbb{D})$, the weighted composition operator uC_ϕ is defined by $(uC_\phi f)(z) = u(z)f(\phi(z))$, $f \in \mathcal{H}(\mathbb{D})$, for more details, see, [1, 3, 16, 18].

For $0 < p, s < \infty$, $-2 < q < \infty$, a function $f \in \mathcal{H}(\mathbb{D})$ is said to belong to the general function space $F(p, q, s)$ if

$$\|f\|_{F(p,q,s)} = |f(0)|^p + \sup_{z \in \mathbb{D}} \int_D |f'(z)|^p (1 - |z|^2)^q (1 - |\psi_a(z)|^2)^s dA(z) < \infty,$$

where $\psi_a(z) = (a - z)/(1 - \bar{a}z)$, $a \in \mathbb{D}$. The space $F(p, q, s)$ was introduced by Zhao in [14]. Since for $q + s \leq -1$, $F(p, q, s)$ is the space of constant functions, we assume that $q + s > -1$. For some results on $F(p, q, s)$ space see, for example, [4, 5, 7, 8, 10, 11, 15, 16, 17, 18].

Let μ be a positive continuous function on $[0, 1)$. We say that μ is normal if there exist two positive numbers a and b with $0 < a < b$, and $\delta \in [0, 1)$ such that (see [6])

$$\frac{\mu(r)}{(1-r)^a} \text{ is decreasing on } [\delta, 1), \lim_{r \rightarrow 1} \frac{\mu(r)}{(1-r)^a} = 0; \frac{\mu(r)}{(1-r)^b} \text{ is increasing on } [\delta, 1), \lim_{r \rightarrow 1} \frac{\mu(r)}{(1-r)^b} = \infty.$$

Let $\mu(z) = \mu(|z|)$ be a normal function on \mathbb{D} . The n th weighted-type space on \mathbb{D} , denoted by $\mathcal{W}_\mu^{(n)} = \mathcal{W}_\mu^{(n)}(\mathbb{D})$ which was introduced by Stević in [9], consists of all $f \in \mathcal{H}(\mathbb{D})$ such that

$$b_{\mathcal{W}_\mu^{(n)}}(f) = \sup_{z \in \mathbb{D}} \mu(z) |f^{(n)}(z)| < \infty.$$

For $n = 0$ the space becomes the weighted-type space $H_\mu^\infty(\mathbb{D})$, for $n = 1$ the Bloch-type space $\mathcal{B}_\mu(\mathbb{D})$ and for $n = 2$ the Zygmund-type space $\mathcal{Z}_\mu(\mathbb{D})$. From now on, we will assume that $n \in \mathbb{N}$. Set

$$\|f\|_{\mathcal{W}_\mu^{(n)}} = \sum_{j=0}^{n-1} |f^{(j)}(0)| + b_{\mathcal{W}_\mu^{(n)}}(f).$$

With this norm the n th weighted-type space becomes a Banach space.

Recently, Fernández in [2] uses Young's functions to define the Bloch-Orlicz space. More precisely, let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a strictly increasing convex function such that $\varphi(0) = 0$ and $\lim_{t \rightarrow \infty} \varphi(t) = \infty$. The Bloch-Orlicz space associated with the function φ , denoted by \mathcal{B}^φ , is the class of all analytic functions f in \mathbb{D} such that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) \varphi(\lambda |f'(z)|) < \infty$$

for some $\lambda > 0$ depending on f . Also, since φ is convex, it is not hard to see that the Minkowski's functional

$$\|f\|_{b^\varphi} = \inf \left\{ k > 0 : S_\varphi \left(\frac{f'}{k} \right) \leq 1 \right\}$$

defines a seminorm for \mathcal{B}^φ , which, in this case, is known as Luxemburgs seminorm, where

$$S_\varphi(f) = \sup_{z \in \mathbb{D}} (1 - |z|^2) \varphi(|f(z)|).$$

In fact, it can be shown that \mathcal{B}^φ is a Banach space with the norm $\|f\|_{\mathcal{B}^\varphi} = |f(0)| + \|f\|_{\mathcal{B}^\varphi}$. For more details, see [2]. We also have that the Bloch-Orlicz space is isometrically equal to the μ -Bloch space, where $\mu(z) = \frac{1}{\varphi^{-1}(\frac{1}{1-|z|^2})}$, $z \in \mathbb{D}$.

Inspired by this, now we define the n th weighted-Orlicz space, which is denoted by $\mathcal{W}_\varphi^{(n)}$, as the class of all analytic function f in \mathbb{D} such that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) \varphi(\lambda |f^{(n)}(z)|) < \infty$$

for some $\lambda > 0$ depending on f . Same as the Bloch-Orlicz space, it is not difficult to see that the Minkowski's functional

$$\|f\|_{w^\varphi} = \inf \left\{ k > 0 : S_\varphi\left(\frac{f^{(n)}}{k}\right) \leq 1 \right\}$$

defines a seminorm for $\mathcal{W}_\varphi^{(n)}$. Furthermore, it can be shown that $\mathcal{W}_\varphi^{(n)}$ is a Banach space with the norm

$$\|f\|_{\mathcal{W}_\varphi^{(n)}} = \sum_{j=0}^{n-1} |f^{(j)}(0)| + \|f\|_{w^\varphi}.$$

In the same way as in the case \mathcal{B}^φ , for any $f \in \mathcal{W}_\varphi^{(n)} \setminus \{0\}$, the relation

$$S_\varphi\left(\frac{f^{(n)}}{\|f\|_{\mathcal{W}_\varphi^{(n)}}}\right) \leq 1$$

holds. Also, as a direct consequence of this, we have that the n th weighted-Orlicz space is isometrically equal to the n th weighted-type space, where $\mu(z) = \frac{1}{\varphi^{-1}(\frac{1}{1-|z|^2})}$, $z \in \mathbb{D}$. Thus, for any $f \in \mathcal{W}_\varphi^{(n)}$, we have

$$\|f\|_{\mathcal{W}_\varphi^{(n)}} = \sum_{j=0}^{n-1} |f^{(j)}(0)| + \sup_{z \in \mathbb{D}} \frac{|f^{(n)}(z)|}{\varphi^{-1}(\frac{1}{1-|z|^2})}.$$

Clearly, for $n = 1$, the n th weighted-Orlicz space $\mathcal{W}_\varphi^{(n)}$ becomes the Bloch-Orlicz space, and for $n = 2$ the Zygmund-Orlicz space. In this paper, we are devoted to investigating the boundedness and compactness of the weighted composition operator uC_ϕ from $F(p, q, s)$ spaces to n th weighted-Orlicz spaces. In what follows, we use the letter C to denote a positive constant whose value may change its value at each occurrence.

2 Auxiliary Results

In this section we formulate some auxiliary results which will be used in the proof of the main results. Lemma 1 and Lemma 2 can be found in [5].

Lemma 1. Assume that $f \in F(p, q, s)$, $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$. Then, for each $n \in \mathbb{N}$, there is a positive constant C , independent of f such that $\|f\|_{\mathcal{B}^{\frac{2+q}{p}}} \leq C \|f\|_{F(p, q, s)}$ and

$$|f^{(n)}(z)| \leq \frac{C \|f\|_{F(p, q, s)}}{(1 - |z|^2)^{\frac{2+q-p}{p} + n}}, \quad z \in \mathbb{D}.$$

Lemma 2. Let $\alpha > 0$ and $f \in \mathcal{B}^\alpha$. Then,

$$|f(z)| \leq \begin{cases} C \|f\|_{\mathcal{B}^\alpha}, & 0 < \alpha < 1, \\ C \log \frac{2}{1-|z|^2} \|f\|_{\mathcal{B}^\alpha}, & \alpha = 1, \\ \frac{C}{(1-|z|^2)^{\alpha-1}} \|f\|_{\mathcal{B}^\alpha}, & \alpha > 1. \end{cases}$$

Lemma 3 and Lemma 4 can be found in [12].

Lemma 3. Assume $a > 0$ and

$$D_{n+1} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a & a+1 & \cdots & a+n \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{j=0}^{n-1}(a+j) & \prod_{j=0}^{n-1}(a+j+1) & \cdots & \prod_{j=0}^{n-1}(a+j+n) \end{vmatrix}.$$

Then, $D_{n+1} = \prod_{j=1}^n j!$.

Lemma 4. Assume $n \in \mathbb{N}$, $u, f \in \mathcal{H}(\mathbb{D})$ and ϕ is an analytic self-map of \mathbb{D} . Then,

$$(u(z)f(\phi(z)))^{(n)} = \sum_{k=0}^n f^{(k)}(\phi(z)) \sum_{l=k}^n C_n^l u^{(n-l)}(z) B_{l,k}(\phi'(z), \phi''(z), \dots, \phi^{(l-k+1)}(z)),$$

where

$$B_{l,k}(\phi'(z), \phi''(z), \dots, \phi^{(l-k+1)}(z)) = \sum_{k_1, \dots, k_l} \frac{l!}{k_1! \cdots k_l!} \prod_{j=1}^l \left(\frac{\phi^{(j)}(z)}{j!} \right)^{k_j}, \quad (1)$$

and the sum in (1) is overall non-negative integer k_1, \dots, k_l satisfying $k_1 + k_2 + \dots + k_l = k$ and $k_1 + 2k_2 + \dots + lk_l = l$.

The next characterization of compactness is proved in a standard way (see, e.g., the proofs of [1], Prop 3.11). Hence we omit it. The following Lemma 6 can be found in [13].

Lemma 5. Suppose that $u \in \mathcal{H}(\mathbb{D})$, $n \in \mathbb{N}$, ϕ is an analytic self-map of \mathbb{D} . Then, $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\varphi^{(n)}$ is compact if and only if $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\varphi^{(n)}$ is bounded and for any bounded sequence $\{f_k\}_{k \in \mathbb{N}}$ in $F(p, q, s)$ which converges to zero uniformly on compact subsets of \mathbb{D} as $k \rightarrow \infty$, we have $\|uC_\phi f_k\|_{\mathcal{W}_\varphi^{(n)}} \rightarrow 0$ as $k \rightarrow \infty$.

Lemma 6. Fix $0 < \alpha < 1$ and let $\{f_k\}_{k \in \mathbb{N}}$ be a bounded sequence in \mathcal{B}^α which converges to zero uniformly on compact subsets of \mathbb{D} as $k \rightarrow \infty$. Then we have

$$\lim_{k \rightarrow \infty} \sup_{z \in \mathbb{D}} |f_k(z)| = 0. \quad (2)$$

3 The Boundedness of $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\varphi^{(n)}$

Theorem 7. Let $u \in \mathcal{H}(\mathbb{D})$, $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, $n \in \mathbb{N}$ and ϕ be an analytic self-map of \mathbb{D} .

(a) If $2 + q < p$, then $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\varphi^{(n)}$ is bounded if and only if

$$M_0 = \sup_{z \in \mathbb{D}} \frac{|u^{(n)}(z)|}{\varphi^{-1}(\frac{1}{1-|z|^2})} < \infty, \quad (3)$$

and

$$M_k = \sup_{z \in \mathbb{D}} \frac{|\sum_{l=k}^n C_n^l u^{(n-l)}(z) B_{l,k}(\phi'(z), \phi''(z), \dots, \phi^{(l-k+1)}(z))|}{\varphi^{-1}(\frac{1}{1-|z|^2})(1-|\phi(z)|^2)^{\frac{2+q-p}{p}+k}} < \infty, \quad (4)$$

where $k = 1, 2, \dots, n$.

(b) If $2 + q = p$, then $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\varphi^{(n)}$ is bounded if and only if (4) holds and

$$M'_0 = \sup_{z \in \mathbb{D}} \frac{|u^{(n)}(z)| \log \frac{2}{1-|\phi(z)|^2}}{\varphi^{-1}(\frac{1}{1-|z|^2})} < \infty. \quad (5)$$

(c) If $2 + q > p$, then $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\varphi^{(n)}$ is bounded if and only if (4) holds and

$$M''_0 = \sup_{z \in \mathbb{D}} \frac{|u^{(n)}(z)|}{\varphi^{-1}(\frac{1}{1-|z|^2})(1-|\phi(z)|^2)^{\frac{2+q-p}{p}}} < \infty. \quad (6)$$

Proof. If $2 + q < p$. Assume that (3) and (4) hold, then for each $f \in \mathcal{W}_\varphi^{(n)} \setminus \{0\}$, by Lemma 1, Lemma 2 and Lemma 4, we have

$$\begin{aligned}
& S_\varphi \left(\frac{(uC_\phi f)^{(n)}(z)}{C \|f\|_{F(p,q,s)}} \right) \\
& \leq \sup_{z \in \mathbb{D}} (1 - |z|^2) \cdot \\
& \quad \varphi \left(\frac{|u^{(n)}(z)| |f(\phi(z))| + \left| \sum_{k=1}^n f^{(k)}(\phi(z)) \sum_{l=k}^n C_n^l u^{(n-l)}(z) B_{l,k}(\phi'(z), \phi''(z), \dots, \phi^{(l-k+1)}(z)) \right|}{C \|f\|_{F(p,q,s)}} \right) \\
& \leq \sup_{z \in \mathbb{D}} (1 - |z|^2) \cdot \\
& \quad \varphi \left(\frac{\varphi^{-1} \left(\frac{1}{1-|z|^2} \right) M_0 |f(\phi(z))| + \varphi^{-1} \left(\frac{1}{1-|z|^2} \right) \sum_{k=1}^n M_k (1 - |\phi(z)|^2)^{\frac{2+q-p}{p} + k} |f^{(k)}(\phi(z))|}{C \|f\|_{F(p,q,s)}} \right) \\
& \leq \sup_{z \in \mathbb{D}} (1 - |z|^2) \varphi \left(\frac{C M_0 \|f\|_{B^{\frac{2+q}{p}}} + \sum_{k=1}^n C_k M_k \|f\|_{F(p,q,s)}}{C \|f\|_{F(p,q,s)}} \varphi^{-1} \left(\frac{1}{1-|z|^2} \right) \right) \\
& \leq \sup_{z \in \mathbb{D}} (1 - |z|^2) \varphi \left(\frac{\sum_{j=0}^n C_j M_j}{C} \varphi^{-1} \left(\frac{1}{1-|z|^2} \right) \right) \leq 1.
\end{aligned}$$

Here $C_j (j = 0, 1, \dots, n)$ are all constants, and $C \geq \sum_{j=0}^n C_j M_j$. Now, we can conclude that there exists a constant C such that $\|uC_\phi f\|_{\mathcal{W}_\varphi^{(n)}} \leq C \|f\|_{F(p,q,s)}$ and the weighted composition operator $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\varphi^{(n)}$ is bounded.

If $2 + q = p$, or $2 + q > p$, from (4) (5), or (4) (6), we can get $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\varphi^{(n)}$ is bounded similarly.

Conversely, suppose that $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\varphi^{(n)}$ is bounded, that is, for all $f \in F(p, q, s)$, there exists a constant C such that $\|uC_\phi f\|_{\mathcal{W}_\varphi^{(n)}} \leq C$. For $\omega \in \mathbb{D}$, and constants c_0, c_1, \dots, c_n , set

$$f_\omega(z) = \sum_{j=0}^n c_j \frac{(1 - |\omega|^2)^{j+1}}{(1 - \bar{\omega}z)^{\alpha+j}}, \quad (7)$$

where $\alpha = \frac{2+q}{p}$. It is well known that $f_\omega \in F(p, q, s)$, and

$$f_\omega(\omega) = \frac{1}{(1 - |\omega|^2)^{\alpha-1}} \sum_{j=0}^n c_j, \quad (8)$$

$$f_\omega^{(l)}(\omega) = \frac{\bar{\omega}^l}{(1 - |\omega|^2)^{\alpha-1+l}} \sum_{j=0}^n c_j \prod_{r=0}^{l-1} (\alpha + j + r), \quad l = 1, 2, \dots, n. \quad (9)$$

We claim that for each $k \in \{1, 2, \dots, n\}$, there are constants c_0, c_1, \dots, c_n such that $\sum_{j=0}^n c_j \neq 0$ and

$$f_\omega^{(k)}(\omega) = \frac{\bar{\omega}^k}{(1 - |\omega|^2)^{\alpha-1+k}}, \quad f_\omega^{(t)}(\omega) = 0, \quad t \in \{0, 1, 2, \dots, n\} \setminus \{k\}. \quad (10)$$

In fact, by (8) and (9), (10) is equivalent to the following system of liner equations

$$\begin{cases} c_0 + c_1 + \dots + c_n = 0, \\ c_0 \alpha + c_1 (\alpha + 1) + \dots + c_n (\alpha + n) = 0, \\ c_0 \alpha (\alpha + 1) + c_1 (\alpha + 1) (\alpha + 2) + \dots + c_n (\alpha + n) (\alpha + n + 1) = 0, \\ \dots \dots \\ c_0 \prod_{r=0}^{k-1} (\alpha + r) + c_1 \prod_{r=0}^{k-1} (\alpha + 1 + r) + \dots + c_n \prod_{r=0}^{k-1} (\alpha + n + r) = 1, \\ \dots \dots \\ c_0 \prod_{r=0}^{n-1} (\alpha + r) + c_1 \prod_{r=0}^{n-1} (\alpha + 1 + r) + \dots + c_n \prod_{r=0}^{n-1} (\alpha + n + r) = 0. \end{cases} \quad (11)$$

By using Lemma 3, we obtain that the determinant of system of linear Eq.(11) is different from zero, from which the claim follows. For each $k \in \{1, 2, \dots, n\}$, we choose the corresponding family of functions that satisfy (10) and denote it by $f_{\omega, k}$. Then, from Lemma 4 and the boundedness of $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\varphi^{(n)}$, for $\omega \in \mathbb{D}$ such that $|\phi(\omega)| > \frac{1}{2}$,

$$\begin{aligned} 1 &\geq S_\varphi \left(\frac{(uC_\phi f_{\phi(\omega), k})^{(n)}(z)}{C} \right) \geq \sup_{|\phi(\omega)| > \frac{1}{2}} (1 - |\omega|^2) \varphi \left(\frac{|(uC_\phi f_{\phi(\omega), k})^{(n)}(\omega)|}{C} \right) \\ &= \sup_{|\phi(\omega)| > \frac{1}{2}} (1 - |\omega|^2) \varphi \left(\frac{|\phi(\omega)|^k \left| \sum_{l=k}^n C_n^l u^{(n-l)}(z) B_{l,k}(\phi'(z), \phi''(z), \dots, \phi^{(l-k+1)}(z)) \right|}{C(1 - |\phi(\omega)|^2)^{\frac{2+q-p}{p} + k}} \right) \end{aligned}$$

It follows that

$$\sup_{|\phi(\omega)| > \frac{1}{2}} \frac{\left| \sum_{l=k}^n C_n^l u^{(n-l)}(z) B_{l,k}(\phi'(z), \phi''(z), \dots, \phi^{(l-k+1)}(z)) \right|}{\varphi^{-1}(\frac{1}{1-|z|^2})(1 - |\phi(z)|^2)^{\frac{2+q-p}{p} + k}} < \infty. \quad (12)$$

By the test functions $f_k(z) = z^k$ ($k = 1, 2, \dots, n$), use the mathematical induction as in [12], we can get that

$$\sup_{z \in \mathbb{D}} \frac{\left| \sum_{l=k}^n C_n^l u^{(n-l)}(z) B_{l,k}(\phi'(z), \phi''(z), \dots, \phi^{(l-k+1)}(z)) \right|}{\varphi^{-1}(\frac{1}{1-|z|^2})} < \infty.$$

Then, for each $k \in \{1, 2, \dots, n\}$,

$$\sup_{|\phi(\omega)| \leq \frac{1}{2}} \frac{\left| \sum_{l=k}^n C_n^l u^{(n-l)}(z) B_{l,k}(\phi'(z), \phi''(z), \dots, \phi^{(l-k+1)}(z)) \right|}{\varphi^{-1}(\frac{1}{1-|z|^2})(1 - |\phi(z)|^2)^{\frac{2+q-p}{p} + k}} < \infty. \quad (13)$$

Combining (12) with (13), we obtain that (4) is necessary for all cases.

If $2 + q < p$, taking $f(z) = 1$, then $(uC_\phi f)(z) = u(z)$, by the boundedness of $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\varphi^{(n)}$, we have

$$S_\varphi \left(\frac{(uC_\phi f)^{(n)}(z)}{C} \right) = \sup_{z \in \mathbb{D}} (1 - |z|^2) \varphi \left(\frac{|u^{(n)}(z)|}{C} \right) \leq 1.$$

It follows that (3) holds.

If $2 + q = p$, for a fixed $\omega \in \mathbb{D}$, set

$$g_\omega(z) = \log \frac{2}{1 - \bar{\omega}z}.$$

Then it is easy to see that $g_\omega \in F(p, q, s)$ and we have

$$g_\omega(\omega) = \log \frac{2}{1 - |\omega|^2}, \quad g_\omega^{(k)}(\omega) = (k-1)! \frac{\bar{\omega}^k}{(1 - |\omega|^2)^k}, \quad k = 1, 2, \dots, n.$$

From Lemma 4 and the boundedness of $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\varphi^{(n)}$, we have

$$\begin{aligned} 1 &\geq S_\varphi \left(\frac{(uC_\phi g_{\phi(\omega)})^{(n)}(z)}{C} \right) \geq \sup_{\omega \in \mathbb{D}} (1 - |\omega|^2) \varphi \left(\frac{|(uC_\phi g_{\phi(\omega)})^{(n)}(\omega)|}{C} \right) \\ &\geq \sup_{\omega \in \mathbb{D}} (1 - |\omega|^2) \cdot \\ &\quad \varphi \left(\frac{|u^{(n)}(\omega) \log \frac{2}{1 - |\phi(\omega)|^2}|}{C} - \sum_{k=1}^n \frac{|\phi(\omega)|^k \left| \sum_{l=k}^n C_n^l u^{(n-l)}(z) B_{l,k}(\phi'(z), \phi''(z), \dots, \phi^{(l-k+1)}(z)) \right|}{C(1 - |\phi(\omega)|^2)^k} \right). \end{aligned}$$

By $M_k < \infty$ and the boundedness of $\phi(\omega)$, it follows that

$$\sup_{\omega \in \mathbb{D}} \frac{|u^{(n)}(\omega) \log \frac{2}{1 - |\phi(\omega)|^2}|}{\varphi^{-1}(\frac{1}{1-|z|^2})} \leq C + \sum_{k=1}^n \frac{\left| \sum_{l=k}^n C_n^l u^{(n-l)}(z) B_{l,k}(\phi'(z), \phi''(z), \dots, \phi^{(l-k+1)}(z)) \right|}{\varphi^{-1}(\frac{1}{1-|z|^2})(1 - |\phi(z)|^2)^k} < \infty.$$

If $2 + q > p$, using the function in (7), and in the system of linear Eq.(11), we can also find c_0, c_1, \dots, c_n and denote the corresponding function $h_\omega(z)$ such that

$$h_\omega(z) = \frac{1}{(1 - |\omega|^2)^{\frac{2+q-p}{p}}}, \quad h_\omega^{(k)}(z) = 0, \quad k = 1, 2, \dots, n.$$

Then from Lemma 4 and the boundedness of $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\phi^{(n)}$, we have

$$\begin{aligned} 1 &\geq S_\phi \left(\frac{(uC_\phi h_{\phi(\omega)})^{(n)}(z)}{C} \right) \geq \sup_{\omega \in \mathbb{D}} (1 - |\omega|^2)^\varphi \left(\frac{|(uC_\phi h_{\phi(\omega)})^{(n)}(\omega)|}{C} \right) \\ &= \sup_{\omega \in \mathbb{D}} (1 - |\omega|^2)^\varphi \left(\frac{|u^{(n)}(\omega)|}{C(1 - |\phi(\omega)|^2)^{\frac{2+q-p}{p}}} \right), \end{aligned}$$

from which we can see that (6) holds. \square

4 The Compactness of $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\phi^{(n)}$

Theorem 8. Let $u \in \mathcal{H}(\mathbb{D})$, $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, $n \in \mathbb{N}$ and ϕ be an analytic self-map of \mathbb{D} .

(a) If $2 + q < p$, then $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\phi^{(n)}$ is compact if and only if $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\phi^{(n)}$ is bounded and

$$\lim_{|\phi(z)| \rightarrow 1} \frac{\left| \sum_{l=k}^n C_n^l u^{(n-l)}(z) B_{l,k}(\phi'(z), \phi''(z), \dots, \phi^{(l-k+1)}(z)) \right|}{\varphi^{-1}(\frac{1}{1-|z|^2})(1 - |\phi(z)|^2)^{\frac{2+q-p}{p}+k}} = 0, \quad (14)$$

where $k = 1, 2, \dots, n$.

(b) If $2 + q = p$, then $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\phi^{(n)}$ is compact if and only if $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\phi^{(n)}$ is bounded, (14) holds and

$$\lim_{|\phi(z)| \rightarrow 1} \frac{|u^{(n)}(z)| \log \frac{2}{1-|\phi(z)|^2}}{\varphi^{-1}(\frac{1}{1-|z|^2})} = 0. \quad (15)$$

(c) If $2 + q > p$, then $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\phi^{(n)}$ is compact if and only if $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\phi^{(n)}$ is bounded, (14) holds and

$$\lim_{|\phi(z)| \rightarrow 1} \frac{|u^{(n)}(z)|}{\varphi^{-1}(\frac{1}{1-|z|^2})(1 - |\phi(z)|^2)^{\frac{2+q-p}{p}}} = 0. \quad (16)$$

Proof. Let $\{f_i\}_{i \in \mathbb{N}}$ be a sequence in $F(p, q, s)$ with $\|f_i\|_{F(p, q, s)} \leq L$, and f_i converges to zero uniformly on compact subsets of \mathbb{D} as $i \rightarrow \infty$. To prove that $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\phi^{(n)}$ is compact, by Lemma 5, we only need to show $\lim_{i \rightarrow \infty} \|uC_\phi f_i\|_{\mathcal{W}_\phi^{(n)}} = 0$.

If $2 + q < p$, suppose that $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\phi^{(n)}$ is bounded and (14) holds, then for given $\epsilon > 0$, there exists a $\delta \in (0, 1)$, when $\delta < |\phi(z)| < 1$, we have

$$\frac{\left| \sum_{l=k}^n C_n^l u^{(n-l)}(z) B_{l,k}(\phi'(z), \phi''(z), \dots, \phi^{(l-k+1)}(z)) \right|}{\varphi^{-1}(\frac{1}{1-|z|^2})(1 - |\phi(z)|^2)^{\frac{2+q-p}{p}+k}} < \epsilon, \quad k = 1, 2, \dots, n. \quad (17)$$

By the proof of the boundedness, we know that $M_0 < \infty$ and

$$\sup_{z \in \mathbb{D}} \frac{\left| \sum_{l=k}^n C_n^l u^{(n-l)}(z) B_{l,k}(\phi'(z), \phi''(z), \dots, \phi^{(l-k+1)}(z)) \right|}{\varphi^{-1}(\frac{1}{1-|z|^2})} \leq C, \quad k = 1, 2, \dots, n.$$

Let $K = \{z \in \mathbb{D}, |\phi(z)| \leq \delta\}$, then by Lemma 1 and (17), we have

$$\begin{aligned}
 & \sup_{z \in \mathbb{D}} \frac{|(uC_\phi f_i)^{(n)}(z)|}{\varphi^{-1}(\frac{1}{1-|z|^2})} \\
 & \leq \sup_{z \in \mathbb{D}} \frac{|u^{(n)}(z)|}{\varphi^{-1}(\frac{1}{1-|z|^2})} |f_i(\phi(z))| \\
 & \quad + \sum_{k=1}^n \sup_{z \in K} \frac{|\sum_{l=k}^n C_n^l u^{(n-l)}(z) B_{l,k}(\phi'(z), \phi''(z), \dots, \phi^{(l-k+1)}(z))|}{\varphi^{-1}(\frac{1}{1-|z|^2})} |f_i^{(k)}(\phi(z))| \\
 & \quad + \sum_{k=1}^n \sup_{z \in \mathbb{D} \setminus K} \frac{|\sum_{l=k}^n C_n^l u^{(n-l)}(z) B_{l,k}(\phi'(z), \phi''(z), \dots, \phi^{(l-k+1)}(z))|}{\varphi^{-1}(\frac{1}{1-|z|^2})} |f_i^{(k)}(\phi(z))| \\
 & \leq M_0 |f_i(\phi(z))| + C \sum_{k=1}^n \sup_{z \in K} |f_i^{(k)}(\phi(z))| \\
 & \quad + \sum_{k=1}^n \sup_{z \in \mathbb{D} \setminus K} \frac{|\sum_{l=k}^n C_n^l u^{(n-l)}(z) B_{l,k}(\phi'(z), \phi''(z), \dots, \phi^{(l-k+1)}(z))| \|f_k\|_{F(p,q,s)}}{\varphi^{-1}(\frac{1}{1-|z|^2}) (1-|\phi(z)|^2)^{\frac{2+q-p}{p}+k}} \\
 & \leq M_0 |f_i(\phi(z))| + nC \sup_{|\omega| \leq \delta} |f_i^{(k)}(\omega)| + nL\epsilon.
 \end{aligned}$$

Since $f_k \in F(p, q, s) \subset \mathcal{B}^{\frac{2+q}{p}}$, by Lemma 6, we have $\lim_{i \rightarrow \infty} \sup_{z \in \mathbb{D}} |f_i(\phi(z))| = 0$. By Cauchy's estimate, we know $\sup_{|\omega| \leq \delta} |f_i^{(k)}(\omega)| \rightarrow 0$, as $i \rightarrow \infty$. On the other hand, since $\{\phi(0)\}$ is also compact subset of \mathbb{D} , we have $\sum_{j=0}^{n-1} |f_i^{(j)}(0)| \rightarrow 0$, as $i \rightarrow \infty$. So $\|uC_\phi f_i\|_{\mathcal{W}_\varphi^{(n)}} \rightarrow 0$, as $i \rightarrow \infty$. Hence $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\varphi^{(n)}$ is compact.

If $2 + q = p$ or $2 + q > p$, assume that $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\varphi^{(n)}$ is bounded, (14), (15) or (14), (16) hold respectively. Then given $\epsilon > 0$, there exists a $\delta \in (0, 1)$, when $\delta < |\phi(z)| < 1$, we have

$$\frac{|u^{(n)}(z)| \log \frac{2}{1-|\phi(z)|^2}}{\varphi^{-1}(\frac{1}{1-|z|^2})} < \epsilon \quad \text{or} \quad \frac{|u^{(n)}(z)|}{\varphi^{-1}(\frac{1}{1-|z|^2}) (1-|\phi(z)|^2)^{\frac{2+q-p}{p}}} < \epsilon.$$

Then by Lemma 1, Lemma 2 and Lemma 5 and similar to the above, we can easily get that $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\varphi^{(n)}$ is compact.

Conversely, assume that $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\varphi^{(n)}$ is compact, then it is clear that $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\varphi^{(n)}$ is bounded. Let $\{z_i\}_{i \in \mathbb{N}}$ be a sequence in \mathbb{D} such that $|\phi(z_i)| \rightarrow 1$, as $i \rightarrow \infty$. (If such a sequence does not exist, then the condition in (14), (15), (16) automatically hold.) Let $f_{\omega,k}(z) (k = 1, 2, \dots, n)$ be as defined in the proof of Theorem 7. Then the sequence $\{f_{\phi(z_i),k}\}$ are bounded in $F(p, q, s)$ and converge to zero uniformly on compact subsets of \mathbb{D} as $i \rightarrow \infty$. By Lemma 5 and the compactness of $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\varphi^{(n)}$, we have

$$\lim_{i \rightarrow \infty} \|uC_\phi f_{\phi(z_i),k}\|_{\mathcal{W}_\varphi^{(n)}} = 0. \tag{18}$$

Then

$$\begin{aligned}
 1 & \geq S_\varphi \left(\frac{(uC_\phi f_{\phi(z_i),k})^{(n)}(z_i)}{\|uC_\phi f_{\phi(z_i),k}\|_{\mathcal{W}_\varphi^{(n)}}} \right) \\
 & \geq (1 - |z_i|^2) \varphi \left(\frac{|\phi(z_i)|^k |\sum_{l=k}^n C_n^l u^{(n-l)}(z_i) B_{l,k}(\phi'(z_i), \phi''(z_i), \dots, \phi^{(l-k+1)}(z_i))|}{\|uC_\phi f_{\phi(z_i),k}\|_{\mathcal{W}_\varphi^{(n)}} (1-|\phi(z_i)|^2)^{\frac{2+q-p}{p}+k}} \right)
 \end{aligned}$$

It follows that

$$\frac{|\phi(z_i)|^k |\sum_{l=k}^n C_n^l u^{(n-l)}(z_i) B_{l,k}(\phi'(z_i), \phi''(z_i), \dots, \phi^{(l-k+1)}(z_i))|}{\varphi^{-1}(\frac{1}{1-|z_i|^2}) (1-|\phi(z_i)|^2)^{\frac{2+q-p}{p}+k}} \leq \|uC_\phi f_{\phi(z_i),k}\|_{\mathcal{W}_\varphi^{(n)}}.$$

$$\begin{aligned} & \lim_{|\phi(z_i)| \rightarrow 1} \frac{\left| \sum_{l=k}^n C_n^l u^{(n-l)}(z_i) B_{l,k}(\phi'(z_i), \phi''(z_i), \dots, \phi^{(l-k+1)}(z_i)) \right|}{\varphi^{-1}\left(\frac{1}{1-|z_i|^2}\right)(1-|\phi(z_i)|^2)^{\frac{2+q-p}{p}+k}} \\ &= \lim_{i \rightarrow \infty} \frac{|\phi(z_i)|^k \left| \sum_{l=k}^n C_n^l u^{(n-l)}(z_i) B_{l,k}(\phi'(z_i), \phi''(z_i), \dots, \phi^{(l-k+1)}(z_i)) \right|}{\varphi^{-1}\left(\frac{1}{1-|z_i|^2}\right)(1-|\phi(z_i)|^2)^{\frac{2+q-p}{p}+k}} = 0, \end{aligned}$$

which implies that (14) is necessary for all cases. If $2+q=p$, set

$$g_i(z) = \left(\log \frac{2}{1-\phi(z_i)z} \right)^2 \left(\log \frac{2}{1-|\phi(z_i)|^2} \right)^{-1}.$$

Then $\{g_i(z)\}$ is a bounded sequence in $F(p, q, s)$ and converges to zero uniformly on compact subsets of \mathbb{D} , and we have

$$g_i(\phi(z_i)) = \log \frac{2}{1-|\phi(z_i)|^2}, \quad g_i^{(k)}(\phi(z_i)) = \frac{2(k-1)! \overline{\phi(z_i)}^k}{(1-|\phi(z_i)|^2)^k} + C_k \frac{\overline{\phi(z_i)}^k}{(1-|\phi(z_i)|^2)^k} \left(\log \frac{2}{1-|\phi(z_i)|^2} \right)^{-1},$$

where $C_k (k=1, 2, \dots, n)$ is constants about k . By Lemma 5 and the compactness of $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\varphi^{(n)}$, we have

$$\lim_{i \rightarrow \infty} \|uC_\phi g_i\|_{\mathcal{W}_\varphi^{(n)}} = 0. \quad (19)$$

Then

$$\begin{aligned} 1 &\geq S_\varphi \left(\frac{(uC_\phi g_i)^{(n)}(z_i)}{\|uC_\phi g_i\|_{\mathcal{W}_\varphi^{(n)}}} \right) \\ &\geq (1-|z_i|^2) \cdot \\ &\quad \varphi \left(\frac{|u^{(n)}(z_i)| \log \frac{2}{1-|\phi(z_i)|^2}}{\|uC_\phi g_i\|_{\mathcal{W}_\varphi^{(n)}}} - \sum_{k=1}^n \frac{C|\phi(z_i)|^k \left| \sum_{l=k}^n C_n^l u^{(n-l)}(z_i) B_{l,k}(\phi'(z_i), \phi''(z_i), \dots, \phi^{(l-k+1)}(z_i)) \right|}{\|uC_\phi g_i\|_{\mathcal{W}_\varphi^{(n)}} (1-|\phi(z_i)|^2)^{\frac{2+q-p}{p}+k}} \right). \end{aligned}$$

It follows that

$$\begin{aligned} & \frac{|u^{(n)}(z_i)| \log \frac{2}{1-|\phi(z_i)|^2}}{\varphi^{-1}\left(\frac{1}{1-|z_i|^2}\right)} \\ &\leq \|uC_\phi g_i\|_{\mathcal{W}_\varphi^{(n)}} + \sum_{k=1}^n \frac{C \left| \sum_{l=k}^n C_n^l u^{(n-l)}(z_i) B_{l,k}(\phi'(z_i), \phi''(z_i), \dots, \phi^{(l-k+1)}(z_i)) \right|}{\varphi^{-1}\left(\frac{1}{1-|z_i|^2}\right)(1-|\phi(z_i)|^2)^{\frac{2+q-p}{p}+k}}. \end{aligned}$$

Then by (14) and (19), we can get (15) holds.

If $2+q>p$, let $h_\omega(z)$ be as defined in the proof of Theorem 7. Then the sequence $\{h_{\phi(z_i)}\}$ is bounded in $F(p, q, s)$ and converges to zero uniformly on compact subsets of \mathbb{D} as $i \rightarrow \infty$. By Lemma 5 and the compactness of $uC_\phi : F(p, q, s) \rightarrow \mathcal{W}_\varphi^{(n)}$, we have

$$\lim_{i \rightarrow \infty} \|uC_\phi h_{\phi(z_i)}\|_{\mathcal{W}_\varphi^{(n)}} = 0. \quad (20)$$

Then

$$1 \geq S_\varphi \left(\frac{(uC_\phi h_{\phi(z_i)})^{(n)}(z_i)}{\|uC_\phi h_{\phi(z_i)}\|_{\mathcal{W}_\varphi^{(n)}}} \right) \geq (1-|z_i|^2) \varphi \left(\frac{|u^{(n)}(z_i)|}{\|uC_\phi h_{\phi(z_i)}\|_{\mathcal{W}_\varphi^{(n)}} (1-|\phi(z_i)|^2)^{\frac{2+q-p}{p}}} \right).$$

It follows that

$$\frac{|u^{(n)}(z_i)|}{\varphi^{-1}\left(\frac{1}{1-|z_i|^2}\right)(1-|\phi(z_i)|^2)^{\frac{2+q-p}{p}}} \leq \|uC_\phi h_{\phi(z_i)}\|_{\mathcal{W}_\varphi^{(n)}}$$

from which we can get (16) holds by (20). \square

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MODIFIED q -DAEHEE NUMBERS AND POLYNOMIALS

DONGKYU LIM

ABSTRACT. The p -adic q -integral was defined by T. Kim to be

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} \frac{q^x}{[p^N]_q} f(x) \quad (\text{see [9, 10]}).$$

From p -adic q -integrals' equations, we can derive various q -extension of Bernoulli polynomials and numbers (see [1-20]). In [4], T. Kim have studied Daehee polynomials and numbers and their applications. Recently, many properties and valuable identities related to Daehee polynomials and numbers are introduced by several authors (see [1-20]). In [11], T. Kim *et al.* introduced the q -analogue of Daehee numbers and polynomials which are called q -Daehee numbers and polynomials. In this paper, we consider the modified q -Daehee numbers and polynomials which are different the q -Daehee numbers and polynomials of T. Kim *et al.* and give some useful properties and identities of those polynomials which are derived the new p -adic q -integral equations.

MSC: 11B68, 11S40, 11S80

KEYWORDS AND PHRASES. Modified q -Daehee number; Modified q -Daehee polynomial; Modified q -Bernoulli number; p -adic q -integral

1. Introduction

The q -Daehee polynomials $D_{n,q}(x)$ are defined and studied by T. Kim *et al.*, the generating function to be

$$(1) \quad \frac{1-q + \frac{1-q}{\log q} \log(1+t)}{1-q-qt} (1+t)^x = \sum_{n=0}^{\infty} D_{n,q}(x) \frac{t^n}{n!} \quad (\text{see [11]}).$$

This generating function for $D_{n,q}(x)$ is related with p -adic q -integral on \mathbb{Z}_p defined by T. Kim (see [9, 10]).

In this paper, we consider modified p -adic q -integration on \mathbb{Z}_p which are used by many authors (see [1-20]). We define modified q -Daehee polynomials $D_n(x|q)$ from modified p -adic q -integrals, and relate $D_n(x|q)$ with modified q -Bernoulli polynomials $B_n(x|q)$.

Throughout this paper, we denote the ring of p -adic integers, the field of p -adic numbers and the completion of algebraic closure of \mathbb{Q}_p by \mathbb{Z}_p , \mathbb{Q}_p and \mathbb{C}_p , respectively. The p -adic norm $|\cdot|_p$ is normalized by $|p|_p = \frac{1}{p}$. We denote the space of uniformly differentiable function on \mathbb{Z}_p by $UD[\mathbb{Z}_p]$. The q -Haar measure is defined as (see [9, 10]) $\mu_q(a+p^m\mathbb{Z}_p) = \frac{q^a}{[p^m]_q}$, where $[x]_q = \frac{1-q^x}{1-q}$. For a function f in $UD[\mathbb{Z}_p]$, the modified p -adic q -integral on \mathbb{Z}_p is given by

$$(2) \quad I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} \frac{q^x}{[p^N]_q} f(x) \quad (\text{see [9-20]}).$$

The bosonic integral on \mathbb{Z}_p is given by $I_1(f) = \lim_{q \rightarrow 1} I_q(f)$.

From (2), we have the following integral identity.

$$(3) \quad qI_q(f_1) - I_q(f) = \frac{q-1}{\log q} f'(0) + (q-1)f(0),$$

where $f_1(x) = f(x+1)$ and $f'(x) = \frac{d}{dx}f(x)$.

In special case, we apply $f(x) = q^{-x}e^{tx}$ on (3), we have

$$(e^t - 1) \int_{\mathbb{Z}_p} q^{-x} e^{xt} d\mu_q(x) = \frac{q-1}{\log q} t.$$

Thus

$$(4) \quad \int_{\mathbb{Z}_p} q^{-x} e^{xt} d\mu_q(x) = \frac{q-1}{\log q} \frac{t}{e^t - 1}.$$

The q -analogue Bernoulli numbers $B_n(q)$ are known as follows:

$$(5) \quad \sum_{n=0}^{\infty} B_n(q) \frac{t^n}{n!} = \frac{q-1}{\log q} \frac{t}{e^t - 1} \quad (\text{see [3, 5, 9]}).$$

Indeed if $q \rightarrow 1$, we have $\lim_{q \rightarrow 1} B_n(q) = B_n$. So we call $B_n(x|q)$ as the n th modified q -Bernoulli polynomials and the generating function to be

$$(6) \quad \frac{q-1}{\log q} \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x|q) \frac{t^n}{n!}.$$

When $x = 0$, $B_n(0|q) = B_n(q)$ are the n th modified q -Bernoulli numbers.

From (3) and (6), we have

$$B_n(x|q) = \int_{\mathbb{Z}_p} q^{-y} (x+y)^n d\mu_q(y).$$

From (6), we note that

$$(7) \quad B_n(x|q) = \sum_{l=0}^n \binom{n}{l} B_l(q) x^{n-l}.$$

For the case $|t|_p \leq p^{-\frac{1}{p-1}}$, the Daehee polynomials are defined as follows:

$$(8) \quad \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!} = \frac{\log(1+t)}{t} (1+t)^x \quad (\text{see [11]}).$$

From p -adic q -integrals' equations, we can derive various q -extension of Bernoulli polynomials and numbers(see [1-20]). In [4], T. Kim have studied Daehee polynomials and numbers and their applications. Recently, many properties and valuable identities related to Daehee polynomials and numbers are introduced by several authors(see [1-20]). In [11], T. Kim *et al.* introduced the q -analogue of Daehee numbers and polynomials which are called q -Daehee numbers and polynomials. In this paper, we consider the modified q -Daehee numbers and polynomials which are different the q -Daehee numbers and polynomials of T. Kim *et al.* and give some useful properties and identities of those polynomials which are derived the new p -adic q -integral equations.

2. Modified q -Daehee numbers and polynomials

Let us now consider the p -adic q -integral representation as follows:

$$(9) \quad \int_{\mathbb{Z}_p} q^{-y}(x+y)_n d\mu_q(y) \quad (n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}),$$

where $(x)_n$ is known as the *Pochhammer symbol* (or *decreasing factorial*) defined by

$$(10) \quad (x)_n = x(x-1) \cdots (x-n+1) = \sum_{k=0}^n S_1(n, k) x^k$$

and here $S_1(n, k)$ is the Stirling number of the first kind (see [4, 11]).

From (9), we have

$$(11) \quad \sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} q^{-y}(x+y)_n d\mu_q(y) \right) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} q^{-y} \left(\sum_{n=0}^{\infty} \binom{x+y}{n} t^n \right) d\mu_q(y) \\ = \int_{\mathbb{Z}_p} q^{-y} (1+t)^{x+y} d\mu_q(y),$$

where $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$.

For $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$, from (3), we have

$$(12) \quad \int_{\mathbb{Z}_p} q^{-y} (1+t)^{x+y} d\mu_q(y) = \frac{q-1}{\log q} \frac{\log(1+t)}{t} (1+t)^x.$$

Let

$$(13) \quad F_q(x, t) = \frac{q-1}{\log q} \frac{\log(1+t)}{t} (1+t)^x = \sum_{n=0}^{\infty} D_n(x|q) \frac{t^n}{n!}.$$

In here the polynomial $D_n(x|q)$ is called modified n th q -Daehee polynomials of the first kind. Moreover, we have

$$(14) \quad D_n(x|q) = \int_{\mathbb{Z}_p} q^{-y}(x+y)_n d\mu_q(y).$$

When $x = 0$, $D_n(0|q) = D_n(q)$ is called modified the n -th q -Daehee numbers.

Notice that $F_q(x, t)$ seems to be a new q -extension of the generating function for Daehee polynomials of the first kind. Therefore, from (8) and the following fact,

$$\lim_{q \rightarrow 1} F_q(x, t) = \frac{\log(1+t)}{t} (1+t)^x.$$

On the other hand, we can derive

$$(15) \quad \sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} q^{-x}(x)_n d\mu_q(x) \right) \frac{t^n}{n!} = \frac{q-1}{\log q} \frac{\log(1+t)}{t} = \sum_{n=0}^{\infty} D_n(q) \frac{t^n}{n!}.$$

From (10) and (15), we have

$$(16) \quad \frac{q-1}{\log q} D_n(x) = D_n(x|q).$$

From (10) and (11), we have

$$(17) \quad \begin{aligned} D_n(x|q) &= \int_{\mathbb{Z}_p} q^{-y} (x+y)_n d\mu_q(y) \\ &= \sum_{k=0}^n S_1(n, k) B_k(x|q). \end{aligned}$$

$B_k(x|q)$ are the modified q -Bernoulli polynomials introduced in (6).

Thus we have the following theorem, which relates modified q -Bernoulli polynomials and modified q -Daehee polynomials.

Theorem 1. For $n, m \in \mathbb{Z}_+$, we have the following equalities.

$$D_n(x|q) = \sum_{k=0}^n S_1(n, k) B_k(x|q)$$

and

$$D_n(q) = \sum_{k=0}^n S_1(n, k) B_k(q).$$

From the generating function of modified q -Daehee polynomials in $D_n(x|q)$ in (13), by replacing t to $e^t - 1$, we have

$$(18) \quad \begin{aligned} \sum_{n=0}^{\infty} D_n(x|q) \frac{(e^t - 1)^n}{n!} &= \sum_{n=0}^{\infty} B_n(x|q) \frac{t^n}{n!} \\ &= \sum_{m=0}^{\infty} D_m(x|q) \sum_{n=0}^{\infty} S_2(n, m) \frac{t^n}{n!}. \end{aligned}$$

Thus by comparing the coefficients of t^n , we have

$$B_n(x|q) = \sum_{m=0}^n D_m(x|q) S_2(n, m).$$

In here, $S_2(n, m)$ is the Stirling number of the second kind defined by the following generating series:

$$(19) \quad \sum_{n=m}^{\infty} S_2(n, m) \frac{t^n}{n!} = \frac{(e^t - 1)^m}{m!} \quad cf.[4, 11].$$

Therefore, we obtain the following theorem.

Theorem 2. For $n, m \in \mathbb{Z}_+$, we have the following identity.

$$B_n(x|q) = \sum_{m=0}^n D_m(x|q) S_2(n, m).$$

The increasing factorial sequence is known as

$$x^{(n)} = x(x+1)(x+2) \cdots (x+n-1) \quad (n \in \mathbb{Z}_+).$$

Let us define the modified q -Daehee numbers of the second kind as follows:

$$(20) \quad \hat{D}_n(q) = \int_{\mathbb{Z}_p} q^{-y} (-y)_n d\mu_q(y) \quad (n \in \mathbb{Z}_+).$$

It is easy to observe that

$$(21) \quad x^{(n)} = (-1)^n (-x)_n = \sum_{k=0}^n S_1(n, k) (-1)^{n-k} x^k.$$

From (20) and (21), we have

$$(22) \quad \begin{aligned} \widehat{D}_n(q) &= \int_{\mathbb{Z}_p} q^{-y} (-y)_n d\mu_q(y) \\ &= \int_{\mathbb{Z}_p} q^{-y} y^{(n)} (-1)^n d\mu_q(y) \\ &= \sum_{k=0}^n S_1(n, k) (-1)^k B_k(q). \end{aligned}$$

Thus, we state the following theorem.

Theorem 3. *The following holds true:*

$$\widehat{D}_n(q) = \sum_{k=0}^n S_1(n, k) (-1)^k B_k(q).$$

Let us now consider the generating function of the modified q -Daehee numbers of the second kind as follows:

$$(23) \quad \begin{aligned} \sum_{n=0}^{\infty} \widehat{D}_n(q) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} q^{-y} (-y)_n d\mu_q(y) \right) \frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} q^{-y} \left(\sum_{n=0}^{\infty} \binom{-y}{n} t^n \right) d\mu_q(y) \\ &= \int_{\mathbb{Z}_p} q^{-y} (1+t)^{-y} d\mu_q(y). \end{aligned}$$

From (23), we denote the generating function for the modified q -Daehee numbers of the second as follows:

$$(24) \quad \widehat{F}_q(t) = \frac{q-1}{\log q} \frac{\log(1+t)}{t} (1+t).$$

Let us consider the modified q -Daehee polynomials of the second kind as follows:

$$(25) \quad \frac{q-1}{\log q} \frac{\log(1+t)}{t} (1+t)^{x+1} = \sum_{n=0}^{\infty} \widehat{D}_n(x|q) \frac{t^n}{n!}.$$

It follows from (25) that

$$(26) \quad \int_{\mathbb{Z}_p} q^{-y} (1+t)^{x-y} d\mu_q(y) = \sum_{n=0}^{\infty} \widehat{D}_n(x|q) \frac{t^n}{n!}.$$

From (26) gives

$$(27) \quad \begin{aligned} \widehat{D}_n(x|q) &= \int_{\mathbb{Z}_p} q^{-y} (x-y)_n d\mu_q(y) \\ &= q^{-1} \sum_{k=0}^n |S_1(n, k)| B_k(x+1|q^{-1}), \end{aligned}$$

where $n \geq 0$ and $|S_1(n, k)|$ is the unsigned stirling numbers of the first kind.

Then, by (27), we have the following theorem.

Theorem 4. *For $n \geq 0$, the following are true.*

$$\widehat{D}_n(x|q) = q^{-1} \sum_{k=0}^n |S_1(n, k)| B_k(x+1|q^{-1}).$$

From the modified q -Bernoulli polynomials in (6),

$$(28) \quad \begin{aligned} q \sum_{n=0}^{\infty} B_n(x|q^{-1}) \frac{t^n}{n!} &= \frac{q-1}{\log q} \frac{t}{e^t - 1} e^{(1-x)t} \\ &= \sum_{n=0}^{\infty} B_n(1-x|q) \frac{t^n}{n!}. \end{aligned}$$

Thus, we have

$$(29) \quad q(-1)^n B_n(x|q^{-1}) = B_n(1-x|q).$$

From (29), the value at $x = 1$, we have

$$q(-1)^n B_n(1|q^{-1}) = B_n(q).$$

On the other hand, we can check easily the following

$$(30) \quad (x+y)_n = (-1)^n (-x-y+n-1)_n$$

and

$$(31) \quad \frac{(x+y)_n}{n!} = (-1)^n \binom{-x+y+n-1}{n}.$$

From (13), (27), (30) and (31), we have

$$(32) \quad \begin{aligned} (-1)^n \frac{D_n(x|q)}{n!} &= \int_{\mathbb{Z}_p} q^{-y} \binom{-x-y+n-1}{n} d\mu_q(y) \\ &= \sum_{m=0}^n \binom{n-1}{n-m} \int_{\mathbb{Z}_p} q^{-y} \binom{-x-y}{m} d\mu_q(y) \\ &= \sum_{m=1}^n \binom{n-1}{m-1} \frac{\widehat{D}_m(-x|q)}{m!} \end{aligned}$$

and

$$(33) \quad \begin{aligned} (-1)^n \frac{\widehat{D}_n(x|q)}{n!} &= (-1)^n \int_{\mathbb{Z}_p} q^{-y} \binom{-x+y}{n} d\mu_q(y) \\ &= \int_{\mathbb{Z}_p} q^{-y} \binom{-x+y+n-1}{n} d\mu_q(y) \\ &= \sum_{m=0}^n \binom{n-1}{n-m} \int_{\mathbb{Z}_p} q^{-y} \binom{-x+y}{m} d\mu_q(y) \\ &= \sum_{m=1}^n \binom{n-1}{m-1} \frac{D_m(-x|q)}{m!}. \end{aligned}$$

Therefore, we get the following theorem, which relates modified q -Daehee polynomials of the first and the second kind.

Theorem 5. For $n \in \mathbb{N}$, the following equality hold true.

$$(-1)^n \frac{D_n(x|q)}{n!} = \sum_{m=1}^n \binom{n-1}{m-1} \frac{\widehat{D}_m(-x|q)}{m!}$$

and

$$(-1)^n \frac{\widehat{D}_n(x|q)}{n!} = \sum_{m=1}^n \binom{n-1}{m-1} \frac{D_m(-x|q)}{m!}.$$

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A class of BVPS for second-order impulsive integro-differential equations of mixed type in Banach space*

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This paper is concerned with a class of boundary value problems for the non-linear mixed impulsive integro-differential equations with the derivative u' and deviating arguments in Banach space by using the cone theory and upper and lower solutions method together with monotone iterative technique. Sufficient conditions are established for the existence of extremal solutions of the given problem.

Keywords Integro-differential equations; cone; upper and lower solutions; monotone iterative technique; Impulsive

Mathematics Subject Classifications (2000) 34B15, 34B37.

1 Introduction

Impulsive differential equations have become more important in recent years in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics etc. and there have appeared many papers (see [1-28]) and the references therein). There has been a significant development in impulse theory. Especially, there is an increasing interest in the study of nonlinear mixed integro-differential

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equations with deviating arguments and multipoint BVPs [7-14] for impulsive differential equations.

In this article, we are concerned with the following BVPs for the nonlinear mixed impulsive integro-differential equations with the derivative u' and deviating arguments in Banach space E :

$$\begin{cases} u''(t) = f(t, u(t), u(\alpha(t)), u'(t), Tu, Su) & t \neq t_k, \quad t \in J = [0, 1] \\ \Delta u(t_k) = Q_k u'(t_k) & k = 1, 2, \dots, m \\ \Delta u'(t_k) = I_k(u'(t_k), u(t_k)) & k = 1, 2, \dots, m \\ u(0) = \lambda_1 u(1) + k_1 & u'(0) = \lambda_2 u'(1) + \lambda_3 \int_0^1 w(s, u(s)) ds + \mu u'(\eta) + k_2 \end{cases} \quad (1.1)$$

where $0 = t_0 < t_1 < t_2 < \dots < t_k < \dots < t_m < t_{m+1} = 1$, $f \in C(J \times E^5, E)$, $I_k \in C(E \times E, E)$, $Q_k \geq 0$, $(Tu)(t) = \int_0^{\beta(t)} k(t, s) u(\gamma(s)) ds$, $(Su)(t) = \int_0^1 h(t, s) u(\delta(s)) ds$, $D = \{(t, s) \in J^2 \mid 0 \leq s \leq \beta(t)\}$, $k \in C(D, R^+)$, and $h \in C(J^2, R^+)$, $w \in (J \times E, E)$, $\alpha, \beta, \gamma, \delta \in C(J, J)$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-)$, $0 \leq \eta \leq 1$, $0 \leq \mu$, $0 < \lambda_1, \lambda_2 < 1$, $0 \leq \lambda_3$, $k_1, k_2 \in E$.

The article is organized as follows. In section 2, we establish comparison principles and lemmas. In Section 3, we prove the existence of the result of minimal and maximal solutions for the first order impulsive differential equations, which nonlinearly involve the operator A by using upper and lower solutions, i.e. Theorem 3.1. In Section 4, we obtain the main results (Theorem 4.1) by applying Theorem 3.1, that is the existence of the theorem of minimal and maximal solutions of (1.1).

2 Preliminaries and lemmas

Let $PC(J, E) = \{x : J \rightarrow E; x(t) \text{ is continuous everywhere except for some } t_k \text{ at which } x(t_k^+) \text{ and } x(t_k^-) \text{ exist and } x(t_k) = x(t_k^-), k = 1, 2, \dots, m\}$; $PC^1(J, E) = \{x \in PC(J, E) : x'(t) \text{ is continuous everywhere except for some } t_k \text{ at which } x'(t_k^+) \text{ and } x'(t_k^-) \text{ exist and } x'(t_k) = x'(t_k^-), k = 1, 2, \dots, m\}$. Evidently, $PC(J, E)$ and $PC^1(J, E)$ are Banach spaces with the norms $\|x\|_{PC} = \sup\{|x(t)| : t \in J\}$ and $\|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\}$. Let $J^- = J \setminus \{t_k, k = 1, 2, \dots, m\}$, $\Omega = PC^1(J, E) \cap C^2(J^-, E)$.

If P is a normal cone in E , then $P_c = \{x \in PC(J, E) \mid x(t) \geq \theta, \forall t \in J\}$ is a normal cone in $PC(J, E)$, $P^* = \{f \in E^* \mid f(x) \geq 0, \forall x \in P\}$ denotes the dual cone of P .

A function $x \in \Omega$ is called a solution of BVPs (1.1) if it satisfies Eq.(1.1). In this paper, we always assume that E is a real Banach space and P is a regular cone in E , and denote $K_0 = \max\{k(t, s), (t, s) \in D\}$ and $H_0 = \max\{h(t, s), (t, s) \in J^2\}$.

We consider the following first order impulsive differential equation in Ba-

nach space E:

$$\begin{cases} x'(t) = f(t, Ax(t), Ax(\alpha(t)), x(t), TAx, S Ax) & t \neq t_k, \quad t \in J = [0, 1] \\ \Delta x(t_k) = I_k(Ax(t_k), x(t_k)) & k = 1, 2, \dots, m \\ x(0) = \lambda_2 x(1) + \lambda_3 \int_0^1 w(s, Ax(s)) ds + \mu x(\eta) + k_2 \end{cases} \quad (2.1)$$

where $f, I_k, T, S, w, Q_k, t_k, \lambda_2, \lambda_3, \mu, k_2$ are defined as (1.1) and

$$Ax(t) = \frac{k_1}{1 - \lambda_1} + \int_0^1 G(t, s)x(s)ds + \sum_{k=1}^m G(t, t_k)Q_k x(t_k)$$

with

$$G(t, s) = \begin{cases} \frac{1}{1 - \lambda_1}, & 0 \leq s \leq t \leq 1 \\ \frac{\lambda_1}{1 - \lambda_1}, & 0 \leq t \leq s \leq 1 \end{cases}$$

Lemma 2.1 Suppose $x \in PC(J, E) \cap C^1(J^-, E)$ satisfies

$$\begin{cases} x'(t) + Mx(t) + M_1 Bx + M_2 Bx(\alpha(t)) + M_3 T Bx + M_4 S Bx \leq 0 & t \neq t_k, \quad t \in J = [0, 1] \\ \Delta x(t_k) \leq -L_k Bx(t_k) & k = 1, 2, \dots, m \\ x(0) \leq \lambda_2 x(1) \end{cases} \quad (2.2)$$

where

$$Bx(t) = \int_0^1 G(t, s)x(s)ds + \sum_{k=1}^m G(t, t_k)Q_k x(t_k)$$

$0 < \lambda_1, \lambda_2 < 1$, $L_k \geq 0$ and constants $M, M_i (i = 1, 2, 3, 4)$ satisfy

$$M > 0, \quad M_i \geq 0, \quad M + (M_1 + M_2 + M_3 K_0 + M_4 H_0 + \sum_{k=1}^m L_k) \left(\frac{1}{1 - \lambda_1} + \sum_{k=1}^m \frac{Q_k}{1 - \lambda_1} \right) \leq \lambda_2. \quad (2.3)$$

then $x(t) \leq \theta$ for $t \in J$. (θ denotes the zero element of E)

Proof. For any given $g \in P^*$, let $y(t) = g(x(t))$, then $y \in PC(J, R) \cap C^1(J^-, R)$ and $y'(t) = g(x'(t))$.

In view of (2.2), we get

$$\begin{cases} y'(t) + My(t) + M_1 By + M_2 By(\alpha(t)) + M_3 T By + M_4 S By \leq 0 & t \neq t_k, \quad t \in J = [0, 1] \\ \Delta y(t_k) \leq -L_k By(t_k) & k = 1, 2, \dots, m \\ y(0) \leq \lambda_2 y(1) \end{cases} \quad (2.4)$$

We will show that $y(t) \leq 0, t \in J$.

(i) Suppose to contrary that $y(t) \geq 0, y(t) \not\equiv 0$ for $t \in J$,

In view of the first inequality of (2.4), we get $y'(t) \leq 0$. And by the second one in (2.4), we obtain that $y(t)$ is decreasing in J . Then $0 \leq y(1) \leq y(t) \leq y(0)$. By the third inequality of (2.4), we have $y(1) > 0$ and $\lambda_2 \geq 1$, which is contradiction.

(ii) Suppose there are $\bar{t}, \underline{t} \in J$ such that $y(\bar{t}) > 0$ and $y(\underline{t}) < 0$.
 Let $y(t_*) = \min_{t \in J} y(t) = -\lambda$, then $\lambda > 0$. By (2.4), we get

$$\begin{aligned} y'(t) &\leq \{M + (M_1 + M_2)(\int_0^1 G(t, s)ds + \sum_{k=1}^m G(t, t_k)Q_k) \\ &\quad + M_3(\int_0^{\beta(t)} K(t, s)[\int_0^1 G(s, r)dr + \sum_{k=1}^m G(s, t_k)Q_k]ds) \\ &\quad + M_4(\int_0^1 H(t, s)[\int_0^1 G(s, r)dr + \sum_{k=1}^m G(s, t_k)Q_k]ds)\}\lambda \\ &\leq \{M + (M_1 + M_2)(\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1}) \\ &\quad + M_3K_0(\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1}) + M_4H_0(\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1})\}\lambda \\ &\leq [M + (M_1 + M_2 + M_3K_0 + M_4H_0)(\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1})]\lambda \quad t \neq t_k \end{aligned}$$

$$\Delta y(t_k) \leq -L_k B y(t_k) \leq \lambda L_k (\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1}) \quad k = 1, 2, \dots, m$$

Case 1 If $t_* \in [0, \bar{t})$, integrating from t_* to \bar{t} , we get

$$\begin{aligned} 0 < y(\bar{t}) &= y(t_*) + \int_{t_*}^{\bar{t}} y'(s)ds + \sum_{t_* \leq t_k < \bar{t}} \Delta y(t_k) \\ &\leq -\lambda + [M + (M_1 + M_2 + M_3K_0 + M_4H_0) \\ &\quad (\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1})]\lambda - \sum_{t_* \leq t_k < \bar{t}} L_k B y(t_k) \\ &\leq -\lambda + [M + (M_1 + M_2 + M_3K_0 + M_4H_0) \\ &\quad (\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1})]\lambda + \lambda \sum_{k=1}^m L_k (\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1}) \end{aligned}$$

Hence

$$1 < [M + (M_1 + M_2 + M_3K_0 + M_4H_0)(\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1})] + \sum_{k=1}^m L_k (\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1})$$

It is contradiction to (2.3).

Case 2 If $t_* \in [\bar{t}, 1]$, we have

$$\begin{aligned} 0 < y(\bar{t}) &= y(0) + \int_0^{\bar{t}} y'(s)ds + \sum_{0 < t_k < \bar{t}} \Delta y(t_k) \\ &\leq y(0) + \int_0^{\bar{t}} [M + (M_1 + M_2 + M_3K_0 + M_4H_0) \\ &\quad (\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1})]\lambda ds + \lambda \sum_{0 < t_k < \bar{t}} L_k (\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1}) \end{aligned}$$

$$\begin{aligned}
y(1) &= u(t^*) + \int_{t_*}^1 y'(s)ds + \sum_{t_* \leq t_k < 1} \Delta y(t_k) \\
&\leq -\lambda + \int_{t_*}^1 [M + (M_1 + M_2 + M_3 K_0 + M_4 H_0) \\
&\quad (\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1})] \lambda ds + \lambda \sum_{t_* \leq t_k < 1} L_k (\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1})
\end{aligned}$$

Hence

$$\begin{aligned}
&-\lambda + \frac{1}{\lambda_2} \int_{t_*}^1 [M + (M_1 + M_2 + M_3 K_0 + M_4 H_0) \\
&\quad (\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1})] \lambda ds + \frac{1}{\lambda_2} \lambda \sum_{t_* \leq t_k < 1} L_k (\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1}) \\
> &-\lambda + \int_{t_*}^1 [M + (M_1 + M_2 + M_3 K_0 + M_4 H_0) \\
&\quad (\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1})] \lambda ds + \lambda \sum_{t_* \leq t_k < 1} L_k (\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1}) \\
\geq &y(1) \geq \frac{1}{\lambda_2} y(0) \\
> &-\frac{1}{\lambda_2} \int_0^{\bar{t}} y'(s)ds - \frac{1}{\lambda_2} \sum_{0 < t_k < \bar{t}} \Delta y(t_k) \\
\geq &-\frac{1}{\lambda_2} \int_0^{\bar{t}} [M + (M_1 + M_2 + M_3 K_0 + M_4 H_0) \\
&\quad (\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1})] \lambda ds - \frac{1}{\lambda_2} \lambda \sum_{0 < t_k < \bar{t}} L_k (\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1}) \\
\geq &-\frac{1}{\lambda_2} \lambda \int_0^{t_*} ([M + (M_1 + M_2 + M_3 K_0 + M_4 H_0) \\
&\quad (\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1})] ds - \frac{1}{\lambda_2} \lambda \sum_{0 < t_k < t_*} L_k (\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1})).
\end{aligned}$$

We obtain that $M + (M_1 + M_2 + M_3 K_0 + M_4 H_0) + \sum_{k=1}^m L_k (\frac{1}{1-\lambda_1} + \sum_{k=1}^m \frac{Q_k}{1-\lambda_1}) >$

λ_2 which is contradiction.

Since $g \in P^*$ is arbitrary, we have $x(t) \leq \theta$, $\forall t \in J$.

We complete the proof.

Lemma 2.2 Assume that (2.3) is satisfied. Let $e_k, a \in E, \sigma \in PC(J, E)$.

Then the linear problem

$$\begin{cases} x'(t) = -Mx(t) - M_1 Ax - M_2 Ax(\alpha(t)) - M_3 TAx - M_4 S Ax + \sigma(t) & t \neq t_k, \quad t \in J = [0, 1] \\ \Delta x(t_k) = -L_k Ax(t_k) + e_k & k = 1, 2, \dots, m \\ x(0) = \lambda_2 x(1) + a \end{cases} \quad (2.5)$$

has a unique solution $x \in PC^1(J, E)$ if and only if $x \in PC(J, E)$ is a solution of the integral equation:

$$\begin{aligned} x(t) = & aDe^{-Mt} + \int_0^1 H(t, s)(\sigma(s) - M_1Ax(s) - M_2Ax(\alpha(s)) \\ & - M_3TAx(s) - M_4SAx(s))ds + \sum_{k=1}^m H(t, t_k)(-L_kAx(t_k) + e_k), \end{aligned} \quad (2.6)$$

where $D = (1 - \lambda_2 e^{-M})^{-1}$,

$$H(t, s) = \begin{cases} De^{-M(t-s)}, & 0 \leq s \leq t \leq 1, \\ D\lambda_2 e^{-M(1+t-s)}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (2.7)$$

Proof. First, differentiating (2.6), we have

$$\begin{aligned} x'(t) = & (aDe^{-Mt} + \int_0^1 H(t, s)(\sigma(s) - M_1Ax(s) - M_2Ax(\alpha(s)) \\ & - M_3TAx(s) - M_4SAx(s))ds + \sum_{k=1}^m H(t, t_k)(-L_kAx(t_k) + e_k))' \\ = & -M(t)[aDe^{-Mt} + \int_0^1 H(t, s)(\sigma(s) - M_1Ax(s) \\ & - M_2Ax(\alpha(s)) - M_3TAx(s) - M_4SAx(s))ds \\ & + \sum_{k=1}^m H(t, t_k)(-L_kAx(t_k) + e_k)] - M_1Ax \\ & - M_2Ax(\alpha(t)) - M_3TAx - M_4SAx + \sigma(t) \\ = & -M(t)x(t) - M_1Ax(t) - M_2Ax(\alpha(t)) - M_3TAx(t) - M_4SAx(t) + \sigma(t) \\ \Delta x(t_k) = & x(t_k^+) - x(t_k^-) \\ = & \sum_{0 < t_j < t_k} \Delta x(t_j) - \sum_{0 < t_j < t_k^-} \Delta x(t_j) \\ = & \sum_{j=1}^k (-L_jAx(t_j) + e_j) - \sum_{j=1}^{k-1} (-L_jAx(t_j) + e_j) \\ = & -L_kAx(t_k) + e_k. \end{aligned}$$

Also

$$\begin{aligned} x(0) = & \lambda_2 D \int_0^1 e^{-M(1-s)}(\sigma(s) - M_1Ax(s) - M_2Ax(\alpha(s)) \\ & - M_3TAx(s) - M_4SAx(s))ds + \lambda_2 D \sum_{k=1}^m e^{-M(1-t_k)} \Delta x(t_k) + aD \\ x(1) = & D \int_0^1 e^{-M(1-s)}(\sigma(s) - M_1Ax(s) - M_2Ax(\alpha(s)) \\ & - M_3TAx(s) - M_4SAx(s))ds + e^{-M} aD \sum_{k=1}^m e^{-M(1-t_k)} \Delta x(t_k) + aD. \end{aligned}$$

It is easy to check that $x(0) = \lambda_2 x(1) + a$.

Hence, we know that (2.6) is a solution of (2.5).

Next we show that the solution of (2.5) is unique. Let x_1, x_2 are the solutions of (2.5) and set $p = x_1 - x_2$, we get

$$\begin{aligned} p' = & x_1' - x_2' \\ = & -Mx_1(t) - M_1Ax_1 - M_2Ax_1(\alpha(t)) - M_3TAx_1 - M_4SAx_1 + \sigma(t) \\ & - (-Mx_2(t) - M_1Ax_2 - M_2Ax_2(\alpha(t)) - M_3TAx_2 - M_4SAx_2 + \sigma(t)) \\ = & -Mp - M_1Ap - M_2Ap(\alpha(t)) - M_3TAp - M_4SAp, \end{aligned}$$

$$\begin{aligned}
\Delta p(t_k) &= \Delta x_1 - \Delta x_2 \\
&= -L_k A x_1(t_k) + e_k - (-L_k A x_2(t_k) + e_k) \\
&= -L_k A p(t_k), \\
p(0) &= x_1(0) - x_2(0) \\
&= \lambda_2 x_1(T) + a - (\lambda_2 x_2(1) + a) \\
&= \lambda_2 p(1).
\end{aligned}$$

In view of Lemma 2.1, we get $p \leq \theta$ which implies $x_1 \leq x_2$. Similarly, we have $x_1 \geq x_2$. Hence $x_1 = x_2$. The proof is complete.

3 Results for first order impulsive differential equation

For convenience, let us list the following conditions:

(H₁) There exist $x_0, y_0 \in PC^1(J, E)$ satisfying

$$\begin{cases}
x'_0(t) \leq f(t, Ax_0(t), Ax_0(\alpha(t)), x_0(t), TA x_0, SA x_0) & t \neq t_k, \quad t \in J = [0, 1] \\
\Delta x_0(t_k) \leq I_k(Ax_0(t_k), x_0(t_k)) & k = 1, 2, \dots, m \\
x_0(0) \leq \lambda_2 x_0(1) + \lambda_3 \int_0^1 w(s, Ax_0(s)) ds + \mu x_0(\eta) + k_2 \\
y'_0(t) \geq f(t, Ay_0(t), Ay_0(\alpha(t)), y_0(t), TA y_0, SA y_0) & t \neq t_k, \quad t \in J = [0, 1] \\
\Delta y_0(t_k) \geq I_k(Ay_0(t_k), y_0(t_k)) & k = 1, 2, \dots, m \\
y_0(0) \geq \lambda_2 y_0(1) + \lambda_3 \int_0^1 w(s, Ay_0(s)) ds + \mu y_0(\eta) + k_2
\end{cases} \quad (3.1)$$

(H₂)

$$\begin{aligned}
&f(t, \bar{x}, \bar{y}, \bar{z}, \bar{u}, \bar{v}) - f(t, x, y, z, u, v) \\
&\geq -M_1(\bar{x} - x) - M_2(\bar{y} - y) - M_3(\bar{z} - z) - M_4(\bar{u} - u) - M_4(\bar{v} - v)
\end{aligned} \quad (3.2)$$

$$I_k(\bar{x}, \bar{z}) - I_k(x, z) \geq -L_k(\bar{x} - x) \quad (3.3)$$

Where $Ax_0 \leq x \leq \bar{x} \leq Ay_0$, $Ax_0(\alpha(t)) \leq y \leq \bar{y} \leq Ay_0(\alpha(t))$, $x_0 \leq z \leq \bar{z} \leq y_0$, $TAx_0 \leq u \leq \bar{u} \leq TAy_0$, $SAx_0 \leq v \leq \bar{v} \leq SAy_0$, $\forall t \in J$.

(H₃) Constants $L_k, M, M_i, i = 1, 2, 3, 4$ satisfy (2.3).

(H₄) Assume that $a(t)$ is non-negative integral function, such that

$$w(t, A\bar{u}) - w(t, Au) \geq a(t)(A\bar{u} - Au) \quad (3.4)$$

Where $x_0 \leq u \leq \bar{u} \leq y_0$.

If $x_0, y_0 \in PC^1(J, E)$ and $x_0 \leq y_0, t \in J$, then the interval $[x_0, y_0]$ denotes the set

$$\{x \in PC^1(J, E) : x_0(t) \leq x(t) \leq y_0(t), t \in J\}$$

Theorem 3.1 Assume the hypotheses (H₁) – (H₄) hold. Then Eq.(2.1) has the extremal solutions $x^*(t), y^*(t) \in [x_0, y_0]$. Moreover there exist two iterative sequences $\{x_n\}$ and $\{y_n\}$ satisfying

$$x_0 \leq x_1 \leq \dots \leq x_n \leq \dots \leq y_n \leq \dots \leq y_1 \leq y_0 \quad (3.5)$$

such that $\{x_n\}, \{y_n\}$ uniformly converge in $PC(J, E) \cap C^1(J^-, E)$ to x^*, y^* , respectively.

Proof. For $z \in [x_0, y_0]$, considering (2.5) with

$$\sigma(t) = f(t, Az(t), Az(\alpha(t)), z, TAz, SAz) + M(t)z(t) + M_1Az(t) + M_2Az(\alpha(t)) + M_3TAz + M_4SAz,$$

$$e_k = I_k(Az(t_k), z(t_k)) + L_kAz(t_k),$$

$$a = \lambda_3 \int_0^T w(s, Az(s)) + \mu z(\eta) ds + k_2.$$

By Lemma 2.2, the BVPS has a unique solution $z \in [x_0, y_0]$.

We define an operator φ by $x = \varphi z$, then φ is an operator from $[x_0, y_0]$ to $PC(J, E)$.

We claim that

$$(a) \ x_0 \leq \varphi x_0, \quad \varphi y_0 \leq y_0,$$

$$(b) \ \varphi \text{ is nondecreasing on } [x_0, y_0].$$

We prove (a), let $x_1 = \varphi x_0$, $p(t) = x_0(t) - x_1(t)$

$$\begin{aligned} p' &= x_0' - x_1' \\ &\leq f(t, Ax_0(t), Ax_0(\alpha(t)), x_0(t), TAx_0, SAx_0) - [f(t, Ax_0(t), Ax_0(\alpha(t)), x_0(t), TAx_0, SAx_0) \\ &\quad + M(x_0(t) - x_1(t)) + M_1(Ax_0(t) - Ax_1(t)) + M_2(Ax_0(\alpha(t)) - Ax_1(\alpha(t))) \\ &\quad + M_3(TAx_0 - TAx_1) + M_4(SAx_0 - SAx_1)] \\ &= -Mp(t) - M_1Ap - M_2Ap(\alpha(t)) - M_3TAp - M_4SAp, \end{aligned}$$

$$\begin{aligned} \Delta p(t_k) &= \Delta x_0(t_k) - \Delta x_1(t_k) \\ &\leq I_k(Ax_0(t_k), x_0(t_k)) - [I_k(Ax_0(t_k), x_0(t_k)) - L_k(Ax_1 - Ax_0)] \\ &= -L_kAp(t_k), \end{aligned}$$

$$\begin{aligned} p(0) &= x_0(0) - x_1(0) \\ &\leq \lambda_2 x_0(1) + \mu u_0(\eta) + \lambda_3 \int_0^1 w(s, Ax_0(s)) ds + k_2 \\ &\quad - (\lambda_2 u_1(1) + \mu u_0(\eta) + \lambda_3 \int_0^1 w(s, Ax_0(s)) ds + k_2) \\ &= \lambda_1 p(1). \end{aligned}$$

By Lemma 2.1, we have $p \leq \theta$. That is $x_0 \leq \varphi x_0$. Similarly, we can prove $\varphi y_0 \leq y_0$.

To prove (b), let $x_1 = \varphi x_0$, $y_1 = \varphi y_0$, $p = x_1 - y_1$, then

$$\begin{aligned} p'(t) &= x_1' - y_1' \\ &= f(t, Ax_0(t), Ax_0(\alpha(t)), x_0(t), TAx_0, SAx_0) + M(x_0(t) - x_1(t)) \\ &\quad + M_1(Ax_0(t) - Ax_1(t)) + M_2(Ax_0(\alpha(t)) - Ax_1(\alpha(t))) \\ &\quad + M_3(TAx_0 - TAx_1) + M_4(SAx_0 - SAx_1) \\ &\quad - [f(t, Ay_0(t), Ay_0(\alpha(t)), y_0(t), T Ay_0, S Ay_0) \\ &\quad + M(y_0(t) - y_1(t)) + M_1(Ay_0(t) - Ay_1(t)) \\ &\quad + M_2(Ay_0(\alpha(t)) - Ay_1(\alpha(t))) + M_3(T Ay_0 - T Ay_1) + M_4(S Ay_0 - S Ay_1)] \\ &\leq -Mp(t) - M_1Ap - M_2Ap(\alpha(t)) - M_3TAp - M_4SAp, \end{aligned}$$

$$\begin{aligned} \Delta p(t_k) &= \Delta x_1(t_k) - \Delta y_1(t_k) \\ &= -L_kAx_1 + I_k(Ax_0(t_k), x_0(t_k)) + L_kAx_0 \\ &\quad - (-L_kAy_1 + I_k(Ay_0(t_k), y_0(t_k)) + L_kAy_0) \\ &\leq -L_k(Ax_0 - Ay_0) + L_kAx_0 - L_kAy_0 - L_kAp \\ &\leq -L_kAp(t_k), \end{aligned}$$

$$\begin{aligned}
 p(0) &= x_1(0) - y_1(0) \\
 &\leq \lambda_2 x_1(1) + \mu x_0(\eta) + \lambda_3 \int_0^1 w(s, Ax_0(s))ds + k_2 \\
 &\quad - (\lambda_2 y_1(1) + \mu y_0(\eta) + \lambda_3 \int_0^1 w(s, Ay_0(s))ds + k_2) \\
 &= \lambda_2 p(1) + \mu(x_0(\eta) - y_0(\eta)) + \lambda_3 \int_0^1 a(s)(Ax_0(s) - Ay_0(s))ds \\
 &\leq \lambda_1 p(1).
 \end{aligned}$$

In view of Lemma 2.1, we know $\varphi x_0 \leq \varphi y_0$. Hence (b) holds.

We define two sequences $\{x_n\}, \{y_n\}$

$$x_{n+1} = \varphi x_n, \quad y_{n+1} = \varphi y_n, \quad (n = 0, 1, 2, \dots)$$

By (a) and (b), we know that (3.5) holds.

And each x_n, y_n satisfies

$$\begin{cases}
 x'_n(t) = f(t, Ax_{n-1}(t), Ax_{n-1}(\alpha(t)), x_{n-1}(t), TA_{n-1}, SA_{n-1}) \\
 \quad + M(x_{n-1}(t) - x_1(t)) + M_1(Ax_{n-1}(t) - Ax_n(t)) + M_2(Ax_{n-1}(\alpha(t)) - Ax_n(\alpha(t))) \\
 \quad + M_3(TAx_{n-1} - TA_{n-1}) + M_4(SAx_{n-1} - SA_{n-1}) \quad t \neq t_k, \quad t \in J = [0, 1] \\
 \Delta x_n(t_k) = -L_k Ax_n(t_k) + I_k(Ax_{n-1}(t_k), x_{n-1}(t_k)) + L_k Ax_{n-1}(t_k) \quad k = 1, 2, \dots, m \\
 x_n(0) = \lambda_2 x_n(1) + \lambda_3 \int_0^1 w(s, Ax_{n-1}(s))ds + \mu x_{n-1}(\eta) + k_2
 \end{cases} \quad (3.6)$$

$$\begin{cases}
 y'_n(t) = f(t, Ay_{n-1}(t), Ay_{n-1}(\alpha(t)), y_{n-1}(t), TA_{n-1}, SA_{n-1}) \\
 \quad + M(y_{n-1}(t) - y_1(t)) + M_1(Ay_{n-1}(t) - Ay_n(t)) + M_2(Ay_{n-1}(\alpha(t)) - Ay_n(\alpha(t))) \\
 \quad + M_3(TAy_{n-1} - TA_{n-1}) + M_4(SAy_{n-1} - SA_{n-1}) \quad t \neq t_k, \quad t \in J = [0, 1] \\
 \Delta y_n(t_k) = -L_k Ay_n(t_k) + I_k(Ay_{n-1}(t_k), y_{n-1}(t_k)) + L_k Ay_{n-1}(t_k) \quad k = 1, 2, \dots, m \\
 y_n(0) = \lambda_2 y_n(1) + \lambda_3 \int_0^1 w(s, Ay_{n-1}(s))ds + \mu y_{n-1}(\eta) + k_2.
 \end{cases} \quad (3.7)$$

By virtue of the regularity of the cone P , we obtain that there exist $x^*, y^* \in [x_0, y_0]$ such that

$$\lim_{n \rightarrow \infty} x_n(t) = x^*(t) \quad \lim_{n \rightarrow \infty} y_n(t) = y^*(t) \quad (3.8)$$

and $\{x_n | n = 1, 2, \dots\}$ is a bounded subset in $PC(J, E)$.

Let $X = \{x_n | n = 1, 2, \dots\}$, $X(t) = \{x_n(t) | n = 1, 2, \dots\}$ $t \in J$, in view of (3.8) we get

$$\alpha(X(t)) = 0 \quad t \in J$$

which implies that $X(t)$ is relatively compact for $t \in J$.

For any $z \in [x_0, y_0]$, by (H_1) (H_2) we have

$$\begin{aligned}
 &x'_0(t) + Mx_0(t) + M_1Ax_0(t) + M_2Ax_0(\alpha(t)) + M_3TAx_0 + M_4SAx_0 \\
 &\leq f(t, Ax_0(t), Ax_0(\alpha(t)), x_0(t), TA_{x_0}, SA_{x_0}) + Mx_0(t) \\
 &\quad + M_1Ax_0(t) + M_2Ax_0(\alpha(t)) + M_3TAx_0 + M_4SAx_0 \\
 &\leq f(t, Az_0(t), Az_0(\alpha(t)), z_0(t), TA_{z_0}, SA_{z_0}) + Mz_0(t) \\
 &\quad + M_1Az_0(t) + M_2Az_0(\alpha(t)) + M_3TAz_0 + M_4SAz_0 \\
 &\leq f(t, Ay_0(t), Ay_0(\alpha(t)), y_0(t), TA_{y_0}, SA_{y_0}) + My_0(t) \\
 &\quad + M_1Ay_0(t) + M_2Ay_0(\alpha(t)) + M_3TAy_0 + M_4SAy_0 \\
 &\leq y'_0(t) + My_0(t) + M_1Ay_0(t) + M_2Ay_0(\alpha(t)) + M_3TAy_0 + M_4SAy_0.
 \end{aligned}$$

In view of the normality of the cone P_c , we get that there exists a constant $C > 0$, such that

$$\|f(t, Az_0(t), Az_0(\alpha(t)), z_0(t), TAz_0, SAz_0) + Mz_0(t) + M_1Az_0(t) + M_2Az_0(\alpha(t)) + M_3TAz_0 + M_4SAz_0\| \leq C,$$

$\forall z \in [x_0, y_0], t \in J$. From (3.5) (3.6), it is obviously to show that $\{x'_n | n = 1, 2, \dots\}$ is a bounded subset in $PC(J, E)$. It follows in view of the mean value theorem that X is equicontinuous on J_k , $k = 0, 1, 2, \dots, m$. So we obtain by virtue of Ascoli-Arzelà's theorem and $\alpha(X(t)) = 0$ that $\alpha(X) = \sup_{t \in J} \alpha(X(t)) = 0$ which implies X is relatively compact in $PC(J, E)$ and so there exists a sequence of $\{x_n(t)\}$ which converges uniformly on J to $x^*(t)$. Since $\{x_n | n = 1, 2, \dots\}$ is nondecreasing and the cone P_c is normal, we get that $\{x_n | n = 1, 2, \dots\}$ itself converges uniformly on J to $x^*(t)$, which implies $x^* \in PC(J, E)$. By the lemma 2.2 and (3.6), we see that x^* satisfies (2.1).

Similarly, we also can prove that y_n converges uniformly on J to $y^*(t)$, and y^* satisfies (2.1).

Finally, we assert that if $z \in [x_0, y_0]$ is any solution of Eq.(2.1), then $x^*(t) \leq z(t) \leq y^*(t)$ on J . We will prove that if $x_n \leq z \leq y_n$, for $n = 0, 1, 2, \dots$, then $x_{n+1}(t) \leq z(t) \leq y_{n+1}(t)$.

Letting $p(t) = x_{n+1}(t) - z(t)$, then

$$\begin{cases} p'(t) \leq -Mp(t) - M_1Ap - M_2Ap(\alpha(t)) - M_3TAp - M_4SAp \leq 0 & t \neq t_k, \quad t \in J = [0, 1] \\ \Delta p(t_k) \leq -L_kAp(t_k) & k = 1, 2, \dots, m \\ p(0) \leq \lambda_2 p(1) \end{cases}$$

By Lemma 2.1, we have $p(t) \leq \theta$ for all $t \in J$, that is $x_{n+1}(t) \leq z(t)$. Similarly, we can prove $z(t) \leq y_{n+1}(t)$ for all $t \in J$. Thus $x_{n+1}(t) \leq z(t) \leq y_{n+1}(t)$ for all $t \in J$, which implies $x^*(t) \leq z(t) \leq y^*(t)$.

The proof is complete.

Remark In (2.1), if $w(s, Ax(s)) = a(s)Ax(s)$, where $a(t)$ is non-negative integral function, then (H_4) is not required in Theorem 3.1, and we have the following theorem.

Theorem 3.2 Suppose that conditions $(H_1) - (H_3)$ are satisfied. In addition that $x_0, y_0 \in PC^1(J, E)$ be such that $x_0 \leq y_0$. Then the conclusion of Theorem 3.1 holds.

The proof is almost similar to theorem 3.1, so we omit it.

4 Results for second order impulsive differential equation

In this section, we prove the existence theorem of maximal and minimal solutions of (1.1) by applying Theorem 3.1 in Section 3.

Let us list other conditions for convenience.

(G₁) There exists $u_0, v_0 \in \Omega$, satisfying $u_0(t) \leq v_0(t)$, $u'_0(t) \leq v'_0(t)$,

$$\begin{cases} u''_0(t) \leq f(t, u_0(t), u_0(\alpha(t)), u_0(t), Tu_0, Su_0) & t \neq t_k, \quad t \in J = [0, 1] \\ \Delta u_0(t_k) = Q_k u'_0(t_k) \\ \Delta u'_0(t_k) \leq I_k(u_0(t_k), u'_0(t_k)) & k = 1, 2, \dots, m \\ u_0(0) = \lambda_1 u_0(1) + k_1 \\ u'_0(0) \leq \lambda_2 u'_0(1) + \lambda_3 \int_0^1 w(s, u_0(s))ds + \mu u'_0(\eta) + k_2 \end{cases} \quad (4.1)$$

and v_0 satisfies inverse inequalities of (4.1)

(G₂)

$$\begin{aligned} & f(t, \bar{x}, \bar{y}, \bar{z}, \bar{u}, SA\bar{v}) - f(t, x, y, z, u, v) \\ & \geq -M_1(\bar{x} - x) - M_2(\bar{y} - y) - M(\bar{z} - z) - M_3(\bar{u} - u) - M_4(\bar{v} - v) \end{aligned} \quad (4.2)$$

$$I_k(\bar{x}, \bar{z}) - I_k(x, z) \geq -L_k(\bar{x} - x) \quad (4.3)$$

Where $u_0 \leq x \leq \bar{x} \leq v_0$, $u_0(\alpha(t)) \leq y \leq \bar{y} \leq v_0(\alpha(t))$, $u'_0 \leq z \leq \bar{z} \leq v'_0$, $Tu_0 \leq u \leq \bar{u} \leq Tv_0$, $Su_0 \leq v \leq \bar{v} \leq Sv_0$, $\forall t \in J$.

(G₃) Constants $L_k, M, M_i, i = 1, 2, 3, 4$ satisfy (2.3).

(G₄) Assume that $a(t)$ is non-negative integral function, such that

$$w(t, \bar{u}) - w(t, u) \geq a(t)(\bar{u} - u) \quad (4.4)$$

Where $u_0 \leq u \leq \bar{u} \leq v_0$.

Let $\Lambda = \{z \in [x_0, y_0] \cap PC^1(J, E) \mid u'_0(t) \leq z'(t) \leq v'_0(t)\}$.

Theorem 4.1 Assume the conditions (G₁) – (G₄) hold. Then Eq.(1.1) has minimal and maximal solutions $u^*, v^* \in \Omega$ in Λ .

Proof. In Eq.(1.1), let $u'(t) = x(t)$. Then (1.1) is equivalent to the following system:

$$\begin{cases} u'(t) = x(t) \\ x'(t) = f(t, u, u(\alpha), x, Tu(t), Su(t)) \\ \Delta u(t_k) = Q_k x(t_k) \\ \Delta x(t_k) = I_k(x(t_k), u(t_k)) \\ u(0) = \lambda_1 u(1) + k_1 \\ x(0) = \lambda_2 x(1) + \lambda_3 \int_0^1 w(s, u(s))ds + \mu x(\eta) + k_2 \end{cases} \quad (4.5)$$

For $x \in PC(J, E)$, the system

$$\begin{cases} u'(t) = x(t) \\ \Delta u(t_k) = Q_k x(t_k) \\ u(0) = \lambda_1 u(1) + k_1 \end{cases} \quad (4.6)$$

has a unique solution $x \in PC(J, E) \cap C^1(J^-, E)$, which satisfies

$$u(t) = \frac{k_1}{1 - \lambda_1} + \int_0^1 G(t, s)x(s)ds + \sum_{k=1}^m G(t, t_k)Q_k x(t_k) \quad (4.7)$$

It is easy to prove, so we omit it .

Define an operator A by $u = Ax(t)$, $t \in J$. It is easy to show that $A : PC(J, E) \cap C^1(J^-, E) \rightarrow \Omega$ is continuous and nondecreasing .

Hence, from (4.5)-(4.7), Eq.(1.1) is transformed into first order boundary value problem (2.1).

Let $x_0 = u'_0$, $y_0 = v'_0$, by (G_1) we have $x_0 \leq y_0$ and

$$u_0(t) = \frac{k_1}{1 - \lambda_1} + \int_0^1 G(t, s)x_0(s)ds + \sum_{k=1}^m G(t, t_k)Q_k x_0(t_k) \quad (4.8)$$

$$v_0(t) = \frac{k_1}{1 - \lambda_1} + \int_0^1 G(t, s)y_0(s)ds + \sum_{k=1}^m G(t, t_k)Q_k y_0(t_k) \quad (4.9)$$

which imply that $u_0 = Ax_0$, $v_0 = Ay_0$, and x_0, y_0 satisfies (H_1) .

By the condition (G_2) (G_4) it is easy to see that (H_2) (H_4) hold .

Therefore, it follows from Theorem 3.1 that (2.1) has minimal and maximal solutions $x^*, y^* \in PC(J, E) \cap C^1(J^-, E)$ in $[x_0, y_0]$.

Let $u^* = Ax^*$, $v^* = Ay^*$, then $u^*, v^* \in \Omega$ and

$$u^*(t) = \frac{k_1}{1 - \lambda_1} + \int_0^1 G(t, s)x^*(s)ds + \sum_{k=1}^m G(t, t_k)Q_k x^*(t_k) \quad (4.10)$$

In view of (4.10), we have

$$\begin{cases} u^{*'}(t) = x^*(t) \\ \Delta u^*(t_k) = Q_k x^*(t_k) \\ u^*(0) = \lambda_1 u^*(1) + k_1 \end{cases} \quad (4.11)$$

The fact that x^* satisfies (2.1) and u^* satisfies (4.11) implies u^* is a solution of (1.1). Similarly, we can prove v^* is a solution of (1.1).

It is easy to show that $u^*, v^* \in \Omega$ are minimal and maximal solutions for (1.1) in Λ . We complete the proof.

Remark In (1.1), if $w(s, x(s)) = a(s)x(s)$, where $a(t)$ is non-negative integral function, then (H_4) is not required in Theorem 4.1, and we have the following theorem.

Theorem 4.2 Suppose that conditions $(G_1) - (G_3)$ are satisfied. Then the conclusion of Theorem 4.1 holds.

The proof is almost similar to theorem 4.1, so we omit it.

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DISTRIBUTION AND SURVIVAL FUNCTIONS WITH APPLICATIONS IN INTUITIONISTIC RANDOM LIE C^* -ALGEBRAS

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ABSTRACT. In this paper, first, we consider the distribution and survival functions and we define intuitionistic random Lie C^* -algebras. As an application, using the fixed point method, we approximate the derivations on intuitionistic random Lie C^* -algebras for the the following additive functional equation

$$\sum_{i=1}^m f\left(mx_i + \sum_{j=1, j \neq i}^m x_j\right) + f\left(\sum_{i=1}^m x_i\right) = 2f\left(\sum_{i=1}^m mx_i\right)$$

for all $m \in \mathbb{N}$ with $m \geq 2$.

1. Introduction

Distribution and survival functions are important in probability theory. In this paper, we use these functions to define intuitionistic random Lie C^* -algebras and find an approximation of an m -variable functional equation.

2. Preliminaries

Now, we give some definitions and lemmas for our main results in this paper.

Definition 2.1. A function $\mu : \mathbb{R} \rightarrow [0, 1]$ is called a *distribution function* if it is left continuous on \mathbb{R} , non-decreasing and

$$\inf_{t \in \mathbb{R}} \mu(t) = 0, \quad \sup_{t \in \mathbb{R}} \mu(t) = 1.$$

We denote by D the family of all measure distribution functions and by H a special element of D defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Forward, $\mu(x)$ is denoted by μ_x .

Definition 2.2. A function $\nu : \mathbb{R} \rightarrow [0, 1]$ is called a *survival function* if it is right continuous on \mathbb{R} , non-increasing and

$$\inf_{t \in \mathbb{R}} \nu(t) = 0, \quad \sup_{t \in \mathbb{R}} \nu(t) = 1.$$

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We denote by B the family of all survival functions and by G a special element of B defined by

$$G(t) = \begin{cases} 1, & \text{if } t \leq 0, \\ 0, & \text{if } t > 0. \end{cases}$$

Forward, $\nu(x)$ is denoted by ν_x .

Lemma 2.3. ([1]) Consider the set L^* and the operation \leq_{L^*} defined by:

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1, x_2 \geq y_2$$

for all $(x_1, x_2), (y_1, y_2) \in L^*$. Then (L^*, \leq_{L^*}) is a complete lattice.

We denote the bottom and the top elements of lattices by $0_{L^*} = (0, 1)$ and $1_{L^*} = (1, 0)$. Classically, the *triangular norm* $* = T$ on $[0, 1]$ is defined as an increasing, commutative and associative mapping $T : [0, 1]^2 \longrightarrow [0, 1]$ satisfying

$$T(1, x) = 1 * x = x$$

for all $x \in [0, 1]$. The *triangular conorm* $S = \diamond$ is defined as an increasing, commutative, associative mapping $S : [0, 1]^2 \longrightarrow [0, 1]$ satisfying $S(0, x) = 0 \diamond x = x$ for all $x \in [0, 1]$.

Using the lattice (L^*, \leq_{L^*}) , these definitions can be straightforwardly extended.

Definition 2.4. ([1]) A *triangular norm* (*t-norm*) on L^* is a mapping $\mathcal{T} : (L^*)^2 \longrightarrow L^*$ satisfying the following conditions:

- (1) for all $x \in L^*$, $\mathcal{T}(x, 1_{L^*}) = x$ (: boundary condition);
- (2) for all $(x, y) \in (L^*)^2$, $\mathcal{T}(x, y) = \mathcal{T}(y, x)$ (: commutativity);
- (3) for all $(x, y, z) \in (L^*)^3$, $\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z)$ (: associativity);
- (4) for all $(x, x', y, y') \in (L^*)^4$, $x \leq_{L^*} x'$ and $y \leq_{L^*} y' \implies \mathcal{T}(x, y) \leq_{L^*} \mathcal{T}(x', y')$ (: monotonicity).

In this paper, $(L^*, \leq_{L^*}, \mathcal{T})$ has an Abelian topological monoid with the top element 1_{L^*} and so \mathcal{T} is a *continuous t-norm*.

Definition 2.5. A continuous *t-norm* \mathcal{T} on L^* is said to be *continuous representable t-norm* if there exist a continuous *t-norm* $*$ and a continuous *t-conorm* \diamond on $[0, 1]$ such that, for all $x = (x_1, x_2), y = (y_1, y_2) \in L^*$,

$$\mathcal{T}(x, y) = (x_1 * y_1, x_2 \diamond y_2).$$

For example,

$$\mathcal{T}(a, b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$$

and

$$\mathbf{M}(a, b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$$

for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ are the continuous representable *t-norm*.

Definition 2.6. (1) A *negator* on L^* is any decreasing mapping $\mathcal{N} : L^* \longrightarrow L^*$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$.

(2) If $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in L^*$, then \mathcal{N} is called an *involution negator*.

(3) A *negator* on $[0, 1]$ is a decreasing mapping $N : [0, 1] \longrightarrow [0, 1]$ satisfying $N(0) = 1$ and $N(1) = 0$, where N_s denotes the *standard negator* on $[0, 1]$ defined by

$$N_s(x) = 1 - x$$

for all $x \in [0, 1]$.

Definition 2.7. Let μ and ν be a distribution function and a survival function from $X \times (0, +\infty)$ to $[0, 1]$ such that $\mu_x(t) + \nu_x(t) \leq 1$ for all $x \in X$ and $t > 0$. The 3-tuple $(X, \mathcal{P}_{\mu, \nu}, \mathcal{T})$ is said to be an *intuitionistic random normed space* (briefly, IRN-space) if X is a vector space, \mathcal{T} is a continuous representable t -norm and $\mathcal{P}_{\mu, \nu}$ is a mapping $X \times (0, +\infty) \rightarrow L^*$ satisfying the following conditions: for all $x, y \in X$ and $t, s > 0$,

- (1) $\mathcal{P}_{\mu, \nu}(x, 0) = 0_{L^*}$;
- (2) $\mathcal{P}_{\mu, \nu}(x, t) = 1_{L^*}$ if and only if $x = 0$;
- (3) $\mathcal{P}_{\mu, \nu}(\alpha x, t) = \mathcal{P}_{\mu, \nu}(x, \frac{t}{\alpha})$ for all $\alpha \neq 0$;
- (4) $\mathcal{P}_{\mu, \nu}(x + y, t + s) \geq_{L^*} \mathcal{T}(\mathcal{P}_{\mu, \nu}(x, t), \mathcal{P}_{\mu, \nu}(y, s))$.

In this case, $\mathcal{P}_{\mu, \nu}$ is called an *intuitionistic random norm*, where

$$\mathcal{P}_{\mu, \nu}(x, t) = (\mu_x(t), \nu_x(t)).$$

Note that, if $(X, \mathcal{P}_{\mu, \nu}, \mathcal{T})$ is an IRN-space and define $\mathcal{P}_{\mu, \nu}(x - y, t) = \mathcal{M}_{\mu, \nu}(x, y, t)$, then

$$(X, \mathcal{M}_{\mu, \nu}, \mathcal{T})$$

is an *intuitionistic Menger spaces*.

Example 2.8. Let $(X, \|\cdot\|)$ be a normed space. Let $\mathcal{T}(a, b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$ for all $a = (a_1, a_2)$, $b = (b_1, b_2) \in L^*$ and μ, ν be a distribution function and a survival function defined by

$$\mathcal{P}_{\mu, \nu}(x, t) = (\mu_x(t), \nu_x(t)) = \left(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|} \right)$$

for all $t \in \mathbb{R}^+$. Then $(X, \mathcal{P}_{\mu, \nu}, \mathcal{T})$ is an IRN-space.

Definition 2.9. (1) A sequence $\{x_n\}$ in an IRN-space $(X, \mathcal{P}_{\mu, \nu}, \mathcal{T})$ is called a *Cauchy sequence* if, for any $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\mathcal{P}_{\mu, \nu}(x_n - x_m, t) >_{L^*} (N_s(\varepsilon), \varepsilon)$$

for all $n, m \geq n_0$, where N_s is the standard negator.

(2) A sequence $\{x_n\}$ in an IRN-space $(X, \mathcal{P}_{\mu, \nu}, \mathcal{T})$ is said to be *convergent* to a point $x \in X$ (denoted by $x_n \xrightarrow{\mathcal{P}_{\mu, \nu}} x$) if $\mathcal{P}_{\mu, \nu}(x_n - x, t) \longrightarrow 1_{L^*}$ as $n \longrightarrow \infty$ for all $t > 0$.

(3) An IRN-space $(X, \mathcal{P}_{\mu, \nu}, \mathcal{T})$ is said to be *complete* if every Cauchy sequence in X is convergent to a point $x \in X$.

Definition 2.10. A *intuitionistic random normed algebra* $(X, \mathcal{P}_{\mu, \nu}, \mathcal{T}, \mathcal{T}')$ is a IRN-space $(X, \mathcal{P}_{\mu, \nu}, \mathcal{T})$ with algebraic structure such that

(4) $\mathcal{P}_{\mu, \nu}(xy, ts) \geq \mathcal{T}'(\mathcal{P}_{\mu, \nu}(x, t), \mathcal{P}_{\mu, \nu}(y, s))$ for all $x, y \in X$ and $t, s > 0$, in which \mathcal{T}' is a continuous representable t -norm.

Every normed algebra $(X, \|\cdot\|)$ defines a random normed algebra (X, μ, T_M, T_P) , where

$$\mathcal{P}_{\mu,\nu}(x, t) = \left(\frac{t}{t + \|x\|}, \frac{\|x\|}{t + \|x\|} \right)$$

for all $t > 0$ if and only if

$$\|xy\| \leq \|x\|\|y\| + s\|y\| + t\|x\|$$

for all $x, y \in X$ and $t, s > 0$. This space is called the *induced random normed algebra* (see [6]). For more properties and example of theory of random normed spaces, we refer to [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24].

Definition 2.11. Let $(\mathcal{U}, \mathcal{P}_{\mu,\nu}, \mathcal{T}, \mathcal{T}')$ be an intuitionistic random Banach algebra. An *involution* on \mathcal{U} is a mapping $u \rightarrow u^*$ from \mathcal{U} into \mathcal{U} satisfying the following conditions:

- (1) $u^{**} = u$ for all $u \in \mathcal{U}$;
- (2) $(\alpha u + \beta v)^* = \bar{\alpha}u^* + \bar{\beta}v^*$ for all $u, v \in \mathcal{U}$ and $\alpha, \beta \in \mathbb{C}$;
- (3) $(uv)^* = v^*u^*$ for all $u, v \in \mathcal{U}$.

If, in addition, $\nu_{u^*u}(ts) = T'(\nu_u(t), \nu_u(s))$ for all $u \in \mathcal{U}$ and $t, s > 0$, then \mathcal{U} is an *intuitionistic random C^* -algebra*.

Now, we recall a fundamental result in fixed point theory.

Let Ω be a set. A function $d : \Omega \times \Omega \rightarrow [0, \infty]$ is called a *generalized metric* on Ω if d satisfies the following conditions:

- (1) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in \Omega$;
- (2) $d(x, y) = d(y, x)$ for all $x, y \in \Omega$;
- (3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in \Omega$.

Theorem 2.12. ([2]) Let (Ω, d) be a complete generalized metric space and let $J : \Omega \rightarrow \Omega$ be a contractive mapping with Lipschitz constant $L < 1$. Then, for each given element $x \in \Omega$, either $d(J^n x, J^{n+1} x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that

- (1) $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;
- (2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;
- (3) y^* is the unique fixed point of J in the set $\Gamma = \{y \in \Omega \mid d(J^{n_0} x, y) < \infty\}$;
- (4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in \Gamma$.

In this paper, using the fixed point method, we approximate the derivations on intuitionistic random Lie C^* -algebras for the the following additive functional equation

$$\sum_{i=1}^m f\left(mx_i + \sum_{j=1, j \neq i}^m x_j\right) + f\left(\sum_{i=1}^m x_i\right) = 2f\left(\sum_{i=1}^m mx_i\right) \quad (2.1)$$

for all $m \in \mathbb{N}$ with $m \geq 2$.

3. Approximation of derivations in intuitionistic random Lie C^* -algebras

In this section, we approximate the derivations on intuitionistic random Lie C^* -algebras (see also [32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 44]).

For any mapping $f : A \rightarrow A$, we define

$$D_\omega f(x_1, \dots, x_m) := \sum_{i=1}^m \mu f\left(mx_i + \sum_{j=1, j \neq i}^m x_j\right) + f\left(\mu \sum_{i=1}^m x_i\right) - 2f\left(\mu \sum_{i=1}^m mx_i\right)$$

for all $\omega \in \mathbb{T}^1 := \{\xi \in \mathbb{C} : |\xi| = 1\}$ and $x_1, \dots, x_m \in A$.

Note that a \mathbb{C} -linear mapping $\delta : A \rightarrow A$ is called a *derivation* on intuitionistic random C^* -algebras if $\delta(xy) = y\delta(x) + x\delta(y)$ and $\delta(x^*) = \delta(x)^*$ for all $x, y \in A$.

Now, we approximate the derivations on intuitionistic random Lie C^* -algebras for the functional equation $D_\omega f(x_1, \dots, x_m) = 0$.

Theorem 3.1. *Let $f : A \rightarrow A$ be a mapping for which there are functions $\varphi : A^m \rightarrow L^*$, $\psi : A^2 \rightarrow L^*$ and $\eta : A \rightarrow L^*$ such that*

$$\mathcal{P}_{\mu, \nu}(D_\omega f(x_1, \dots, x_m), t) \geq_L \varphi(x_1, \dots, x_m, t), \quad (3.1)$$

$$\lim_{j \rightarrow \infty} \varphi(m^j x_1, \dots, m^j x_m, m^j t) = 1_{\mathcal{L}}, \quad (3.2)$$

$$\mathcal{P}_{\mu, \nu}(f(xy) - xf(y) - xf(y), t) \geq_L \psi(x, y, t), \quad (3.3)$$

$$\lim_{j \rightarrow \infty} \psi(m^j x, m^j y, m^{2j} t) = 1_{\mathcal{L}}, \quad (3.4)$$

$$\mathcal{P}_{\mu, \nu}(f(x^*) - f(x)^*, t) \geq_L \eta(x, t), \quad (3.5)$$

$$\lim_{j \rightarrow \infty} \eta(m^j x, m^j t) = 1_{\mathcal{L}} \quad (3.6)$$

for all $\omega \in \mathbb{T}^1$, $x_1, \dots, x_m, x, y \in A$ and $t > 0$. If there exists $R < 1$ such that

$$\varphi(mx, 0, \dots, 0, mRt) \geq_L \varphi(x, 0, \dots, 0, t) \quad (3.7)$$

for all $x \in A$ and $t > 0$, then there exists a unique derivation $\delta : A \rightarrow A$ such that

$$\mathcal{P}_{\mu, \nu}(f(x) - \delta(x), t) \geq_L \varphi(x, 0, \dots, 0, (m - mR)t) \quad (3.8)$$

for all $x \in A$ and $t > 0$.

Proof. Consider the set $X := \{g : A \rightarrow A\}$ and introduce the *generalized metric* on X defined by

$$d(g, h) = \inf\{C \in \mathbb{R}_+ : \mathcal{P}_{\mu, \nu}(g(x) - h(x), Ct) \geq_L \varphi(x, 0, \dots, 0, t), \forall x \in A, t > 0\}.$$

It is easy to show that (X, d) is complete.

Now, we consider the linear mapping $J : X \rightarrow X$ such that

$$Jg(x) := \frac{1}{m}g(mx)$$

for all $x \in A$. By Theorem 3.1 of [46],

$$d(Jg, Jh) \leq Rd(g, h)$$

for all $g, h \in X$. Letting $\omega = 1$, $x = x_1$ and $x_2 = \dots = x_m = 0$ in (3.1), we have

$$\mathcal{P}_{\mu, \nu}(f(mx) - mf(x), t) \geq_L \varphi(x, 0, \dots, 0, t) \quad (3.9)$$

for all $x \in A$ and $t > 0$ and so

$$\mathcal{P}_{\mu, \nu}\left(f(x) - \frac{1}{m}f(mx), t\right) \geq_L \varphi(x, 0, \dots, 0, mt)$$

for all $x \in A$ and $t > 0$. Hence $d(f, Jf) \leq \frac{1}{m}$. By Theorem 2.12, there exists a mapping $\delta : A \rightarrow A$ such that

(1) δ is a fixed point of J , i.e.,

$$\delta(mx) = m\delta(x) \quad (3.10)$$

for all $x \in A$. The mapping δ is a unique fixed point of J in the set

$$Y = \{g \in X : d(f, g) < \infty\}.$$

This implies that δ is a unique mapping satisfying (3.10) such that there exists $C \in (0, \infty)$ satisfying

$$\mathcal{P}_{\mu, \nu}(\delta(x) - f(x), Ct) \geq_L \varphi(x, 0, \dots, 0, t)$$

for all $x \in A$ and $t > 0$.

(2) $d(J^n f, \delta) \rightarrow 0$ as $n \rightarrow \infty$. This implies the equality

$$\lim_{n \rightarrow \infty} \frac{f(m^n x)}{m^n} = \delta(x) \quad (3.11)$$

for all $x \in A$.

(3) $d(f, \delta) \leq \frac{1}{1-R}d(f, Jf)$, which implies the inequality $d(f, \delta) \leq \frac{1}{m-mR}$. This implies that the inequality (3.8) holds.

Thus it follows from (3.1), (3.2) and (3.11) that

$$\begin{aligned} & \mathcal{P}_{\mu, \nu} \left(\sum_{i=1}^m \delta \left(mx_i + \sum_{j=1, j \neq i}^m x_j \right) + \delta \left(\sum_{i=1}^m x_i \right) - 2\delta \left(\sum_{i=1}^m mx_i \right), t \right) \\ &= \lim_{n \rightarrow \infty} \mathcal{P}_{\mu, \nu} \left(\sum_{i=1}^m f \left(m^{n+1}x_i + \sum_{j=1, j \neq i}^m m^n x_j \right) + f \left(\sum_{i=1}^m m^n x_i \right) - 2f \left(\sum_{i=1}^m m^{n+1}x_i \right), m^n t \right) \\ &\leq_L \lim_{n \rightarrow \infty} \varphi(m^n x_1, \dots, m^n x_m, m^n t) \\ &= 1_{\mathcal{L}} \end{aligned}$$

for all $x_1, \dots, x_m \in A$ and $t > 0$ and so

$$\sum_{i=1}^m \delta \left(mx_i + \sum_{j=1, j \neq i}^m x_j \right) + \delta \left(\sum_{i=1}^m x_i \right) = 2\delta \left(\sum_{i=1}^m mx_i \right)$$

for all $x_1, \dots, x_m \in A$.

By a similar method to above, we get

$$\omega\delta(mx) = \delta(m\omega x)$$

for all $\omega \in \mathbb{T}^1$ and $x \in A$. Thus one can show that the mapping $H : A \rightarrow A$ is \mathbb{C} -linear.

Also, it follows from (3.3), (3.4) and (3.11) that

$$\begin{aligned} & \mathcal{P}_{\mu, \nu}(\delta(xy) - y\delta(x) - x\delta(y), t) \\ &= \lim_{n \rightarrow \infty} \mathcal{P}_{\mu, \nu}(f(m^n xy) - m^n yf(m^n x) - m^n xf(m^n y), m^n t) \\ &\leq \lim_{n \rightarrow \infty} \psi(m^n x, m^n y, m^{2n}t) \\ &= 1_{\mathcal{L}} \end{aligned}$$

for all $x, y \in A$ and so

$$\delta(xy) = y\delta(x) + x\delta(y)$$

for all $x, y \in A$. Thus $\delta : A \rightarrow A$ is a derivation satisfying (3.7), as desired.

Also, Similarly, by (3.5), (3.6) and (3.11), we have $\delta(x^*) = \delta(x)^*$. This completes the proof. \square

4. Approximation of derivations on intuitionistic random Lie C^* -algebras

An intuitionistic random C^* -algebra \mathcal{C} , endowed with the Lie product

$$[x, y] := \frac{xy - yx}{2}$$

in \mathcal{C} , is called a *intuitionistic random Lie C^* -algebra*.

Definition 4.1. Let A and B be intuitionistic random Lie C^* -algebras. A \mathbb{C} -linear mapping $\delta : A \rightarrow A$ is called an *intuitionistic random Lie C^* -algebra derivation* if

$$\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$$

for all $x, y \in \mathcal{A}$.

Throughout this Section, assume that A is an intuitionistic random Lie C^* -algebra with norm $\mathcal{P}_{\mu, \nu}$.

Now, we approximate the derivations on intuitionistic random Lie C^* -algebras for the functional equation

$$D_\omega f(x_1, \dots, x_m) = 0.$$

Theorem 4.2. Let $f : A \rightarrow A$ be a mapping for which there are functions $\varphi : A^m \rightarrow L^*$ and $\psi : A^2 \rightarrow L^*$ such that

$$\lim_{j \rightarrow \infty} \varphi(m^j x_1, \dots, m^j x_m, m^j t) = 1_{\mathcal{L}}, \quad (4.1)$$

$$\mathcal{P}_{\mu, \nu}(D_\omega f(x_1, \dots, x_m), t) \geq_L \varphi(x_1, \dots, x_m, t), \quad (4.2)$$

$$\mathcal{P}_{\mu, \nu}(f([x, y]) - [f(x), y] - [x, f(y)], t) \geq_L \psi(x, y, t), \quad (4.3)$$

$$\lim_{j \rightarrow \infty} \psi(m^j x, m^j y, m^{2j} t) = 1_{\mathcal{L}} \quad (4.4)$$

for all $\omega \in \mathbb{T}^1$, $x_1, \dots, x_m, x, y \in A$ and $t > 0$. If there exists $R < 1$ such that

$$\varphi(mx, 0, \dots, 0, mx) \geq_L \varphi(x, 0, \dots, 0, t)$$

for all $x \in A$ and $t > 0$, then there exists a unique homomorphism $\delta : A \rightarrow A$ such that

$$\mathcal{P}_{\mu, \nu}(f(x) - \delta(x), t) \geq_L \varphi(x, 0, \dots, 0, (m - mR)t) \quad (4.5)$$

for all $x \in A$ and $t > 0$.

Proof. By the same reasoning as the proof of Theorem 3.1, we can find the mapping $\delta : A \rightarrow A$ given by

$$\delta(x) = \lim_{n \rightarrow \infty} \frac{f(m^n x)}{m^n}$$

for all $x \in A$. It follows from (4.3) that

$$\begin{aligned} & \mathcal{P}_{\mu,\nu}(\delta([x, y]) - [\delta(x), y] - [x, \delta(y)], t) \\ &= \lim_{n \rightarrow \infty} \mathcal{P}_{\mu,\nu}(f(m^{2n}[x, y]) - [f(m^n x), \cdot m^n y] - [m^n x, f(m^n y)], m^{2n}t) \\ &\geq_L \lim_{n \rightarrow \infty} \psi(m^n x, m^n y, m^{2n}t) = 1_{\mathcal{L}} \end{aligned}$$

for all $x, y \in A$ and $t > 0$ and so

$$\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$$

for all $x, y \in A$. Thus $\delta : A \rightarrow B$ is an intuitionistic random Lie C^* -algebra derivation satisfying (4.5). This completes the proof. \square

Corollary 4.3. *Let $0 < r < 1$ and θ be nonnegative real numbers and $f : A \rightarrow A$ be a mapping such that*

$$\begin{aligned} & \mathcal{P}_{\mu,\nu}(D_{\omega}f(x_1, \dots, x_m), t) \\ &\geq_L \left(\frac{t}{t + \theta(\|x_1\|_A^r + \|x_2\|_A^r + \dots + \|x_m\|_A^r)}, \frac{\theta(\|x_1\|_A^r + \|x_2\|_A^r + \dots + \|x_m\|_A^r)}{t + \theta(\|x_1\|_A^r + \|x_2\|_A^r + \dots + \|x_m\|_A^r)} \right), \\ & \mathcal{P}_{\mu,\nu}(f([x, y]) - [f(x), y] - [x, f(y)], t) \\ &\geq_L \left(\frac{t}{t + \theta \cdot \|x\|_A^r \cdot \|y\|_A^r}, \frac{\theta \cdot \|x\|_A^r \cdot \|y\|_A^r}{t + \theta \cdot \|x\|_A^r \cdot \|y\|_A^r} \right) \end{aligned}$$

for all $\omega \in \mathbb{T}^1$, $x_1, \dots, x_m, x, y \in A$ and $t > 0$. Then there exists a unique derivation $\delta : A \rightarrow A$ such that

$$\mathcal{P}_{\mu,\nu}(f(x) - \delta(x), t) \leq_L \left(\frac{t}{t + \frac{\theta}{m-m^r}\|x\|_A^r}, \frac{\frac{\theta}{m-m^r}\|x\|_A^r}{t + \frac{\theta}{m-m^r}\|x\|_A^r} \right)$$

for all $x \in A$ and $t > 0$.

Proof. The proof follows from Theorem 4.2 by taking

$$\begin{aligned} & \varphi(x_1, \dots, x_m, t) \\ &= \left(\frac{t}{t + \theta(\|x_1\|_A^r + \|x_2\|_A^r + \dots + \|x_m\|_A^r)}, \frac{\theta(\|x_1\|_A^r + \|x_2\|_A^r + \dots + \|x_m\|_A^r)}{t + \theta(\|x_1\|_A^r + \|x_2\|_A^r + \dots + \|x_m\|_A^r)} \right), \\ & \psi(x, y, t) := \left(\frac{t}{t + \theta \cdot \|x\|_A^r \cdot \|y\|_A^r}, \frac{\theta \cdot \|x\|_A^r \cdot \|y\|_A^r}{t + \theta \cdot \|x\|_A^r \cdot \|y\|_A^r} \right) \end{aligned}$$

and

$$R = m^{r-1}$$

for all $x_1, \dots, x_m, x, y \in A$ and $t > 0$. This completes the proof. \square

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CUBIC ρ -FUNCTIONAL INEQUALITY AND QUARTIC ρ -FUNCTIONAL INEQUALITY

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ABSTRACT. In this paper, we solve the following cubic ρ -functional inequality

$$\begin{aligned} & \|f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x)\| \\ & \leq \left\| \rho \left(4f\left(x + \frac{y}{2}\right) + 4f\left(x - \frac{y}{2}\right) - f(x+y) - f(x-y) - 6f(x) \right) \right\|, \end{aligned} \quad (0.1)$$

where ρ is a fixed complex number with $|\rho| < 2$, and the quartic ρ -functional inequality

$$\begin{aligned} & \|f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y)\| \\ & \leq \left\| \rho \left(8f\left(x + \frac{y}{2}\right) + 8f\left(x - \frac{y}{2}\right) - 2f(x+y) - 2f(x-y) - 12f(x) + 3f(y) \right) \right\|, \end{aligned} \quad (0.2)$$

where ρ is a fixed complex number with $|\rho| < 2$.

Using the direct method, we prove the Hyers-Ulam stability of the cubic ρ -functional inequality (0.1) and the quartic ρ -functional inequality (0.2) in complex Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations originated from a question of Ulam [17] concerning the stability of group homomorphisms. Hyers [8] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [12] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

In [9], Jun and Kim considered the following cubic functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x). \quad (1.1)$$

It is easy to show that the function $f(x) = x^3$ satisfies the functional equation (1.1), which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*. We can define the following Jensen type cubic functional equation

$$4f\left(x + \frac{y}{2}\right) + 4f\left(x - \frac{y}{2}\right) = f(x+y) + f(x-y) + 6f(x).$$

Note that if $f(2x) = 8f(x)$ then the Jensen type cubic functional equation is equivalent to the cubic functional equation (1.1).

In [10], Lee et al. considered the following quartic functional equation

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y). \quad (1.2)$$

It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.2), which is called a *quartic functional equation* and every solution of the quartic functional equation is said

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to be a *quartic mapping*. We can define the following Jensen type quartic functional equation

$$8f\left(x + \frac{y}{2}\right) + 8f\left(x - \frac{y}{2}\right) = 2f(x+y) + 2f(x-y) + 12f(x) - 3f(y).$$

Note that if $f(2x) = 16f(x)$ then the Jensen type quartic functional equation is equivalent to the quartic functional equation (1.2).

Recently, considerable attention has been increasing to the problem of the Hyers-Ulam stability of functional equations. Several Hyers-Ulam stability results concerning Cauchy, Jensen, quadratic, cubic and quartic functional equations have been investigated in [1, 3, 13, 14, 15, 16, 18].

In [6], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\| \quad (1.3)$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

Gilányi [7] and Fechner [4] proved the Hyers-Ulam stability of the functional inequality (1.3). Park, Cho and Han [11] proved the Hyers-Ulam stability of additive functional inequalities.

In Section 3, we solve the cubic ρ -functional inequality (0.1) and prove the Hyers-Ulam stability of the cubic ρ -functional inequality (0.1) in complex Banach spaces.

In Section 4, we solve the quartic ρ -functional inequality (0.2) and prove the Hyers-Ulam stability of the quartic ρ -functional inequality (0.2) in complex Banach spaces.

Throughout this paper, assume that X is a complex normed space and that Y is a complex Banach space.

2. CUBIC ρ -FUNCTIONAL INEQUALITY (0.1)

Throughout this section, assume that ρ is a fixed complex number with $|\rho| < 2$.

In this section, we solve and investigate the cubic ρ -functional inequality (0.1) in complex normed spaces.

Lemma 2.1. *Let X and Y be vector spaces. A mapping $f : X \rightarrow Y$ satisfies $f(2x) = 8f(x)$ and*

$$4f\left(x + \frac{y}{2}\right) + 4f\left(x - \frac{y}{2}\right) = f(x+y) + f(x-y) + 6f(x)$$

if and only if the mapping $f : X \rightarrow Y$ is a cubic mapping.

Proof. One can easily prove it. We omit the proof. □

Lemma 2.2. *If a mapping $f : X \rightarrow Y$ satisfies*

$$\begin{aligned} & \|f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x)\| \\ & \leq \left\| \rho \left(4f\left(x + \frac{y}{2}\right) + 4f\left(x - \frac{y}{2}\right) - f(x+y) - f(x-y) - 6f(x) \right) \right\| \end{aligned} \quad (2.1)$$

for all $x, y \in X$, then $f : X \rightarrow Y$ is cubic.

Proof. Assume that $f : X \rightarrow Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $\| -14f(0) \| \leq |\rho| \|0\| = 0$. So $f(0) = 0$.

Letting $y = 0$ in (2.1), we get $\|2f(2x) - 16f(x)\| \leq 0$ and so $f(2x) = 8f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{8}f(x) \quad (2.2)$$

for all $x \in X$.

CUBIC AND QUARTIC ρ -FUNCTIONAL INEQUALITIES

It follows from (2.1) and (2.2) that

$$\begin{aligned} & \|f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x)\| \\ & \leq \left\| \rho \left(4f\left(x + \frac{y}{2}\right) + 4f\left(x - \frac{y}{2}\right) - f(x+y) - f(x-y) - 6f(x) \right) \right\| \\ & = \frac{|\rho|}{2} \|f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x)\| \end{aligned}$$

and so

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x)$$

for all $x, y \in X$, since $|\rho| < 2$. So $f : X \rightarrow Y$ is cubic. \square

We prove the Hyers-Ulam stability of the cubic ρ -functional inequality (2.1) in complex Banach spaces.

Theorem 2.3. *Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function and let $f : X \rightarrow Y$ be a mapping such that*

$$\Psi(x, y) := \sum_{j=1}^{\infty} 8^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty, \quad (2.3)$$

$$\begin{aligned} & \|f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x)\| \\ & \leq \left\| \rho \left(4f\left(x + \frac{y}{2}\right) + 4f\left(x - \frac{y}{2}\right) - f(x+y) - f(x-y) - 6f(x) \right) \right\| + \varphi(x, y) \end{aligned} \quad (2.4)$$

for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \rightarrow Y$ such that

$$\|f(x) - C(x)\| \leq \frac{1}{16} \Psi(x, 0) \quad (2.5)$$

for all $x \in X$.

Proof. Letting $y = 0$ in (2.4), we get

$$\|2f(2x) - 16f(x)\| \leq \varphi(x, 0) \quad (2.6)$$

and so $\|f(x) - 8f\left(\frac{x}{2}\right)\| \leq \frac{1}{2}\varphi\left(\frac{x}{2}, 0\right)$ for all $x \in X$. So

$$\begin{aligned} \left\| 8^l f\left(\frac{x}{2^l}\right) - 8^m f\left(\frac{x}{2^m}\right) \right\| & \leq \sum_{j=l+1}^m \left\| 8^j f\left(\frac{x}{2^j}\right) - 8^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \\ & \leq \frac{1}{16} \sum_{j=l+1}^m 8^j \varphi\left(\frac{x}{2^j}, 0\right) \end{aligned} \quad (2.7)$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.7) that the sequence $\{8^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{8^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $C : X \rightarrow Y$ by

$$C(x) := \lim_{n \rightarrow \infty} 8^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (2.7), we get (2.5).

It follows from (2.3) and (2.4) that

$$\begin{aligned}
& \|C(2x+y) + C(2x-y) - 2C(x+y) - 2C(x-y) - 12C(x)\| \\
&= \lim_{n \rightarrow \infty} 8^n \left\| f\left(\frac{2x+y}{2^n}\right) + f\left(\frac{2x-y}{2^n}\right) - 2f\left(\frac{x+y}{2^n}\right) - 2f\left(\frac{x-y}{2^n}\right) - 12f\left(\frac{x}{2^n}\right) \right\| \\
&\leq \lim_{n \rightarrow \infty} 8^n |\rho| \left\| 4f\left(\frac{2x+y}{2^{n+1}}\right) + 4f\left(\frac{2x-y}{2^{n+1}}\right) - f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x-y}{2^n}\right) - 6f\left(\frac{x}{2^n}\right) \right\| \\
&\quad + \lim_{n \rightarrow \infty} 8^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \\
&= \left\| \rho \left(4C\left(x + \frac{y}{2}\right) + 4C\left(x - \frac{y}{2}\right) - C(x+y) - C(x-y) - 6C(x) \right) \right\|
\end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned}
& \|C(2x+y) + C(2x-y) - 2C(x+y) - 2C(x-y) - 12C(x)\| \\
&\leq \left\| \rho \left(4C\left(x + \frac{y}{2}\right) + 4C\left(x - \frac{y}{2}\right) - C(x+y) - C(x-y) - 6C(x) \right) \right\|
\end{aligned}$$

for all $x, y, z \in X$. By Lemma 2.2, the mapping $C : X \rightarrow Y$ is cubic.

Now, let $T : X \rightarrow Y$ be another cubic mapping satisfying (2.5). Then we have

$$\begin{aligned}
\|C(x) - T(x)\| &= \left\| 8^q C\left(\frac{x}{2^q}\right) - 8^q T\left(\frac{x}{2^q}\right) \right\| \\
&\leq \left\| 8^q C\left(\frac{x}{2^q}\right) - 8^q f\left(\frac{x}{2^q}\right) \right\| + \left\| 8^q T\left(\frac{x}{2^q}\right) - 8^q f\left(\frac{x}{2^q}\right) \right\| \\
&\leq \frac{2}{16} \cdot 8^q \Psi\left(\frac{x}{2^q}, 0\right),
\end{aligned}$$

which tends to zero as $q \rightarrow \infty$ for all $x \in X$. So we can conclude that $C(x) = T(x)$ for all $x \in X$. This proves the uniqueness of C . Thus the mapping $C : X \rightarrow Y$ is a unique cubic mapping satisfying (2.5). \square

Corollary 2.4. Let $r > 3$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be a mapping such that

$$\begin{aligned}
& \|f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x)\| \\
&\leq \left\| \rho \left(4f\left(x + \frac{y}{2}\right) + 4f\left(x - \frac{y}{2}\right) - f(x+y) - f(x-y) - 6f(x) \right) \right\| + \theta(\|x\|^r + \|y\|^r)
\end{aligned} \tag{2.8}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \rightarrow Y$ such that

$$\|f(x) - C(x)\| \leq \frac{\theta}{2^{r+1} - 16} \|x\|^r$$

for all $x \in X$.

Theorem 2.5. Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function and let $f : X \rightarrow Y$ be a mapping satisfying (2.4) and

$$\Psi(x, y) := \sum_{j=0}^{\infty} \frac{1}{8^j} \varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \rightarrow Y$ such that

$$\|f(x) - C(x)\| \leq \frac{1}{16} \Psi(x, 0) \tag{2.9}$$

for all $x \in X$.

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Proof. It follows from (2.6) that

$$\left\| f(x) - \frac{1}{8}f(2x) \right\| \leq \frac{1}{16}\varphi(x, 0)$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| \frac{1}{8^l}f(2^l x) - \frac{1}{8^m}f(2^m x) \right\| &\leq \sum_{j=l}^{m-1} \left\| \frac{1}{8^j}f(2^j x) - \frac{1}{8^{j+1}}f(2^{j+1} x) \right\| \\ &\leq \frac{1}{16} \sum_{j=l}^{m-1} \frac{1}{8^j} \varphi(2^j x, 0) \end{aligned} \quad (2.10)$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.10) that the sequence $\{\frac{1}{8^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{8^n}f(2^n x)\}$ converges. So one can define the mapping $C : X \rightarrow Y$ by

$$C(x) := \lim_{n \rightarrow \infty} \frac{1}{8^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (2.10), we get (2.9).

The rest of the proof is similar to the proof of Theorem 2.3. \square

Corollary 2.6. *Let $r < 3$ and θ be positive real numbers, and let $f : X \rightarrow Y$ be a mapping satisfying (2.8). Then there exists a unique cubic mapping $C : X \rightarrow Y$ such that*

$$\|f(x) - C(x)\| \leq \frac{\theta}{16 - 2^{r+1}} \|x\|^r \quad (2.11)$$

for all $x \in X$.

Remark 2.7. If ρ is a real number such that $-2 < \rho < 2$ and Y is a real Banach space, then all the assertions in this section remain valid.

3. QUARTIC ρ -FUNCTIONAL INEQUALITY (0.2)

Throughout this section, assume that ρ is a fixed complex number with $|\rho| < 2$.

In this section, we solve and investigate the quartic ρ -functional inequality (0.2) in complex normed spaces.

Lemma 3.1. *Let X and Y be vector spaces. An even mapping $f : X \rightarrow Y$ satisfies*

$$8f\left(x + \frac{y}{2}\right) + 8f\left(x - \frac{y}{2}\right) = 2f(x+y) + 2f(x-y) + 12f(x) - 3f(y) \quad (3.1)$$

if and only if the mapping $f : X \rightarrow Y$ is a quartic mapping.

Proof. Sufficiency. Assume that $f : X \rightarrow Y$ satisfies (3.1)

Letting $x = y = 0$ in (3.1), we have $16f(0) = 13f(0)$. So $f(0) = 0$.

Letting $x = 0$ in (3.1), we get $16f(\frac{y}{2}) = f(y)$ for all $y \in X$. So $f : X \rightarrow Y$ satisfies the quartic functional equation.

Necessity. Assume that $f : X \rightarrow Y$ is a quartic mapping. Then $f(2x) = 16f(x)$ for all $x \in X$. So $f : X \rightarrow Y$ satisfies (3.1). \square

Lemma 3.2. *If a mapping $f : X \rightarrow Y$ satisfies*

$$\begin{aligned} &\|f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y)\| \\ &\leq \left\| \rho \left(8f\left(x + \frac{y}{2}\right) + 8f\left(x - \frac{y}{2}\right) - 2f(x+y) - 2f(x-y) - 12f(x) + 3f(y) \right) \right\| \end{aligned} \quad (3.2)$$

for all $x, y \in X$, then the mapping $f : X \rightarrow Y$ is quartic.

Proof. Assume that $f : X \rightarrow Y$ satisfies (3.2).

Letting $x = y = 0$ in (3.2), we get $\|24f(0)\| \leq |\rho|\|3f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.2), we get

$$\|2f(2x) - 32f(x)\| \leq 0 \quad (3.3)$$

and so

$$f\left(\frac{x}{2}\right) = \frac{1}{16}f(x) \quad (3.4)$$

for all $x \in X$.

It follows from (3.2) and (3.4) that

$$\begin{aligned} & \|f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y)\| \\ & \leq \left\| \rho \left(8f\left(x + \frac{y}{2}\right) + 8f\left(x - \frac{y}{2}\right) - 2f(x+y) - 2f(x-y) - 12f(x) + 3f(y) \right) \right\| \\ & = \frac{|\rho|}{2} \|f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y)\| \end{aligned}$$

and so

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y)$$

for all $x, y \in X$, since $|\rho| < 2$. So $f : X \rightarrow Y$ is quartic. \square

We prove the Hyers-Ulam stability of the quartic ρ -functional inequality (3.2) in complex Banach spaces.

Theorem 3.3. Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function and let $f : X \rightarrow Y$ be a mapping satisfying $f(0) = 0$,

$$\Psi(x, y) := \sum_{j=1}^{\infty} 16^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty,$$

$$\begin{aligned} & \|f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y)\| \quad (3.5) \\ & \leq \left\| \rho \left(8f\left(x + \frac{y}{2}\right) + 8f\left(x - \frac{y}{2}\right) - 2f(x+y) - 2f(x-y) - 12f(x) + 3f(y) \right) \right\| + \varphi(x, y) \end{aligned}$$

for all $x, y \in X$. Then there exists a unique quartic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{32} \Psi(x, 0) \quad (3.6)$$

for all $x \in X$.

Proof. Letting $y = 0$ in (3.5), we get

$$\|2f(2x) - 32f(x)\| \leq \varphi(x, 0) \quad (3.7)$$

and so $\|f(x) - 16f\left(\frac{x}{2}\right)\| \leq \frac{1}{2}\varphi\left(\frac{x}{2}, 0\right)$ for all $x \in X$. So

$$\begin{aligned} \left\| 16^l f\left(\frac{x}{2^l}\right) - 16^m f\left(\frac{x}{2^m}\right) \right\| & \leq \sum_{j=l+1}^m \left\| 16^j f\left(\frac{x}{2^j}\right) - 16^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \\ & \leq \frac{1}{32} \sum_{j=l+1}^m 16^j \varphi\left(\frac{x}{2^j}, 0\right) \end{aligned} \quad (3.8)$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.8) that the sequence $\{16^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{16^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \rightarrow Y$ by

$$Q(x) := \lim_{n \rightarrow \infty} 16^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (3.8), we get (3.6).

The rest of the proof is similar to the proof of Theorem 2.3. \square

Corollary 3.4. *Let $r > 4$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be a mapping such that*

$$\begin{aligned} & \|f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y)\| \\ & \leq \left\| \rho \left(8f\left(x + \frac{y}{2}\right) + 8f\left(x - \frac{y}{2}\right) - 2f(x+y) - 2f(x-y) - 12f(x) + 3f(y) \right) \right\| \\ & \quad + \theta(\|x\|^r + \|y\|^r) \end{aligned} \quad (3.9)$$

for all $x, y \in X$. Then there exists a unique quartic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{\theta}{2^{r+1} - 32} \|x\|^r$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (3.9), we get $\|24f(0)\| \leq |\rho| \|3f(0)\|$, So $f(0) = 0$. Letting $\varphi(x, y) := \theta(\|x\|^r + \|y\|^r)$ in Theorem 3.3, we obtain the desired result. \square

Theorem 3.5. *Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function and let $f : X \rightarrow Y$ be a mapping satisfying $f(0) = 0$, (3.5) and*

$$\Psi(x, y) := \sum_{j=0}^{\infty} \frac{1}{16^j} \varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in X$. Then there exists a unique quartic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{32} \Psi(x, 0) \quad (3.10)$$

for all $x \in X$.

Proof. It follows from (3.7) that

$$\left\| f(x) - \frac{1}{16} f(2x) \right\| \leq \frac{1}{32} \varphi(x, 0)$$

for all $x \in X$. Hence

$$\begin{aligned} \left\| \frac{1}{16^l} f(2^l x) - \frac{1}{16^m} f(2^m x) \right\| & \leq \sum_{j=l}^{m-1} \left\| \frac{1}{16^j} f(2^j x) - \frac{1}{16^{j+1}} f(2^{j+1} x) \right\| \\ & \leq \frac{1}{32} \sum_{j=l}^{m-1} \frac{1}{16^j} \varphi(2^j x, 0) \end{aligned} \quad (3.11)$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.11) that the sequence $\{\frac{1}{16^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{16^n} f(2^n x)\}$ converges. So one can define the mapping $Q : X \rightarrow Y$ by

$$Q(x) := \lim_{n \rightarrow \infty} \frac{1}{16^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (3.11), we get (3.10).

The rest of the proof is similar to the proof of Theorem 2.3. \square

Corollary 3.6. *Let $r < 4$ and θ be positive real numbers, and let $f : X \rightarrow Y$ be a mapping satisfying (3.9). Then there exists a unique quartic mapping $Q : X \rightarrow Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{\theta}{32 - 2^{r+1}} \|x\|^r$$

for all $x \in X$.

Remark 3.7. If ρ is a real number such that $-2 < \rho < 2$ and Y is a real Banach space, then all the assertions in this section remain valid.

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Complex Valued G_b -Metric Spaces

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Abstract

In this paper, we introduce the concept of complex valued G_b -metric spaces. We also prove Banach contraction principle and Kannan's fixed point theorem in this space. Our result generalizes some well-known results in the fixed point theory.

Keywords: Complex valued G_b -metric space, fixed point, Banach contraction principle, Kannan's fixed point theorem.

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1 Introduction

The concept of a metric space was introduced by Frechet [11]. Then many mathematicians study of fixed points of contractive mappings. After the introduction of Banach contraction principle, the study of existence and uniqueness of fixed points and common fixed points have been a major area of interest. In a number of generalized metric spaces, many researchers proved the Banach fixed point theorem.

Bakhtin [6] presented b -metric spaces as a generalization of metric spaces. He also proved generalized Banach contraction principle in b -metric spaces. After that, many papers related to variational principle for single-valued and multi-valued operators have studied in b -metric spaces (see [7, 8, 9, 10, 18]). Azam et al. [4] defined the notion of complex valued metric spaces and gave common fixed point result for mappings. Rao et al. [21] introduced the complex valued b -metric spaces. Mustafa and Sims [13] presented the notion of G -metric spaces. Many researchers [1, 2, 3, 12, 14, 15, 19, 20, 22, 23, 25] obtained common fixed point results for G -metric spaces. The concept of G_b -metric space was given in [5]. Mustafa et al. [16] prove some coupled coincidence fixed point theorems for nonlinear (ψ, φ) -weakly contractive mappings in partially ordered G_b -metric spaces. Other important studies on G_b -metric spaces, see [17, 24].

In this work, our aim is to prove Banach contraction principle and Kannan's fixed point theorem in complex valued G_b -metric spaces. For this purpose, we give new definitions and additional theorems with proofs.

2 Preliminaries

In this section, we recall some properties of G_b -metric spaces.

Definition 2.1. [5]. Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a mapping $G : X \times X \times X \rightarrow \mathbb{R}^+$ satisfies:

- (G_b1) $G(x, y, z) = 0$ if $x = y = z$;
- (G_b2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
- (G_b3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;
- (G_b4) $G(x, y, z) = G(p\{x, y, z\})$, where p is a permutation of x, y, z ;
- (G_b5) $G(x, y, z) \leq s(G(x, a, a) + G(a, y, z))$ for all $x, y, z, a \in X$ (rectangle inequality).

Then, G is called a generalized b -metric and (X, G) is called a generalized b -metric or a G_b -metric space.

Note that each G -metric space is a G_b -metric space with $s = 1$.

Proposition 2.2. [5]. Let X be a G_b -metric space. Then for each $x, y, z, a \in X$ it follows that:

- (i) if $G(x, y, z) = 0$ then $x = y = z$,
- (ii) $G(x, y, z) \leq s(G(x, x, y) + G(x, x, z))$,
- (iii) $G(x, y, y) \leq 2sG(y, x, x)$,
- (iv) $G(x, y, z) \leq s(G(x, a, z) + G(a, y, z))$.

Definition 2.3. [5]. Let X be a G_b -metric space. A sequence $\{x_n\}$ in X is said to be:

- G_b -Cauchy if for each $\epsilon > 0$, there exists a positive integer n_0 such that for all $m, n, l \geq n_0$, $G(x_n, x_m, x_l) < \epsilon$,
- G_b -convergent to a point $x \in X$ if for each $\epsilon > 0$, there exists a positive integer n_0 such that for all $m, n \geq n_0$, $G(x_n, x_m, x) < \epsilon$.

Proposition 2.4. [5]. Let X be a G_b -metric space.

- (1) The sequence $\{x_n\}$ is G_b -Cauchy.
- (2) For any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $m, n \geq n_0$.

Proposition 2.5. [5]. Let X be a G_b -metric space. The following are equivalent:

- (1) $\{x_n\}$ is G_b -convergent to x .
- (2) $G(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$.
- (3) $G(x_n, x, x) \rightarrow 0$ as $n \rightarrow \infty$.

Definition 2.6. [5]. A G_b -metric space X is called complete if every G_b -Cauchy sequence is G_b -convergent in X .

The complex metric space was initiated by Azam et al. [4]. Let \mathbb{C} be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order \preceq on \mathbb{C} as follows:

$$z_1 \preceq z_2 \text{ if and only if } \operatorname{Re}(z_1) \leq \operatorname{Re}(z_2) \text{ and } \operatorname{Im}(z_1) \leq \operatorname{Im}(z_2).$$

It follows that $z_1 \preceq z_2$ if one of the following conditions is satisfied:

- (C₁) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$,
- (C₂) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$,
- (C₃) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$,
- (C₄) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$.

Particularly, we write $z_1 \prec z_2$ if $z_1 \neq z_2$ and one of (C₂), (C₃) and (C₄) is satisfied and we write $z_1 \prec z_2$ if only (C₄) is satisfied. The following statements hold:

- (1) If $a, b \in \mathbb{R}$ with $a \leq b$, then $az \prec bz$ for all $z \in \mathbb{C}$.
- (2) If $0 \preceq z_1 \preceq z_2$, then $|z_1| < |z_2|$.
- (3) If $z_1 \preceq z_2$ and $z_2 \prec z_3$, then $z_1 \prec z_3$.

3 Complex Valued G_b -Metric Spaces

In this section, we define the complex valued G_b -metric space.

Definition 3.1. Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that a mapping $G : X \times X \times X \rightarrow \mathbb{C}$ satisfies:

- (CG_b1) $G(x, y, z) = 0$ if $x = y = z$;
- (CG_b2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
- (CG_b3) $G(x, x, y) \preceq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;
- (CG_b4) $G(x, y, z) = G(p\{x, y, z\})$, where p is a permutation of x, y, z ;
- (CG_b5) $G(x, y, z) \preceq s(G(x, a, a) + G(a, y, z))$ for all $x, y, z, a \in X$ (rectangle inequality).

Then, G is called a complex valued G_b -metric and (X, G) is called a complex valued G_b -metric space.

From (CG_b5), we have the following proposition.

Proposition 3.2. Let (X, G) be a complex valued G_b -metric space. Then for any $x, y, z \in X$,

- $G(x, y, z) \preceq s(G(x, x, y) + G(x, x, z))$,
- $G(x, y, y) \preceq 2sG(y, x, y)$.

Definition 3.3. Let (X, G) be a complex valued G_b -metric space, let $\{x_n\}$ be a sequence in X .

- (i) $\{x_n\}$ is complex valued G_b -convergent to x if for every $a \in \mathbb{C}$ with $0 \prec a$, there exists $k \in \mathbb{N}$ such that $G(x, x_n, x_m) \prec a$ for all $n, m \geq k$.
- (ii) A sequence $\{x_n\}$ is called complex valued G_b -Cauchy if for every $a \in \mathbb{C}$ with $0 \prec a$, there exists $k \in \mathbb{N}$ such that $G(x_n, x_m, x_l) \prec a$ for all $n, m, l \geq k$.
- (iii) If every complex valued G_b -Cauchy sequence is complex valued G_b -convergent in (X, G) , then (X, G) is said to be complex valued G_b -complete.

Proposition 3.4. Let (X, G) be a complex valued G_b -metric space and $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is complex valued G_b -convergent to x if and only if $|G(x, x_n, x_m)| \rightarrow 0$ as $n, m \rightarrow \infty$.

Proof. (\Rightarrow) Assume that $\{x_n\}$ is complex valued G_b -convergent to x and let

$$a = \frac{\epsilon}{\sqrt{2}} + i \frac{\epsilon}{\sqrt{2}}$$

for a real number $\epsilon > 0$. Then we have $0 \prec a \in \mathbb{C}$ and there is a natural number k such that $G(x, x_n, x_m) \prec a$ for all $n, m \geq k$. Thus, $|G(x, x_n, x_m)| < |a| = \epsilon$ for all $n, m \geq k$ and so $|G(x, x_n, x_m)| \rightarrow 0$ as $n, m \rightarrow \infty$.

(\Leftarrow) Suppose that $|G(x, x_n, x_m)| \rightarrow 0$ as $n, m \rightarrow \infty$. For a given $a \in \mathbb{C}$ with $0 \prec a$, there exists a real number $\delta > 0$ such that for $z \in \mathbb{C}$

$$|z| < \delta \Rightarrow z \prec a.$$

Considering δ , we have a natural number k such that $|G(x, x_n, x_m)| < \delta$ for all $n, m \geq k$. This means that $G(x, x_n, x_m) \prec a$ for all $n, m \geq k$, i.e., $\{x_n\}$ is complex valued G_b -convergent to x . \square

From Propositions 3.2 and 3.4, we can prove the following theorem.

Theorem 3.5. Let (X, G) be a complex valued G_b -metric space, then for a sequence $\{x_n\}$ in X and point $x \in X$, the following are equivalent:

- (1) $\{x_n\}$ is complex valued G_b -convergent to x .
- (2) $|G(x_n, x_n, x)| \rightarrow 0$ as $n \rightarrow \infty$.
- (3) $|G(x_n, x, x)| \rightarrow 0$ as $n \rightarrow \infty$.
- (4) $|G(x_m, x_n, x)| \rightarrow 0$ as $m, n \rightarrow \infty$.

Proof. (1) \Rightarrow (2) It is clear from Proposition 3.4.

(2) \Rightarrow (3) By Proposition 3.2, we have

$$G(x_n, x, x) \preceq s(G(x_n, x_n, x) + G(x_n, x, x))$$

and using (2), we get

$$|G(x_n, x, x)| \rightarrow 0$$

as $n \rightarrow \infty$.

(3) \Rightarrow (4) If we use (CG_b4) and Proposition 3.2, then

$$\begin{aligned} G(x_m, x_n, x) &= G(x, x_m, x_n) \preceq s(G(x, x, x_m) + G(x, x, x_n)) \\ &= s(G(x_m, x, x) + G(x_n, x, x)) \end{aligned}$$

and $|G(x_m, x_n, x)| \rightarrow 0$ as $m, n \rightarrow \infty$.

(4) \Rightarrow (1) We will use the equivalence in Proposition 3.4, (CG_b3) and (CG_b4) . Since

$$\begin{aligned} G(x, x_n, x_m) &= G(x_m, x, x_n) \preceq s(G(x_m, x_m, x) + G(x_m, x_m, x_n)) \\ &\preceq s(G(x_m, x_n, x)) \end{aligned}$$

and $|G(x_m, x_n, x)| \rightarrow 0$ as $m, n \rightarrow \infty$, we obtain $|G(x, x_n, x_m)| \rightarrow 0$ and this completes the proof. \square

Theorem 3.6. Let (X, G) be a complex valued G_b -metric space and $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is complex valued G_b -Cauchy sequence if and only if $|G(x_n, x_m, x_l)| \rightarrow 0$ as $n, m, l \rightarrow \infty$.

Proof. (\Rightarrow) Let $\{x_n\}$ be complex valued G_b -Cauchy sequence and

$$b = \frac{\epsilon}{\sqrt{2}} + i \frac{\epsilon}{\sqrt{2}}$$

where $\epsilon > 0$ is a real number. Then $0 \prec b \in \mathbb{C}$ and there is a natural number k such that $G(x_n, x_m, x_l) \prec b$ for all $n, m, l \geq k$. Therefore, we get $|G(x_n, x_m, x_l)| < |b| = \epsilon$ for all $n, m, l \geq k$ and the required result.

(\Leftarrow) Assume that $|G(x_n, x_m, x_l)| \rightarrow 0$ as $n, m, l \rightarrow \infty$. Then there exists a real number $\gamma > 0$ such that for $z \in \mathbb{C}$

$$|z| < \gamma \text{ implies } z \prec b$$

where $b \in \mathbb{C}$ with $0 \prec b$. For this γ , there is a natural number k such that $|G(x_n, x_m, x_l)| < \gamma$ for all $n, m, l \geq k$. This means that $G(x_n, x_m, x_l) \prec b$ for all $n, m, l \geq k$. Hence $\{x_n\}$ is complex valued G_b -Cauchy sequence. \square

We prove the contraction principle in complex valued G_b -metric spaces as follows:

Theorem 3.7. *Let (X, G) be a complete complex valued G_b -metric space with coefficient $s > 1$ and $T : X \rightarrow X$ be a mapping satisfying:*

$$G(Tx, Ty, Tz) \lesssim kG(x, y, z) \quad (3.1)$$

for all $x, y, z \in X$, where $k \in [0, \frac{1}{s})$. Then T has a unique fixed point.

Proof. Let T satisfy (3.1), $x_0 \in X$ be an arbitrary point and define the sequence $\{x_n\}$ by $x_n = T^n x_0$. From (3.1), we obtain

$$G(x_n, x_{n+1}, x_{n+1}) \lesssim kG(x_{n-1}, x_n, x_n). \quad (3.2)$$

Using again (3.1), we have

$$G(x_{n-1}, x_n, x_n) \lesssim kG(x_{n-2}, x_{n-1}, x_{n-1})$$

and by (3.2), we get

$$G(x_n, x_{n+1}, x_{n+1}) \lesssim k^2 G(x_{n-2}, x_{n-1}, x_{n-1}).$$

If we continue in this way, we find

$$G(x_n, x_{n+1}, x_{n+1}) \lesssim k^n G(x_0, x_1, x_1). \quad (3.3)$$

Using (CG_b5) and (3.3) for all $n, m \in \mathbb{N}$ with $n < m$,

$$\begin{aligned} G(x_n, x_m, x_m) &\lesssim s[G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_m, x_m)] \\ &\lesssim s[G(x_n, x_{n+1}, x_{n+1})] + s^2[G(x_{n+1}, x_{n+2}, x_{n+2}) \\ &\quad + G(x_{n+2}, x_m, x_m)] \\ &\lesssim s[G(x_n, x_{n+1}, x_{n+1})] + s^2[G(x_{n+1}, x_{n+2}, x_{n+2})] + \\ &\quad s^3[G(x_{n+2}, x_{n+3}, x_{n+3})] + \dots + s^{m-n}G(x_{m-1}, x_m, x_m)] \\ &\lesssim (sk^n + s^2k^{n+1} + s^3k^{n+2} + \dots + s^{m-n}k^{m-1})G(x_0, x_1, x_1) \\ &\lesssim sk^n[1 + sk + (sk)^2 + (sk)^3 + \dots + (sk)^{m-n-1}]G(x_0, x_1, x_1) \\ &\lesssim \frac{sk^n}{1 - sk}G(x_0, x_1, x_1). \end{aligned}$$

Thus, we obtain

$$|G(x_n, x_m, x_m)| \leq \frac{sk^n}{1 - sk}|G(x_0, x_1, x_1)|.$$

Since $k \in [0, \frac{1}{s})$ where $s > 1$, taking limits as $n \rightarrow \infty$, then

$$\frac{sk^n}{1 - sk}|G(x_0, x_1, x_1)| \rightarrow 0.$$

This means that

$$|G(x_n, x_m, x_m)| \rightarrow 0.$$

By Proposition 3.2, we get

$$G(x_n, x_m, x_l) \lesssim G(x_n, x_m, x_m) + G(x_l, x_m, x_m)$$

for $n, m, l \in \mathbb{N}$. Thus,

$$|G(x_n, x_m, x_l)| \leq |G(x_n, x_m, x_m)| + |G(x_l, x_m, x_m)|.$$

If we take limit as $n, m, l \rightarrow \infty$, we obtain $|G(x_n, x_m, x_l)| \rightarrow 0$. So $\{x_n\}$ is complex valued G_b -Cauchy sequence by Theorem 3.6. Completeness of (X, G) gives us that there is an element $u \in X$ such that $\{x_n\}$ is complex valued G_b -convergent to u .

To prove $Tu = u$, we will assume the contrary. From (3.1), we obtain

$$G(x_{n+1}, Tu, Tu) \lesssim kG(x_n, u, u)$$

and

$$|G(x_{n+1}, Tu, Tu)| \leq k|G(x_n, u, u)|.$$

If we take the limit as $n \geq \infty$, we get

$$|G(u, Tu, Tu)| \leq k|G(u, u, u)|,$$

which is a contradiction since $k \in [0, \frac{1}{s})$. As a result, $Tu = u$.

Lastly, we prove the uniqueness. Let $w \neq u$ be another fixed point of T . Using (3.1),

$$G(z, w, w) = G(Tz, Tw, Tw) \lesssim kG(z, w, w).$$

and

$$|G(z, w, w)| \leq k|G(z, w, w)|.$$

Since $k \in [0, \frac{1}{s})$, we have $|G(z, w, w)| \leq 0$. Thus, $u = w$ and so u is a unique fixed point of T . \square

Example 3.8. Let $X = [-1, 1]$ and $G : X \times X \times X \rightarrow \mathbb{C}$ be defined as follows:

$$G(x, y, z) = |x - y| + |y - z| + |z - x|$$

for all $x, y, z \in X$. (X, G) is complex valued G -metric space [12]. Define

$$G_*(x, y, z) = G(x, y, z)^2.$$

G_* is a complex valued G_b -metric with $s = 2$ (see [5]). If we define $T : X \rightarrow X$ as $Tx = \frac{x}{3}$, then T satisfies the following condition for all $x, y, z \in X$:

$$G(Tx, Ty, Tz) = G\left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right) = \frac{1}{3}G(x, y, z) \lesssim kG(x, y, z)$$

where $k \in [\frac{1}{3}, \frac{1}{s})$, $s > 1$. Thus $x = 0$ is the unique fixed point of T in X .

We will prove Kannan's fixed point theorem for complex valued G_b -metric spaces.

Theorem 3.9. Let (X, G) be a complete complex valued G_b -metric space and the mapping $T : X \rightarrow X$ satisfies for every $x, y \in X$

$$G(Tx, Ty, Ty) \lesssim \alpha[G(x, Tx, Tx) + G(y, Ty, Ty)] \quad (3.4)$$

where $\alpha \in [0, \frac{1}{2})$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. We shall show that $\{x_n\}$ is G_b -Cauchy sequence. If $x_n = x_{n+1}$, then x_n is the fixed point of T . Thus, suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. Setting $G(x_n, x_{n+1}, x_{n+1}) = G_n$, it follows from (3.4) that

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+1}) &= G(Tx_{n-1}, Tx_n, Tx_n) \\ &\lesssim \alpha[G(x_{n-1}, Tx_{n-1}, Tx_{n-1}) + G(x_n, Tx_n, Tx_n)] \\ &\lesssim \alpha[G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1})] \\ &\lesssim \alpha[G_{n-1} + G_n] \\ G_n &\lesssim \frac{\alpha}{1-\alpha}G_{n-1} = \beta G_{n-1}, \end{aligned}$$

where $\beta = \frac{\alpha}{1-\alpha} < 1$ as $\alpha \in [0, \frac{1}{2})$. If we repeat this process, then we get

$$G_n \lesssim \beta^n G_0. \quad (3.5)$$

We can also suppose that x_0 is not a periodic point. If $x_n = x_0$, then we have

$$G_0 \lesssim \beta^n G_0.$$

Since $\beta < 1$, then $1 - \beta^n < 1$ and

$$(1 - \beta^n)|G_0| \leq 0 \Rightarrow |G_0| = 0.$$

It follows that x_0 is a fixed point of T . Therefore in the sequel of proof we can assume $T^n x_0 \neq x_0$ for $n = 1, 2, 3, \dots$. From inequality (3.4), we obtain

$$\begin{aligned} G(T^n x_0, T^{n+m} x_0, T^{n+m} x_0) &\lesssim \alpha[G(T^{n-1} x_0, T^{n+m} x_0, T^{n+m} x_0) \\ &\quad + G(T^{n+m-1} x_0, T^{n+m} x_0, T^{n+m} x_0)] \\ &\lesssim \alpha[\beta^{n-1} G(x_0, Tx_0, Tx_0) + \beta^{n+m-1} G(x_0, Tx_0, Tx_0)]. \end{aligned}$$

So, $|G(x_n, x_{n+m}, x_{n+m})| \rightarrow 0$ as $n \rightarrow \infty$. It implies that $\{x_n\}$ is a G_b -Cauchy in X . By the completeness of X , there exists $u \in X$ such that $x_n \rightarrow u$. From (CG_b5), we get

$$\begin{aligned} G(Tu, u, u) &\lesssim s[G(Tu, T^{n+1} x_0, T^{n+1} x_0) + G(T^{n+1} x_0, u, u)] \\ &\lesssim s[\alpha[G(u, Tu, Tu) + G(T^n x_0, T^{n+1} x_0, T^{n+1} x_0)]] + sG(T^{n+1} x_0, u, u) \\ &\lesssim s\alpha[G(u, Tu, Tu) + s\alpha G(T^n x_0, T^{n+1} x_0, T^{n+1} x_0)] + sG(T^{n+1} x_0, u, u). \end{aligned}$$

Letting $n \rightarrow \infty$, since $s\alpha < 1$ and $x_n \rightarrow u$, we have $|G(Tu, u, u)| \rightarrow 0$, i.e., $u = Tu$.

Now we show that T has a unique fixed point. For this, assume that there exists another point v in X such that $v = Tv$. Now,

$$\begin{aligned} G(v, u, u) &\lesssim G(Tv, Tu, Tu) \\ &\lesssim \alpha[G(v, Tv, Tv) + G(u, Tu, Tu)] \\ &\lesssim \alpha[G(v, v, v) + G(u, u, u)] \\ &\lesssim 0. \end{aligned}$$

Hence, we conclude that $u = v$. □

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Finite Difference approximations for the Two-side Space-time Fractional Advection-diffusion Equations*

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Abstract

Fractional order advection-diffusion equation is viewed as generalizations of classical diffusion equations, treating super-diffusive flow processes. In this paper, we present a new weighted finite difference approximation for the equation with initial and boundary conditions in a finite domain. Using mathematical induction, we prove that the weighted finite difference approximation is conditionally stable and convergent. Numerical computations are presented which demonstrate the effectiveness of the method and confirm the theoretical claims.

Keywords: Fractional order advection-diffusion equation; Weighted finite difference approximation; Stability; Convergence.

1 INTRODUCTION

In recent years, fractional differential equations have attracted much attention. Many important phenomena in physics [1, 2, 3], finance [4, 5], hydrology [6], engineering [7], mathematics [8] and material science are well described by differential equations of fractional order. These fractional order models tend to be more appropriate than the traditional integer-order models. So, the fractional derivatives are considered to be a very powerful and useful tool.

The fractional advection-diffusion equation provides a useful description of transport dynamics in complex systems which are governed by anomalous diffusion and non-exponential relaxation [9]. In this paper, we consider a special case

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of anomalous diffusion, the two-sided space-time fractional advection-diffusion equation can be written in the following way

$$\frac{\partial^\beta u(x,t)}{\partial t^\beta} = -v(x) \frac{\partial u(x,t)}{\partial x} + d_+(x) \frac{\partial^\alpha u(x,t)}{\partial_+ x^\alpha} + d_-(x) \frac{\partial^\alpha u(x,t)}{\partial_- x^\alpha} + f(x,t), \quad x \in [L, R], t \in (0, T], \quad (1)$$

$$u(L, t) = 0, u(R, t) = \varphi(t), \quad t \in [0, T], \quad (2)$$

$$u(x, 0) = u_0(x), \quad x \in (L, R], \quad (3)$$

where α and β are parameters describing the order of the space- and time-fractional derivatives, respectively, physical considerations restrict $0 < \beta < 1, 1 < \alpha < 2$. The functions $v(x, t), d_+(x, t)$ and $d_-(x, t)$ are all non-negative, bounded and $d_+(x, t), d_-(x, t) \geq v(x, t)$.

The left-sided (+) and the right-sided (−) Riemann-Liouville fractional derivatives of order α of a function $u(x, t)$ are defined as follows

$$\frac{\partial^\alpha u(x, t)}{\partial_+ x^\alpha} = \frac{1}{\Gamma(n - \alpha)} \frac{\partial^n}{\partial x^n} \int_L^x \frac{u(\xi, t)}{(x - \xi)^{\alpha+1-n}} d\xi \quad (4)$$

and

$$\frac{\partial^\alpha u(x, t)}{\partial_- x^\alpha} = \frac{(-1)^n}{\Gamma(n - \alpha)} \frac{\partial^n}{\partial x^n} \int_x^R \frac{u(\xi, t)}{(x - \xi)^{\alpha+1-n}} d\xi, \quad (5)$$

where n is an integer such that $n - 1 < \alpha \leq n$. The time derivative $\frac{\partial^\beta u(x, t)}{\partial t^\beta}$ is given by a Caputo fractional derivative

$$\frac{\partial^\beta u(x, t)}{\partial t^\beta} = \frac{1}{\Gamma(1 - \beta)} \int_0^t (t - \eta)^{-\beta} \frac{\partial u(x, \eta)}{\partial \eta} d\eta, \quad (6)$$

where $\Gamma(\cdot)$ is the gamma function.

As is well known, the fractional order differential operator is a nonlocal operator, which requires more involved computational schemes for its handling. Finite difference schemes for fractional partial differential equations are more complex than partial differential equations [1, 2, 4, 10, 11, 12, 13, 14]. It should note the following work for fractional advection-diffusion equation. Su et al. [13] presented a Crank-Nicolson type finite difference scheme for two-sided space fractional advection-diffusion equation. Liu et al. [14] considered a space-time fractional advection-diffusion with Caputo time fractional derivative and Riemann-Liouville space fractional derivatives. In this paper, we present a new weighted finite difference approximation for the equation.

The rest of the paper is as follows. In Section 2, we derive the new weighted finite difference approximation (NWFD) for the fractional advection-diffusion equation. The convergence and stability of the finite difference scheme is given in Section 3, where we apply discrete energy method. In Section 4, numerical results are shown which confirm that the numerical method is effective.

2 NEW WEIGHTED FINITE DIFFERENCE SCHEME

To present the numerical approximation scheme, we give some notations: τ is the time step, u_j^n be the numerical solution at (x_i, t_n) for $x_j = L + ih, t_n = n\tau, j = 0, 1, \dots, J, n = 0, 1, \dots, N$.

The shifted Grünwald formula is applied to discretize the left-handed fractional derivative and right-handed fractional derivative [15],

$$\frac{\partial^\alpha u(x_i, t_n)}{\partial_+ x^\alpha} = \frac{1}{h^\alpha} \sum_{j=0}^{i+1} g_j u(x_i - (j-1)h, t_n) + o(h), \quad (7)$$

$$\frac{\partial^\alpha u(x_i, t_n)}{\partial_- x^\alpha} = \frac{1}{h^\alpha} \sum_{j=0}^{N-i+1} g_j u(x_i + (j-1)h, t_n) + o(h), \quad (8)$$

where the Grünwald coefficients are defined by

$$g_0 = 1, g_j = (1 - \frac{\alpha+1}{j})g_{j-1}, \quad j = 1, 2, 3, \dots$$

Adopting the discrete scheme in [15], we discretize the Caputo time fractional derivative as,

$$\frac{\partial^\beta u(x_i, t_n)}{\partial t^\beta} = \frac{\tau^{1-\beta}}{\Gamma(2-\beta)} \sum_{j=0}^n \frac{u(x_i, t_{n+1-j}) - u(x_i, t_{n-j})}{\tau} \sigma_j + o(\tau),$$

where $\sigma_j = (j+1)^{1-\beta} - j^{1-\beta}$.

Now we replace (1) with the following weighted finite difference approximation:

$$\begin{aligned} & \frac{\tau^{1-\beta}}{\Gamma(2-\beta)} \sum_{j=0}^n \frac{u_i^{n+1-j} - u_i^{n-j}}{\tau} \sigma_j = -v_i [\theta \frac{u_{i+1}^n - u_{i-1}^n}{2h} \\ & + (1-\theta) \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2h}] + \frac{d_{+i}}{h^\alpha} [\theta \sum_{k=0}^{i+1} g_k u_{i-k+1}^n \\ & + (1-\theta) \sum_{k=0}^{i+1} g_k u_{i-k+1}^{n+1}] + \frac{d_{-i}}{h^\alpha} [\theta \sum_{k=0}^{N-i+1} g_k u_{i+k-1}^n \\ & + (1-\theta) \sum_{k=0}^{N-i+1} g_k u_{i+k-1}^{n+1}] + \theta f_i^n + (1-\theta) f_i^{n+1}, \end{aligned} \quad (9)$$

for $i = 1, 2, \dots, J-1, n = 0, 1, \dots, N-1$, where θ is the weighting parameter subjected to $0 \leq \theta \leq 1$. When $\theta = 0, 1, \frac{1}{2}$, we get the space-time fractional implicit, explicit, Crank-Nicolson type difference scheme, respectively.

The above equation (9) can be simplified, for $n = 0$,

$$\begin{aligned}
& -(1-\theta)(\xi_i + \eta_i g_2 + \zeta_i)u_{i-1}^1 - (1-\theta)\eta_i \sum_{k=3}^{i+1} g_k u_{i-k+1}^1 \\
& -(1-\theta)\zeta_i \sum_{k=3}^{J-i+1} g_k u_{i+k-1}^1 + (1-\theta)(\xi_i - \eta_i - \zeta_i g_2)u_{i+1}^1 \\
& + [1 - (1-\theta)(\eta_i g_1 + \zeta_i g_1)]u_i^1 = \theta(\xi_i + \eta_i g_2 + \zeta_i)u_{i-1}^0 \\
& + [1 + \theta(\eta_i g_1 + \zeta_i g_1)]u_i^0 + \theta(-\xi_i + \eta_i + \zeta_i g_2)u_{i+1}^0 \\
& + \theta\eta_i \sum_{k=3}^{i+1} g_k u_{i-k+1}^0 + \theta\zeta_i \sum_{k=3}^{J-i+1} g_k u_{i+k-1}^0 \\
& + \Gamma(1-\beta)\tau^\beta(\theta f_i^0 + (1-\theta)f_i^1), \tag{10}
\end{aligned}$$

and for $n > 0$,

$$\begin{aligned}
& -(1-\theta)(\xi_i + \eta_i g_2 + \zeta_i)u_{i-1}^{n+1} - (1-\theta)\eta_i \sum_{k=3}^{i+1} g_k u_{i-k+1}^{n+1} \\
& -(1-\theta)\zeta_i \sum_{k=3}^{J-i+1} g_k u_{i+k-1}^{n+1} + (1-\theta)(\xi_i - \eta_i - \zeta_i g_2)u_{i+1}^{n+1} \\
& + [1 - (1-\theta)(\eta_i g_1 + \zeta_i g_1)]u_i^{n+1} = \theta(\xi_i + \eta_i g_2 + \zeta_i)u_{i-1}^n \\
& + [2 - 2^{1-\beta} + \theta(\eta_i g_1 + \zeta_i g_1)]u_i^n + \theta(-\xi_i + \eta_i + \zeta_i g_2)u_{i+1}^n \\
& + \theta\eta_i \sum_{k=3}^{i+1} g_k u_{i-k+1}^n + \theta\zeta_i \sum_{k=3}^{J-i+1} g_k u_{i+k-1}^n + \sum_{j=1}^{n-1} d_j u_i^{n-j} \\
& + u_i^0 \sigma_n + \Gamma(1-\beta)\tau^\beta(\theta f_i^n + (1-\theta)f_i^{n+1}), \tag{11}
\end{aligned}$$

and Dirichlet boundary conditions

$$u_0^n = 0, u_J^n = \varphi(t_n), \quad n = 1, 2, \dots, N-1,$$

and initial conditions

$$u_i^0 = u_0(x_i), \quad i = 0, 1, \dots, J,$$

where $\xi_i = \frac{v_i \tau^\beta \Gamma(2-\beta)}{2h}$, $\eta_i = \frac{d_{+i} \tau^\beta \Gamma(2-\beta)}{h^\alpha}$, $\zeta_i = \frac{d_{-i} \tau^\beta \Gamma(2-\beta)}{h^\alpha}$ and $d_j = \sigma_{j+1} - \sigma_j$, $j = 1, 2, \dots, n-1$.

The numerical method (10) and (11) can be written in the matrix form:

$$\begin{aligned}
AU^1 &= B_0 U^0 + Q^0, \\
AU^{n+1} &= BU^n + d_1 U^{n-1} + \dots + d_{n-1} U^1 + \sigma_n U^0 + Q^n,
\end{aligned}$$

where

$$U^n = (u_1^n, u_2^n, \dots, u_{J-1}^n)^T,$$

$$\begin{aligned}
U^0 &= [u_0(x_1), u_0(x_2), \dots, u_0(x_{J-1})]^T, \\
b &= (\eta_{J-1} + \zeta_{J-1}g_2)[(1-\theta)u_J^{n+1} + \theta u_J^n], \\
F^n &= (f_1^n, f_2^n, \dots, f_{J-1}^n + b)^T, \\
E &= (\zeta_1 g_J, \zeta_2 g_{J-1}, \dots, \zeta_{J-1} g_2)^T, \\
Q^n &= \Gamma(2-\beta)\tau^\beta(\theta F^n + (1-\theta)F^{n+1}) \\
&\quad + (1-\theta)U_J^{n+1}E + \theta U_J^n E,
\end{aligned}$$

and matrix $A = (A_{ij})_{(J-1) \times (J-1)}$ is defined as follows:

$$A_{ij} = \begin{cases} -(1-\theta)(\xi_i + \eta_i g_2 + \zeta_i), & j = i-1, \\ 1 - (1-\theta)(\eta_i g_1 + \zeta_i g_1), & j = i, \\ (1-\theta)(\xi_i - \eta_i - \zeta_i g_2), & j = i+1, \\ -(1-\theta)\eta_i g_{i+1-j}, & j = 1, 2, \dots, i-2, \\ -(1-\theta)\zeta_i g_{j+1-i}, & j = i+2, i+3, \dots, J-1. \end{cases}$$

It is obvious that matrix A is strictly dominant, the system defined by (10) and (11) has unique solution.

3 STABILITY AND CONVERGENCE

In this section, we investigate the stability and convergence of the numerical scheme (9).

Theorem 1 For

$$\frac{\theta \alpha \Gamma(2-\beta)\tau^\beta}{h^\alpha} \max_{x \in [L, R]} (d_+(x) + d_-(x)) \leq 2 - 2^{1-\beta}, \quad (12)$$

the weighted finite difference scheme (9) for solving equation (1)-(3) is stable.

Proof. Let $u_i^n, \tilde{u}_i^n (i = 1, 2, \dots, J, n = 0, 1, 2, \dots, N-1)$ be the numerical solutions of (9) corresponding to the initial data u_i^0 and \tilde{u}_i^0 , respectively. Let $\varepsilon_i^n = \tilde{u}_i^n - u_i^n$, the stability condition is equivalent to

$$\|E^n\|_\infty \leq \|E^0\|_\infty, \quad n = 0, 1, \dots, N-1, \quad (13)$$

where $E^n = (\varepsilon_1^n, \varepsilon_2^n, \dots, \varepsilon_{J-1}^n)$. We will use mathematical induction to get the above result.

For $n = 0$, we have

$$\begin{aligned}
& -(1-\theta)[(\xi_i + \eta_i g_2 + \zeta_i)\varepsilon_{i-1}^1 + \eta_i \sum_{k=3}^{i+1} g_k \varepsilon_{i-k+1}^1 \\
& + \zeta_i \sum_{k=3}^{J-i+1} g_k \varepsilon_{i+k-1}^1 - (\xi_i - \eta_i - \zeta_i g_2)\varepsilon_{i+1}^1]
\end{aligned}$$

$$\begin{aligned}
& +[1 - (1 - \theta)(\eta_i g_1 + \zeta_i g_1)]\varepsilon_i^1 = \theta[(\xi_i + \eta_i g_2 + \zeta_i)\varepsilon_{i-1}^0 \\
& + \zeta_i \sum_{k=3}^{J-i+1} g_k \varepsilon_{i+k-1}^0 + (-\xi_i + \eta_i + \zeta_i g_2)\varepsilon_{i+1}^0 \\
& + \eta_i \sum_{k=3}^{i+1} g_k \varepsilon_{i-k+1}^0] + [1 + \theta(\eta_i g_1 + \zeta_i g_1)]\varepsilon_i^0, \tag{14}
\end{aligned}$$

for $n > 0$,

$$\begin{aligned}
& -(1 - \theta)[(\xi_i + \eta_i g_2 + \zeta_i)\varepsilon_{i-1}^{n+1} + \eta_i \sum_{k=3}^{i+1} g_k \varepsilon_{i-k+1}^{n+1} \\
& + \zeta_i \sum_{k=3}^{J-i+1} g_k \varepsilon_{i+k-1}^{n+1} - (\xi_i - \eta_i - \zeta_i g_2)\varepsilon_{i+1}^{n+1}] \\
& + [1 - (1 - \theta)(\eta_i g_1 + \zeta_i g_1)]\varepsilon_i^{n+1} = \sum_{j=1}^{n-1} d_j \varepsilon_i^{n-j} \\
& + \sigma_n \varepsilon_i^0 + \theta[(-\xi_i + \eta_i + \zeta_i g_2)\varepsilon_{i+1}^n + \eta_i \sum_{k=3}^{i+1} g_k \varepsilon_{i-k+1}^n \\
& + \zeta_i \sum_{k=3}^{J-i+1} g_k \varepsilon_{i+k-1}^n + (\xi_i + \eta_i g_2 + \zeta_i)\varepsilon_{i-1}^n] \\
& + [2 - 2^{1-\beta} + \theta(\eta_i g_1 + \zeta_i g_1)]\varepsilon_i^n. \tag{15}
\end{aligned}$$

Note that $d_+(x, t), d_-(x, t) \geq v(x, t)$, we have

$$\xi_i - \eta_i - \zeta_i g_2 \leq 0. \tag{16}$$

In fact, if $n = 0$, suppose $|\varepsilon_l^1| = \max_{1 \leq i \leq J-1} |\varepsilon_i^1|$, note that $\xi_i, \eta_i, \zeta_i > 0$ and for any integer number m , $\sum_{j=0}^m g_j < 0$, from (12), (16), we derive

$$\begin{aligned}
\|E^1\|_\infty &= |\varepsilon_l^1| \leq -(1 - \theta)\eta_l \sum_{k=0}^{l+1} g_k |\varepsilon_l^1| + |\varepsilon_l^1| - (1 - \theta)\zeta_l \sum_{k=0}^{J-l+1} |\varepsilon_l^1| \\
&\leq |-(1 - \theta)[(\xi_l + \eta_l g_2 + \zeta_l)\varepsilon_{l-1}^1 + \zeta_l \sum_{k=3}^{J-l+1} g_k \varepsilon_{l+k-1}^1 \\
&\quad + (\eta_l + \zeta_l g_2 - \xi_l)\varepsilon_{l+1}^1 + \eta_l \sum_{k=3}^{l+1} g_k \varepsilon_{l-k+1}^1] \\
&\quad + [1 - (1 - \theta)(\eta_l g_1 + \zeta_l g_1)]\varepsilon_l^1| \\
&\leq \theta[(\xi_l + \eta_l g_2 + \zeta_l)|\varepsilon_{l-1}^0| + \zeta_l \sum_{k=3}^{J-l+1} g_k |\varepsilon_{l+k-1}^0|
\end{aligned}$$

$$\begin{aligned}
& +(\eta_l + \zeta_l g_2)|\varepsilon_{l+1}^0| + \eta_l \sum_{k=3}^{l+1} g_k |\varepsilon_{l-k+1}^0| \\
& +[1 - \theta(\xi_l - \eta_l g_1 - \zeta_l g_1)]|\varepsilon_l^0| \leq \|E^0\|_\infty,
\end{aligned}$$

Suppose that $\|E^n\|_\infty \leq \|E^0\|_\infty$, $n = 1, 2, \dots, s$, then when $n = s + 1$, let $|\varepsilon_l^{s+1}| = \max_{1 \leq i \leq J-1} |\varepsilon_i^{s+1}|$. Similar to former estimate, we obtain

$$\begin{aligned}
\|E^{s+1}\|_\infty & \leq |(1 - \theta)[(\xi_l + \eta_l g_2 + \zeta_l)\varepsilon_{l-1}^{n+1} + \eta_l \sum_{k=3}^{l+1} g_k \varepsilon_{l-k+1}^{n+1} \\
& + \zeta_l \sum_{k=3}^{J-l+1} g_k \varepsilon_{l+k-1}^{n+1} - (\xi_l - \eta_l - \zeta_l g_2)\varepsilon_{l+1}^{n+1}] \\
& + [1 - (1 - \theta)(\eta_l g_1 + \zeta_l g_1)]\varepsilon_l^{n+1}| \\
& \leq \theta(\xi_l + \eta_l g_2 + \zeta_l)|\varepsilon_{l-1}^s| + \theta(-\xi_l + \eta_l + \zeta_l g_2)|\varepsilon_{l+1}^s| \\
& + [2 - 2^{1-\beta} + \theta(\eta_l g_1 + \zeta_l g_1)]|\varepsilon_l^s| + \theta \eta_l \sum_{k=3}^{l+1} g_k |\varepsilon_{l-k+1}^s| \\
& + \theta \zeta_l \sum_{k=3}^{J-l+1} g_k |\varepsilon_{l+k-1}^s| + \sum_{j=1}^{s-1} d_j |\varepsilon_l^{s-j}| + \sigma_s |\varepsilon_l^0| \\
& \leq \|E^0\|_\infty.
\end{aligned}$$

Hence, $\|E^{s+1}\|_\infty \leq \|E^0\|_\infty$. The proof is completed.

Theorem 2 Suppose that $u(x, t)$ is the sufficiently smooth solution of (1)-(3) and u_i^k is the difference solution of difference scheme (9). If the condition (12) is satisfied, then

$$\|u(x_i, t_n) - u_i^n\|_\infty \leq M \sigma_{n-1}^{-1} (\tau^{1+\beta} + \tau^\beta h),$$

where M is a positive constant.

Proof. Define $e_i^n = u(x_i, t_n) - u_i^n$ and $e^n = (e_1^n, e_2^n, \dots, e_{J-1}^n)$. Notice that $e_j^0 = 0$, we have: when $n = 0$,

$$\begin{aligned}
& -(1 - \theta)[(\xi_i + \eta_i g_2 + \zeta_i)e_{i-1}^1 + \eta_i \sum_{k=3}^{i+1} g_k e_{i-k+1}^1 \\
& + \zeta_i \sum_{k=3}^{J-i+1} g_k e_{i+k-1}^1 - (\xi_i - \eta_i - \zeta_i g_2)e_{i+1}^1] \\
& + [1 - (1 - \theta)(\eta_i g_1 + \zeta_i g_1)]e_i^1 = R_i^1,
\end{aligned} \tag{17}$$

when $n > 0$,

$$-(1 - \theta)[(\xi_i + \eta_i g_2 + \zeta_i)e_{i-1}^{n+1} + \eta_i \sum_{k=3}^{i+1} g_k e_{i-k+1}^{n+1}$$

$$\begin{aligned}
& +\zeta_i \sum_{k=3}^{J-i+1} g_k e_{i+k-1}^{n+1} - (\xi_i - \eta_i - \zeta_i g_2) e_{i+1}^{n+1}] \\
& + [1 - (1 - \theta)(\eta_i g_1 + \zeta_i g_1)] e_i^{n+1} - \theta \eta_i \sum_{k=3}^{i+1} g_k e_{i-k+1}^n \\
& - [2 - 2^{1-\beta} + \theta(\eta_i g_1 + \zeta_i g_1)] e_i^n - \theta \zeta_i \sum_{k=3}^{J-i+1} g_k e_{i+k-1}^n \\
& - \theta(\xi_i + \eta_i g_2 + \zeta_i) e_{i-1}^n - \sum_{j=1}^{n-1} d_j e_i^{n-j} \\
& - \theta(-\xi_i + \eta_i + \zeta_i g_2) e_{i+1}^n = R_i^{n+1}, \tag{18}
\end{aligned}$$

where R_i^{n+1} is the truncation error of difference scheme (9). Furthermore, there exists a positive constant M independent of step sizes such that $|R_i^{n+1}| \leq M(\tau^{1+\beta} + \tau^\beta h)$.

We will prove by inductive method. Let $|e_l^1| = \max_{1 \leq i \leq J-1} |e_i^1|$. If $k = 1$, subject to the condition (12), based on (17), we have

$$\begin{aligned}
\|e^1\|_\infty & \leq |-(1 - \theta)[(\xi_i + \eta_i g_2 + \zeta_i) e_{i-1}^1 + \eta_i \sum_{k=3}^{i+1} g_k e_{i-k+1}^1] \\
& + \zeta_i \sum_{k=3}^{J-i+1} g_k e_{i+k-1}^1 - (\xi_i - \eta_i - \zeta_i g_2) e_{i+1}^1] \\
& + [1 - (1 - \theta)(\eta_i g_1 + \zeta_i g_1)] e_i^1| \\
& \leq M(\tau^{1+\beta} + \tau^\beta h) = \sigma_0^{-1} M(\tau^{1+\beta} + \tau^\beta h).
\end{aligned}$$

Assume that $\|e^n\|_\infty \leq M \sigma_{n-1}^{-1}(\tau^{1+\beta} + \tau^\beta h)$, $n = 1, 2, \dots, s$, then when $n = s + 1$, let $|e_l^{s+1}| = \max_{1 \leq i \leq J-1} |e_i^{s+1}|$, notice that $\sigma_j^{-1} < \sigma_k^{-1}$, $j = 0, 1, \dots, k - 1$. Similarly, we obtain

$$\begin{aligned}
\|e^{s+1}\|_\infty & \leq d_1 \|e^s\|_\infty + \sum_{j=1}^{n-1} d_j \|e^{s-j}\|_\infty + M(\tau^{1+\beta} + \tau^\beta h) \\
& \leq (d_1 \sigma_{s-1}^{-1} + d_2 \sigma_{s-1}^{-1} + \dots + d_s \sigma_0^{-1} + 1) M(\tau^{1+\beta} + \tau^\beta h) \\
& \leq \sigma_s^{-1} M(\tau^{1+\beta} + \tau^\beta h).
\end{aligned}$$

Thus, the proof is completed.

In additional, since

$$\lim_{n \rightarrow \infty} \frac{\sigma_n^{-1}}{n^\beta} = \lim_{n \rightarrow \infty} \frac{n^{-1}}{(1 - \beta)n^{-1}} = \frac{1}{1 - \beta}.$$

there is a constant C_1 for which

$$\|e^n\|_\infty \leq C_1 n^\beta (\tau^{1+\beta} + \tau^\beta h).$$

and $n\tau \leq T$ is finite, we obtain the following result.

Theorem 3 *Under the conditions of Theorem 2, then numerical solution converges to exact solution as h and τ tend to zero. Furthermore there exists positive constant $C > 0$, such that*

$$\|u(x_i, t_n) - u_i^n\| \leq C(\tau + h),$$

where $i = 1, 2, \dots, J-1; n = 1, 2, \dots, N$.

4 NUMERICAL RESULTS

In this section, the following two-sided space-time fractional advection-diffusion equation in a bounded domain is considered in [15]:

$$\begin{aligned} \frac{\partial^{0.6} u(x, t)}{\partial t^{0.6}} &= -\frac{\partial u(x, t)}{\partial x} + d_+(x, t) \frac{\partial^{1.6} u(x, t)}{\partial_+ x^{1.6}} \\ &+ d_-(x, t) \frac{\partial^{1.6} u(x, t)}{\partial_- x^{1.6}} + f(x, t), \quad (x, t) \in [0, 1] \times [0, 1] \\ u(0, t) &= 0, \quad u(1, t) = 1 + 4t^2, \quad t \in [0, 1], \\ u(x, 0) &= x^2, \quad x \in [0, 1], \end{aligned}$$

where $d_+(x, t) = \frac{2}{5}\Gamma(0.4)x^{0.6}$, $d_-(x, t) = 5\Gamma(0.4)(1-x)^{1.6}$, and $f(x, t) = \frac{100}{7\Gamma(0.4)}x^2t^{1.4} + (1 + 4t^2)(-25x^2 + 40x - 12)$. The exact solution is $u(x, t) = (1 + 4t^2)x^2$.

Table 1: The error $\max |u_i^k - u(x_i, t^k)|$ for the IWFDMS with $\theta = 1$

N	J	State	The error
10	10	Divergence	1.1305e+019
100	10	Divergence	2.3237e+163
10000	10	Divergence	Infinity
30000	10	Convergence	1.3230

Table 1 shows the maximum absolute numerical error between the exact solution and the numerical solution obtained by NWFDMS with $\theta = 1$. From Table 1, it can see that our scheme is conditionally stable.

Table 2 and Table 3 show the maximum absolute error, at time $t = 1.0$, between the exact analytical solution and the numerical solution obtained by NWFDMS with $\theta = 1/2$ and $\theta = 0$, respectively.

Table 4 and Table 5 show the comparison of maximum absolute numerical error of the weighted finite difference scheme in [12] (WFDMS) and new weighted finite difference (NWFDMS). We can see that the NNWDM is more accurate than WFDMS at $\theta = 0$, but at $\theta = 0.4$ is opposite. From the above five tables, it can be seen that the numerical tests are in excellent agreement with theoretical analysis.

Table 2: The error and convergence rate for the scheme with $\theta = 1/2$

N	J	Maximum error	Convergence rate
200	200	0.0809	-
400	400	0.0486	1.6646
800	800	0.0298	1.6309
1600	1600	0.0055	1.6022

Table 3: The error and convergence rate for the scheme with $\theta = 0$

N	J	Maximum error	Convergence rate
200	200	0.0415	-
400	400	0.0209	1.9378
800	800	0.0107	1.9533
1600	1600	0.0054	1.9815

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Table 4: The comparison of two schemes with $\theta = 0$

N	J	NWFDM	WFDM
50	50	0.1514	0.1522
100	100	0.0783	0.0797
150	150	0.0533	0.0545
200	200	0.0405	0.0415

Table 5: The comparison of two schemes with $\theta = 0.4$

N	J	NWFDM	WFDM
50	50	0.2198	0.1498
100	100	0.1242	0.0785
150	150	0.0899	0.0537
200	200	0.0717	0.0409

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A modified Newton-Shamanskii method for a nonsymmetric algebraic Riccati equation*

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Abstract

The non-symmetric algebraic Riccati equation arising in transport theory can be rewritten as a vector equation and the minimal positive solution of the non-symmetric algebraic Riccati equation can be obtained by solving the vector equation. In this paper, based on the Newton-Shamanskii method, we propose a new iterative method called modified Newton-Shamanskii method for solving the vector equation. Some convergence results are presented. The convergence analysis shows that sequence of vectors generated by the modified Newton-Shamanskii method is monotonically increasing and converges to the minimal positive solution of the vector equation. Finally, numerical experiments are presented to illustrate the performance of the modified Newton-Shamanskii method.

Key words: non-symmetric algebraic Riccati equation; M -matrix; transport theory; minimal positive solution; modified Newton-Shamanskii method.

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1 Introduction

For convenience, firstly, we give some definitions and notations. For any matrices $A = [a_{i,j}]$ and $B = [b_{i,j}] \in R^{m \times n}$, we write $A \geq B$ ($A > B$) if $a_{i,j} \geq b_{i,j}$ ($a_{i,j} > b_{i,j}$)

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holds for all i, j . The Hadamard product of A and B is defined by $A \circ B = [a_{i,j} \cdot b_{i,j}]$. I denotes the identity matrix with appropriate dimension. The superscript T denotes the transpose of a vector or a matrix. We denote the norm by $\|\cdot\|$ for a vector or a matrix.

In this paper we are interested in iteratively solving the following nonsymmetric algebraic Riccati equation (NARE) arising in transport theory (see [3–5, 21] and the references cited therein):

$$XCX - XE - AX + B = 0, \quad (1.1)$$

where $A, B, C, E \in R^{n \times n}$ have the following special form:

$$A = \Delta - eq^T, \quad B = ee^T, \quad C = qq^T, \quad E = D - qe^T. \quad (1.2)$$

Here and in the following, $e = (1, 1, \dots, 1)^T$, $q = (q_1, q_2, \dots, q_n)^T$ with $q_i = c_i/2\omega_i$,

$$\begin{cases} \Delta = \text{diag}(\delta_1, \delta_2, \dots, \delta_n) & \text{with } \delta_i = \frac{1}{c\omega_i(1+\alpha)}, \\ D = \text{diag}(d_1, d_2, \dots, d_n) & \text{with } d_i = \frac{1}{c\omega_i(1-\alpha)}, \end{cases} \quad (1.3)$$

and

$$0 < c \leq 1, \quad 0 \leq \alpha < 1, \quad 0 < \omega_n < \dots < \omega_2 < \omega_1 < 1 \quad (1.4)$$

$\sum_{i=1}^n c_i = 1$, $c_i > 0$, $i = 1, 2, \dots, n$.

The form of the Riccati equation (1.1) arises in Markov models [22] and in nuclear physics [3, 24], and it has many positive solutions in the componentwise sense. There have been a lot of studies about algebraic properties [11, 21] and iterative methods for the nonnegative solution of the nonsymmetric algebraic Riccati equations (1.1), including the basic fixed-point iterations [5–8, 19], the doubling algorithm [9], the Schur method [23, 28], the Matrix Sign Function method [13, 25] and the alternately linearized implicit iteration method [15], and so on; see related references therein. The existence of positive solutions of (1.1) has been shown in [3] and [4], but only the minimal positive solution is physically meaningful. So it is important to develop some effective and efficient procedures to compute the minimal positive solution of Equation (1.1).

Recently, Lu [10] has shown that the matrix equation (1.1) is equivalent to a vector equation and has developed a simple and efficient iterative procedure to compute the minimal positive solution of (1.1). The fixed-point iteration methods were further studied in [14, 16] for solving the vector equation. In [14] Bai, Gao and Lu proposed two nonlinear splitting iteration methods: the nonlinear block Jacobi and the nonlinear block Gauss-Seidel iteration methods. In [16] Bao, Lin and Wei proposed a modified simple iteration method for solving the vector equation. Furthermore, the convergence rates of various fixed-point iterations [10, 14, 16] were determined and compared in [20].

The Newton method has been presented and analyzed by Lu for solving the vector equation in [12]. It has been shown that the Newton method for the vector equation is more simple and efficient than using the corresponding Newton method directly for the original Riccati equation (1.1). Li, Huang and Zhang present a relaxed Newton-like method [17] for solving the vector equation. Especially, in [18] Lin and Bao applied the Newton-Shamanskii method [2, 26] to solve the vector equation.

Based on the Newton-Shamanskii method [18], in this paper, we propose a modified Newton-Shamanskii method to solve the vector equation. The convergence analysis shows that the sequence of vectors generated by the new iterative method is monotonically increasing and converges to the minimal positive solution of the vector equation, which can be used to obtain the minimal positive solution of the original Riccati equation. Our method extends the recent work done by Lu [12] and Lin and Bao [18].

Now, we give the definition of Z -matrix and M -matrix, and also give the following two Lemmas which will be used later.

Definition 1 [1] *A real square matrix A is called a Z -matrix if all its off-diagonal elements are non-positive. Any Z -matrix A can be written as $A = sI - B$ with $B \geq 0$, $s > 0$.*

Definition 2 [1] *Any matrix A of the form $A = sI - B$ for which $s > \rho(B)$, the spectral radius of B , is called an M -matrix.*

Lemma 1.1 [1] *For a Z -matrix A , the following statements are equivalent:*

- (1) *A is a nonsingular M -matrix;*
- (2) *A is nonsingular and $A^{-1} \geq 0$;*
- (3) *$Av > 0$ for some vector $v \geq 0$.*

Lemma 1.2 [1] *Let $A \in R^{n \times n}$ be a nonsingular M -matrix. If $B \in R^{n \times n}$ is a Z -matrix and satisfies the relation $B \geq A$, then $B \in R^{n \times n}$ is also a nonsingular M -matrix.*

The rest of the paper is organized as follows. In Section 2, we review the Newton-Shamanskii method and some useful results, and present the modified Newton-Shamanskii method. Some convergence results are given in Section 3. Section 4 and 5 give numerical experiments and conclusions, respectively.

2 The modified Newton-Shamanskii method

It has been shown in [10, 12] that the solution of (1.1) must have the following form:

$$X = T \circ (uv^T) = (uv^T) \circ T,$$

where $T = [t_{i,j}] = [1/(\delta_i + d_j)]$ and u, v are two vectors, which satisfy the vector equations:

$$\begin{cases} u = u \circ (Pv) + e, \\ v = v \circ (\tilde{P}u) + e, \end{cases} \quad (2.1)$$

where $P = [p_{i,j}] = [q_j/(\delta_i + d_j)]$, $\tilde{P} = [\tilde{p}_{i,j}] = [q_j/(\delta_j + d_i)]$. Define $w = [u^T, v^T]^T$. The equation (2.1) can be rewritten equivalently as

$$f(w) = w - w \circ \mathcal{P}w - e = 0, \quad (2.2)$$

where

$$\mathcal{P} = \begin{bmatrix} 0 & P \\ \tilde{P} & 0 \end{bmatrix}.$$

The minimal positive solution of (1.1) can be obtained via computing the minimal positive solution of the vector equation (2.2).

The Newton method presented by Lu in [12] for the vector equation (2.2) is the following:

$$w_{k+1} = w_k - f'(w_k)^{-1}f(w_k), \quad k = 0, 1, 2, \dots$$

where for any $w \in R^{2n}$, the Jacobian matrix $f'(w)$ of $f(w)$ is given by

$$f'(w) = I_{2n} - G(w), \quad \text{with } G(w) = \begin{bmatrix} G_1(v) & H_1(u) \\ H_2(v) & G_2(u) \end{bmatrix} \quad (2.3)$$

where $G_1(v) = \text{diag}(Pv)$, $G_2(u) = \text{diag}(\tilde{P}u)$, $H_1(u) = [u \circ p_1, u \circ p_2, \dots, u \circ p_n]$ and $H_2(v) = [v \circ \tilde{p}_1, v \circ \tilde{p}_2, \dots, v \circ \tilde{p}_n]$. For $i = 1, 2, \dots, n$, p_i and \tilde{p}_i are the i th column of P and \tilde{P} , respectively. Obviously, when $w > 0$, $G(w) \geq 0$ and $f'(w)$ is a Z -matrix.

The Newton-Shamanskii method for solving the vector equation (2.2) is given in [18] as follows:

Algorithm 2.1 (Newton-Shamanskii method) *For a given $m \geq 1$ and $k = 0, 1, 2, \dots$,*

$$\begin{cases} \tilde{w}_{k,1} = w_k - f'(w_k)^{-1}f(w_k), \\ \tilde{w}_{k,p+1} = \tilde{w}_{k,p} - f'(w_k)^{-1}f(\tilde{w}_{k,p}), \quad 1 \leq p \leq m-1, \\ w_{k+1} = \tilde{w}_{k,m}. \end{cases} \quad (2.4)$$

It has been shown in [18] that the Newton-Shamanskii method has a better convergence than the Newton method [12]. However, if the inversion of the Jacobian matrix $f'(w)$ is difficult to compute, the Newton-Shamanskii method may converge slowly. Hence, based on the Newton-Shamanskii method, we propose the following modified Newton-Shamanskii method:

Algorithm 2.2 (Modified Newton-Shamanskii method) *For a given $m \geq 1$ and $k = 0, 1, 2, \dots$, the Modified Newton-Shamanskii method is defined as follows:*

$$\begin{cases} \tilde{w}_{k,1} = w_k - T_k^{-1} f(w_k), \\ \tilde{w}_{k,p+1} = \tilde{w}_{k,p} - T_k^{-1} f(\tilde{w}_{k,p}), & 1 \leq p \leq m-1, \\ w_{k+1} = \tilde{w}_{k,m}. \end{cases} \quad (2.5)$$

where T_k is a Z -matrix and $T_k \geq f'(w_k)$.

Remark 2.1 When $T_k = f'(w_k)$, the modified Newton-Shamanskii method becomes the Newton-Shamanskii method [18]. When $m = 1$ and $T_k = f'(w_k)$, the modified Newton-Shamanskii method becomes the Newton method [12].

Before we give the convergence analysis of the Modified Newton-Shamanskii method, let us now state some results which are indispensable for our subsequent discussions.

Lemma 2.1 [18] *For any vectors $w_1, w_2 \in R^{2n}$, $f'(w_1) - f'(w_2) = G(w_2 - w_1)$. Furthermore, if $w_2 > w_1$, we have $f'(w_1) - f'(w_2) = G(w_2 - w_1) \geq 0$.*

Here and in the subsequent section, for convenience, $[f''(w)y]y$ is defined as $f''(w)y^2$. Let

$$f''(w)y = [L_1y, L_2y, \dots, L_{2n}y]^T \in R^{2n \times 2n},$$

where $L_i \in R^{2n \times 2n}$, $y \in R^{2n}$ and for $k = 1, 2, \dots, n$,

$$L_k = \begin{bmatrix} 0 & (-e_k P_k^T) \\ (-e_k P_k^T)^T & 0 \end{bmatrix}, \quad L_{n+k} = \begin{bmatrix} 0 & (-\tilde{P}_k e_k^T) \\ (-\tilde{P}_k e_k^T)^T & 0 \end{bmatrix}$$

with $e_k^T = (0, \dots, 0, 1, 0, \dots)$, P_k^T and \tilde{P}_k^T are the k th rows of the matrices P and \tilde{P} , respectively.

Lemma 2.2 [12] *For any vectors $w_+, w \in R^{2n}$, we have*

$$f(w_+) = f(w) + f'(w)(w_+ - w) + \frac{1}{2}f''(w)(w_+ - w, w_+ - w). \quad (2.6)$$

In particular, if $w_+ = w_$, the minimal positive solution of (2.2), then*

$$0 = f(w) + f'(w)(w_* - w) + \frac{1}{2}f''(w)(w_* - w, w_* - w). \quad (2.7)$$

Furthermore, for any $y > 0$ or $y < 0$,

$$f''(w)y^2 < 0 \quad (2.8)$$

and $f''(w)y^2$ is independent of w .

Because of the independence, in the following, we denote the operator $f''(w)$ by \mathcal{L} , i.e., $\mathcal{L}(y, y) = f''(w)(y, y)$ for any $y \in R^{2n}$. By (2.7), we have

$$f(w) = f'(w)(w - w_*) - \frac{1}{2}\mathcal{L}(w - w_*, w - w_*), \quad (2.9)$$

$$f'(w)(w - w_*) = f(w) + \frac{1}{2}\mathcal{L}(w - w_*, w - w_*). \quad (2.10)$$

Lemma 2.3 [12] *If $0 \leq w < w_*$ and $f(w) < 0$, then $f'(w)$ is a nonsingular M -matrix.*

3 Convergence analysis of the Modified Newton-Shamanskii method

Now, we analyse convergence of the modified Newton-Shamanskii method (2.5).

Theorem 3.1 *Given a vector $w_k \in R^{2n}$. $\tilde{w}_{k,1}, \tilde{w}_{k,2}, \dots, \tilde{w}_{k,m}, w_{k+1}$ are obtained by the modified Newton-Shamanskii method (2.5). If $w_k < w_*$ and $f(w_k) < 0$, then, $f'(w_k)$ is a nonsingular M -matrix, moreover,*

- (1) $w_k < \tilde{w}_{k,1} < \tilde{w}_{k,2} < \dots < \tilde{w}_{k,m} = w_{k+1} < w_*$;
- (2) $f(\tilde{w}_{k,p}) < 0$ for $p = 1, 2, \dots, m$;
- (3) $f'(\tilde{w}_{k,p})$ is a nonsingular M -matrix for $p = 1, 2, \dots, m$.

Therefore, $w_{k+1} < w_$, $f(w_{k+1}) < 0$ and $f'(w_{k+1})$ is a nonsingular M -matrix.*

Proof. Since $w_k < w_*$ and $f(w_k) < 0$, by Lemma 2.3, we can easily obtain that $f'(w_k)$ is a nonsingular M -matrix. By Lemma 1.2, we can conclude that T_k is also a nonsingular M -matrix. Now, we prove the theorem by mathematical induction. Define the error vectors $\tilde{e}_{k,i} = \tilde{w}_{k,i} - w_*$ and $e_k = w_k - w_*$, then $e_k < 0$. For $p = 1$, we have $\tilde{w}_{k,1} = w_k - T_k^{-1}f(w_k)$. Since $f(w_k) < 0$ and T_k is also a nonsingular M -matrix, then $\tilde{w}_{k,1} > w_k$ by Lemma 1.1.

By Eqs. (2.5) and (2.9), we obtain

$$\begin{aligned} \tilde{e}_{k,1} &= e_k - T_k^{-1}f(w_k) \\ &= e_k - T_k^{-1}[f'(w_k)e_k - \frac{1}{2}\mathcal{L}(e_k, e_k)] \\ &= T_k^{-1}[T_k - f'(w_k)]e_k + \frac{1}{2}T_k^{-1}\mathcal{L}(e_k, e_k) < 0. \end{aligned} \quad (3.1)$$

Thus, $\tilde{w}_{k,1} < w_*$.

By Eq. (2.6) and Lemma 1.1, we have

$$\begin{aligned} f(\tilde{w}_{k,1}) &= f(w_k - T_k^{-1}f(w_k)) \\ &= f(w_k) - f'(w_k)T_k^{-1}f(w_k) + \frac{1}{2}\mathcal{L}(T_k^{-1}f(w_k), T_k^{-1}f(w_k)) \\ &= [T_k - f'(w_k)]T_k^{-1}f(w_k) + \frac{1}{2}\mathcal{L}(T_k^{-1}f(w_k), T_k^{-1}f(w_k)) < 0. \end{aligned} \quad (3.2)$$

By Lemma 2.3, it can be concluded that $f'(\tilde{w}_{k,1})$ is a nonsingular M -matrix. Therefore, the results hold for $p = 1$.

Assume the results are true for $1 \leq p \leq t$. Then, for $p = t + 1$, we have $\tilde{w}_{k,t+1} = \tilde{w}_{k,t} - T_k^{-1}f(\tilde{w}_{k,t})$. Since $f(\tilde{w}_{k,t}) < 0$ and T_k is a nonsingular M -matrix, then $\tilde{w}_{k,t+1} > \tilde{w}_{k,t}$.

Since $w_k < \tilde{w}_{k,1} < \tilde{w}_{k,2} < \dots < \tilde{w}_{k,t}$, by Lemma 2.1, we have $f'(w_k) > f'(\tilde{w}_{k,1}) > f'(\tilde{w}_{k,2}) > \dots > f'(\tilde{w}_{k,t})$. Therefore,

$$T_k - f'(\tilde{w}_{k,t}) > \dots > T_k - f'(\tilde{w}_{k,1}) > T_k - f'(w_k) \geq 0.$$

By Eqs. (2.5) and (2.9), we have the following error vectors equation

$$\begin{aligned} \tilde{e}_{k,t+1} &= \tilde{e}_{k,t} - T_k^{-1}f(\tilde{w}_{k,t}) \\ &= \tilde{e}_{k,t} - T_k^{-1}[f'(\tilde{w}_{k,t})\tilde{e}_{k,t} - \frac{1}{2}\mathcal{L}(\tilde{e}_{k,t}, \tilde{e}_{k,t})] \\ &= T_k^{-1}[T_k - f'(\tilde{w}_{k,t})]\tilde{e}_{k,t} + \frac{1}{2}T_k^{-1}\mathcal{L}(\tilde{e}_{k,t}, \tilde{e}_{k,t}) < 0. \end{aligned} \quad (3.3)$$

Therefore, $\tilde{w}_{k,t+1} < w_*$.

Similarly, by Eq. (2.6) and Lemma 1.1, we have

$$\begin{aligned} f(\tilde{w}_{k,t+1}) &= f(\tilde{w}_{k,t} - T_k^{-1}f(\tilde{w}_{k,t})) \\ &= f(\tilde{w}_{k,t}) - f'(\tilde{w}_{k,t})T_k^{-1}f(\tilde{w}_{k,t}) + \frac{1}{2}\mathcal{L}(T_k^{-1}f(\tilde{w}_{k,t}), T_k^{-1}f(\tilde{w}_{k,t})) \\ &= [T_k - f'(\tilde{w}_{k,t})]T_k^{-1}f(\tilde{w}_{k,t}) + \frac{1}{2}\mathcal{L}(T_k^{-1}f(\tilde{w}_{k,t}), T_k^{-1}f(\tilde{w}_{k,t})) < 0. \end{aligned} \quad (3.4)$$

By Lemma 2.3, we have that $f'(\tilde{w}_{k,t+1})$ is a nonsingular M -matrix. Therefore, the results hold for $p = t + 1$. Hence, by the principle of mathematical induction, the proof of the theorem is completed. \square

In practical computation, we should choose T_k such that the iteration step (2.5) is less expensive to implement. For any $w_k \in R^{2n}$, according to the structure of the Jacobian $f'(w_k)$, T_k may be chosen as

$$T_k = I_{2n} - \begin{bmatrix} G_1(v_k) & 0 \\ 0 & G_2(u_k) \end{bmatrix} \quad (3.5)$$

or

$$T_k = I_{2n} - \begin{bmatrix} G_1(v_k) & H_1(u_k) \\ 0 & G_2(u_k) \end{bmatrix}. \quad (3.6)$$

Another choice for T_k is

$$T_k = I_{2n} - \begin{bmatrix} G_1(v_k) & 0 \\ H_2(v_k) & G_2(u_k) \end{bmatrix}.$$

Numerical experiments show that the performance for this choice is almost the same as that for T_k given by (3.6).

The following theorem provides some results concerning the convergence of the modified Newton-Shamanskii method for the vector equation (2.2).

Theorem 3.2 *Let w_* be the minimal positive solution of the vector equation (2.2). The sequence of the vector sets $\{w_k, \tilde{w}_{k,1}, \tilde{w}_{k,2}, \dots, \tilde{w}_{k,m}\}$ obtained by the modified Newton-Shamanskii method (2.5) with the initial vector $w_0 = 0$ is well defined. For all $k \geq 0$ and $1 \leq p \leq m$, we have*

- (1) $f(w_k) < 0$ and $f(\tilde{w}_{k,p}) < 0$;
- (2) $f'(w_k)$ and $f'(\tilde{w}_{k,p})$ are nonsingular M -matrices;
- (3) $w_0 < \tilde{w}_{0,1} < \tilde{w}_{0,2} < \dots < \tilde{w}_{0,m} = w_1 < \tilde{w}_{1,1} < \tilde{w}_{1,2} < \dots < \tilde{w}_{1,m} = w_2 < \dots < \tilde{w}_{k-1,m} = w_k < \tilde{w}_{k,1} < \dots < \tilde{w}_{k,m} = w_{k+1} < \dots < w_*$.

Furthermore, we have

$$\lim_{k \rightarrow \infty} w_k = w_*.$$

Proof. This theorem can also be proved by mathematical induction. The proof is similar to that of the Theorem 1 in [18]. Therefore, it is omitted. \square

4 Numerical experiments

In this section, we give numerical experiments to illustrate the performance of the modified Newton-Shamanskii method presented in Section 3 with two different choices of the matrix T_k . Let NS denote the Newton-Shamanskii iterative method [18], MNS1 and MNS2 denote the modified Newton-Shamanskii iterative method (2.5) with T_k given by (3.5) and (3.6), respectively. In order to show numerically the performance of the modified Newton-Shamanskii iterative method, we list the number of iteration steps (denoted as IT), the CPU time in seconds (denoted as CPU), and relative residual error (denoted as ERR). The residual error is defined by

$$\text{ERR} = \max \left\{ \frac{\|u_{k+1} - u_k\|_2}{\|u_{k+1}\|_2}, \frac{\|v_{k+1} - v_k\|_2}{\|v_{k+1}\|_2} \right\},$$

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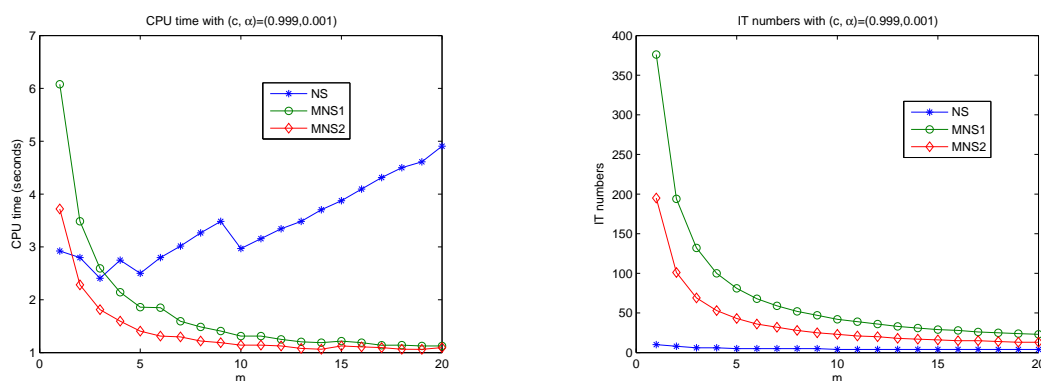


Figure 1: *CPU time and IT numbers for $(c, \alpha) = (0.999, 0.001)$ and $n = 512$ with different m . Left: CPU time; right: IT numbers*

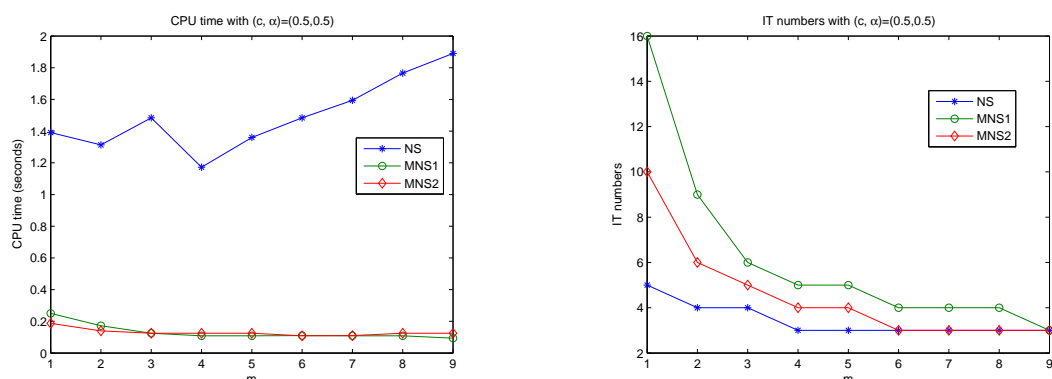


Figure 2: *CPU time and IT numbers for $(c, \alpha) = (0.5, 0.5)$ and $n = 512$ with different m . Left: CPU time; right: IT numbers*

where $\|\cdot\|_2$ is the 2-norm for a vector. For comparison, every experiment is repeated 5 times, and the average of the 5 CPU times is shown here. All the experiments are run in MATLAB 7.0 on a personal computer with Intel(R) Pentium(R) D 3.00GHz CPU and 0.99 GB memory, and all iterations are terminated once the current iterate satisfies $\text{ERR} \leq n \cdot \text{eps}$, where $\text{eps} = 1 \times 10^{-16}$.

In the test example, the constants c_i and w_i , $i = 1, 2, \dots, n$, are given by the numerical quadrature formula on the interval $[0, 1]$, which are obtained by dividing $[0, 1]$ into $\frac{n}{4}$ subintervals of equal length and applying a Gauss-Legendre quadrature [27] with 4 nodes to each subinterval; see the Example 5.2 in [6]

We test several different values (c, α) . In Table 1, for $n = 512$ with different m and pairs of (c, α) , and in Table 2, for the fixed $(c, \alpha) = (0.99, 0.01)$ with different n , we list ITs, CPUs and ERRs for the NS method and MNS methods, respectively. Figure 1 and

Table 1: Numerical results for $n = 512$ and different pairs of (c, α)

m	method		(c, α)			
			$(0.999, 0.001)$	$(0.99, 0.01)$	$(0.9, 0.1)$	$(0.5, 0.5)$
1	NS	IT	10	9	7	5
		CPU	2.9380	2.6100	2.2810	1.6090
		ERR	2.1776e-15	1.5433e-15	1.4280e-15	1.5773e-14
	MNS1	IT	376	130	43	16
		CPU	5.9370	2.0630	0.7820	0.2810
		ERR	4.7938e-14	4.3618e-14	2.7158e-14	7.0829e-15
	MNS2	IT	195	69	24	10
		CPU	3.7500	1.3430	0.5150	0.2190
		ERR	4.7717e-014	3.3654e-14	1.6087e-14	1.7311e-15
3	NS	IT	6	5	5	4
		CPU	2.5310	2.0630	2.0780	1.7190
		ERR	5.3953e-15	4.6570e-14	1.2318e-15	1.0553e-15
	MNS1	IT	132	46	16	6
		CPU	2.5780	0.9370	0.3130	0.1410
		ERR	4.3397e-14	3.7357e-14	1.0270e-14	2.6302e-14
	MNS2	IT	69	25	10	5
		CPU	1.8440	0.6720	0.2810	0.1410
		ERR	4.4170e-14	2.9843e-14	8.7831e-16	1.5640e-16
6	NS	IT	5	4	4	3
		CPU	2.9220	2.2340	2.2810	1.7350
		ERR	1.9497e-15	1.7274e-15	1.3919e-15	1.0832e-15
	MNS1	IT	68	24	9	4
		CPU	1.9060	0.6100	0.2340	0.1100
		ERR	4.7883e-14	4.0900e-14	3.3139e-15	2.9604e-16
	MNS2	IT	36	14	6	3
		CPU	1.3280	0.5160	0.2340	0.1250
		ERR	4.7025e-14	8.5873e-15	5.5104e-16	2.1332e-15
12	NS	IT	4	4	3	3
		CPU	3.4530	3.5160	2.5630	2.6410
		ERR	1.9512e-15	1.6885e-15	1.3243e-15	1.1225e-15
	MNS1	IT	36	13	5	3
		CPU	1.2660	0.4680	0.1880	0.1250
		ERR	2.9584e-14	2.6812e-14	1.9980e-14	1.64101e-16
	MNS2	IT	20	8	4	3
		CPU	1.2190	0.4530	0.2180	0.2030
		ERR	1.1402e-14	4.8204e-15	5.5981e-16	1.6410e-16

Table 2: Numerical results for $(c, \alpha) = (0.99, 0.01)$ and different n, m

m	method		n				
			64	128	256	512	1024
1	NS	IT	9	9	9	9	9
		CPU	0.0310	0.0620	0.4220	2.6100	16.9060
		ERR	6.8597e-16	9.5495e-16	1.1845e-15	1.5433e-15	2.7143e-15
	MNS1	IT	140	136	133	130	126
		CPU	0.0630	0.0940	0.4850	2.0630	7.4530
		ERR	5.3918e-15	1.2247e-14	2.3157e-14	4.3618e-14	1.0212e-14
	MNS2	IT	73	72	70	69	67
		CPU	0.0160	0.0630	0.2970	1.3430	4.7180
		ERR	6.1723e-15	9.44380e-15	2.1990e-14	3.3654e-14	7.8688e-14
5	NS	IT	5	5	5	5	5
		CPU	0.0160	0.0630	0.4220	2.3750	14.6560
		ERR	8.1022e-16	8.6594e-16	1.2226e-15	1.6581e-15	2.1565e-15
	MNS1	IT	31	30	29	29	28
		CPU	0.0150	0.0310	0.1410	0.6410	2.3590
		ERR	2.7024e-15	7.6059e-15	2.2028e-14	2.1877e-14	6.3491e-14
	MNS2	IT	17	17	16	16	16
		CPU	0.0160	0.0310	0.0930	0.5160	1.9530
		ERR	2.4804e-15	2.4814e-15	2.0690e-14	2.0656e-14	2.0698e-14
10	NS	IT	4	4	4	4	4
		CPU	0.0160	0.0780	0.5000	2.7500	16.6090
		ERR	7.2604e-16	8.1207e-16	1.1571e-15	1.5460e-15	2.2290e-15
	MNS1	IT	16	16	16	15	15
		CPU	0.0150	0.0150	0.0780	0.4680	1.7350
		ERR	6.0508e-15	5.8374e-15	6.0658e-15	4.9045e-14	4.8935e-14
	MNS2	IT	10	10	9	9	9
		CPU	0.0160	0.0320	0.0630	0.4380	1.6400
		ERR	5.6442e-16	4.2340e-16	1.4346e-14	1.3941e-14	1.3698e-14

Figure 2 describe the CPU time and IT numbers of those methods when $n = 512$ for $(c, \alpha) = (0.999, 0.001)$ and $(c, \alpha) = (0.5, 0.5)$. From these Tables and Figures, we can see that the optimal choice of m for the modified Newton-Shamanskii method is larger when $(c, \alpha) = (0.999, 0.001)$, compared with $(c, \alpha) = (0.5, 0.5)$. Obviously, compared with the Newton-Shamanskii iterative method, though the iterations number of the modified Newton-Shamanskii iterative method is more, according to the CPU time, we can find that the modified Newton-Shamanskii iterative method outperforms the Newton-Shamanskii iterative method. Among these methods, the MNS2 method is the best one.

5 Conclusion

In this paper, based on the Newton-Shamanskii method, we have proposed a modified Newton-Shamanskii method for solving the minimal positive solution of the nonsymmetric algebraic Riccati equation arising in transport theory and have given the convergence analysis. The convergence analysis shows that the iteration sequence generated by the modified Newton-Shamanskii method is monotonically increasing and converges to the minimal positive solution of the vector equation. Numerical experiments show that the modified Newton-Shamanskii method has a better performance than the Newton-Shamanskii method for the nonsymmetric algebraic Riccati equation. We find that when T_k is chosen as the block triangular of the Jacobian matrix, the modified Newton-Shamanskii method has a better convergence rate. The choice of the matrix T_k impacts the convergence rate of the modified Newton-Shamanskii method, hence, the determination of the optimum matrix T_k such that the modified Newton-Shamanskii method has a better convergence rate needs further to be studied.

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Hesitant fuzzy filters and hesitant fuzzy G -filters in residuated lattices

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Abstract.

Characterizations of a hesitant fuzzy filter in a residuated lattice are considered. Given a hesitant fuzzy set, a new hesitant fuzzy filter of a residuated lattice is constructed. The notion of a hesitant fuzzy G -filter of a residuated lattice is introduced, and its characterizations are discussed. Conditions for a hesitant fuzzy filter to be a hesitant fuzzy G -filter are provided. Finally, the extension property of a hesitant fuzzy G -filter is established.

1. INTRODUCTION

The notions of Atanassov's intuitionistic fuzzy sets, type 2 fuzzy sets and fuzzy multisets etc. are a generalization of fuzzy sets. As another generalization of fuzzy sets, Torra and Narukawa [5] and Torra [6] introduced the notion of hesitant fuzzy sets and discussed the relationship between hesitant fuzzy sets and intuitionistic fuzzy sets. Xia and Xu [11] studied hesitant fuzzy information aggregation techniques and their application in decision making. They developed some hesitant fuzzy operational rules based on the interconnection between the hesitant fuzzy set and the intuitionistic fuzzy set. Xu and Xia [12] proposed a variety of distance measures for hesitant fuzzy sets, and investigated the connections of the aforementioned distance measures and further developed a number of hesitant ordered weighted distance measures and hesitant ordered weighted similarity measures. Xu and Xia [13] defined the distance and correlation measures for hesitant fuzzy information and then considered their properties in detail. Wei [9] investigated the hesitant fuzzy multiple attribute decision making problems in which the attributes are in different priority level.

Residuated lattices are a non-classical logic system which is a formal and useful tool for computer science to deal with uncertain and fuzzy information. Filter theory, which is an important notion, in residuated lattices is studied by Shen and Zhang [4] and Zhu and Xu [15]. Wei [10] introduced the notion of hesitant fuzzy (implicative, regular and Boolean) filters in residuated lattice, and discussed its properties.

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In this paper, we deal with further properties of a hesitant fuzzy filter in a residuated lattice. We consider characterizations of a hesitant fuzzy filter in a residuated lattice. Given a hesitant fuzzy set, we construct a new hesitant fuzzy filter of a residuated lattice. We introduce the notion of a hesitant fuzzy G -filter of a residuated lattice, and discuss its characterizations. We provide conditions for a hesitant fuzzy filter to be a hesitant fuzzy G -filter. Finally, we establish the extension property of a hesitant fuzzy G -filter.

2. PRELIMINARIES

Definition 2.1 ([1, 2, 3]). A *residuated lattice* is an algebra $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that

- (1) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice.
- (2) $(L, \odot, 1)$ is a commutative monoid.
- (3) \odot and \rightarrow form an adjoint pair, that is,

$$(\forall x, y, z \in L) (x \leq y \rightarrow z \Leftrightarrow x \odot y \leq z).$$

In a residuated lattice L , the ordering \leq and negation \neg are defined as follows:

$$(\forall x, y \in L) (x \leq y \Leftrightarrow x \wedge y = x \Leftrightarrow x \vee y = y \Leftrightarrow x \rightarrow y = 1)$$

and $\neg x = x \rightarrow 0$ for all $x \in L$.

Proposition 2.2 ([1, 2, 3, 7, 8]). In a residuated lattice L , the following properties are valid.

- (2.1) $1 \rightarrow x = x, x \rightarrow 1 = 1, x \rightarrow x = 1, 0 \rightarrow x = 1, x \rightarrow (y \rightarrow x) = 1.$
- (2.2) $y \leq (y \rightarrow x) \rightarrow x.$
- (2.3) $x \leq y \rightarrow z \Leftrightarrow y \leq x \rightarrow z.$
- (2.4) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z = y \rightarrow (x \rightarrow z).$
- (2.5) $x \leq y \Rightarrow z \rightarrow x \leq z \rightarrow y, y \rightarrow z \leq x \rightarrow z.$
- (2.6) $z \rightarrow y \leq (x \rightarrow z) \rightarrow (x \rightarrow y), z \rightarrow y \leq (y \rightarrow x) \rightarrow (z \rightarrow x).$
- (2.7) $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z.$
- (2.8) $x \odot y \leq x \wedge y.$
- (2.9) $x \leq y \Rightarrow x \odot z \leq y \odot z.$
- (2.10) $y \rightarrow z \leq x \vee y \rightarrow x \vee z.$
- (2.11) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z).$

Definition 2.3 ([4]). A nonempty subset F of a residuated lattice L is called a *filter* of L if it satisfies the conditions:

$$(2.12) \quad (\forall x, y \in L) (x, y \in F \Rightarrow x \odot y \in F).$$

$$(2.13) \quad (\forall x, y \in L) (x \in F, x \leq y \Rightarrow y \in F).$$

Proposition 2.4 ([4]). A nonempty subset F of a residuated lattice L is a filter of L if and only if it satisfies:

$$(2.14) \quad 1 \in F.$$

$$(2.15) \quad (\forall x \in F) (\forall y \in L) (x \rightarrow y \in F \Rightarrow y \in F).$$

3. HESITANT FUZZY FILTERS

Let E be a reference set. A *hesitant fuzzy set* on E (see [6]) is defined in terms of a function h that when applied to E returns a subset of $[0, 1]$, that is, $h : E \rightarrow \mathcal{P}([0, 1])$.

In what follows, we take a residuated lattice L as a reference set.

Definition 3.1 ([10]). A hesitant fuzzy set h on L is called a *hesitant fuzzy filter* of L if it satisfies:

$$(3.1) \quad (\forall x, y \in L) (x \leq y \Rightarrow h(x) \subseteq h(y)),$$

$$(3.2) \quad (\forall x, y \in L) (h(x) \cap h(y) \subseteq h(x \odot y)).$$

Example 3.2. Let $L = [0, 1]$ be a subset of \mathbb{R} . For any $a, b \in L$, define

$$a \vee b = \max\{a, b\}, \quad a \wedge b = \min\{a, b\},$$

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b, \\ (1 - a) \vee b & \text{otherwise,} \end{cases}$$

and

$$a \odot b = \begin{cases} 0 & \text{if } a + b \leq 1, \\ a \wedge b & \text{otherwise.} \end{cases}$$

Then $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a residuated lattice (see [15]). We define a hesitant fuzzy set

$$h : L \rightarrow \mathcal{P}([0, 1]), \quad x \mapsto \begin{cases} (0.2, 0.7) & \text{if } x \in (c, 1] \text{ where } 0.5 \leq c \leq 1, \\ (0.3, 0.6] & \text{otherwise.} \end{cases}$$

It is routine to verify that h is a hesitant fuzzy filter of L .

Example 3.3. Let $L = \{0, a, b, c, d, 1\}$ be a set with the lattice diagram appears in Figure 1.

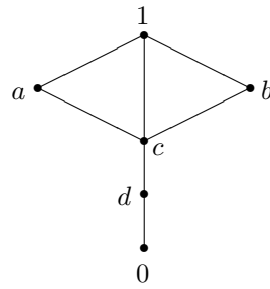


Figure 1

Consider two operation ' \odot ' and ' \rightarrow ' shown in Table 1 and Table 2, respectively.

TABLE 1. Cayley table for the binary operation ' \odot '

\odot	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	c	c	0	a
b	0	c	b	c	d	b
c	0	c	c	c	0	c
d	0	0	d	0	0	d
1	0	a	b	c	d	1

TABLE 2. Cayley table for the binary operation ' \rightarrow '

\rightarrow	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	b	b	d	1
b	0	a	1	a	d	1
c	d	1	1	1	d	1
d	a	1	1	1	1	1
1	0	a	b	c	d	1

Then $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a residuated lattice. We define a hesitant fuzzy set

$$h : L \rightarrow \mathcal{P}([0, 1]), \quad x \mapsto \begin{cases} [0.2, 0.9] & \text{if } x \in \{1, a\}, \\ (0.3, 0.8] & \text{otherwise.} \end{cases}$$

It is routine to verify that h is a hesitant fuzzy filter of L .

Wei [10] provided a characterization of a hesitant fuzzy filter as follows.

Lemma 3.4 ([10]). *A hesitant fuzzy set h on L is a hesitant fuzzy filter of L if and only if it satisfies*

$$(3.3) \quad (\forall x \in L) (h(x) \subseteq h(1)).$$

$$(3.4) \quad (\forall x, y \in L) (h(x) \cap h(x \rightarrow y) \subseteq h(y)).$$

We provide other characterizations of a hesitant fuzzy filter.

Theorem 3.5. *A hesitant fuzzy set h on L is a hesitant fuzzy filter of L if and only if it satisfies:*

$$(3.5) \quad (\forall x, y, z \in L) (x \leq y \rightarrow z \Rightarrow h(x) \cap h(y) \subseteq h(z)).$$

Proof. Assume that h is a hesitant fuzzy filter of L . Let $x, y, z \in L$ be such that $x \leq y \rightarrow z$. Then $h(x) \subseteq h(y \rightarrow z)$ by (3.1), and so

$$h(z) \supseteq h(y) \cap h(y \rightarrow z) \supseteq h(x) \cap h(y)$$

by (3.4).

Conversely let h be a hesitant fuzzy set on L satisfying (3.5). Since $x \leq x \rightarrow 1$ for all $x \in L$, it follows from (3.5) that

$$h(1) \supseteq h(x) \cap h(x) = h(x)$$

for all $x \in L$. Since $x \rightarrow y \leq x \rightarrow y$ for all $x, y \in L$, we have

$$h(y) \supseteq h(x) \cap h(x \rightarrow y)$$

for all $x, y \in L$. Hence h is a hesitant fuzzy filter of L . \square

Theorem 3.6. *A hesitant fuzzy set h on L is a hesitant fuzzy filter of L if and only if h satisfies the condition (3.3) and*

$$(3.6) \quad (\forall x, y, z \in L) (h(x \rightarrow (y \rightarrow z)) \cap h(y) \subseteq h(x \rightarrow z)).$$

Proof. Assume that h is a hesitant fuzzy filter of L . Then the condition (3.3) is valid. Using (2.4) and (3.4), we have

$$\begin{aligned} h(x \rightarrow z) &\supseteq h(y) \cap h(y \rightarrow (x \rightarrow z)) \\ &= h(y) \cap h(x \rightarrow (y \rightarrow z)) \end{aligned}$$

for all $x, y, z \in L$.

Conversely, let h be a hesitant fuzzy set on L satisfying (3.3) and (3.6). Taking $x := 1$ in (3.6) and using (2.1), we get

$$\begin{aligned} h(z) &= h(1 \rightarrow z) \supseteq h(1 \rightarrow (y \rightarrow z)) \cap h(y) \\ &= h(y \rightarrow z) \cap h(y) \end{aligned}$$

for all $y, z \in L$. Thus h is a hesitant fuzzy filter of L by Lemma 3.4. \square

Lemma 3.7. *Every hesitant fuzzy filter h on L satisfies the following condition:*

$$(3.7) \quad (\forall a, x \in L) (h(a) \subseteq h((a \rightarrow x) \rightarrow x)).$$

Proof. If we take $y = (a \rightarrow x) \rightarrow x$ and $x = a$ in (3.4), then

$$\begin{aligned} h((a \rightarrow x) \rightarrow x) &\supseteq h(a) \cap h(a \rightarrow ((a \rightarrow x) \rightarrow x)) \\ &= h(a) \cap h((a \rightarrow x) \rightarrow (a \rightarrow x)) \\ &= h(a) \cap h(1) = h(a). \end{aligned}$$

This completes the proof. \square

Theorem 3.8. *A hesitant fuzzy set h on L is a hesitant fuzzy filter of L if and only if it satisfies the following conditions:*

$$(3.8) \quad (\forall x, y \in L) (h(x) \subseteq h(y \rightarrow x)),$$

$$(3.9) \quad (\forall x, a, b \in L) (h(a) \cap h(b) \subseteq h((a \rightarrow (b \rightarrow x)) \rightarrow x)).$$

Proof. Assume that h is a hesitant fuzzy filter of L . Using (2.1), (3.3) and (3.4), we have

$$h(y \rightarrow x) \supseteq h(x) \cap h(x \rightarrow (y \rightarrow x)) = h(x) \cap h(1) = h(x)$$

for all $x, y \in L$. Using (3.6) and (3.7), we get

$$h((a \rightarrow (b \rightarrow x)) \rightarrow x) \supseteq h((a \rightarrow (b \rightarrow x)) \rightarrow (b \rightarrow x)) \cap h(b) \supseteq h(a) \cap h(b)$$

for all $a, b, x \in L$.

Conversely, let h be a hesitant fuzzy set on L satisfying two conditions (3.8) and (3.9). If we take $y := x$ in (3.8), then $h(x) \subseteq h(x \rightarrow x) = h(1)$ for all $x \in L$. Using (3.9) induces

$$h(y) = h(1 \rightarrow y) = h((x \rightarrow y) \rightarrow (x \rightarrow y)) \rightarrow y \supseteq h(x \rightarrow y) \cap h(x)$$

for all $x, y \in L$. Therefore h is a hesitant fuzzy filter of L by Lemma 3.4. \square

Theorem 3.9. *A hesitant fuzzy set h on L is a hesitant fuzzy filter of L if and only if the set*

$$h_\tau := \{x \in L \mid \tau \subseteq h(x)\}$$

is a filter of L for all $\tau \in \mathcal{P}([0, 1])$ with $h_\tau \neq \emptyset$.

Proof. Assume that h is a hesitant fuzzy filter of L . Let $x, y \in L$ and $\tau \in \mathcal{P}([0, 1])$ be such that $x \in h_\tau$ and $x \rightarrow y \in h_\tau$. Then $\tau \subseteq h(x)$ and $\tau \subseteq h(x \rightarrow y)$. It follows from (3.3) and (3.4) that $h(1) \supseteq h(x) \supseteq \tau$ and $h(y) \supseteq h(x) \cap h(x \rightarrow y) \supseteq \tau$ and so that $1 \in h_\tau$ and $y \in h_\tau$. Hence h_τ is a filter of L by Proposition 2.4.

Conversely, suppose that h_τ is a filter of L for all $\tau \in \mathcal{P}([0, 1])$ with $h_\tau \neq \emptyset$. For any $x \in L$, let $h(x) = \delta$. Then $x \in h_\delta$ and h_δ is a filter of L . Hence $1 \in h_\delta$ and so $h(x) = \delta \subseteq h(1)$. For

any $x, y \in L$, let $h(x) = \delta_x$ and $h(x \rightarrow y) = \delta_{x \rightarrow y}$. If we take $\delta = \delta_x \cap \delta_{x \rightarrow y}$, then $x \in h_\delta$ and $x \rightarrow y \in h_\delta$ which imply that $y \in h_\delta$. Thus

$$h(x) \cap h(x \rightarrow y) = \delta_x \cap \delta_{x \rightarrow y} = \delta \subseteq h(y).$$

Therefore h is a hesitant fuzzy filter of L by Lemma 3.4. \square

Theorem 3.10. For a hesitant fuzzy set h on L , let \tilde{h} be a hesitant fuzzy set on L defined by

$$\tilde{h} : L \rightarrow \mathcal{P}([0, 1]), \quad x \mapsto \begin{cases} h(x) & \text{if } x \in h_\tau, \\ \emptyset & \text{otherwise,} \end{cases}$$

where $\tau \in \mathcal{P}([0, 1]) \setminus \{\emptyset\}$. If h is a hesitant fuzzy filter of L , then so is \tilde{h} .

Proof. Suppose that h is a hesitant fuzzy filter of L . Then h_τ is a filter of L for all $\tau \in \mathcal{P}([0, 1])$ with $h_\tau \neq \emptyset$ by Theorem 3.9. Thus $1 \in h_\tau$, and so $\tilde{h}(1) = h(1) \supseteq h(x) \supseteq \tilde{h}(x)$ for all $x \in L$. Let $x, y \in L$. If $x \in h_\tau$ and $x \rightarrow y \in h_\tau$, then $y \in h_\tau$. Hence

$$\tilde{h}(x) \cap \tilde{h}(x \rightarrow y) = h(x) \cap h(x \rightarrow y) \subseteq h(y) = \tilde{h}(y).$$

If $x \notin h_\tau$ or $x \rightarrow y \notin h_\tau$, then $\tilde{h}(x) = \emptyset$ or $\tilde{h}(x \rightarrow y) = \emptyset$. Thus

$$\tilde{h}(x) \cap \tilde{h}(x \rightarrow y) = \emptyset \subseteq \tilde{h}(y).$$

Therefore \tilde{h} is a hesitant fuzzy filter of L . \square

Theorem 3.11. If h is a hesitant fuzzy filter of L , then the set

$$\Gamma_a := \{x \in L \mid h(a) \subseteq h(x)\}$$

is a filter of L for every $a \in L$.

Proof. Since $h(1) \supseteq h(a)$ for all $a \in L$, we have $1 \in \Gamma_a$. Let $x, y \in L$ be such that $x \in \Gamma_a$ and $x \rightarrow y \in \Gamma_a$. Then $h(x) \supseteq h(a)$ and $h(x \rightarrow y) \supseteq h(a)$. Since h is a hesitant fuzzy filter of L , it follows from (3.4) that

$$h(y) \supseteq h(x) \cap h(x \rightarrow y) \supseteq h(a)$$

so that $y \in \Gamma_a$. Hence Γ_a is a filter of L by Proposition 2.4. \square

Theorem 3.12. Let $a \in L$ and let h be a hesitant fuzzy set on L . Then

(1) If Γ_a is a filter of L , then h satisfies the following condition:

$$(3.10) \quad (\forall x, y \in L) (h(a) \subseteq h(x) \cap h(x \rightarrow y) \Rightarrow h(a) \subseteq h(y)).$$

(2) If h satisfies (3.3) and (3.10), then Γ_a is a filter of L .

Proof. (1) Assume that Γ_a is a filter of L . Let $x, y \in L$ be such that

$$h(a) \subseteq h(x) \cap h(x \rightarrow y).$$

Then $x \rightarrow y \in \Gamma_a$ and $x \in \Gamma_a$. Using (2.15), we have $y \in \Gamma_a$ and so $h(y) \supseteq h(a)$.

(2) Suppose that h satisfies (3.3) and (3.10). From (3.3) it follows that $1 \in \Gamma_a$. Let $x, y \in L$ be such that $x \in \Gamma_a$ and $x \rightarrow y \in \Gamma_a$. Then $h(a) \subseteq h(x)$ and $h(a) \subseteq h(x \rightarrow y)$, which imply that $h(a) \subseteq h(x) \cap h(x \rightarrow y)$. Thus $h(a) \subseteq h(y)$ by (3.10), and so $y \in \Gamma_a$. Therefore Γ_a is a filter of L by Proposition 2.4. \square

Definition 3.13 ([14]). A nonempty subset F of L is called a *G-filter* of L if it is a filter of L that satisfies the following condition:

$$(3.11) \quad (\forall x, y \in L) ((x \odot x) \rightarrow y \in F \Rightarrow x \rightarrow y \in F).$$

We consider the hesitant fuzzification of *G*-filters.

Definition 3.14. A hesitant fuzzy set h on L is called a *hesitant fuzzy G-filter* of L if it is a hesitant fuzzy filter of L that satisfies:

$$(3.12) \quad (\forall x, y \in L) (h((x \odot x) \rightarrow y) \subseteq h(x \rightarrow y)).$$

Note that the condition (3.12) is equivalent to the following condition:

$$(3.13) \quad (\forall x, y \in L) (h(x \rightarrow (x \rightarrow y)) \subseteq h(x \rightarrow y)).$$

Example 3.15. The hesitant fuzzy filter h in Example 3.3 is a hesitant fuzzy *G*-filter of L .

Lemma 3.16. Every hesitant fuzzy filter h of L satisfies the following condition:

$$(3.14) \quad (\forall x, y, z \in L) (h(x \rightarrow (y \rightarrow z)) \cap h(x \rightarrow y) \subseteq h(x \rightarrow (x \rightarrow z))).$$

Proof. Let $x, y, z \in L$. Using (2.4) and (2.6), we have

$$x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z) \leq (x \rightarrow y) \rightarrow (x \rightarrow (x \rightarrow z)).$$

It follows from Theorem 3.5 that

$$h(x \rightarrow (y \rightarrow z)) \cap h(x \rightarrow y) \subseteq h(x \rightarrow (x \rightarrow z)).$$

This completes the proof. \square

Theorem 3.17. Let h be a hesitant fuzzy set on L . Then h is a hesitant fuzzy *G*-filter of L if and only if it is a hesitant fuzzy filter of L that satisfies the following condition:

$$(3.15) \quad (\forall x, y, z \in L) (h(x \rightarrow (y \rightarrow z)) \cap h(x \rightarrow y) \subseteq h(x \rightarrow z)).$$

Proof. Assume that h is a hesitant fuzzy G -filter of L . Then h is a hesitant fuzzy filter of L . Note that $x \leq 1 = (x \rightarrow y) \rightarrow (x \rightarrow y)$, and thus $x \rightarrow y \leq x \rightarrow (x \rightarrow y)$ for all $x, y \in L$. It follows from (3.1) that $h(x \rightarrow y) \subseteq h(x \rightarrow (x \rightarrow y))$. Combining this and (3.13), we have

$$(3.16) \quad h(x \rightarrow y) = h(x \rightarrow (x \rightarrow y))$$

for all $x, y \in L$. Using (3.14) and (3.16), we have

$$h(x \rightarrow (y \rightarrow z)) \cap h(x \rightarrow y) \subseteq h(x \rightarrow z)$$

for all $x, y, z \in L$.

Conversely, let h be a hesitant fuzzy filter of L that satisfies the condition (3.15). If we put $y = x$ and $z = y$ in (3.15) and use (2.1) and (3.3), then

$$\begin{aligned} h(x \rightarrow y) &\supseteq h(x \rightarrow (x \rightarrow y)) \cap h(x \rightarrow x) \\ &= h(x \rightarrow (x \rightarrow y)) \cap h(1) \\ &= h(x \rightarrow (x \rightarrow y)) \end{aligned}$$

for all $x, y \in L$. Therefore h is a hesitant fuzzy G -filter of L . \square

Theorem 3.18. *Let h be a hesitant fuzzy filter of L . Then h is a hesitant fuzzy G -filter of L if and only if the following condition holds:*

$$(3.17) \quad (\forall x \in L) (h(x \rightarrow (x \odot x)) = h(1)).$$

Proof. Assume that h satisfies the condition (3.17) and let $x, y \in L$. Since

$$x \rightarrow (x \rightarrow y) = (x \odot x) \rightarrow y \leq (x \rightarrow (x \odot x)) \rightarrow (x \rightarrow y)$$

by (2.4) and (2.6), it follows from (3.1) that

$$h(x \rightarrow (x \rightarrow y)) \subseteq h((x \rightarrow (x \odot x)) \rightarrow (x \rightarrow y)).$$

Hence, we have

$$\begin{aligned} h(x \rightarrow y) &\supseteq h((x \rightarrow (x \odot x)) \rightarrow (x \rightarrow y)) \cap h(x \rightarrow (x \odot x)) \\ &\supseteq h(x \rightarrow (x \rightarrow y)) \cap h(x \rightarrow (x \odot x)) \\ &= h(x \rightarrow (x \rightarrow y)) \cap h(1) \\ &= h(x \rightarrow (x \rightarrow y)) \end{aligned}$$

by using (3.4), (3.17) and (3.3). Hence h is a hesitant fuzzy G -filter of L . \square

Theorem 3.19. (Extension property) *Let h and g be hesitant fuzzy filters of L such that $h \subseteq g$, i.e., $h(x) \subseteq g(x)$ for all $x \in L$ and $h(1) = g(1)$. If h is a hesitant fuzzy G -filter of L , then so is g .*

Proof. Assume that h is a hesitant fuzzy G -filter of L . Using (2.4) and (2.1), we have

$$x \rightarrow (x \rightarrow ((x \rightarrow (x \rightarrow y)) \rightarrow y)) = (x \rightarrow (x \rightarrow y)) \rightarrow (x \rightarrow (x \rightarrow y)) = 1$$

for all $x, y \in L$. Thus

$$\begin{aligned} g(x \rightarrow ((x \rightarrow (x \rightarrow y)) \rightarrow y)) &\supseteq h(x \rightarrow ((x \rightarrow (x \rightarrow y)) \rightarrow y)) \\ &= h(x \rightarrow (x \rightarrow ((x \rightarrow (x \rightarrow y)) \rightarrow y))) \\ &= h(1) = g(1) \end{aligned}$$

by hypotheses and (3.16), and so

$$g(x \rightarrow ((x \rightarrow (x \rightarrow y)) \rightarrow y)) = g(1)$$

for all $x, y \in L$ by (3.3). Since g is a hesitant fuzzy filter of L , it follows from (3.4), (2.4) and (3.3) that

$$\begin{aligned} g(x \rightarrow y) &\supseteq g(x \rightarrow (x \rightarrow y)) \cap g((x \rightarrow (x \rightarrow y)) \rightarrow (x \rightarrow y)) \\ &= g(x \rightarrow (x \rightarrow y)) \cap g(x \rightarrow ((x \rightarrow (x \rightarrow y)) \rightarrow y)) \\ &= g(x \rightarrow (x \rightarrow y)) \cap g(1) \\ &= g(x \rightarrow (x \rightarrow y)) \end{aligned}$$

for all $x, y \in L$. Therefore g is a hesitant fuzzy G -filter of L . □

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Some new Chebyshev type quantum integral inequalities on finite intervals

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Abstract: By using the two parameters of deformation q_1 and q_2 , we establish some new Chebyshev type quantum integral inequalities on finite intervals. Furthermore, we also consider their relevance with other related known results.

Keywords: Chebyshev type inequalities; quantum integral inequalities; synchronous (asynchronous) functions

2010 Mathematics Subject Classification: 34A08; 26D10; 26D15

1 Introduction

Let us start by considering the following celebrated Chebyshev functional (see [1]):

$$T(f, g, p, q) = \left(\int_a^b q(x) dx \right) \left(\int_a^b p(x) f(x) g(x) dx \right) + \left(\int_a^b p(x) dx \right) \left(\int_a^b q(x) f(x) g(x) dx \right) \\ - \left(\int_a^b q(x) f(x) dx \right) \left(\int_a^b p(x) g(x) dx \right) - \left(\int_a^b p(x) f(x) dx \right) \left(\int_a^b q(x) g(x) dx \right), \quad (1.1)$$

where $f, g : [a, b] \rightarrow \mathbb{R}$ are two integrable functions on $[a, b]$ and $p(x)$ and $q(x)$ are positive integrable functions on $[a, b]$. If f and g are *synchronous* on $[a, b]$, that is,

$$(f(x) - f(y))(g(x) - g(y)) \geq 0,$$

for any $x, y \in [a, b]$, then we have (see, e.g., [2, 3])

$$T(f, g, p, q) \geq 0, \quad (1.2)$$

The inequality in (1.2) is reversed if f and g are *asynchronous* on $[a, b]$, that is,

$$(f(x) - f(y))(g(x) - g(y)) \leq 0,$$

for any $x, y \in [a, b]$. If $p(x) = q(x)$ for any $x, y \in [a, b]$, we get the Chebyshev inequality, see [1].

Here we should point out that the Chebyshev functional (1.1) has attracted many researchers attention mainly due to its distinguished applications in numerical quadrature, probability and statistical problems and transform theory. At the same time, the Chebyshev functional (1.1) has also been employed to yield a number of integral inequalities, see [4, 5, 6, 7, 8, 9, 10, 11, 12, 13] and the references therein.

The integral inequalities can be applied for the study of qualitative and quantitative properties of integrals, see [14, 15, 16, 17]. In order to generalize and spread the existing inequalities, we specify two ways to overcome the problems which ensue from the general definition of q -integral. In [18], Gauchman has introduced the restricted q -integral over $[a, b]$. In [19], Stanković, Rajković and Marinković have introduced the definition of the q -integral of the Riemann type. In [18], Gauchman gave the q -analogues of the well-known inequalities in the integral calculus, as Chebyshev, Grüss, Hermite-Hadamard for all the types of the q -integrals. In [19], Stanković, Rajković and Marinković obtained some new q -Chebyshev, q -Grüss, q -Hermite-Hadamard type inequalities. In [20, 21], by using the weighted q -integral Montgomery identity, Yang and Liu and Yang established the weighted q -Chebyshev-Grüss type inequalities for single and double integrals, respectively. Recently, Tariboon

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and Ntouyas [22] introduced the quantum calculus on finite intervals, they extended the Hölder, Hermite-Hadamard, trapezoid, Ostrowski, Cauchy-Bunyakovsky-Schwarz, Grüss, and Grüss-Čebyšev integral inequalities to quantum calculus on finite intervals in the paper [23].

Motivated by the results mentioned above, by using the two parameters of deformation q_1 and q_2 , we establish some new Chebyshev type quantum integral inequalities on finite intervals. Furthermore, we also obtain their relevance with other related known results.

2 Preliminaries

Let $J := [a, b] \subset \mathbb{R}$, $K := [c, d] \subset \mathbb{R}$, $J_0 := (a, b)$ be interval and $0 < q, q_1, q_2 < 1$ be a constant. We give the definition q -derivative of a function $f : J \rightarrow \mathbb{R}$ at a point $x \in J$ on $[a, b]$ as follows.

Definition 2.1 ([22]). Assume $f : J \rightarrow \mathbb{R}$ is a continuous function and let $x \in J$. Then the expression

$${}_a D_q f(x) = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \quad x \neq a, \quad {}_a D_q f(a) = \lim_{x \rightarrow a} {}_a D_q f(x), \quad (2.1)$$

is called the q -derivative on J of function f at x .

We say that f is q -differentiable on J provided ${}_a D_q f(x)$ exists for all $x \in J$. Note that if $a = 0$ in (2.1), then ${}_0 D_q f = D_q f$, where D_q is the well-known q -derivative of the function $f(x)$ defined by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}.$$

For more details, see [24].

Definition 2.2 ([22]). Assume $f : J \rightarrow \mathbb{R}$ is a continuous function. Then the q -integral on J is defined by

$$I_q^a f(x) = \int_a^x f(t) {}_a d_q t = (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a), \quad (2.2)$$

for $x \in J$. Moreover, if $c \in (a, x)$ then the definite q -integral on J is defined by

$$\begin{aligned} \int_c^x f(t) {}_a d_q t &= \int_a^x f(t) {}_a d_q t - \int_a^c f(t) {}_a d_q t \\ &= (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) - (1-q)(c-a) \sum_{n=0}^{\infty} q^n f(q^n c + (1-q^n)a). \end{aligned}$$

Note that if $a = 0$, then (2.2) reduces to the classical q -integral of a function $f(x)$ defined by (see [24])

$$\int_0^x f(t) {}_0 d_q t = (1-q)x \sum_{n=0}^{\infty} q^n f(q^n x), \quad \forall x \in [0, \infty).$$

Lemma 2.3. Assume $f, g : J \rightarrow \mathbb{R}$ are two continuous functions and $f(t) \leq g(t)$ for all $t \in J$. Then

$$\int_a^x f(t) {}_a d_q t \leq \int_a^x g(t) {}_a d_q t. \quad (2.3)$$

Proof. For $x \in J$, then $q^n x + (1-q^n)a \in J$. Because $f, g : J \rightarrow \mathbb{R}$ are two continuous functions and $f(t) \leq g(t)$ for all $t \in J$. Then

$$f(q^n x + (1-q^n)a) \leq g(q^n x + (1-q^n)a). \quad (2.4)$$

Summing from 0 to ∞ with respect to n and multiplying both sides of (2.4) by $(1-q)(x-a) \geq 0$, then we get

$$\begin{aligned} \int_a^x f(t) {}_a d_q t &= (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) \\ &\leq (1-q)(x-a) \sum_{n=0}^{\infty} q^n g(q^n x + (1-q^n)a) = \int_a^x g(t) {}_a d_q t, \end{aligned}$$

which implies (2.3). The proof is completed. \square

3 Chebyshev type quantum integral inequalities

In this section, we establish some new Chebyshev type quantum integral inequalities on finite intervals. For the sake of simplicity, we always assume that in this paper all of quantum integral exist and

$$I_q^a(uf)(b) = \int_a^b u(t)f(t)_a d_q t \quad \text{and} \quad I_q^a(ufg)(b) = \int_a^b u(t)f(t)g(t)_a d_q t.$$

Lemma 3.1. *Let f and g be two continuous and synchronous functions on J and let $u, v : J \rightarrow [0, \infty)$ be two continuous functions. Then the following inequality holds true*

$$I_q^a u(b) I_q^a(vfg)(b) + I_q^a v(b) I_q^a(ufg)(b) \geq I_q^a(uf)(b) I_q^a(vg)(b) + I_q^a(vf)(b) I_q^a(ug)(b). \quad (3.1)$$

Proof. Since f and g be two continuous and synchronous functions on J , then for all $\tau, \rho \in J$, we have

$$(f(\tau) - f(\rho))(g(\tau) - g(\rho)) \geq 0. \quad (3.2)$$

By (3.2), we write

$$f(\tau)g(\tau) + f(\rho)g(\rho) \geq f(\tau)g(\rho) + f(\rho)g(\tau). \quad (3.3)$$

Multiplying both sides of (3.3) by $v(\tau)$ and integrating the resulting identity with respect to τ from a to b , then we obtain

$$I_q^a(vfg)(b) + f(\rho)g(\rho)I_q^a v(b) \geq g(\rho)I_q^a(vf)(b) + f(\rho)I_q^a(vg)(b). \quad (3.4)$$

Multiplying both side of (3.4) by $u(\rho)$ and integrating the resulting identity with respect to ρ from a to b , then we get

$$I_q^a u(b) I_q^a(vfg)(b) + I_q^a v(b) I_q^a(ufg)(b) \geq I_q^a(uf)(b) I_q^a(vg)(b) + I_q^a(vf)(b) I_q^a(ug)(b),$$

which implies (3.1). \square

Theorem 3.2. *Let f and g be two continuous and synchronous functions on J and let $\phi, \varphi, \psi : J \rightarrow [0, \infty)$ be three continuous functions. Then the following inequality holds true*

$$\begin{aligned} & 2I_q^a \phi(b) (I_q^a \varphi(b) I_q^a(\psi fg)(b) + I_q^a \psi(b) I_q^a(\varphi fg)(b)) + 2I_q^a \varphi(b) I_q^a \psi(b) I_q^a(\phi fg)(b) \\ & \geq I_q^a \phi(b) (I_q^a(\varphi f)(b) I_q^a(\psi g)(b) + I_q^a(\psi f)(b) I_q^a(\varphi g)(b)) + I_q^a \varphi(b) (I_q^a(\phi f)(b) I_q^a(\psi g)(b) \\ & \quad + I_q^a(\psi f)(b) I_q^a(\varphi g)(b)) + I_q^a \psi(b) (I_q^a(\phi f)(b) I_q^a(\varphi g)(b) + I_q^a(\varphi f)(b) I_q^a(\phi g)(b)). \end{aligned} \quad (3.5)$$

Proof. Putting $u = \varphi$, $v = \psi$ and using Lemma 3.1, we can write

$$I_q^a \varphi(b) I_q^a(\psi fg)(b) + I_q^a \psi(b) I_q^a(\varphi fg)(b) \geq I_q^a(\varphi f)(b) I_q^a(\psi g)(b) + I_q^a(\psi f)(b) I_q^a(\varphi g)(b). \quad (3.6)$$

Multiplying both sides of (3.6) by $I_q^a \phi(b)$, we obtain

$$I_q^a \phi(b) (I_q^a \varphi(b) I_q^a(\psi fg)(b) + I_q^a \psi(b) I_q^a(\varphi fg)(b)) \geq I_q^a \phi(b) (I_q^a(\varphi f)(b) I_q^a(\psi g)(b) + I_q^a(\psi f)(b) I_q^a(\varphi g)(b)). \quad (3.7)$$

Putting $u = \phi$, $v = \psi$ and using Lemma 3.1, we can write

$$I_q^a \phi(b) I_q^a(\psi fg)(b) + I_q^a \psi(b) I_q^a(\phi fg)(b) \geq I_q^a(\phi f)(b) I_q^a(\psi g)(b) + I_q^a(\psi f)(b) I_q^a(\phi g)(b). \quad (3.8)$$

Multiplying both sides of (3.8) by $I_q^a \varphi(b)$, we obtain

$$I_q^a \varphi(b) (I_q^a \phi(b) I_q^a(\psi fg)(b) + I_q^a \psi(b) I_q^a(\phi fg)(b)) \geq I_q^a \varphi(b) (I_q^a(\phi f)(b) I_q^a(\psi g)(b) + I_q^a(\psi f)(b) I_q^a(\phi g)(b)). \quad (3.9)$$

With the same arguments as before, we can get

$$I_q^a \psi(b) (I_q^a \phi(b) I_q^a(\varphi fg)(b) + I_q^a \varphi(b) I_q^a(\phi fg)(b)) \geq I_q^a \psi(b) (I_q^a(\phi f)(b) I_q^a(\varphi g)(b) + I_q^a(\varphi f)(b) I_q^a(\phi g)(b)). \quad (3.10)$$

The required inequality (3.5) follows on adding the inequalities (3.7), (3.9) and (3.10). \square

Lemma 3.3. Let f and g be two continuous and synchronous functions on $J \cup K$ and let $u, v : J \cup K \rightarrow [0, \infty)$ be two continuous functions. Then the following inequality holds true

$$I_{q_1}^a u(b) I_{q_2}^c (vfg)(d) + I_{q_2}^c v(d) I_{q_1}^a (ufg)(b) \geq I_{q_1}^a (uf)(b) I_{q_2}^c (vg)(d) + I_{q_2}^c (vf)(d) I_{q_1}^a (ug)(b). \quad (3.11)$$

Proof. Multiplying both sides of (3.3) by $v(\rho)$ and q_2 -integrating the resulting inequality obtained with respect to ρ from c to d , then we have

$$f(\tau)g(\tau)I_{q_2}^c v(d) + I_{q_2}^c (vfg)(d) \geq f(\tau)I_{q_2}^c (vg)(d) + g(\tau)I_{q_2}^c (vf)(d). \quad (3.12)$$

Multiplying both sides of (3.12) by $u(\tau)$ and q_1 -integrating the resulting identity with respect to τ from a to b , then we obtain

$$I_{q_1}^a u(b) I_{q_2}^c (vfg)(d) + I_{q_2}^c v(d) I_{q_1}^a (ufg)(b) \geq I_{q_1}^a (uf)(b) I_{q_2}^c (vg)(d) + I_{q_2}^c (vf)(d) I_{q_1}^a (ug)(b),$$

which implies (3.11). \square

Theorem 3.4. Let f and g be two continuous and synchronous functions on $J \cup K$ and let $\phi, \varphi, \psi : J \cup K \rightarrow [0, \infty)$ be three continuous functions. Then the following inequality holds true

$$\begin{aligned} & I_{q_1}^a \phi(b) (I_{q_1}^a \psi(b) I_{q_2}^c (\varphi fg)(d) + 2I_{q_1}^a \varphi(b) I_{q_2}^c (\psi fg)(d) + I_{q_2}^c \psi(d) I_{q_1}^a (\varphi fg)(b)) \\ & + (I_{q_1}^a \varphi(b) I_{q_2}^c \psi(d) + I_{q_2}^c \varphi(d) I_{q_1}^a \psi(b)) I_{q_1}^a (\phi fg)(b) \geq I_{q_1}^a \phi(b) (I_{q_1}^a (\varphi f)(b) I_{q_2}^c (\psi g)(d) + I_{q_2}^c (\varphi f)(d) I_{q_1}^a (\varphi g)(b)) \\ & + I_{q_1}^a \varphi(b) (I_{q_1}^a (\phi f)(b) I_{q_2}^c (\psi g)(d) + I_{q_2}^c (\psi f)(d) I_{q_1}^a (\phi g)(b)) + I_{q_1}^a \psi(b) (I_{q_1}^a (\phi f)(b) I_{q_2}^c (\varphi g)(d) + I_{q_2}^c (\varphi f)(d) I_{q_1}^a (\phi g)(b)). \end{aligned} \quad (3.13)$$

Proof. Putting $u = \varphi$, $v = \psi$ and using Lemma 3.3, we can write

$$I_{q_1}^a \varphi(b) I_{q_2}^c (\psi fg)(d) + I_{q_2}^c \psi(d) I_{q_1}^a (\varphi fg)(b) \geq I_{q_1}^a (\varphi f)(b) I_{q_2}^c (\psi g)(d) + I_{q_2}^c (\psi f)(d) I_{q_1}^a (\varphi g)(b). \quad (3.14)$$

Multiplying both sides of (3.14) by $I_{q_1}^a \phi(b)$, we obtain

$$I_{q_1}^a \phi(b) (I_{q_1}^a \varphi(b) I_{q_2}^c (\psi fg)(d) + I_{q_2}^c \psi(d) I_{q_1}^a (\varphi fg)(b)) \geq I_{q_1}^a \phi(b) (I_{q_1}^a (\varphi f)(b) I_{q_2}^c (\psi g)(d) + I_{q_2}^c (\psi f)(d) I_{q_1}^a (\varphi g)(b)), \quad (3.15)$$

Putting $u = \phi$, $v = \psi$ and using Lemma 3.3, we can write

$$I_{q_1}^a \phi(b) I_{q_2}^c (\psi fg)(d) + I_{q_2}^c \psi(d) I_{q_1}^a (\phi fg)(b) \geq I_{q_1}^a (\phi f)(b) I_{q_2}^c (\psi g)(d) + I_{q_2}^c (\psi f)(d) I_{q_1}^a (\phi g)(b). \quad (3.16)$$

Multiplying both sides of (3.16) by $I_{q_1}^a \varphi(b)$, we obtain

$$I_{q_1}^a \varphi(b) (I_{q_1}^a \phi(b) I_{q_2}^c (\psi fg)(d) + I_{q_2}^c \psi(d) I_{q_1}^a (\phi fg)(b)) \geq I_{q_1}^a \varphi(b) (I_{q_1}^a (\phi f)(b) I_{q_2}^c (\psi g)(d) + I_{q_2}^c (\psi f)(d) I_{q_1}^a (\phi g)(b)), \quad (3.17)$$

With the same arguments as before, we can get

$$I_{q_1}^a \psi(b) (I_{q_1}^a \phi(b) I_{q_2}^c (\varphi fg)(d) + I_{q_2}^c \phi(d) I_{q_1}^a (\phi fg)(b)) \geq I_{q_1}^a \psi(b) (I_{q_1}^a (\phi f)(b) I_{q_2}^c (\varphi g)(d) + I_{q_2}^c (\varphi f)(d) I_{q_1}^a (\phi g)(b)), \quad (3.18)$$

The required inequality (3.14) follows on adding the inequalities (3.15), (3.17) and (3.18). \square

Remark 3.5. The inequalities (3.5) and (3.13) are reversed in the following cases: (a) The functions f and g are synchronous on $J \cup K$. (b) The functions ϕ , φ and ψ are negative on $J \cup K$. (c) Two of the functions ϕ , φ and ψ are positive and the third one is negative on $J \cup K$.

Theorem 3.6. Let f, g, h be three continuous and synchronous functions on $J \cup K$ and let $u : J \cup K \rightarrow [0, \infty)$ be a continuous function. Then the following inequality holds true

$$\begin{aligned} & I_{q_1}^a u(b) I_{q_2}^c (ufgh)(d) + I_{q_1}^a (uh)(b) I_{q_2}^c (ufg)(d) + I_{q_1}^a (ufg)(b) I_{q_2}^c (uh)(d) + I_{q_1}^a (ufgh)(b) I_{q_2}^c u(d) \\ & \geq I_{q_1}^a (uf)(b) I_{q_2}^c (ugh)(d) + I_{q_1}^a (ug)(b) I_{q_2}^c (ufh)(d) + I_{q_1}^a (ugh)(b) I_{q_2}^c (uf)(d) + I_{q_1}^a (ufh)(b) I_{q_2}^c (ug)(d). \end{aligned} \quad (3.19)$$

Proof. Let f, g, h be three continuous and synchronous functions on $J \cup K$, then for all $\tau, \rho \in J \cup K$, we have

$$(f(\tau) - f(\rho))(g(\tau) - g(\rho))(h(\tau) + h(\rho)) \geq 0,$$

which implies that

$$\begin{aligned} f(\tau)g(\tau)h(\tau) + f(\rho)g(\rho)h(\rho) + f(\tau)g(\tau)h(\rho) + f(\rho)g(\rho)h(\tau) \\ \geq f(\tau)g(\rho)h(\tau) + f(\tau)g(\rho)h(\rho) + f(\rho)g(\tau)h(\tau) + f(\rho)g(\tau)h(\rho). \end{aligned} \quad (3.20)$$

Multiplying both sides of (3.20) by $u(\tau)$ and q_2 -integrating the resulting identity with respect to τ from c to d , then we obtain

$$\begin{aligned} I_{q_2}^c(ufgh)(d) + f(\rho)g(\rho)h(\rho)I_{q_2}^c u(d) + h(\rho)I_{q_2}^c(ufg)(d) + f(\rho)g(\rho)I_{q_2}^c(uh)(d) \\ \geq g(\rho)I_{q_2}^c(ufh)(d) + g(\rho)h(\rho)I_{q_2}^c(uf)(d) + f(\rho)I_{q_2}^c(ugh)(d) + f(\rho)h(\rho)I_{q_2}^c(ug)(d). \end{aligned} \quad (3.21)$$

Multiplying both sides of (3.21) by $u(\rho)$ and q_1 -integrating the resulting inequality obtained with respect to ρ from c to d , then we have

$$\begin{aligned} I_{q_1}^a u(b)I_{q_2}^c(ufgh)(d) + I_{q_1}^a(uh)(b)I_{q_2}^c(ufg)(d) + I_{q_1}^a(ufg)(b)I_{q_2}^c(uh)(d) + I_{q_1}^a(ufgh)(b)I_{q_2}^c u(d) \\ \geq I_{q_1}^a(uf)(b)I_{q_2}^c(ugh)(d) + I_{q_1}^a(ug)(b)I_{q_2}^c(ufh)(d) + I_{q_1}^a(ugh)(b)I_{q_2}^c(uf)(d) + I_{q_1}^a(ufh)(b)I_{q_2}^c(ug)(d), \end{aligned}$$

which implies ((3.19)). \square

Theorem 3.7. Let f, g, h be three continuous and synchronous functions on $J \cup K$ and let $u, v : J \cup K \rightarrow [0, \infty)$ be two continuous functions. Then the following inequality holds true

$$\begin{aligned} I_{q_1}^a u(b)I_{q_2}^c(vfgh)(d) + I_{q_1}^a(uh)(b)I_{q_2}^c(vfg)(d) + I_{q_1}^a(ufg)(b)I_{q_2}^c(vh)(d) + I_{q_1}^a(ufgh)(b)I_{q_2}^c v(d) \\ \geq I_{q_1}^a(uf)(b)I_{q_2}^c(vgh)(d) + I_{q_1}^a(ug)(b)I_{q_2}^c(vfh)(d) + I_{q_1}^a(ugh)(b)I_{q_2}^c(vf)(d) + I_{q_1}^a(ufh)(b)I_{q_2}^c(vg)(d). \end{aligned} \quad (3.22)$$

Proof. Multiplying both sides of (3.20) by $v(\tau)$ and integrating the resulting identity with respect to τ from c to d , then we obtain

$$\begin{aligned} I_{q_2}^c(vfgh)(d) + f(\rho)g(\rho)h(\rho)I_{q_2}^c u(d) + h(\rho)I_{q_2}^c(vfg)(d) + f(\rho)g(\rho)I_{q_2}^c(vh)(d) \\ \geq g(\rho)I_{q_2}^c(vfh)(d) + g(\rho)h(\rho)I_{q_2}^c(vf)(d) + f(\rho)I_{q_2}^c(vgh)(d) + f(\rho)h(\rho)I_{q_2}^c(vg)(d). \end{aligned} \quad (3.23)$$

Multiplying both sides of (3.23) by $u(\rho)$ and integrating the resulting inequality obtained with respect to ρ from a to b , then we have

$$\begin{aligned} I_{q_1}^a u(b)I_{q_2}^c(vfgh)(d) + I_{q_1}^a(uh)(b)I_{q_2}^c(vfg)(d) + I_{q_1}^a(ufg)(b)I_{q_2}^c(vh)(d) + I_{q_1}^a(ufgh)(b)I_{q_2}^c v(d) \\ \geq I_{q_1}^a(uf)(b)I_{q_2}^c(vgh)(d) + I_{q_1}^a(ug)(b)I_{q_2}^c(vfh)(d) + I_{q_1}^a(ugh)(b)I_{q_2}^c(vf)(d) + I_{q_1}^a(ufh)(b)I_{q_2}^c(vg)(d), \end{aligned}$$

which implies (3.22). \square

Remark 3.8. It may be noted that the inequalities in (3.19) and (3.22) are reversed if functions f, g and h are asynchronous. It is also easily seen that the special case $u = v$ of (3.22) in Theorem 3.7 reduces to that in Theorem 3.6.

4 Other quantum integral inequalities

The first class are the inequalities related to Cauchy's inequality.

Theorem 4.1. Let ϕ, f and g be three continuous functions on J . Then the following inequality holds true

$$[T(\phi, f, g)]^2 \leq T(\phi, f, f)T(\phi, g, g), \quad (4.1)$$

where $T(\phi, f, g) = I_q^a \phi(b)I_q^a(\phi fg)(b) - I_q^a(\phi f)(b)I_q^a(\phi g)(b)$.

Proof. By simple computation, we have the following fact that

$$T(\phi, f, g) = \frac{1}{2} \int_a^b \int_a^b \phi(\rho)\phi(\tau)[f(\rho) - f(\tau)][g(\rho) - g(\tau)]_a d_q \rho_a d_q \tau. \quad (4.2)$$

From (4.2) and weighted Cauchy's inequality, we easily obtain (4.1). \square

Lemma 4.2. *Let f and h be two continuous functions on J and let $\phi : J \rightarrow [0, \infty)$ be a continuous function. Then the following inequality holds true*

$$m[g(\rho) - g(\tau)] \leq f(\rho) - f(\tau) \leq M[g(\rho) - g(\tau)], \quad \forall \rho, \tau \in J, \quad (4.3)$$

where m and M are given real numbers. Then for all $t > 0$ and $\nu > 0$, we have

$$T(\phi, f, f) + mMT(\phi, g, g) \leq (m + M)T(\phi, f, g), \quad (4.4)$$

where $T(\phi, f, g)$ is defined as in Theorem 4.1.

Proof. If we use the condition (4.3), we get

$$(M[g(\rho) - g(\tau)] - [f(\rho) - f(\tau)])([f(\rho) - f(\tau)] - m[g(\rho) - g(\tau)]) \geq 0, \quad \forall \rho, \tau \in J. \quad (4.5)$$

From (4.5) and through simple computation, we have

$$[f(\rho) - f(\tau)]^2 + mM[g(\rho) - g(\tau)]^2 \leq (m + M)[f(\rho) - f(\tau)][g(\rho) - g(\tau)]. \quad (4.6)$$

Multiplying both sides of (4.6) by $\phi(\rho)\phi(\tau)$ and integrating the resulting identity with respect to ρ and τ from a to b , we deduce the required inequality (4.4). \square

Theorem 4.3. *Let f, g, ϕ be defined as in Lemma 4.2 and $0 < m \leq M < \infty$. Then the following inequalities hold true*

$$T(\phi, f, f)T(\phi, g, g) \leq \frac{(m + M)^2}{4mM} [T(\phi, f, g)]^2, \quad (4.7)$$

$$0 \leq \sqrt{T(\phi, f, f)T(\phi, g, g)} - T(\phi, f, g) \leq \frac{(\sqrt{M} - \sqrt{m})^2}{2\sqrt{mM}} T(\phi, f, g), \quad (4.8)$$

and

$$0 \leq T(\phi, f, f)T(\phi, g, g) - [T(\phi, f, g)]^2 \leq \frac{(M - m)^2}{4mM} [T(\phi, f, g)]^2, \quad (4.9)$$

where $T(\phi, f, g)$ is defined as in Theorem 4.1.

Proof. We use the following elementary inequality

$$2xy \leq \alpha x^2 + \frac{1}{\alpha} y^2, \quad \forall x, y \geq 0, \quad \alpha > 0,$$

to get, for the choices

$$\alpha = \sqrt{mM} > 0, \quad x = \sqrt{T(\phi, g, g)} \geq 0, \quad y = \sqrt{T(\phi, f, f)} \geq 0$$

the following inequality

$$2\sqrt{T(\phi, f, f)T(\phi, g, g)} \leq \sqrt{mM}T(\phi, g, g) + \frac{1}{\sqrt{mM}}T(\phi, f, f). \quad (4.10)$$

Using (4.4) and (4.10), we deduce

$$2\sqrt{T(\phi, f, f)T(\phi, g, g)} \leq \frac{m + M}{\sqrt{mM}} T(\phi, f, g).$$

which is clearly equivalent to (4.7). By a few transformations of (4.7), similarly, we obtain (4.8) and (4.9). \square

The second class are the inequalities related to Hölder's inequality.

Theorem 4.4. *Let $\phi : J \rightarrow [0, \infty)$ be a continuous function on J and $f, g : J \rightarrow (0, \infty)$ be two continuous functions on J such that $0 < m \leq f^\alpha(\tau)/g^\beta(\tau) \leq M < \infty$ on J . If $1/\alpha + 1/\beta = 1$ with $\alpha > 1$, then the following inequality holds true*

$$(I_q^a(\phi f^\alpha)(b))^{\frac{1}{\alpha}} (I_q^a(\phi g^\beta)(b))^{\frac{1}{\beta}} \leq \left(\frac{M}{m}\right)^{\frac{1}{\alpha\beta}} I_q^a(\phi fg)(b). \quad (4.11)$$

Proof. Since $f^\alpha(\tau)/g^\beta(\tau) \leq M$, then $f^{\alpha/\beta} \leq M^{1/\beta} g$. Multiplying by $\phi f > 0$, it follows that

$$\phi f^\alpha = \phi f^{1+\frac{\alpha}{\beta}} \leq M^{\frac{1}{\beta}} \phi fg$$

and integrating the above inequality from a to b , then we have

$$(I_q^a(\phi f^\alpha)(b))^{\frac{1}{\alpha}} \leq M^{\frac{1}{\alpha\beta}} (I_q^a(\phi fg)(b))^{\frac{1}{\alpha}}. \quad (4.12)$$

On the other hand, since $m \leq f^\alpha(\tau)/g^\beta(\tau)$, then $f \geq m^{1/\alpha} g^{\beta/\alpha}$. Multiplying by $\phi g > 0$, it follows that

$$\phi fg \geq m^{\frac{1}{\alpha}} \phi g^{1+\frac{\beta}{\alpha}} = m^{\frac{1}{\alpha}} \phi g^\beta.$$

Integrating the above inequality from a to b , we obtain that

$$(I_q^a(\phi fg)(b))^{\frac{1}{\beta}} \geq m^{\frac{1}{\alpha\beta}} (I_q^a(\phi g^\alpha)(b))^{\frac{1}{\beta}}. \quad (4.13)$$

Combining (4.12) and (4.13), we have the desired inequality (4.11). The proof is completed. \square

Theorem 4.5. *Suppose that $1/\alpha + 1/\beta = 1/\gamma$ with $\alpha, \beta, \gamma > 0$. Let $\phi : J \rightarrow [0, \infty)$ be a continuous function on J and $f, g : J \rightarrow (0, \infty)$ be two continuous functions on J . If $0 < m \leq f^\gamma(\tau)/g^{\beta\gamma/\alpha}(\tau) \leq M < \infty$ for any $\tau \in J$, then the following inequalities hold true*

$$(M - m)I_q^a(\phi f^\alpha)(b) + (mM^{\frac{\alpha}{\gamma}} - Mm^{\frac{\alpha}{\gamma}})I_q^a(\phi g^\beta)(b) \leq (M^{\frac{\alpha}{\gamma}} - m^{\frac{\alpha}{\gamma}})I_q^a(\phi f^\gamma g^\gamma)(b), \quad (4.14)$$

and

$$\alpha^{\frac{1}{\alpha}} \beta^{\frac{1}{\beta}} \gamma^{-\frac{1}{\gamma}} \frac{(M - m)^{\frac{1}{\alpha}} (mM^{\frac{\alpha}{\gamma}} - Mm^{\frac{\alpha}{\gamma}})^{\frac{1}{\beta}}}{(M^{\frac{\alpha}{\gamma}} - m^{\frac{\alpha}{\gamma}})^{\frac{1}{\gamma}}} (I_q^a(\phi f^\alpha))^{\frac{1}{\alpha}} (I_q^a(\phi g^\beta))^{\frac{1}{\beta}} \leq (I_q^a(\phi f^\gamma g^\gamma))^{\frac{1}{\gamma}}. \quad (4.15)$$

Proof. If $0 < m \leq x^\gamma/y^{\beta\gamma/\alpha} \leq M < \infty$, then the following inequality is valid (see [25]):

$$(M - m)x^\alpha + (mM^{\frac{\alpha}{\gamma}} - Mm^{\frac{\alpha}{\gamma}})y^\beta \leq (M^{\frac{\alpha}{\gamma}} - m^{\frac{\alpha}{\gamma}})x^\gamma y^\gamma. \quad (4.16)$$

Substituting in the inequality (4.16) $x \rightarrow f(\tau)$ and $y \rightarrow g(\tau)$, and multiplying both sides of the obtained result by $\phi(\tau)$ and integrating the resulting identity with respect to τ from a to b , we obtain (4.14).

Now, rewrite (4.14) in the form

$$\left(\frac{\gamma}{\alpha}\right) \left(\left(\frac{\alpha}{\gamma}\right) (M - m)I_q^a(\phi f^\alpha)(b)\right) + \left(\frac{\gamma}{\beta}\right) \left(\left(\frac{\beta}{\gamma}\right) (mM^{\frac{\alpha}{\gamma}} - Mm^{\frac{\alpha}{\gamma}})I_q^a(\phi g^\beta)(b)\right) \leq (M^{\frac{\alpha}{\gamma}} - m^{\frac{\alpha}{\gamma}})I_q^a(\phi f^\gamma g^\gamma)(b), \quad (4.17)$$

and applying arithmetic-geometric inequality on the left-hand side of (4.17) we get (4.15). \square

The next class are the inequalities related to Minkowsky's inequality.

Theorem 4.6. *Let $p \geq 1$ and $\phi : J \rightarrow [0, \infty)$ be a continuous function on J and $f, g : J \rightarrow (0, \infty)$ be two continuous functions on J . If $0 < m \leq f(\tau)/g(\tau) \leq M < \infty$ for any $\tau \in J$, then we have*

$$(I_q^a(\phi f^p)(b))^{\frac{1}{p}} + (I_q^a(\phi g^p)(b))^{\frac{1}{p}} \leq \frac{1 + M(m + 1)}{(m + 1)(M + 1)} (I_q^a(\phi(f + g)^p)(b))^{\frac{1}{p}}. \quad (4.18)$$

Proof. Using the condition $f(\tau)/g(\tau) \leq M$ for any $\tau \in J$, we can get

$$(M+1)^p f^p(\tau) \leq M^p (f+g)^p(\tau). \quad (4.19)$$

Multiplying both sides of (4.19) by $\phi(\tau)$ and integrating the resulting inequalities with respect to τ from a to b , we obtain

$$(M+1)^p I_q^a(f^p)(b) \leq M^p I_q^a((f+g)^p)(b).$$

Hence, we can write

$$(I_q^a(\phi f^p)(b))^{\frac{1}{p}} \leq \frac{M}{M+1} (I_q^a(\phi(f+g)^p)(b))^{\frac{1}{p}}. \quad (4.20)$$

On the other hand, using the condition $m \leq f(\tau)/g(\tau)$, we can get

$$(m+1)^p g^p(\tau) \leq (f+g)^p(\tau). \quad (4.21)$$

Multiplying both sides of (4.21) by $\phi(\tau)$ and integrating the resulting inequalities with respect to τ from a to b , we obtain

$$(m+1)^p I_q^a(\phi g^p)(b) \leq I_q^a(\phi(f+g)^p)(b).$$

Hence, we can write

$$(I_q^a(\phi g^p)(b))^{\frac{1}{p}} \leq \frac{1}{m+1} (I_q^a(\phi(f+g)^p)(b))^{\frac{1}{p}}. \quad (4.22)$$

Adding the inequalities (4.20) and (4.22), we obtain the inequality (4.19). \square

Theorem 4.7. Let $p \geq 1$ and $\phi : J \rightarrow [0, \infty)$ be a continuous function on J and $f, g : J \rightarrow (0, \infty)$ be two continuous functions on J . If $0 < m \leq f(\tau)/g(\tau) \leq M < \infty$ for any $\tau \in J$, then we have

$$(I_q^a(\phi f^p)(b))^{\frac{2}{p}} + (I_q^a(\phi g^p)(b))^{\frac{2}{p}} \geq \left(\frac{(m+1)(M+1)}{M} - 2 \right) (I_q^a(\phi f^p)(b))^{\frac{1}{p}} (I_q^a(\phi g^p)(b))^{\frac{1}{p}}. \quad (4.23)$$

Proof. Multiplying the inequalities (4.20) and (4.22), we obtain

$$\frac{(m+1)(M+1)}{M} (I_q^a(\phi f^p)(b))^{\frac{1}{p}} (I_q^a(\phi g^p)(b))^{\frac{1}{p}} \leq ((I_q^a(\phi(f+g)^p)(b))^{\frac{1}{p}})^2. \quad (4.24)$$

Applying Minkowski's inequality to the right hand side of (4.24), we get

$$\begin{aligned} ((I_q^a(\phi(f+g)^p)(b))^{\frac{1}{p}})^2 &\leq ((I_q^a(\phi f^p)(b))^{\frac{1}{p}} + (I_q^a(\phi g^p)(b))^{\frac{1}{p}})^2 \\ &= (I_q^a(\phi f^p)(b))^{\frac{2}{p}} + (I_q^a(\phi g^p)(b))^{\frac{2}{p}} + 2(I_q^a(\phi f^p)(b))^{\frac{1}{p}} (I_q^a(\phi g^p)(b))^{\frac{1}{p}}. \end{aligned} \quad (4.25)$$

Combining (4.24) and (4.25), we obtain (4.23). \square

Theorem 4.8. Suppose that $1/\alpha + 1/\beta = 1$ with $\alpha > 1$. Let $\phi : J \rightarrow [0, \infty)$ be a continuous function on J and $f, g : J \rightarrow (0, \infty)$ be two continuous functions on J . If $0 < m \leq f(\tau)/g(\tau) \leq M < \infty$ for any $\tau \in J$, then the following inequality holds true

$$I_q^a(\phi f g)(b) \leq \frac{2^\alpha}{\alpha} \left(\frac{M}{M+1} \right)^\alpha \left(\frac{I_q^a(\phi f^\alpha)(b) + I_q^a(\phi g^\alpha)(b)}{2} \right) + \frac{2^\beta}{\beta} \left(\frac{1}{m+1} \right)^\beta \left(\frac{I_q^a(\phi f^\beta)(b) + I_q^a(\phi g^\beta)(b)}{2} \right) \quad (4.26)$$

Proof. From $m \leq f(\tau)/g(\tau) \leq M$ for any $\tau \in J$, we have

$$f(\tau) \leq \frac{M}{M+1} (f(\tau) + g(\tau)), \quad g(\tau) \leq \frac{1}{m+1} (f(\tau) + g(\tau)). \quad (4.27)$$

From (4.27) and the following Young-type inequality

$$xy \leq \frac{1}{\alpha}x^\alpha + \frac{1}{\beta}y^\beta, \quad \forall x, y \geq 0, \alpha > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1,$$

we obtain

$$\begin{aligned} I_q^a(\phi f g)(b) &\leq \frac{1}{\alpha} I_q^a(\phi f^\alpha)(b) + \frac{1}{\beta} I_q^a(\phi g^\beta)(b) \\ &\leq \frac{1}{\alpha} \left(\frac{M}{M+1} \right)^\alpha I_q^a(\phi(f+g)^\alpha)(b) + \frac{1}{\beta} \left(\frac{1}{m+1} \right)^\beta I_q^a(\phi(f+g)^\beta)(b). \end{aligned} \quad (4.28)$$

Using the elementary inequality $(c+d)^\alpha \leq 2^{\alpha-1}(c^\alpha + d^\alpha)$, $(\alpha > 1$ and $c, d > 0)$ in (4.28), we get

$$\begin{aligned} I_q^a(\phi f g)(b) &\leq \frac{1}{\alpha} \left(\frac{M}{M+1} \right)^\alpha 2^{\alpha-1} I_q^a(\phi(f^\alpha + g^\alpha))(b) + \frac{1}{\beta} \left(\frac{1}{m+1} \right)^\beta 2^{\beta-1} I_q^a(\phi(f^\beta + g^\beta))(b) \\ &= \frac{2^\alpha}{\alpha} \left(\frac{M}{M+1} \right)^\alpha \left(\frac{I_q^a(\phi f^\alpha)(b) + I_q^a(\phi g^\alpha)(b)}{2} \right) + \frac{2^\beta}{\beta} \left(\frac{1}{m+1} \right)^\beta \left(\frac{I_q^a(\phi f^\beta)(b) + I_q^a(\phi g^\beta)(b)}{2} \right). \end{aligned}$$

This completes the proof of the inequality in (4.26). \square

Theorem 4.9. Suppose that $1/\alpha + 1/\beta = 1$ with $\alpha, \beta > 0$. Let $\phi : J \rightarrow [0, \infty)$ be a continuous function on J and $f, g : J \rightarrow (0, \infty)$ be two continuous functions on J . If $0 < m \leq f(\tau)/(f(\tau) + g(\tau)) \leq M < \infty$ and $0 < m \leq g(\tau)/(f(\tau) + g(\tau)) \leq M < \infty$ for any $\tau \in J$, then we have

$$(I_q^a(\phi(f+g)^\alpha)(b))^{\frac{1}{\alpha}} \geq \alpha^{\frac{1}{\alpha}} \beta^{\frac{1}{\beta}} \frac{(M-m)^{\frac{1}{\alpha}} (mM^\alpha - Mm^\alpha)^{\frac{1}{\beta}}}{M^\alpha - m^\alpha} ((I_q^a(\phi f^\alpha)(b))^{\frac{1}{\alpha}} + (I_q^a(\phi g^\alpha)(b))^{\frac{1}{\alpha}}). \quad (4.29)$$

Proof. Due to (4.15) with $\gamma = 1$ of Theorem 4.5, $m \leq f(\tau)/g^{\beta/\alpha}(\tau) \leq M$ for any $\tau \in J$, we have

$$\alpha^{\frac{1}{\alpha}} \beta^{\frac{1}{\beta}} \frac{(M-m)^{\frac{1}{\alpha}} (mM^\alpha - Mm^\alpha)^{\frac{1}{\beta}}}{M^\alpha - m^\alpha} (I_q^a(\phi f^\alpha)(b))^{\frac{1}{\alpha}} (I_q^a(\phi g^\beta)(b))^{\frac{1}{\beta}} \leq I_q^a(\phi f g)(b). \quad (4.30)$$

By simple computation, we have

$$I_q^a(\phi(f+g)^\alpha)(b) = I_q^a(\phi f(f+g)^{\alpha-1})(b) + I_q^a(\phi g(f+g)^{\alpha-1})(b). \quad (4.31)$$

From $m \leq f(\tau)/(f(\tau) + g(\tau)) \leq M$ and $m \leq g(\tau)/(f(\tau) + g(\tau)) \leq M$ for any $\tau \in J$, we have $m \leq f(\tau)/((f(\tau) + g(\tau))^{\alpha-1})^{\beta/\alpha} \leq M$ and $m \leq g(\tau)/((f(\tau) + g(\tau))^{\alpha-1})^{\beta/\alpha} \leq M$ for any $\tau \in J$. Applying (4.30) on right hand of (4.31), we get

$$\begin{aligned} I_q^a(\phi f(f+g)^{\alpha-1})(b) &\geq \alpha^{\frac{1}{\alpha}} \beta^{\frac{1}{\beta}} \frac{(M-m)^{\frac{1}{\alpha}} (mM^\alpha - Mm^\alpha)^{\frac{1}{\beta}}}{M^\alpha - m^\alpha} [I_q^a(\phi f^\alpha)(b)]^{\frac{1}{\alpha}} [I_q^a(\phi(f+g)^{(\alpha-1)\beta})(b)]^{\frac{1}{\beta}}, \\ I_q^a(\phi g(f+g)^{\alpha-1})(b) &\geq \alpha^{\frac{1}{\alpha}} \beta^{\frac{1}{\beta}} \frac{(M-m)^{\frac{1}{\alpha}} (mM^\alpha - Mm^\alpha)^{\frac{1}{\beta}}}{M^\alpha - m^\alpha} [I_q^a(\phi g^\alpha)(b)]^{\frac{1}{\alpha}} [I_q^a(\phi(f+g)^{(\alpha-1)\beta})(b)]^{\frac{1}{\beta}}. \end{aligned} \quad (4.32)$$

Using (4.31) and adding two inequalities of (4.32), we obtain

$$I_q^a(\phi(f+g)^\alpha)(b) \geq \alpha^{\frac{1}{\alpha}} \beta^{\frac{1}{\beta}} \frac{(M-m)^{\frac{1}{\alpha}} (mM^\alpha - Mm^\alpha)^{\frac{1}{\beta}}}{M^\alpha - m^\alpha} ((I_q^a(\phi f^\alpha)(b))^{\frac{1}{\alpha}} + (I_q^a(\phi g^\alpha)(b))^{\frac{1}{\alpha}}) (I_q^a(\phi(f+g)^\alpha)(b))^{\frac{1}{\beta}}. \quad (4.33)$$

Dividing both sides of (4.33) by $(I_q^a(\phi(f+g)^\alpha)(b))^{\frac{1}{\beta}}$, we get (4.29). \square

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A FIRST ORDER DIFFERENTIAL SUBORDINATION AND ITS APPLICATIONS

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ABSTRACT. In this paper we use the differential subordinations techniques to obtain some properties of functions belonging to the class of analytic functions in the open unit disc \mathbb{U} . Also, some properties of the class of two fixed points in \mathbb{U} , are also discussed. Furthermore, some interesting results of Hurwitz Lerch Zeta function and Digamma function are obtained.

1. INTRODUCTION

Let A_k denote the class of functions f of the form

$$(1.1) \quad f(z) = z + \sum_{m=k+1}^{\infty} a_m z^m \quad (k \in \mathbb{N} = \{1, 2, \dots\}),$$

which are analytic in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$. Also, let $\mathcal{H}[a, k]$ denote the class of analytic functions in \mathbb{U} in the form

$$(1.2) \quad r(z) = a + \sum_{m=k}^{\infty} a_m z^m \quad (z \in \mathbb{U}),$$

for $a \in \mathbb{C}$ (\mathbb{C} is the complex plane).

Usually the analytic functions with the normalization $f(0) = 0 = f'(0) - 1$ is studied. Moreover, we can obtain interesting results by using the Montel's normalization of f (cf. [16], [6]) as follows

$$(1.3) \quad f(z)|_{z=0} = 0 \quad \text{and} \quad \left. \frac{f(z)}{z} \right|_{z=\rho} = 1,$$

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where ρ is a fixed point in \mathbb{U} . We see that, when $\rho = 0$, we get the classical normalization in \mathbb{U} . We denote by $A_{k,\rho}$ the class of functions f in A_k with Montel's normalization. The class $A_{k,\rho}$ will be called the class of functions f with two fixed points.

A function f in the class A_k is said to be in the class $R_k(\alpha)$ if it satisfies

$$(1.4) \quad \operatorname{Re} \left(\frac{f(z)}{z} \right) > \alpha \quad (z \in \mathbb{U}),$$

for some α ($0 \leq \alpha < 1$). The classes $R_1(\alpha) = \mathcal{C}(\alpha)$, and $R_1(0) = \mathcal{C}(0)$, were earlier studied by Èzrohi [7] and MacGregor [19], respectively. Further; a function f in the class A_k is said to be in the class $P_k(\alpha)$ if it satisfies

$$(1.5) \quad \operatorname{Re} \left(f'(z) \right) > \alpha \quad (z \in \mathbb{U}),$$

for some α ($0 \leq \alpha < 1$). The class $P_1(0) = \mathcal{B}(0)$, was earlier studied by Yamaguchi [28]. For some α ($0 \leq \alpha < 1$), $\lambda \neq 0$ with $\operatorname{Re}(\lambda) \geq 0$ and $z \in \mathbb{U}$ we write :

$$(1.6) \quad R_k^1(\alpha, \lambda) := \left\{ f(z) \in A_k : \operatorname{Re} \left((1 - \lambda) \frac{f(z)}{z} + \lambda f'(z) \right) > \alpha \right\}$$

and

$$(1.7) \quad R_k^2(\alpha, \lambda) := \left\{ f(z) \in A_k : \operatorname{Re} \left(f'(z) + \lambda z f''(z) \right) > \alpha \right\}.$$

We note that

- (i) $R_k^1(\alpha, 1) = P_k(\alpha)$,
- (ii) $f \in R_k^2(\alpha, \lambda)$ if and only if $zf' \in R_k^1(\alpha, \lambda)$.

Now, if $f \in A_k$, we define the function $G_k(\mu, \gamma; z)$ by

$$(1.8) \quad G_k(\mu, \gamma; z) := \frac{(f'(z))^\mu (f(z))^{1-\mu}}{\gamma z^{1-\mu}} \left((\gamma - 1) + (1 - \mu) \frac{zf'(z)}{f(z)} + \mu \left(1 + \frac{zf''(z)}{f'(z)} \right) \right),$$

with $\frac{z^{1-\mu}}{(f'(z))^\mu (f(z))^{1-\mu}} \neq 0$, for $\mu \in \mathbb{R}$, $\gamma \neq 0$ with $\operatorname{Re}(\gamma) \geq 0$ and $z \in \mathbb{U}$.

Let $H_k(\mu, \gamma, \alpha)$ denote the class of functions f satisfying the condition

$$(1.9) \quad \operatorname{Re}(G_k(\mu, \gamma; z)) > \alpha \quad (z \in \mathbb{U}),$$

for some $\alpha(0 \leq \alpha < 1)$ and $G_k(\mu, \gamma; z)$ defined by (1.8).

Also, we note that

(i) For $\lambda \neq 0$ with $\operatorname{Re}(\lambda) \geq 0$

$$(1.10) \quad H_k\left(0, \frac{1}{\lambda}, \alpha\right) = R_k^1(\alpha, \lambda) \quad \text{and} \quad H_k\left(1, \frac{1}{\lambda}, \alpha\right) = R_k^2(\alpha, \lambda).$$

(ii) One can define the $R_k^1(\alpha, \lambda)$ for $\lambda = 0$. Therefore we may use the following relations

$$(1.11) \quad R_k(\alpha) = R_k^1(\alpha, 0) = \lim_{\lambda \rightarrow 0} H_k\left(0, \frac{1}{\lambda}, \alpha\right),$$

and

$$(1.12) \quad R_k^2(\alpha, 0) = \lim_{\lambda \rightarrow 0} H_k\left(1, \frac{1}{\lambda}, \alpha\right).$$

A general Hurwitz- Lerch Zeta function (or Lerch transcendent) $\Phi(z, s, b)$ (cf., e.g., [24, Section 2.5, P. 121]) is the analytic continuation of the series

$$(1.13) \quad \Phi(z, s, b) = \sum_{k=0}^{\infty} \frac{z^k}{(k+b)^s},$$

which converges for b ($b \in \mathbb{C} \setminus \mathbb{Z}_0^-, \mathbb{Z}_0^- = \{0, -1, -2, \dots\}$) if z and $s \in \mathbb{C}$ are any complex numbers with either $z \in \mathbb{U}$, or $|z| = 1$ and $\operatorname{Re}(s) > 1$. See also [2, Section 1.11].

Many authors obtained several properties of $\Phi(z, s, b)$, for example, Attiya and Hakami [1], Cho *et al.* [3], Choi and *et al.* [5], Ferreira and López [9], Guillera and Sondow [10, Section 2], Gupta *et al.* [11], Kutbi and Attiya ([13],[14]), Luo and Srivastava [15], Owa and Attiya [21], Prajapat and Bulboacă [22], Srivastava and Attiya [23], Srivastava *et al.* [25] and Wang *et al.* [27].

Moreover, the Digamma function (or Psi) (cf., e.g., [24, Section 1.2, P. 13]) is the logarithmic derivative of the classical gamma function,

defined by

$$(1.14) \quad \Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -C - \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{z+n} \right),$$

with the Euler constant $C = 0.57721566\dots$. See also [2, Section 1.7] and [18, Section 5.1]. Several properties of Ψ can be found in [17], [4], [8] and [26].

We shall also need the following definitions

Definition 1.1. Let f and F be analytic functions. The function f is said to be *subordinate* to F , written $f(z) \prec F(z)$, if there exists a function w analytic in \mathbb{U} , with $w(0) = 0$ and $|w(z)| \leq 1$, and such that $f(z) = F(w(z))$. If F is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$.

Definition 1.2. Let $\Psi : \mathbb{C}^3 \times \mathbb{U} \rightarrow \mathbb{C}$ and let h be univalent in \mathbb{U} . If $q \in \mathcal{H}[a, k]$ satisfies the differential subordination

$$(1.15) \quad \Psi(p(z), z p'(z), z^2 p''(z); z) \prec h(z) \quad (z \in \mathbb{U}),$$

then q will be called (a, k) -*solution*. The univalent function s is called (a, k) -*dominant*, if $q(z) \prec s(z)$ for all q satisfying (1.15), (a, k) -*dominant* $\bar{s}(z) \prec s(z)$ for all (a, k) -*dominant* s of (1.15) is said to be *the best* (a, k) -*dominant* of (1.15).

In this paper, using the technique of differential subordination, some properties of functions in the class $H_k(\mu, \gamma, \alpha)$ are obtained. Furthermore, some properties of the class of two fixed points in \mathbb{U} , are also introduced. Some applications to *Analytic Number Theory* are also discussed.

2. THE CLASS $H_k(\mu, \gamma, \alpha)$ WITH FIRST ORDER DIFFERENTIAL SUBORDINATION

To prove our results, we need the following theorem due to Hallenbeck and Ruscheweyh [12] (see also Miller and Mocanu [20, P. 71]).

Theorem 2.1. Let h be convex univalent in \mathbb{U} , with $h(0) = a$, $\gamma \neq 0$ and $\operatorname{Re}(\gamma) \geq 0$. If $q \in \mathcal{H}[a, k]$ and

$$(2.1) \quad q(z) + \frac{z q'(z)}{\gamma} \prec h(z),$$

then

$$(2.2) \quad q(z) \prec S(z) \prec h(z),$$

where

$$(2.3) \quad S(z) = \frac{\gamma}{k z^{\frac{\gamma}{k}}} \int_0^z h(t) t^{\frac{\gamma}{k}-1} dt.$$

The function S is a convex univalent and is *the best* (a, k) -domain int.

Now, we prove

Theorem 2.2. Let γ be a complex number satisfying $\gamma \neq 0$ with $\operatorname{Re}(\gamma) \geq 0$. If $q \in \mathcal{H}[1, k]$ and

$$(2.4) \quad \operatorname{Re} \left(q(z) + \frac{z q'(z)}{\gamma} \right) > \alpha,$$

then

$$(2.5) \quad \operatorname{Re}(q(z)) > (2\alpha - 1) + 2(1 - \alpha) {}_2F_1 \left(1, \frac{|\gamma|^2}{k \operatorname{Re}(\gamma)}; \frac{|\gamma|^2}{k \operatorname{Re}(\gamma)} + 1, -1 \right) \quad (z \in \mathbb{U}),$$

for some α ($0 \leq \alpha < 1$) and $\lambda \neq 0$ with $\operatorname{Re}(\lambda) > 0$. The constant $(2\alpha - 1) + 2(1 - \alpha) {}_2F_1 \left(1, \frac{|\gamma|^2}{k \operatorname{Re}(\gamma)}; \frac{|\gamma|^2}{k \operatorname{Re}(\gamma)} + 1, -1 \right)$ is the best possible.

Proof. If we take the convex univalent function h defined by

$$(2.6) \quad h(z) = \frac{1 + (2\alpha - 1)z}{1 + z} \quad (0 \leq \alpha < 1),$$

then, we have

$$(2.7) \quad q(z) + \frac{z q'(z)}{\gamma} \prec h(z),$$

where h is defined by (2.6) satisfying $h(0) = 1$.

Applying Theorem 2.1, then

$$(2.8) \quad q(z) \prec S(z),$$

where the convex function S defined by

$$S(z) = \frac{\gamma}{k z^{\frac{\gamma}{k}}} \int_0^z \frac{1 + (2\alpha - 1)t}{1 + t} t^{\frac{\gamma}{k}-1} dt,$$

$$(2.9) \quad = (2\alpha - 1) + 2(1 - \alpha) \int_0^1 \frac{dt}{1 + t^{\frac{k}{\gamma}} z},$$

since $\operatorname{Re}(h(z)) > 0$ and $S(z) \prec h(z)$, we have $\operatorname{Re}(S(z)) > 0$.
Also, since

$$(2.10) \quad \inf_{z \in \mathbb{U}} \operatorname{Re} \left(\frac{1}{1 + t^{\frac{k}{\gamma}} z} \right) = \frac{1}{1 + t^{\frac{k}{\gamma}} \frac{k \operatorname{Re}(\gamma)}{|\gamma|^2}} \quad (0 \leq t \leq 1).$$

Hence

$$(2.11) \quad \begin{aligned} \inf_{z \in \mathbb{U}} \operatorname{Re}(S(z)) &= (2\alpha - 1) + 2(1 - \alpha) \int_0^1 \frac{dt}{1 + t^{\frac{k}{\gamma}} \frac{k \operatorname{Re}(\gamma)}{|\gamma|^2}} \\ &= (2\alpha - 1) + 2(1 - \alpha) {}_2F_1 \left(1, \frac{|\gamma|^2}{k \operatorname{Re}(\gamma)}; \frac{|\gamma|^2}{k \operatorname{Re}(\gamma)} + 1, -1 \right). \end{aligned}$$

Therefore, the constant $(2\alpha - 1) + 2(1 - \alpha) {}_2F_1 \left(1, \frac{|\gamma|^2}{k \operatorname{Re}(\gamma)}; \frac{|\gamma|^2}{k \operatorname{Re}(\gamma)} + 1, -1 \right)$ cannot be replaced by a larger one, which completes the proof of Theorem 2.2. \square

Theorem 2.3. Let the function f defined by (1.1) be in the class $H_k(\mu, \gamma, \alpha)$, then

$$(2.12) \quad \operatorname{Re} \left(\frac{(f'(z))^\mu (f(z))^{1-\mu}}{z^{1-\mu}} \right) > (2\alpha - 1) + 2(1 - \alpha) {}_2F_1 \left(1, \frac{|\gamma|^2}{k \operatorname{Re}(\gamma)}; \frac{|\gamma|^2}{k \operatorname{Re}(\gamma)} + 1, -1 \right) \quad (z \in \mathbb{U}),$$

for some α ($0 \leq \alpha < 1$), $\mu \in \mathbb{R}$ and $\gamma \neq 0$ with $\operatorname{Re}(\gamma) \geq 0$. The constant $(2\alpha - 1) + 2(1 - \alpha) {}_2F_1 \left(1, \frac{|\gamma|^2}{k \operatorname{Re}(\gamma)}; \frac{|\gamma|^2}{k \operatorname{Re}(\gamma)} + 1, -1 \right)$ is the best possible.

Proof. Defining the function

$$(2.13) \quad q(z) = \frac{(f'(z))^\mu (f(z))^{1-\mu}}{z^{1-\mu}} \quad (z \in \mathbb{U}),$$

then, we have $q \in \mathcal{H}[1, k]$.

Taking the logarithmic differentiation in both sides of (2.13), we have

$$(2.14) \quad q(z) + \frac{z q'(z)}{\gamma} = G_k(\mu, \gamma; z),$$

since $f \in H_k(\mu, \gamma, \alpha)$, then

$$(2.15) \quad \operatorname{Re} \left(q(z) + \frac{z q'(z)}{\gamma} \right) > \alpha.$$

Therefore, we have (2.12) by applying Theorem 2.2. \square

Putting $\mu = 0$ and $\gamma = 1/\lambda$ ($\lambda \neq 0$; $\operatorname{Re}(\lambda) \geq 0$), in Theorem 2.3, we have

Corollary 2.1. Let the function f defined by (1.1) be in the class $R_k^1(\alpha, \lambda)$, then

$$(2.16) \quad \operatorname{Re} \left(\frac{f(z)}{z} \right) > (2\alpha - 1) + 2(1 - \alpha) {}_2F_1 \left(1, \frac{1}{k \operatorname{Re}(\lambda)}; \frac{1}{k \operatorname{Re}(\lambda)} + 1, -1 \right) \quad (z \in \mathbb{U}),$$

for some α ($0 \leq \alpha < 1$) and $\lambda \neq 0$ with $\operatorname{Re}(\lambda) \geq 0$. The constant $(2\alpha - 1) + 2(1 - \alpha) {}_2F_1 \left(1, \frac{1}{k \operatorname{Re}(\lambda)}; \frac{1}{k \operatorname{Re}(\lambda)} + 1, -1 \right)$ is the best possible.

Assuming $\mu = 1$ and $\gamma = 1/\lambda$ ($\lambda \neq 0$; $\operatorname{Re}(\lambda) \geq 0$), in Theorem 2.3 or putting zf' instead of f , in Corollary (2.1), we have

Corollary 2.2. Let the function f defined by (1.1) be in the class $R_k^2(\alpha, \lambda)$, then

$$(2.17) \quad \operatorname{Re} (f'(z)) > (2\alpha - 1) + 2(1 - \alpha) {}_2F_1 \left(1, \frac{1}{k \operatorname{Re}(\lambda)}; \frac{1}{k \operatorname{Re}(\lambda)} + 1, -1 \right) \quad (z \in \mathbb{U}),$$

for some α ($0 \leq \alpha < 1$) and $\lambda \neq 0$ with $\operatorname{Re}(\lambda) \geq 0$. The constant $(2\alpha - 1) + 2(1 - \alpha) {}_2F_1 \left(1, \frac{1}{k \operatorname{Re}(\lambda)}; \frac{1}{k \operatorname{Re}(\lambda)} + 1, -1 \right)$ is the best possible.

Corollary 2.3. Let the function f defined by (1.1) be in the class $A_{k,\rho}$ of functions f with two fixed points. Also, let f be in the class $R_k^1(\alpha, \lambda)$, then

$$(2.18) \quad \operatorname{Re} \left(\sum_{m=k+2}^{\infty} a_m z^k (z^{m-k-1} - \rho^{m-k-1}) \right) > 2(1 - \alpha) \left({}_2F_1 \left(1, \frac{1}{k \operatorname{Re}(\lambda)}; \frac{1}{k \operatorname{Re}(\lambda)} + 1, -1 \right) - 1 \right) \quad (z \in \mathbb{U}),$$

for some α ($0 \leq \alpha < 1$), ρ is a fixed point in \mathbb{U} defined in (1.3) and $\lambda \neq 0$ with $\operatorname{Re}(\lambda) \geq 0$. The constant $2(1 - \alpha) \left({}_2F_1 \left(1, \frac{1}{k \operatorname{Re}(\lambda)}; \frac{1}{k \operatorname{Re}(\lambda)} + 1, -1 \right) - 1 \right)$ is the best possible.

Proof. Since $f \in A_{k,\rho}$, then we have

$$(2.19) \quad a_{k+1} = - \sum_{m=k+2}^{\infty} a_m \rho^{m-k-1},$$

therefore, the function $f(z)/z$, takes the form

$$(2.20) \quad \frac{f(z)}{z} = 1 + \left(\sum_{m=k+2}^{\infty} a_m z^k (z^{m-k-2} - \rho^{m-k-1}) \right),$$

Then, we have the Corollary by applying Corollary 2.1. \square

By using the same technique in Corollary 2.3, we have

Corollary 2.4. Let the function f defined by (1.1) be in the class $A_{k,\rho}$ of functions f with two fixed points. Also, let f be in the class $R_k^2(\alpha, \lambda)$, then

$$(2.21) \quad \operatorname{Re} \left(\sum_{m=k+2}^{\infty} a_m z^k (m z^{m-k-1} - \rho^{m-k-1}) \right) > 2(1-\alpha) \left({}_2F_1 \left(1, \frac{1}{k \operatorname{Re}(\lambda)}; \frac{1}{k \operatorname{Re}(\lambda)} + 1, -1 \right) - 1 \right) \quad (z \in \mathbb{U}),$$

for some α ($0 \leq \alpha < 1$), ρ is a fixed point in \mathbb{U} defined in (1.3) and $\lambda \neq 0$ with $\operatorname{Re}(\lambda) \geq 0$. The constant ${}_2F_1 \left(1, \frac{1}{k \operatorname{Re}(\lambda)}; \frac{1}{k \operatorname{Re}(\lambda)} + 1, -1 \right)$ is the best possible.

3. SOME APPLICATIONS IN ANALYTIC NUMBER THEORY

In this section we need the following lemma due to Guillera and Sondow [10].

Lemma 3.1. For $z \in \mathbb{C} - [1, \infty)$ and $\delta > 0$, we have

$$(3.1) \quad \int_0^1 \int_0^1 \frac{-(xy)^{\delta-1}}{(1-xyz) \ln xy} dx dy = \Phi(z, 1, \delta),$$

and

$$(3.2) \quad \int_0^1 \int_0^1 \frac{-(xy)^{\delta-1}}{(1+xy) \ln xy} dx dy = \frac{1}{2} \left\{ \Psi \left(\frac{\delta}{2} + \frac{1}{2} \right) - \Psi \left(\frac{\delta}{2} \right) \right\},$$

where $\Phi(z, s, \delta)$ and $\Psi(\delta)$ defined by (1.13) and (1.14) respectively

Corollary 3.1. Let $\Phi(z, s, b)$ be the Lerch transcendental function defined by (1.13), then

$$(3.3) \quad \operatorname{Re}(\Phi(z, 1, \delta)) > \frac{1}{2} \left(\Psi \left(\frac{\delta}{2} + \frac{1}{2} \right) - \Psi \left(\frac{\delta}{2} \right) \right) \quad (|z| < 1; \delta > 0),$$

and this result is the best possible.

Proof. We can show that the function

$$(3.4) \quad g(z) = z \left((2\alpha - 1) + \frac{2(1-\alpha)}{\lambda} \int_0^1 \frac{t^{\frac{1-\lambda}{\lambda}} dt}{1-tz} \right) \quad (0 \leq \alpha < 1; z \in \mathbb{U}),$$

is a member in the class $R_1^1(\alpha, \lambda)$. Using (3.4) and (1.13) we obtain

$$(3.5) \quad g(z) = z \left\{ (2\alpha - 1) + \frac{2(1-\alpha)}{\lambda} \Phi \left(z, 1, \frac{1}{\lambda} \right) \right\} \quad (0 \leq \alpha < 1; z \in \mathbb{U}),$$

which a member in the class $R_1^1(\alpha, \lambda)$, where $\Phi(z, s, b)$ is the Lerch transcendental function defined by (1.13).

Using (2.16) and (3.5), we readily obtain the following property with $\lambda > 0$; real

$$(3.6) \quad \operatorname{Re} \left(\Phi \left(z, 1, \frac{1}{\lambda} \right) \right) > \lambda \int_0^1 \frac{dt}{1+t^\lambda} \quad (|z| < 1),$$

which is equivalent to

$$(3.7) \quad \operatorname{Re}(\Phi(z, 1, \delta)) > \Phi(-1, 1, \delta), \quad (|z| < 1; \delta > 0),$$

the constant $\Phi(-1, 1, \delta)$, cannot be replaced by a larger one.

Using (1.14) and (3.7), we have

$$(3.8) \quad \operatorname{Re}(\phi(z, 1, \delta)) > \frac{1}{2} \left(\Psi \left(\frac{\delta}{2} + \frac{1}{2} \right) - \Psi \left(\frac{\delta}{2} \right) \right) \quad (|z| < 1; \delta > 0),$$

this result is the best possible in general, which completes the proof of Corollary 3.2 . \square

Using (3.2) and (3.3), we have the following corollary

Corollary 3.2. For $\delta > 0$, we have

$$(3.9) \quad \int_0^1 \int_0^1 \frac{-(x-y)^{\delta-1}}{(1+xy) \ln xy} dx dy < \operatorname{Re}(\phi(z, 1, \delta)) \quad (|z| < 1) .$$

This result is the best possible.

Using (3.1) and (3.3), we have the following corollary

Corollary 3.4. For $\delta > 0$, we have

$$(3.10) \quad \operatorname{Re} \left(\int_0^1 \int_0^1 \frac{-(x-y)^{\delta-1}}{(1-xy-z) \ln xy} dx dy \right) > \frac{1}{2} \left(\Psi \left(\frac{\delta}{2} + \frac{1}{2} \right) - \Psi \left(\frac{\delta}{2} \right) \right) \quad (|z| < 1).$$

This result is the best possible.

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Approximation by a complex summation-integral type operators in compact disks

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Abstract. In this paper we introduce a kind of complex summation-integral type operators and study the approximation properties of these operators. We obtain a Voronovskaja-type result with quantitative estimate for these operators attached to analytic functions on compact disks. We also study the exact order of approximation. More important, our results show the overconvergence phenomenon for these complex operators.

Keywords: Complex summation-integral type operators; Voronovskaja-type result; Exact order of approximation; Simultaneous approximation; Overconvergence

Mathematical subject classification: 30E10, 41A25, 41A36

1. Introduction

In 1986, some approximation properties of complex Bernstein polynomials in compact disks were initially studied by Lorentz [15]. Very recently, the problem of the approximation of complex operators has been causing great concern, which is becoming a hot topic of research. A Voronovskaja-type result with quantitative estimate for complex Bernstein polynomials in compact disks was obtained by Gal [2]. Also, in [1, 3-14, 16-19] similar results for complex Bernstein-Kantorovich polynomials, Bernstein-Stancu polynomials, Kantorovich-Schurer polynomials, Kantorovich-Stancu polynomials, complex Favard-Szász-Mirakjan operators, complex Beta operators of first kind, complex Baskakov-Stancu operators, complex Bernstein-Durrmeyer polynomials, complex Bernstein-Durrmeyer operators based on Jacobi weights, complex genuine Durrmeyer Stancu polynomials and complex q -Durrmeyer type operators were obtained.

The aim of the present article is to obtain approximation results for a kind of complex summation-integral type operators (introduced and studied in the case of real variable by Ren [20]), which are defined for $f : [0, 1] \rightarrow \mathbf{C}$ continuous

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on $[0, 1]$ by

$$M_n(f; z) := p_{n,0}(z)f(0) + \sum_{k=1}^{n-1} p_{n,k}(z)L_{n,k}(f) + p_{n,n}(z)f(1), \quad (1)$$

where $z \in \mathbf{C}$, $n = 1, 2, \dots$, $p_{n,k}(z) = \binom{n}{k} z^k (1-z)^{n-k}$ is Bernstein basis function, and $L_{n,k}(f) = \frac{1}{B(n(n-k), nk)} \int_0^1 t^{nk-1} (1-t)^{n(n-k)-1} f(t) dt$, $B(x, y)$ is Beta function.

2. Auxiliary results

In the sequel, we shall need the following auxiliary results.

Lemma 1. Let $m, n \in \mathbf{N}$, $z \in \mathbf{C}$, we have $M_n(t^m; z)$ is a polynomial of degree less than or equal to $\min(m, n)$ and

$$M_n(t^m; z) = \frac{(n^2 - 1)!}{(n^2 + m - 1)!} \sum_{s=1}^m c_s(m) n^{2s} B_n(t^s; z),$$

where $c_s(m) \geq 0$ are constants depending on m and

$$B_n(f; z) = \sum_{k=0}^n p_{n,k}(z) f\left(\frac{k}{n}\right).$$

Proof. By the definition of Beta function, for all $m, n \in \mathbf{N}$, $z \in \mathbf{C}$, we have

$$M_n(t^m; z) = \frac{(n^2 - 1)!}{(n^2 + m - 1)!} \sum_{k=1}^{n-1} p_{n,k}(z) \frac{(nk + m - 1)!}{(nk - 1)!} + z^n.$$

Considering the definition of the $B_n(f; z)$, for any $m \in \mathbf{N}$, applying the principle of mathematical induction, we immediately obtain the desired conclusion.

Let $m = 0, 1, 2$, by Lemma 1, we have

$$\begin{aligned} M_n(1; z) &= 1; \\ M_n(t; z) &= z; \\ M_n(t^2; z) &= \frac{n(n-1)}{n^2+1} z^2 + \frac{n+1}{n^2+1} z. \end{aligned}$$

Lemma 2. For all $m, n \in \mathbf{N}$ we can get the equality

$$\frac{(n^2 - 1)!}{(n^2 + m - 1)!} \sum_{s=1}^m c_s(m) n^{2s} = 1.$$

Proof. For all $m, n \in \mathbf{N}$, by Lemma 1 we have

$$M_n(t^m; 1) = \frac{(n^2 - 1)!}{(n^2 + m - 1)!} \sum_{s=1}^m c_s(m) n^{2s}.$$

On the other hand, we have $p_{n,k}(1) = 0, k = 0, 1, 2, \dots, n-1$, and $p_{n,n}(1) = 1$. So, by the formula (1) and using these above value, we have $M_n(t^m; 1) = 1$, which implies that we get desired conclusion.

Corollary 1. Let $e_m(t) = t^m, m \in \mathbb{N} \cup \{0\}, z \in \mathbb{C}, n \in \mathbb{N}$, for all $|z| \leq r, r \geq 1$, we have $|M_n(e_m; z)| \leq r^m$.

Proof. Since $M_n(e_0; z) = 1$, therefor this result is established for $m = 0$. When $m \in \mathbb{N}$, by using the methods Gal [5], p. 61, proof of Theorem 1.5.6, we have $|B_n(t^s; z)| \leq r^s$. Thus, for all $m \in \mathbb{N}$ and $|z| \leq r$, the proof follows directly by Lemma 1 and Lemma 2.

Lemma 3. Let $e_m(t) = t^m, m \in \mathbb{N} \cup \{0\}$ and $z \in \mathbb{C}$, we have

$$M_n(e_{m+1}; z) = \frac{nz(1-z)}{n^2+m} (M_n(e_m; z))' + \frac{m+n^2z}{n^2+m} M_n(e_m; z). \quad (2)$$

Proof. By Lemma 1, we have $M_n(e_0; z) = 1$ and $M_n(e_1; z) = z$, therefore, this result is established for $m = 0$. Now let $m \in \mathbb{N}$, in view of

$$z(1-z)[p_{n,k}(z)]' = (k-nz)p_{n,k}(z),$$

it follows that

$$\begin{aligned} & z(1-z)(M_n(e_m; z))' \\ &= \sum_{k=1}^{n-1} (k-nz)p_{n,k}(z)L_{n,k}(t^m) + nz^n(1-z) \\ &= \sum_{k=1}^{n-1} \left[\frac{(n^2+m)(nk+m)}{n(n^2+m)} - \frac{m}{n} \right] p_{n,k}(z)L_{n,k}(t^m) + nz^n - nzM_n(e_m; z) \\ &= \frac{n^2+m}{n} \sum_{k=1}^{n-1} p_{n,k}(z)L_{n,k}(t^{m+1}) - \frac{m}{n} \sum_{k=1}^{n-1} p_{n,k}(z)L_{n,k}(t^m) + nz^n - nzM_n(e_m; z) \\ &= \frac{n^2+m}{n} M_n(e_{m+1}; z) - \frac{m}{n} M_n(e_m; z) - nzM_n(e_m; z) \\ &= \frac{n^2+m}{n} M_n(e_{m+1}; z) - \frac{m+n^2z}{n} M_n(e_m; z), \end{aligned}$$

which implies the recurrence in the statement.

Lemma 4. Let $m, n \in \mathbb{N}, z \in \mathbb{C}, e_m(z) = z^m, S_{n,m}(z) := M_n(e_m; z) - z^m$, we have

$$\begin{aligned} S_{n,m}(z) &= \frac{nz(1-z)}{n^2+m-1} (M_n(e_{m-1}; z))' + \frac{m-1+n^2z}{n^2+m-1} S_{n,m-1}(z) \\ &\quad + \frac{m-1+n^2z}{n^2+m-1} z^{m-1} - z^m \end{aligned} \quad (3)$$

Proof. Using the recurrence formula (2), by simple calculation, we can easily get the recurrence (3), the proof is omitted.

3. Main results

The first main result is expressed by the following upper estimates.

Theorem 1. Let $1 \leq r \leq R$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. Suppose that $f : D_R \rightarrow \mathbf{C}$ is analytic in D_R , i.e. $f(z) = \sum_{m=0}^{\infty} c_m z^m$ for all $z \in D_R$.

(i) for all $|z| \leq r$ and $n \in \mathbf{N}$, we have

$$|M_n(f; z) - f(z)| \leq \frac{K_r(f)}{n},$$

where $K_r(f) = (1+r) \sum_{m=1}^{\infty} |c_m| m(m-1) r^{m-1} < \infty$.

(ii) (Simultaneous approximation) If $1 \leq r < r_1 < R$ are arbitrary fixed, then for all $|z| \leq r$ and $n, p \in \mathbf{N}$ we have

$$|(M_n(f; z))^{(p)} - f^{(p)}(z)| \leq \frac{K_{r_1}(f) p! r_1}{n(r_1 - r)^{p+1}},$$

where $K_{r_1}(f)$ is defined as at the above point (i).

Proof. Taking $e_m(z) = z^m$, by hypothesis that $f(z)$ is analytic in D_R , i.e. $f(z) = \sum_{m=0}^{\infty} c_m z^m$ for all $z \in D_R$, it is easy for us to obtain

$$M_n(f; z) = \sum_{m=0}^{\infty} c_m M_n(e_m; z),$$

therefore, we get

$$\begin{aligned} |M_n(f; z) - f(z)| &\leq \sum_{m=0}^{\infty} |c_m| \cdot |M_n(e_m; z) - e_m(z)| \\ &= \sum_{m=1}^{\infty} |c_m| \cdot |M_n(e_m; z) - e_m(z)|, \end{aligned}$$

as $M_n(e_0; z) = e_0(z) = 1$.

(i) For $m \in \mathbf{N}$, taking into account that $M_n(e_{m-1}; z)$ is a polynomial degree $\leq \min(m-1, n)$, by the well-known Bernstein inequality and Corollary 1, we get

$$|(M_n(e_{m-1}; z))'| \leq \frac{m-1}{r} \max\{|M_n(e_{m-1}; z)| : |z| \leq r\} \leq (m-1)r^{m-2}.$$

On the one hand, when $m = 1$, for $|z| \leq r$, by Lemma 1, we have

$$|M_n(e_1; z) - e_1(z)| = (1+r) \frac{m(m-1)}{n} r^{m-1}.$$

On the other hand, when $m \geq 2$, for $n \in \mathbf{N}$, $|z| \leq r$, $r \geq 1$, using the

recurrence formula (3) and the above inequality, we have

$$\begin{aligned} |M_n(e_m; z) - e_m(z)| &= |S_{n,m}(z)| \\ &\leq \frac{r(1+r)}{n}(m-1)r^{m-2} + r|S_{n,m-1}(z)| \\ &\quad + \frac{m-1}{n}(1+r)r^{m-1} \\ &= \frac{2(m-1)}{n}(1+r)r^{m-1} + r|S_{n,m-1}(z)|. \end{aligned}$$

By writing the last inequality, for $m = 2, \dots$, we easily obtain step by step the following

$$\begin{aligned} |M_n(e_m; z) - e_m(z)| &\leq r \left(r|S_{n,m-2}(z)| + \frac{2(m-2)}{n}(1+r)r^{m-2} \right) \\ &\quad + \frac{2(m-1)}{n}(1+r)r^{m-1} \\ &= r^2|S_{n,m-2}(z)| + \frac{2(m-2) + 2(m-1)}{n}(1+r)r^{m-1} \\ &\leq \dots \leq (1+r) \frac{m(m-1)}{n} r^{m-1}. \end{aligned}$$

In conclusion, for any $m, n \in \mathbf{N}$, $|z| \leq r$, $r \geq 1$, we have

$$|M_n(e_m; z) - e_m(z)| \leq (1+r) \frac{m(m-1)}{n} r^{m-1},$$

it follows that

$$|M_n(f; z) - f(z)| \leq \frac{1+r}{n} \sum_{m=1}^{\infty} |c_m| m(m-1) r^{m-1}.$$

By assuming that $f(z)$ is analytic in D_R , we have $f^{(2)}(z) = \sum_{m=2}^{\infty} c_m m(m-1) z^{m-2}$ and the series is absolutely convergent in $|z| \leq r$, so we get $\sum_{m=2}^{\infty} |c_m| m(m-1) r^{m-2} < \infty$, which implies $K_r(f) = (1+r) \sum_{m=1}^{\infty} |c_m| m(m-1) r^{m-1} < \infty$.

(ii) For the simultaneous approximation, denoting by Γ the circle of radius $r_1 > r$ and center 0, since for any $|z| \leq r$ and $v \in \Gamma$, we have $|v - z| \geq r_1 - r$, by Cauchy's formulas it follows that for all $|z| \leq r$ and $n \in \mathbf{N}$, we have

$$\begin{aligned} |(M_n(f; z))^{(p)} - f^{(p)}(z)| &= \frac{p!}{2\pi} \left| \int_{\Gamma} \frac{M_n(f; v) - f(v)}{(v-z)^{p+1}} dv \right| \\ &\leq \frac{K_{r_1}(f)}{n} \frac{p!}{2\pi} \frac{2\pi r_1}{(r_1-r)^{p+1}} \\ &= \frac{K_{r_1}(f)}{n} \cdot \frac{p! r_1}{(r_1-r)^{p+1}}, \end{aligned}$$

which proves the theorem.

Theorem 2. Let $R > 1$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. Suppose that $f : D_R \rightarrow \mathbf{C}$

is analytic in D_R , i.e. $f(z) = \sum_{k=0}^{\infty} c_k z^k$ for all $z \in D_R$. For any fixed $r \in [1, R]$ and all $n \in \mathbb{N}$, $|z| \leq r$, we have

$$\left| M_n(f; z) - f(z) - \frac{(n+1)z(1-z)f''(z)}{2(n^2+1)} \right| \leq \frac{M_r(f)}{n^2}. \quad (4)$$

where $M_r(f) = \sum_{k=2}^{\infty} |c_k|(k-1)F_{k,r}r^k$ and $F_{k,r} = 10k^3 - 30k^2 + 39k - 16 + 4(k-2)(k-1)^2(1+r)$.

Proof. For all $z \in D_R$, we have

$$\begin{aligned} & \left| M_n(f; z) - f(z) - \frac{(n+1)z(1-z)f''(z)}{2(n^2+1)} \right| \\ & \leq \sum_{k=2}^{\infty} |c_k| \left| M_n(e_k; z) - e_k(z) - \frac{(n+1)k(k-1)(1-z)z^{k-1}}{2(n^2+1)} \right|. \end{aligned}$$

Denoting

$$E_{k,n}(z) = M_n(e_k; z) - e_k(z) - \frac{(n+1)k(k-1)(1-z)z^{k-1}}{2(n^2+1)},$$

it is obvious that $E_{k,n}(z)$ is a polynomial of degree less than or equal to k . For all $k \geq 2$, by simple computation and the use of Lemma 3, we can get

$$E_{k,n}(z) = \frac{nz(1-z)}{n^2+k-1}(E_{k-1,n}(z))' + \frac{k-1+n^2z}{n^2+k-1}E_{k-1,n}(z) + G_{k,n}(z), \quad (5)$$

where $G_{k,n}(z) = \frac{z^{k-2}}{n^2+k-1}(z^2A_{k,n} + zB_{k,n} + C_{k,n})$ and $A_{k,n} = -n(k-1) + \frac{n(n+1)(k-1)^2(k-2)}{2(n^2+1)} + n^2 - \frac{n^2(n+1)(k-1)(k-2)}{2(n^2+1)} - (n^2+k-1) + \frac{(n+1)k(k-1)(n^2+k-1)}{2(n^2+1)}$, $B_{k,n} = n(k-1) - \frac{n(n+1)(k-1)^2(k-2)}{2(n^2+1)} - \frac{n(n+1)(k-1)(k-2)^2}{2(n^2+1)} + k-1 - \frac{(n+1)(k-1)^2(k-2)}{2(n^2+1)} + \frac{n^2(n+1)(k-1)(k-2)}{2(n^2+1)} - \frac{(n+1)k(k-1)(n^2+k-1)}{2(n^2+1)}$, $C_{k,n} = \frac{n(n+1)(k-1)(k-2)^2}{2(n^2+1)} + \frac{(n+1)(k-1)^2(k-2)}{2(n^2+1)}$.

For all $k \geq 2$, $n \in \mathbb{N}$ and $|z| \leq r$, $r \geq 1$, we easily obtain

$$|C_{k,n}| \leq (k-1)(k-2)(2k-3),$$

it follows that

$$\left| \frac{z^{k-2}C_{k,n}}{n^2+k-1} \right| \leq \frac{(2k^3 - 9k^2 + 13k - 6)r^k}{n^2}.$$

By simple computation, we have $B_{k,n} = \frac{1}{2(n^2+1)}\{2n(k-1) - n(n+1)(k-1)^2(k-2) - n(n+1)(k-1)(k-2)^2 + 2(n^2+1)(k-1) - (n+1)(k-1)^2(k-2) + n^2(k-1)(k-2) - nk(k-1)^2 - n^2k(k-1) - k(k-1)^2\}$, so, we can get

$$\left| \frac{z^{k-1}B_{k,n}}{n^2+k-1} \right| \leq \frac{(5k^3 - 15k^2 + 18k - 8)r^k}{n^2}.$$

By simple computation, we have $A_{k,n} = \frac{1}{2(n^2+1)}\{-2n(k-1) + n(n+1)(k-1)^2(k-2) + 2n^2 - n^2(k-1)(k-2) - 2(n^2k+k-1) + nk(k-1)^2 + n^2k(k-1) + k(k-1)^2\}$, so, we can get

$$\left| \frac{z^kA_{k,n}}{n^2+k-1} \right| \leq \frac{(3k^3 - 6k^2 + 8k - 2)r^k}{n^2}.$$

Thus, for all $k \geq 2$, $n \in \mathbb{N}$ and $|z| \leq r$, $r \geq 1$, we can obtain

$$|G_{k,n}(z)| \leq \frac{r^k}{n^2} D_k,$$

where $D_k = 10k^3 - 30k^2 + 39k - 16$.

By formula (5), for all $k \geq 2$, $n \in \mathbb{N}$ and $|z| \leq r$, $r \geq 1$, we have

$$|E_{k,n}(z)| \leq \frac{r(1+r)}{n} |(E_{k-1,n}(z))'| + r|E_{k-1,n}(z)| + |G_{k,n}(z)|.$$

Using the estimate in the proof of Theorem 1 (i), we get

$$|M_n(e_k; z) - e_k(z)| \leq \frac{1+r}{n} k(k-1)r^{k-1},$$

for all $k, n \in \mathbb{N}$, $|z| \leq r$, $r \geq 1$.

So, denote $\|f\|_r = \max\{|f(z)|; |z| \leq r\}$, we have

$$\begin{aligned} |(E_{k-1,n}(z))'| &\leq \frac{k-1}{r} \|E_{k-1,n}\|_r \\ &\leq \frac{k-1}{r} \left[\|M_n(e_{k-1}; \cdot) - e_{k-1}\|_r + \left\| \frac{(n+1)(k-1)(k-2)(1-e_1)e_{k-2}}{2(n^2+1)} \right\|_r \right] \\ &\leq \frac{k-1}{r} \left[\frac{(k-1)(k-2)(1+r)r^{k-2}}{n} + \frac{(k-1)(k-2)(1+r)r^{k-2}}{n} \right] \\ &\leq \frac{4(k-2)(k-1)^2 r^{k-1}}{n}, \end{aligned}$$

for all $n \in \mathbb{N}$, $k \geq 2$ and $|z| \leq r$, $r \geq 1$.

It follows

$$\begin{aligned} |E_{k,n}(z)| &\leq \frac{4(k-2)(k-1)^2(1+r)r^k}{n^2} + r|E_{k-1,n}(z)| + \frac{r^k}{n^2} D_k \\ &:= \frac{r^k}{n^2} F_{k,r} + r|E_{k-1,n}(z)|, \end{aligned}$$

where $F_{k,r}$ is a polynomial of degree 3 in k defined as $F_{k,r} = D_k + 4(k-2)(k-1)^2(1+r)$, D_k is expressed in the above.

Since $E_{0,n}(q; z) = E_{1,n}(q; z) = 0$ for any $z \in \mathbb{C}$, therefore, by writing the last inequality for $k = 2, 3, \dots$, we easily obtain step by step the following

$$|E_{k,n}(z)| \leq \frac{r^k}{n^2} \sum_{j=2}^k F_{j,r} \leq \frac{(k-1)F_{k,r}r^k}{n^2}.$$

As a conclusion, we have

$$\begin{aligned} \left| M_n(f; z) - f(z) - \frac{(n+1)z(1-z)f''(z)}{2(n^2+1)} \right| &\leq \sum_{k=2}^{\infty} |c_k| |E_{k,n}(q; z)| \\ &\leq \frac{1}{n^2} \sum_{k=2}^{\infty} |c_k| (k-1)F_{k,r}r^k. \end{aligned}$$

As $f^{(4)}(z) = \sum_{k=4}^{\infty} c_k k(k-1)(k-2)(k-3)z^{k-4}$ and the series is absolutely convergent in $|z| \leq r$, it easily follows that $\sum_{k=4}^{\infty} |c_k| k(k-1)(k-2)(k-3)r^{k-4} < \infty$, which implies that $\sum_{k=2}^{\infty} |c_k| (k-1)F_{k,r} r^k < \infty$, this completes the proof of theorem.

In the following theorem, we will obtain the exact order in approximation.

Theorem 3. Let $R > 1$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. Suppose that $f : D_R \rightarrow \mathbf{C}$ is analytic in D_R . If f is not a polynomial of degree ≤ 1 , then for any $r \in [1, R)$ we have

$$\|M_n(f; \cdot) - f\|_r \geq \frac{C_r(f)}{n}, \quad n \in \mathbf{N},$$

where $\|f\|_r = \max\{|f(z)|; |z| \leq r\}$ and the constant $C_r(f) > 0$ depends on f, r but it is independent of n .

Proof. Denote $e_1(z) = z$ and

$$H_n(f; z) = M_n(f; z) - f(z) - \frac{(n+1)z(1-z)}{2(n^2+1)} f''(z).$$

For all $z \in D_R$ and $n \in \mathbf{N}$, we have

$$M_n(f; z) - f(z) = \frac{n+1}{2(n^2+1)} \left\{ z(1-z)f''(z) + \frac{2(n^2+1)}{n^2(n+1)} [n^2 H_n(f; z)] \right\}.$$

In view of the property: $\|F+G\|_r \geq \|F\|_r - \|G\|_r \geq \|F\|_r - \|G\|_r$, it follows

$$\|M_n(f; \cdot) - f\|_r \geq \frac{n+1}{2(n^2+1)} \left\{ \|e_1(1-e_1)f''\|_r - \frac{2(n^2+1)}{n^2(n+1)} [n^2 \|H_n(f; \cdot)\|_r] \right\}.$$

Considering the hypothesis that f is not a polynomial of degree ≤ 1 in D_R , we have

$$\|e_1(1-e_1)f''\|_r > 0.$$

Indeed, supposing the contrary, it follows that

$$z(1-z)f''(z) = 0, \quad \text{for all } z \in \overline{D_r}.$$

By hypothesis that $f(z)$ is analytic in D_R , we can denote $f(z) = \sum_{k=0}^{\infty} c_k z^k$, the identification of the coefficients method immediately leads to $c_k = 0, k = 2, 3, \dots$. This implies that f is a polynomial of degree ≤ 1 on $\overline{D_r}$, a contradiction with the hypothesis.

Using the inequality (4), we get

$$n^2 \|H_n(f; \cdot)\|_r \leq M_r(f),$$

therefore, there exists an index n_0 depending only on f, r , such that for all $n \geq n_0$, we have

$$\|e_1(1-e_1)f''\|_r - \frac{2(n^2+1)}{n^2(n+1)} [n^2 \|H_n(f; \cdot)\|_r] \geq \frac{1}{2} \|e_1(1-e_1)f''\|_r,$$

which implies

$$\|M_n(f; \cdot) - f\|_r \geq \frac{n+1}{4(n^2+1)} \|e_1(1-e_1)f''\|_r \geq \frac{1}{4n} \|e_1(1-e_1)f''\|_r, \text{ for all } n \geq n_0.$$

For $n \in \{1, 2, \dots, n_0 - 1\}$, we have

$$\|M_n(f; \cdot) - f\|_r \geq \frac{W_{r,n}(f)}{n},$$

where $W_{r,n}(f) = n\|M_n(f; \cdot) - f\|_r > 0$.

As a conclusion, we have

$$\|M_n(f; \cdot) - f\|_r \geq \frac{C_r(f)}{n}, \text{ for all } n \in \mathbf{N},$$

where

$$C_r(f) = \min \left\{ W_{r,1}(f), W_{r,2}(f), \dots, W_{r,n_0-1}(f), \frac{1}{4} \|e_1(1-e_1)f''\|_r \right\},$$

this complete the proof.

Combining Theorem 3 with Theorem 1, we get the following result.

Corollary 2. Let $R > 1$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. Suppose that $f : D_R \rightarrow \mathbf{C}$ is analytic in D_R . If f is not a polynomial of degree ≤ 1 , then for any $r \in [1, R)$ we have

$$\|M_n(f; \cdot) - f\|_r \asymp \frac{1}{n}, \quad n \in \mathbf{N},$$

where $\|f\|_r = \max\{|f(z)|; |z| \leq r\}$ and the constants in the equivalence depend on f, r but it is independent of n .

Theorem 4. Let $R > 1$, $D_R = \{z \in \mathbf{C} : |z| < R\}$. Suppose that $f : D_R \rightarrow \mathbf{C}$ is analytic in D_R . Also, let $1 \leq r < r_1 < R$ and $p \in \mathbf{N}$ be fixed. If f is not a polynomial of degree $\leq \max\{1, p-1\}$, then we have

$$\|(M_n(f; \cdot))^{(p)} - f^{(p)}\|_r \asymp \frac{1}{n}, \quad n \in \mathbf{N},$$

where $\|f\|_r = \max\{|f(z)|; |z| \leq r\}$ and the constants in the equivalence depend on f, r, r_1, p , but it is independent of n .

Proof. Taking into account the upper estimate in Theorem 1, it remains to prove the lower estimate only.

Denoting by Γ the circle of radius $r_1 > r$ and center 0, by the Cauchy's formula it follows that for all $|z| \leq r$ and $n \in \mathbf{N}$ we have

$$M_n^{(p)}(f; z) - f^{(p)}(z) = \frac{p!}{2\pi i} \int_{\Gamma} \frac{M_n(f; v) - f(v)}{(v-z)^{p+1}} dv.$$

Keeping the notation there for $H_n(f; z)$, for all $n \in \mathbf{N}$, we have

$$M_n(f; z) - f(z) = \frac{n+1}{2(n^2+1)} \left\{ z(1-z)f''(z) + \frac{2(n^2+1)}{n^2(n+1)} [n^2 H_n(f; z)] \right\}.$$

By using Cauchy's formula, for all $v \in \Gamma$, we get

$$M_n^{(p)}(f; z) - f^{(p)}(z) = \frac{n+1}{2(n^2+1)} \{ [z(1-z)f''(z)]^p + \frac{2(n^2+1)}{n^2(n+1)} \frac{p!}{2\pi i} \int_{\Gamma} \frac{n^2 H_n(f; v)}{(v-z)^{p+1}} dv \},$$

passing now to $\|\cdot\|_r$ and denoting $e_1(z) = z$, it follows

$$\|M_n^{(p)}(f; \cdot) - f^{(p)}\|_r \geq \frac{n+1}{2(n^2+1)} \left\{ \left\| [e_1(1-e_1)f'']^{(p)} \right\|_r - \frac{2(n^2+1)}{n^2(n+1)} \left\| \frac{p!}{2\pi i} \int_{\Gamma} \frac{n^2 H_n(f; v)}{(v-\cdot)^{p+1}} dv \right\|_r \right\},$$

Since for any $|z| \leq r$ and $v \in \Gamma$ we have $|v-z| \geq r_1-r$, so, by using Theorem 2, we get

$$\left\| \frac{p!}{2\pi i} \int_{\Gamma} \frac{n^2 H_n(f; v)}{(v-\cdot)^{p+1}} dv \right\|_r \leq \frac{p!}{2\pi} \frac{2\pi r_1 n^2 \|H_n(f; \cdot)\|_{r_1}}{(r_1-r)^{p+1}} \leq \frac{M_{r_1}(f)p!r_1}{(r_1-r)^{p+1}}.$$

By hypothesis on f , we have

$$\|[e_1(1-e_1)f'']^{(p)}\|_r > 0.$$

Indeed, supposing the contrary, it follows that $\|[e_1(1-e_1)f'']^{(p)}\|_r = 0$, that is $z(1-z)f''(z)$ is a polynomial of degree $\leq p-1$. let $p=1$ and $p=2$, then the analyticity of f obviously implies that f is a polynomial of degree $\leq 1 = \max(1, p-1)$, a contradiction.

Now let $p \geq 3$, then the analyticity of f obviously implies that f is a polynomial of degree $\leq p-1 = \max(1, p-1)$, a contradiction with the hypothesis.

In conclusion, $\|[e_1(1-e_1)f'']^{(p)}\|_r > 0$ and in continuation, reasoning exactly as in the proof of Theorem 3, we can get the desired conclusion.

Remark 1. If we use King's approach to consider King type modification of the complex extension of the operators which was given by (1), we will obtain better approximation (cf. [21-23]).

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On the Convergence of Mann and Ishikawa Type Iterations in the Class of Quasi Contractive Operators

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Abstract

In this paper, we introduce two new iteration schemes, namely modified Mann and modified Ishikawa to approximate the fixed points of quasi contractive operators on a normed space. Various test problems are presented to reveal the validity and high efficiency of these iterative schemes.

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Key Words: Quasi contraction, fixed point, strong convergence.

1 Introduction and preliminaries

In the last few decades, various researchers have explored the fixed points of contractive type operators in metric spaces, Hilbert spaces and different classes of Banach spaces, see [1] and references there in. To approximate unique fixed point of strict contractive type operators, Picard iterative scheme can be used effectively [1, 10, 15, 16]. But this scheme does not generally converge for the operators with slightly weaker contractive

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conditions. For such operators, Mann iteration [13] (cf. [6, 14]), Ishikawa iteration [7] and Krasnosel'okii iteration [11] (cf. [3]) are much useful.

Let E be a normed space and C a nonempty convex subset of E . Let $T : C \rightarrow C$ be an operator and $\{\alpha_n\}$ and $\{\beta_n\}$ sequences of real numbers in $[0, 1]$.

The Mann iteration [13] is defined by the sequence $\{x_n\}_{n=0}^\infty$ as

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_nTx_n, \quad n \geq 0. \quad (1.1)$$

The sequence $\{x_n\}_{n=0}^\infty$ defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nTy_n, \quad n \geq 0, \\ y_n &= (1 - \beta_n)x_n + \beta_nTx_n, \quad n \geq 0 \end{aligned} \quad (1.2)$$

is called Ishikawa iteration [7].

It is noticeable that for $\alpha_n = \lambda$ (constant), the iterative procedure (1.1) turn into Krasnosel'okii iteration. Also for $\beta_n = 0$, Ishikawa iteration(1.2) reduces to Mann iteration (1.1).

Definition 1.1. Let (X, d) be a metric space and $a \in (0, 1)$. A mapping $T : X \rightarrow X$ satisfying

$$d(Tx, Ty) \leq ad(x, y) \quad \text{for all } x, y \in X \quad (1.3)$$

is called a contraction.

The following theorem is the classical Banach's contraction principle and of fundamental importance in the study of Fixed Point Theory.

Theorem 1.2. Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a contraction. Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^\infty$ defined by

$$x_{n+1} = Tx_n, \quad n \geq 0 \quad (1.4)$$

converges to p for any $x_0 \in X$.

The contraction in the above theorem forces T to be continuous. Despite this condition, Theorem 1.2 has many applications in solving the nonlinear equation $f(x) = 0$. Kannan [9] developed a fixed point theorem by relaxing the condition of continuity of T . He produced the following by taking b in $(0, \frac{1}{2})$:

$$d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)] \quad \text{for all } x, y \in X. \quad (1.5)$$

Chatterjea [4] obtained a similar result by considering $c \in (0, \frac{1}{2})$ as follows:

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)] \quad \text{for all } x, y \in X. \quad (1.6)$$

In 1972, Zamfirescu [17] proved the following very interesting and important fixed point theorem by taking into account (1.3), (1.5) and (1.6).

Theorem 1.3. Let (X, d) be a metric space and $T : X \rightarrow X$ be a mapping for which there exist real numbers a, b and c satisfying $0 < a < 1$, $0 < b$ and $c < \frac{1}{2}$ such that for each $x, y \in X$, at least one of the following is true:

- (z_1) $d(Tx, Ty) \leq ad(x, y)$,
- (z_2) $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$,
- (z_3) $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \quad n \geq 0$$

converges to p for any $x_0 \in X$.

An operator $T : X \rightarrow X$ satisfying the contractive conditions (z_1), (z_2) and (z_3) is called Zamferescu operator.

In 1974, Ćirić [5] obtained a more general contraction to approximate unique fixed point with the help of Picard iteration: there exists $0 < h < 1$ such that for all $x, y \in X$

$$d(Tx, Ty) \leq h \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \quad (1.7)$$

Definition 1.4. Let (X, d) be a metric space and $T : X \rightarrow X$ a mapping satisfying (1.7). Then T is called quasi contraction.

A new class of operators on an arbitrary Banach space E , satisfying

$$\|Tx - Ty\| \leq \delta \|x - y\| + 2\delta \|x - Tx\| \quad \text{for all } x, y \in E, 0 \leq \delta < 1, \quad (1.8)$$

was established by Berinde [2] in 2004. He approximated fixed points of this class of operators via Ishikawa iteration.

It is well known that a nonlinear equation $f(x) = 0$ can be expressed in terms of fixed point iteration method as follows:

$$x = Tx. \quad (1.9)$$

Taking up the technique of [8], if $T'x \neq 1$, $\theta \neq -1$, it can easily be seen by adding θx to both sides of (1.9) that

$$x = \frac{\theta x + Tx}{1 + \theta} = T_{\theta}x. \quad (1.10)$$

So as to make (1.10) to be efficient, we can choose $T'_{\theta}x = 0$, which gives

$$\theta = -T'x. \quad (1.11)$$

Now we are in a position to define modified Mann and modified Ishikawa iterative schemes.

Replacing Tx_n and Ty_n in (1.1) and (1.2) with $T_{\theta}x_n$ and $T_{\theta}y_n$, respectively, we get

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_{\theta}x_n \quad (1.12)$$

and

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T_{\theta}y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T_{\theta}x_n. \end{aligned} \quad (1.13)$$

Using (1.10) in (1.12) and (1.13) and also the error term, we obtain

$$x_{n+1} = \left(1 - \frac{1}{1+\theta}\alpha_n\right)x_n + \frac{1}{1+\theta}\alpha_nTx_n + \mu_n \quad (1.14)$$

and

$$\begin{aligned} x_{n+1} &= \left[1 - \frac{\alpha_n}{1+\theta} \left(1 + \frac{\theta\beta_n}{1+\theta}\right)\right]x_n + \frac{\alpha_n}{1+\theta} \left[\frac{\theta\beta_n}{1+\theta}Tx_n + Ty_n\right] + \mu_n, \\ y_n &= \left(1 - \frac{1}{1+\theta}\beta_n\right)x_n + \frac{1}{1+\theta}\beta_nTx_n + \nu_n. \end{aligned} \quad (1.15)$$

We call the procedures defined in (1.14) and (1.15), the modified Mann and modified Ishikawa iterative procedures. It is obvious that (1.14) and (1.15) without error term reduce to (1.1) and (1.2), respectively for $\theta = 0$.

In this paper, we have proved the strong convergence of quasi contractive operator T satisfying (1.14) and (1.15) in the setting of normed space. We also present some test problems to compare the iterative procedures defined in (1.1), (1.2), (1.14) and (1.15). The numerical results obtained demonstrate the high performance and efficiency of modified Mann and modified Ishikawa iterative processes.

We use the following lemma in the sequel.

Lemma 1.5. ([12]) *Let $\{r_n\}$, $\{s_n\}$, $\{t_n\}$ and $\{k_n\}$ be the sequences of nonnegative numbers satisfying*

$$r_{n+1} \leq (1 - s_n)r_n + s_nt_n + k_n, \quad n \geq 0.$$

If $\sum_{n=0}^{\infty} s_n = \infty$ and $\lim_{n \rightarrow \infty} t_n = 0$ and $\sum_{n=0}^{\infty} k_n < \infty$ hold, then $\lim_{n \rightarrow \infty} r_n = 0$.

2 Main results

Assuming that the operator T has at least one fixed point, we prove the convergence theorems for iterative procedures (1.14) and (1.15).

Theorem 2.1. *Let C be a nonempty closed convex subset of a normed space E and $T : C \rightarrow C$ be an operator satisfying (1.8). For arbitrary $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by the iterative process (1.14) satisfying $\theta > -1$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\|\mu_n\| = o(\alpha_n)$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T .*

Proof. Let p be the fixed point of the operator T . We consider

$$\begin{aligned} &\|x_{n+1} - p\| \\ &= \left\| \left(1 - \frac{1}{1+\theta}\alpha_n\right)x_n + \frac{1}{1+\theta}\alpha_nTx_n + \mu_n - p \right\| \\ &= \left\| \left(1 - \frac{1}{1+\theta}\alpha_n\right)x_n + \frac{1}{1+\theta}\alpha_nTx_n + \mu_n - \left(1 - \frac{1}{1+\theta}\alpha_n + \frac{1}{1+\theta}\alpha_n\right)p \right\| \\ &= \left\| \left(1 - \frac{1}{1+\theta}\alpha_n\right)(x_n - p) + \frac{1}{1+\theta}\alpha_n(Tx_n - p) + \mu_n \right\| \\ &\leq \left(1 - \frac{1}{1+\theta}\alpha_n\right)\|x_n - p\| + \frac{1}{1+\theta}\alpha_n\|Tx_n - p\| + \|\mu_n\|. \end{aligned} \quad (2.1)$$

Substituting $y = x_n$ and $x = p$ in (1.8), we get

$$\|Tx_n - p\| \leq \delta \|x_n - p\|.$$

Thus (2.1) implies

$$\begin{aligned} \|x_{n+1} - p\| &\leq \left(1 - \frac{1}{1+\theta}\alpha_n\right) \|x_n - p\| + \frac{\delta}{1+\theta}\alpha_n \|x_n - p\| + \|\mu_n\| \\ &= \left(1 - \frac{1}{1+\theta}\alpha_n + \frac{\delta}{1+\theta}\alpha_n\right) \|x_n - p\| + \|\mu_n\| \\ &= \left(1 - \frac{1-\delta}{1+\theta}\alpha_n\right) \|x_n - p\| + \|\mu_n\|. \end{aligned}$$

Using Lemma 1.5 and the fact that $0 \leq \delta < 1$, $0 \leq \alpha_n \leq 1$, $\theta > -1$, $\|\mu_n\| = o(\alpha_n)$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, we obtain

$$\lim_{n \rightarrow \infty} \|x_n - p\| = 0.$$

Hence $x_n \rightarrow p$. This completes the proof. \square

Taking $\theta = 0$ in the setting of normed space and the contraction condition (1.8), we obtain the following corollary.

Corollary 2.2. *Let C be a nonempty closed convex subset of a normed space E and $T : C \rightarrow C$ be an operator satisfying (1.8). For arbitrary $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by the iterative process (1.1) satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T .*

Now we prove the convergence of modified Ishikawa iterative process in the form of the following theorem.

Theorem 2.3. *Let C a nonempty closed convex subset of a normed space E and $T : C \rightarrow C$ be an operator satisfying (1.8). For arbitrary $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by the iterative process (1.15) satisfying $\theta > -1$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\|\nu_n\| = o(\alpha_n)$ and $\|\mu_n\| = o(\alpha_n)$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T .*

Proof. Let p be the fixed point of the operator T . We consider

$$\begin{aligned} &\|x_{n+1} - p\| \\ &= \|(1 - \alpha_n)x_n + \alpha_n T_\theta y_n + \mu_n - p\| \\ &= \left\| (1 - \alpha_n)x_n + \alpha_n \left(\frac{\theta y_n + T y_n}{1 + \theta} \right) + \mu_n - p \right\| \\ &\leq (1 - \alpha_n) \|x_n - p\| + \alpha_n \left\| \frac{T y_n + \theta y_n}{1 + \theta} - p \right\| + \|\mu_n\| \\ &= (1 - \alpha_n) \|x_n - p\| + \alpha_n \left\| \frac{(T y_n - p) + \theta (y_n - p)}{1 + \theta} \right\| + \|\mu_n\| \\ &\leq (1 - \alpha_n) \|x_n - p\| + \frac{\alpha_n}{1 + \theta} \|T y_n - p\| + \frac{\theta \alpha_n}{1 + \theta} \|y_n - p\| + \|\mu_n\|. \end{aligned} \quad (2.2)$$

Substituting $x = p$ and $y = y_n$ in (1.8), we get

$$\|Ty_n - p\| \leq \delta \|y_n - p\|. \quad (2.3)$$

Thus (2.2) implies

$$\begin{aligned} & \|x_{n+1} - p\| \\ & \leq (1 - \alpha_n) \|x_n - p\| + \frac{\delta \alpha_n}{1 + \theta} \|y_n - p\| + \frac{\theta \alpha_n}{1 + \theta} \|y_n - p\| + \|\mu_n\|. \\ & = (1 - \alpha_n) \|x_n - p\| + \frac{\delta + \theta}{1 + \theta} \alpha_n \|y_n - p\| + \|\mu_n\|. \end{aligned} \quad (2.4)$$

Consider

$$\begin{aligned} \|y_n - p\| &= \left\| \left(1 - \frac{1}{1 + \theta} \beta_n\right) x_n + \frac{1}{1 + \theta} \beta_n T x_n + \nu_n - p \right\| \\ &= \left\| \left(1 - \frac{1}{1 + \theta} \beta_n\right) (x_n - p) + \frac{1}{1 + \theta} \beta_n (T x_n - p) + \nu_n \right\| \\ &\leq \left(1 - \frac{1}{1 + \theta} \beta_n\right) \|x_n - p\| + \frac{1}{1 + \theta} \beta_n \|T x_n - p\| + \|\nu_n\|. \end{aligned} \quad (2.5)$$

Substituting $x = p$ and $y = x_n$ in (1.8), we get

$$\|T x_n - p\| \leq \delta \|x_n - p\|. \quad (2.6)$$

Thus (2.5) implies

$$\begin{aligned} \|y_n - p\| &\leq \left(1 - \frac{1}{1 + \theta} \beta_n\right) \|x_n - p\| + \frac{\delta}{1 + \theta} \beta_n \|x_n - p\| + \|\nu_n\| \\ &= \left(1 - \frac{1 - \delta}{1 + \theta} \beta_n\right) \|x_n - p\| + \|\nu_n\|. \end{aligned} \quad (2.7)$$

Using (2.7) in (2.4), we get

$$\begin{aligned} & \|x_{n+1} - p\| \\ & \leq (1 - \alpha_n) \|x_n - p\| + \frac{\delta + \theta}{1 + \theta} \alpha_n \left[\left(1 - \frac{1 - \delta}{1 + \theta} \beta_n\right) \|x_n - p\| + \|\nu_n\| \right] + \|\mu_n\| \\ & = \left[1 - \alpha_n + \frac{\delta + \theta}{1 + \theta} \alpha_n \left(1 - \frac{1 - \delta}{1 + \theta} \beta_n\right) \right] \|x_n - p\| + \frac{\delta + \theta}{1 + \theta} \alpha_n \|\nu_n\| + \|\mu_n\| \\ & = \left[1 - \alpha_n \left\{ 1 - \frac{\delta + \theta}{1 + \theta} \left(1 - \frac{1 - \delta}{1 + \theta} \beta_n\right) \right\} \right] \|x_n - p\| \\ & \quad + \frac{\delta + \theta}{1 + \theta} \alpha_n \|\nu_n\| + \|\mu_n\|. \end{aligned} \quad (2.8)$$

Let

$$\begin{aligned}
 A_n &= 1 - \alpha_n \left[1 - \frac{\delta + \theta}{1 + \theta} \left(1 - \frac{1 - \delta}{1 + \theta} \beta_n \right) \right] \\
 &= 1 - \alpha_n \left(1 - \frac{\delta + \theta}{1 + \theta} + \frac{(\delta + \theta)(1 - \delta)}{(1 + \theta)^2} \beta_n \right) \\
 &= 1 - \alpha_n \left(1 - \frac{(1 + \theta)(\delta + \theta) - (\delta + \theta)(1 - \delta)\beta_n}{(1 + \theta)^2} \right) \\
 &= 1 - \alpha_n \left(\frac{(1 + \theta)^2 - (1 + \theta)(\delta + \theta) + (\delta + \theta)(1 - \delta)\beta_n}{(1 + \theta)^2} \right) \\
 &= 1 - \alpha_n \left(\frac{(1 - \delta)(1 + \theta) + (\delta + \theta)(1 - \delta)\beta_n}{(1 + \theta)^2} \right) \\
 &= 1 - \frac{(1 - \delta)}{(1 + \theta)} \alpha_n \left(\frac{(1 + \theta) + (\delta + \theta)\beta_n}{(1 + \theta)} \right) \\
 &= 1 - \frac{(1 - \delta)}{(1 + \theta)} \alpha_n \left(1 + \frac{\delta + \theta}{1 + \theta} \beta_n \right). \tag{2.9}
 \end{aligned}$$

Since $\beta_n \geq 0$, $0 \leq \delta < 1$ and $\theta > -1$, therefore $\frac{\delta + \theta}{1 + \theta} \beta_n \geq 0$ and $1 + \frac{\delta + \theta}{1 + \theta} \beta_n \geq 1$.

Hence (2.9) gives

$$A_n = 1 - \frac{1 - \delta}{1 + \theta} \alpha_n \left(1 + \frac{\delta + \theta}{1 + \theta} \beta_n \right) \leq 1 - \frac{1 - \delta}{1 + \theta} \alpha_n.$$

Thus from (2.8), we get

$$\|x_{n+1} - p\| \leq \left(1 - \frac{1 - \delta}{1 + \theta} \alpha_n \right) \|x_n - p\| + \frac{\delta + \theta}{1 + \theta} \alpha_n \|\nu_n\| + \|\mu_n\|.$$

With the help of Lemma 1.5 and using the fact that $0 \leq \delta < 1$, $0 < \alpha_n < 1$, $\theta > -1$, $\|\nu_n\| = o(\alpha_n)$, $\|\mu_n\| = o(\alpha_n)$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, we get

$$\lim_{n \rightarrow \infty} \|x_n - p\| = 0.$$

Consequently, $x_n \rightarrow p \in F$ and this completes the proof. \square

Corollary 2.4. *Let C a nonempty closed convex subset of a normed space E and $T : C \rightarrow C$ be an operator satisfying (1.8). For arbitrary $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by the iterative process (1.2) satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T .*

The above corollary in fact is the generalization of Theorem 2 of Berinde [2] in the context of a normed space and the contraction condition (1.8).

3 Applications

In this section, we consider various test problems to apply Mann (M), modified Mann (MM), Ishikawa (I) and modified Ishikawa (MI) iterative procedures for the estimation

of fixed points. The data in the following table indicates the rapidness of convergence in each problem. We make use of Maple software and 10^{-3} tolerance for the purpose. Here we denote the number of iterations (NI).

Tx	θ	α_n	β_n	x_0	Method	NI	$x[k]$	Tx	$ x[k] - Tx $
$3 - x^2$	$2x$	$\frac{1}{\sqrt{n+1}}$	$\frac{1}{\sqrt{n+1}}$	1	M	9	1.3044	1.2985	0.0059
					MM	4	1.3047	1.2977	0.0070
					I	22	1.3009	1.3076	0.0067
					MI	2	1.3009	1.3077	0.0068
$3^{(1-x)} - \cos x$	$\ln 3(3^{(1-x)}) - \sin x$	$\frac{1}{1+n}$	$\frac{1}{1+n}$	0.5	M	4	0.6657	0.6572	0.0085
					MM	4	0.6576	0.6652	0.0076
					I	11	0.6570	0.6658	0.0088
					MI	1	0.6588	0.6641	0.0053
$1 - x - \cos x$	$1 - \sin x$	$\frac{1}{\sqrt{n+1}}$	$\frac{1}{\sqrt{n+1}}$	0.1	M	4	0.0000	-0.0000	0.0000
					MM	1	-0.0026	0.0026	0.0052
					I	6	0.0037	-0.0036	0.0073
					MI	1	0.0001	-0.0001	0.0002
$\cos x - e^x + 1$	$\sin x + e^x$	$\frac{1}{\sqrt{n+1}}$	$\frac{1}{\sqrt{n+1}}$	0.4	M	5	0.4120	0.4066	0.0054
					MM	1	0.4101	0.4100	0.0001
					I	12	0.4076	0.4150	0.0074
					MI	1	0.4101	0.4101	0.0000
$1 - \frac{x}{2}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{n+1}}$	$\frac{1}{\sqrt{n+1}}$	0.5	M	2	0.6616	0.6692	0.0076
					MM	1	0.6667	0.6667	0.0000
					I	3	0.6616	0.6692	0.0076
					MI	1	0.6667	0.6667	0.0000
$\frac{x}{2}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{n+1}}$	$\frac{1}{\sqrt{n+1}}$	0.5	M	12	0.0181	0.0091	0.0091
					MM	1	0.0000	0.0000	0.0000
					I	6	0.0154	0.0077	0.0077
					MI	1	0.0000	0.0000	0.0000
$e^{(1-x)^2} - 1$	$2(1-x)e^{(1-x)^2}$	$\frac{1}{1+n}$	$\frac{1}{1+n}$	0.5	M	4	0.4160	0.4065	0.0095
					MM	2	0.4089	0.4182	0.0093
					I	13	0.4159	0.4067	0.0092
					MI	1	0.4136	0.4104	0.0032

4 Conclusion

We have developed two new iterative schemes, namely modified Maan and modified Ishikawa. The convergence theorems for our proposed schemes have been proved. In Section 2, the table provides comparison between Mann, modified Mann, Ishikawa and modified Ishikawa iterative procedures. Our results clearly indicate that how rapidly our proposed methods converge to the fixed points. In some given test problems, due to large difference in number of iterations, it is obvious that modified Mann and modified Ishikawa iterative schemes require very little time to produce fixed point.

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MEAN ERGODIC THEOREMS FOR SEMIGROUPS OF LINEAR OPERATORS IN p -BANACH SPACES

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ABSTRACT. In this paper, by using the Rode's method, we extend Yosida's theorem to semigroups of linear operators in p -Banach spaces. Our paper is motivated from ideas in [7].

1. Introduction

In 1938, Yosida [14] Proved the following mean ergodic theorem for linear operators: Let E be a real Banach space and T be a linear operator of E into itself such that there exists a constant C with $\|T^n\| \leq C$ for $n = 1, 2, 3, \dots$, and T is weakly completely continuous, i.e., T maps the closed unite ball of E into a weakly compact subset of E . Then, the Cesaro mean

$$S_n x = \frac{1}{n} \sum_{k=1}^n T^k x$$

converges strongly as $n \rightarrow +\infty$ to a fixed point of T for each $x \in E$.

On the other hand, in 1975, Baillon [1] proved the following nonlinear ergodic theorem: Let X be a Banach space and C a closed convex subset of X . The mapping $T : C \rightarrow C$ is called nonexpansive on C if

$$\|Tx - Ty\| \leq \|x - y\| \quad \forall x, y \in C.$$

Let $F(T)$ be the set of fixed point of T . If X is strictly convex, $F(T)$ is closed and convex. In [1, 4], Baillon proved the first nonlinear ergodic theorem such that if X is a real Hilbert space and $F(T) \neq \emptyset$, then for each $x \in C$, the sequence $\{S_n x\}$ defined by

$$S_n x = \left(\frac{1}{n}\right)(x + Tx + \dots + T^{n-1}x)$$

converges weakly to a fixed point of T . It was also shown by Pazy [8] that if X a real Hilbert space and $S_n x$ converges weakly to $y \in C$, then $y \in F(T)$.

Recently, Rode [10] and Takahashi [13] tried to extend this nonlinear ergodic theorem to semigroup, generalizing the Cesaro means on $N = \{1, 2, \dots\}$, such that the corresponding sequence of mappings converges to a projection onto the set of common fixed points. In this paper, by using the Rode's method, we extend Yosida's theorem to semigroups of linear operators in p -Banach spaces. The proofs employ the methods of Yosida [14], Greenleaf [5], Rode [10] and Takahashi [6, 12]. Our paper is motivated from ideas in [7]

2. p -Norm

Definition 2.1. ([3, 11]) Let X be a real linear space. A function $\|\cdot\| : X \rightarrow \mathbb{R}$ is a quasi-norm (valuation) if it satisfies the following conditions :

(1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;

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- (2) $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$;
 (3) There is a constant $M \geq 1$ such that $\|x + y\| \leq M(\|x\| + \|y\|)$ for all $x, y \in X$.
 Then $(X, \|\cdot\|)$ is called a quasi-normed space. The smallest possible M is called the modulus of concavity of $\|\cdot\|$. A quasi-Banach space is a complete quasi-normed space.

A quasi-norm $\|\cdot\|$ is called a p -norm $0 < p < 1$ if

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a p -Banach space.

By the Aoki-Rolewicz [11], each quasi-norm is equivalent to some p -norm (see also [9]).

Since it is much easier to work with p -norm, henceforth we restrict our attention mainly to p -norms.

3. Preliminaries and lemmas

Let E a real p -Banach space and let E^* be the conjugate space of E , that is, the space of all continuous linear functionals on E . The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^* \rangle$. We denote by $\text{co}D$ the convex hull of D , $\overline{\text{co}}D$ the closure of $\text{co}D$.

Let U be a linear continuous operator of E into itself. Then, we denote by U^* the conjugate operator of U .

Assumption (A). Let $(E, \|\cdot\|_p)$ be a p -Banach space and $\{T_t : t \in G\}$, be a family of linear continuous operators of a real Banach space E into itself such that there exist a real number C with $\|T_t\|_p \leq C$ for all $t \in G$ and the weak closure of $\{T_t x : t \in G\}$ is weakly compact, for each $x \in E$. The index set G is a topological semigroup such that $T_{st} = T_s T_t$ for all $s, t \in G$ and T is continuous with respect to the weak operator topology : $\langle T_s x, x^* \rangle \rightarrow \langle T_t x, x^* \rangle$ for all $x \in E$ and $x^* \in E^*$ if $s \rightarrow t$ in G .

We denote by $m(G)$ the p -Banach space of all bounded continuous real valued functions on the topological semigroup G with the p -norm. For each $s \in G$ and $f \in m(G)$, we define elements $l_s f$ and $r_s f$ in $m(G)$ given by $l_s f(t) = f(st)$ and $r_s f(t) = f(ts)$ for all $t \in G$. An element $\mu \in m(G)^*$ (the conjugate space of $m(G)$) is called a mean on G if $\|\mu\|_p = \mu(1) = 1$. A mean μ on G is called left (right) invariant if $\mu(l_s f) = \mu(f)$ ($\mu(r_s f) = \mu(f)$) for all $f \in m(G)$ and $s \in G$. An invariant mean is a left and right invariant mean. We know that $\mu \in m(G)^*$ is a mean on G if and only if

$$\inf\{f(t) : t \in G\} \leq \mu(f) \leq \sup\{f(t) : t \in G\}$$

for every $f \in m(G)$; see [4, 5, 9].

Let $\{T_t : t \in G\}$ be a family of linear continuous operators of E into itself satisfying the assumption (A) and μ be a mean on G . Fix $x \in E$. Then, for $x^* \in E^*$, the real valued function $t \rightarrow \langle T_t x, x^* \rangle$ is in $m(G)$. Denote by $\mu_t \langle T_t x, x^* \rangle$ the value of μ at this function. By linearity of μ and of $\langle \cdot, \cdot \rangle$, this is linear in x^* ; moreover, since

$$|\mu_t \langle T_t x, x^* \rangle| \leq \|\mu\|_p \cdot \sup_t |\langle T_t x, x^* \rangle| \leq \sup_t \|T_t x\|_p \cdot \|x^*\|_p \leq C \cdot \|x\|_p \cdot \|x^*\|_p,$$

it is continuous in x^* . Hence $\mu_t \langle T_t x, \cdot \rangle$ is an element of E^{**} . So it follows from weak compactness of $\overline{\text{co}}\{T_t x : t \in G\}$ that $\mu_t \langle T_t x, x^* \rangle = \langle T_\mu x, x^* \rangle$ for every $x^* \in E^*$.

Put $K = \overline{\text{co}}\{T_t x : t \in G\}$ and suppose that the element $\mu_t \langle T_t x, \cdot \rangle$ is not contained in the $n(K)$, where n is the natural embedding of the p -Banach space E into its second conjugate space E^{**} . Since the convex set $n(K)$ is compact in the *weak** topology of E^{**} , there exists an element $y^* \in E^*$ such that

$$\mu_t \langle T_t x, y^* \rangle < \inf\{\langle y^*, z^{**} \rangle : z^{**} \in n(K)\}$$

Hence we have

$$\mu_t < T_t x, y^* > < \inf\{< y^*, z^{**} > : z^{**} \in n(k)\} \leq \inf\{< T_t x, y^* > : t \in G\} \leq \mu_t < T_t x, y^* > .$$

This is a contradiction. Thus, for a mean μ on G , we can define a linear continuous operator T_μ of E into itself such that $\|T_\mu\|_p \leq C$, $T_\mu x \in \overline{\text{co}}\{T_t x : t \in G\}$ for all $x \in E$, and $\mu_t < T_t x, x^* > = < T_\mu x, x^* >$ for all $x \in E$ and $x^* \in E^*$. we denote by $F(G)$ the set all common fixed points of the mappings T_t , $t \in G$.

Lemma 3.1. Assume that a left invariant mean μ exists on G . Then $T_\mu(E) \subset F(G)$. Especially, $F(G)$ is not empty.

Proof. Let $x \in E$ and μ be a left invariant mean on G . Then since, for $s \in G$ and x^* ,

$$\begin{aligned} < T_s T_\mu x, x^* > &= < T_\mu x, T_s^* x^* > = \mu_t < T_t x, T_s^* x^* > = \mu_t < T_s T_t x, x^* > \\ &= \mu_t < T_{st} x, x^* > = \mu_t < T_t x, x^* > = < T_\mu x, x^* >, \end{aligned}$$

we have $T_s T_\mu x = T_\mu x$. Hence $T_\mu(E) \subset F(G)$. \square

Lemma 3.2. Let λ be an invariant mean on G . Then $T_\lambda T_s = T_s T_\lambda = T_\lambda$ for each $s \in G$ and $T_\lambda T_\mu = T_\mu T_\lambda = T_\lambda$ for each mean μ on G . Especially, T_λ is a projection of E onto $F(G)$.

Proof. Let $s \in G$. Since

$$< T_\lambda T_s x, x^* > = \lambda_t < T_t T_s x, x^* > = \lambda_t < T_{ts} x, x^* > = \lambda_t < T_t x, x^* > = < T_\lambda x, x^* >$$

for $x \in E$ and $x^* \in E^*$, we have $T_\lambda T_s = T_\lambda$. It follows from Lemma 3.1 that $T_s T_\lambda = T_\lambda$ for each $s \in G$. Let μ_j be a mean on G . Then, since

$$< T_\mu T_\lambda x, x^* > = \mu_t < T_t T_\lambda x, x^* > = \mu_t < T_\lambda x, x^* > = < T_\lambda x, x^* >$$

and

$$\begin{aligned} < T_\lambda T_\mu x, x^* > &= < T_\mu x, T_\lambda^* x^* > = \mu_t < T_t x, T_\lambda^* x^* > = \mu_t < T_\lambda T_t x, x^* > \\ &= \mu_t < T_\lambda x, x^* > = < T_\lambda x, x^* > \end{aligned}$$

for $x \in E$ and $x^* \in E^*$, we have $T_\mu T_\lambda = T_\lambda T_\mu = T_\lambda$. Putting $\mu = \lambda$, we have $T_\lambda^2 = T_\lambda$ and hence T_λ is a projection of E onto $F(G)$. \square

As a direct consequence of Lemma 3.2, we have the following.

Lemma 3.3. Let μ and λ be invariant means on G . Then $T_\mu = T_\lambda$.

Lemma 3.4. Assume that an invariant mean exists on G . Then, for each $x \in E$, the set $\overline{\text{co}}\{T_t x : t \in G\} \cap F(G)$ consists of a single point.

Proof. Let $x \in E$ and μ be an invariant mean on G . Then, we know that $T_\mu x \in F(G)$ and $T_\mu x \in \overline{\text{co}}\{T_t x : t \in G\}$. So, we show that $\overline{\text{co}}\{T_t x : t \in G\} \cap F(G) = \{T_\mu x\}$. Let $x_0 \in \overline{\text{co}}\{T_t x : t \in G\} \cap F(G)$ and $\epsilon > 0$. Then, for $x^* \in E^*$, there exists an element $\sum_{i=1}^n \alpha_i T_{t_i} x$ in the set $\text{co}\{T_t x : t \in G\}$ such that $\epsilon > C \cdot \|x^*\|_p \cdot \|\sum_{i=1}^n \alpha_i T_{t_i} x - x_0\|_p$. Hence we have

$$\begin{aligned} \epsilon &> C \cdot \|x^*\|_p \cdot \left\| \sum_{i=1}^n \alpha_i T_{t_i} x - x_0 \right\|_p \geq \sup_t \|T_t\|_p \cdot \left\| \sum_{i=1}^n \alpha_i T_{t_i} x - x_0 \right\|_p \cdot \|x^*\|_p \\ &\geq \sup_t \left\| \sum_{i=1}^n \alpha_i T_{j,t} T_{j,t_i} x - x_0 \right\|_j \cdot \|x^*\|_j \geq \left| < \sum_{i=1}^n \alpha_i T_t T_{t_i} x - x_0, x^* > \right| \\ &= \left| \sum_{i=1}^n \alpha_i \mu_t < T_{tt_i} x - x_0, x^* > \right| = |\mu_t < T_t x - x_0, x^* >| = |< T_\mu x - x_0, x^* >|. \end{aligned}$$

Since ϵ is arbitrary, we have $< T_\mu x, x^* > = < x_0, x^* >$ for every $x^* \in E^*$ and hence $T_\mu x = x_0$. \square

4. Ergodic Theorems

Now we can prove mean ergodic theorems for semigroups of linear continuous operators in p -Banach space.

Theorem 4.1. *Let $\{T_t : t \in G\}$ be a family of linear continuous operators in a real p -Banach space E satisfying Assumption (A). If a net $\{\mu^\alpha : \alpha \in I\}$ of means on G is asymptotically invariant, i.e.,*

$$\mu^\alpha - r_s^* \mu^\alpha \quad \text{and} \quad \mu^\alpha - l_s^* \mu^\alpha$$

converge to 0 in the weak topology of $m(G)^*$ for each $s \in G$, then there exists a projection Q of E onto $F(G)$ such that $\|Q\|_p \leq C$, $T_{\mu^\alpha} x$ converges weakly to Qx for each $x \in E$, $QT_t = T_t Q = Q$ for each $t \in G$, and $Qx \in \overline{\text{co}}\{T_t x : t \in G\}$ for each $x \in E$. Furthermore, the projection Q onto $F(G)$ is the same for all asymptotically invariant nets.*

Proof. Let μ be a cluster point of net $\{\mu^\alpha : \alpha \in I\}$ in the weak* topology of $m(G)^*$. Then μ is an invariant mean on G . Hence, by Lemma 3.2, T_μ is a projection of E onto $F(G)$ such that $\|T_\mu\|_p \leq C$, $T_\mu T_t = T_t T_\mu = T_\mu$ for each $t \in G$ and $T_\mu x \in \overline{\text{co}}\{T_t x : t \in G\}$ for each $x \in E$. Setting $Q = T_\mu$, we show that $T_{\mu^\alpha} x$ converges weakly to Qx for each $x \in E$. Since $T_{\mu^\alpha} x \in \overline{\text{co}}\{T_t x : t \in G\}$ for all $\alpha \in I$ and $\overline{\text{co}}\{T_t x : t \in G\}$ is weakly compact, there exists a subnet $\{T_{\mu^\beta} x : \beta \in J\}$ of $\{T_{\mu^\alpha} x : \alpha \in I\}$ such that $T_{\mu^\beta} x$ converges weakly to an element $x_0 \in \overline{\text{co}}\{T_t x : t \in G\}$. To show that $T_{\mu^\alpha} x$ converges weakly to Qx , it is sufficient to show $x_0 = Qx$. Let $x^* \in E^*$ and $s \in G$. since $T_{\mu^\beta} x \rightarrow x_0$ weakly, we have $\mu_t^\beta \langle T_t x, x^* \rangle \rightarrow \langle x_0, x^* \rangle$ and $\mu_t^\beta \langle T_t x, T_s^* x^* \rangle \rightarrow \langle x_0, T_s^* x^* \rangle = \langle T_s x_0, x^* \rangle$. On the other hand, since $\mu^\beta - l_s^* \mu^\beta \rightarrow 0$ in the weak* topology, we have

$$\begin{aligned} \mu_t^\beta \langle T_t x, x^* \rangle - l_s^* \mu_t^\beta \langle T_t x, x^* \rangle &= \mu_t^\beta \langle T_t x, x^* \rangle - \mu_t^\beta \langle T_{st} x, x^* \rangle \\ &= \mu_t^\beta \langle T_t x, x^* \rangle - \mu_t^\beta \langle T_t x, T_s^* x^* \rangle \\ &\rightarrow 0. \end{aligned}$$

Hence, we have $\langle x_0, x^* \rangle = \langle T_s x_0, x^* \rangle$ and hence $x_0 \in F(G)$. So, we obtain $Qx = T_\mu x = x_0$ by Lemma 3.4. That the projection Q is the same for all asymptotically invariant nets is obvious from Lemma 3.3. \square

As a direct consequence of Theorem 4.1, we have the following.

Corollary 4.2. *Let $\{T_t : t \in G\}$ be as in Theorem 4.1 and assume that an invariant mean exists on G . Then, there exists a projection Q of E onto F such that $\|Q\|_p \leq C$, $QT_t = T_t Q = Q$ for each $t \in G$ and $Qx \in \overline{\text{co}}\{T_t x : t \in G\}$ for each $x \in E$*

Theorem 4.3. *Let $\{T_t : t \in G\}$ be as in Theorem 4.1. If a net $\{\mu^\alpha : \alpha \in I\}$ of means on G is asymptotically invariant and further $\mu^\alpha - r_s^* \mu^\alpha$ converges to 0 in the strong topology of $m(G)^*$, then exists a projection Q of E onto $F(G)$ such that $\|Q\|_p \leq C$, $T_{\mu^\alpha} x$ converges strongly to Qx for each $x \in E$, $QT_t = T_t Q = Q$ for each $t \in G$, and $Qx \in \overline{\text{co}}\{T_t x : t \in G\}$ for each $x \in E$.*

Proof. As in the proof of Theorem 4.1, let $Q = T_\mu$, where μ is a cluster point of the net $\{\mu^\alpha : \alpha \in I\}$ in the weak* topology of $m(G)^*$. Then we show that $T_{\mu^\alpha} x$ converges strongly to Qx for each $x \in E$.

Let $E_0 = \overline{\text{co}}\{y - T_t y : y \in E, t \in G\}$. Then, for any $z \in E_0$, $T_{\mu^\alpha} z$ converges strongly to 0. In fact, if $z = y - T_s y$, then since, for any $y^* \in E^*$,

$$\begin{aligned} |\langle T_{\mu^\alpha} z, y^* \rangle| &= |\mu_t^\alpha \langle T_t(y - T_s y), y^* \rangle| = |\mu_t^\alpha \langle T_t y, y^* \rangle - \mu_t^\alpha \langle T_{ts} y, y^* \rangle| \\ &= |(\mu_t^\alpha - r_s^* \mu_t^\alpha) \langle T_t y, y^* \rangle| \leq \|\mu^\alpha - r_s^* \mu^\alpha\|_p \cdot \sup_t |\langle T_t y, y^* \rangle| \\ &\leq \|\mu^\alpha - r_s^* \mu^\alpha\|_p \cdot C \cdot \|y\|_p \cdot \|y^*\|_p, \end{aligned}$$

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we have $\|T_{\mu^\alpha} z\|_p \leq C \cdot \|\mu^\alpha - r_s^* \mu^\alpha\|_p \cdot \|y\|_p$. Using this inequality, we show that $T_{\mu^\alpha} z$ converges strongly to 0 for any $z \in E_0$. Let z be any element of E_0 and ϵ be any positive number. By the definition of E_0 , there exists an element $\sum_{i=1}^n a_i(y_i - T_{s_i} y_i) \epsilon$ in the set $co\{y - T_s y : y \in E, s \in G\}$ such that $\epsilon > 2C \cdot \|z - \sum_{i=1}^n a_i(y_i - T_{s_i} y_i)\|_p$. On the other hand, from $\|\mu^\alpha - r_s^* \mu^\alpha\|_p \rightarrow 0$ for all $s \in G$, there exists $\alpha_0 \in I$ such that, for all $\alpha \geq \alpha_0$ and $i = 1, 2, \dots, n$,

$$\epsilon > \|\mu^\alpha - r_{s_i}^* \mu^\alpha\|_p \cdot 2C \|y_i\|_p.$$

This implies

$$\begin{aligned} \|T_{\mu^\alpha} z\|_p &\leq \|T_{\mu^\alpha} z - T_{\mu^\alpha}(\sum_{i=1}^n a_i(y_i - T_{s_i} y_i))\|_p + \|T_{\mu^\alpha}(\sum_{i=1}^n a_i(y_i - T_{s_i} y_i))\|_p \\ &\leq \|T_{\mu^\alpha}\|_p \cdot \|z - \sum_{i=1}^n a_i(y_i - T_{s_i} y_i)\|_p + |\sum_{i=1}^n a_i|^P \|T_{\mu^\alpha}(y_i - T_{s_i} y_i)\|_p \\ &\leq C \cdot \|z - \sum_{i=1}^n a_i(y_i - T_{s_i} y_i)\|_p + \sup_i \|\mu^\alpha - r_{s_i}^* \mu^\alpha\|_p \cdot C \cdot \|y_i\|_p \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence $T_{\mu^\alpha} z$ converges strongly to 0 for each $z \in E_0$.

Next, assume that $x - T_\mu x$ for some $x \in E$ is not contained in the set E_0 . Then, by the Hahn-Banach theorem, there exists a linear continuous functional y^* such that $\langle x - T_\mu x, y^* \rangle = 1$ and $\langle z, y^* \rangle = 0$ for all $z \in E_0$. and so since $x - T_t x \in E_0$ for all $t \in G$, we have

$$\langle x - T_{\mu_j} x, y^* \rangle = \mu_t \langle x - T_t x, y^* \rangle = 0.$$

This is a contradiction. Hence $x - T_\mu x$ for all $x \in E$ are contained in E_0 . Therefore, we have $T_{\mu^\alpha} x - T_\mu x = T_{\mu^\alpha}(x - T_\mu x)$ converges strongly to 0 for all $x \in E$. This completes the proof. \square

By using Theorem 4.3, we can obtain the following corollary.

Corollary 4.4. *Let E be a real p -Banach space and T be a linear operator of E into itself such that exists a constant C with $\|T^n\|_p \leq C$ for $n = 1, 2, \dots$. Assume that T is weakly completely continuous, i.e., T maps the closed unit ball of E into a weakly compact subset of E . Then there exists a projection Q of E onto the set $F(T)$ of all fixed points of T such that $\|Q\|_p \leq C$, the Cesaro means $S_n = \frac{1}{n} \sum_{k=1}^n T^k x$ converges strongly to Qx for each $x \in E$, and $TQ = QT = Q$.*

Proof. Let $x \in E$. Then, since $\{T^n x : n = 1, 2, \dots\} = T(\{T^{n-1} x : n = 1, 2, \dots\}) \subset T(B(0, \|x\|_p \cdot (c+1)))$, where $B(x, r)$ means the closed ball with center x and radius r , the weak closure of $\{T^n x : n = 1, 2, \dots\}$ is weakly compact. On the other hand, let $G = \{1, 2, 3, \dots\}$ with the discrete topology and μ^n be a mean on G such that $\mu^n(f) = \sum_{i=1}^n (\frac{1}{n}) f(i)$ for each $f \in m(G)$. Then, it is obvious that $\|\mu^n - r_k^* \mu^n\|_p \leq \frac{2k}{n} \rightarrow 0$ for all $k \in G$. So, it follows from Theorem 4.3 that the corollary is true. \square

If $G = [0, \infty)$ with the natural topology, then we obtain the corresponding result.

Corollary 4.5. *Let E be a real p -Banach space and $\{T_t : t \in [0, \infty)\}$ be a family of linear operators of E into itself satisfying Assumption (A). Then there exists a projection Q of E onto $F(G)$ such that $\|Q\|_p \leq C$, $\frac{1}{T} \int_0^T T \int_t x dt$ converges strongly to Qx for each $x \in E$, and $T_t Q = QT_t = Q$ for each $t \in [0, \infty)$.*

Remark. $\frac{1}{T} \int_0^T T \int_t x dt$ is a weak vector valued integral with respect to means on $G = [0, \infty)$. As in Section IV of Rode [10], we can also obtain the strong convergence of the

sequences

$$(1-r) \sum_{k=1}^{\infty} r^k T^k x, \quad r \rightarrow 1-$$

and

$$\lambda \int_0^{\infty} e^{-\lambda t} T_t x dt, \quad \lambda \rightarrow 0+.$$

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Fixed Point Results for Ćirić type α - η -GF-Contractions

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Abstract: The aim of this paper is to establish some new fixed point results for Ćirić type α - η -GF-contraction in a complete metric space. We extend the concept of F -contraction and introduce the notion Ćirić type α - η -GF-contraction. An example is given to demonstrate the novelty of our work.

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1 Introduction

In metric fixed point theory the contractive conditions on underlying functions play an important role for finding solutions of fixed point problems. Banach contraction principle [4] is a fundamental result in metric fixed point theory. Due to its importance and simplicity, several authors have generalized/extended it in different directions. In 1973, Geraghty [9] studied a generalization of Banach contraction principle. Ćirić [5], introduced quasi contraction theorem, which generalized Banach contraction principle. Over the years, Banach contraction theorem has been generalized in different ways by several mathematicians (see [1-24]).

In 2012, Samet et al. [22], introduced a concept of $\alpha - \psi$ - contractive type

mappings and established various fixed point theorems for mappings in complete metric spaces. Afterwards Karapinar et al. [16], refined the notion and obtained various fixed point results. Hussain et al. [12], extended the concept of α -admissible mappings and obtained useful fixed point theorems. Subsequently, Abdeljawad [1] introduced pair of α -admissible mappings satisfying new sufficient contractive conditions different from those in [12, 22], and proved fixed point and common fixed point theorems. Lately, Salimi et al. [21], modified the concept of $\alpha - \psi$ - contractive mapping and established fixed point results.

Definition 1 ([22]). Let $T : X \rightarrow X$ and $\alpha : X \times X \rightarrow [0, +\infty)$. We say that T is α -admissible if $x, y \in X$, $\alpha(x, y) \geq 1$ implies that $\alpha(Tx, Ty) \geq 1$.

Definition 2 ([21]). Let $T : X \rightarrow X$ and $\alpha, \eta : X \times X \rightarrow [0, +\infty)$ be two functions. We say that T is α -admissible mapping with respect to η if $x, y \in X$, $\alpha(x, y) \geq \eta(x, y)$ implies that $\alpha(Tx, Ty) \geq \eta(Tx, Ty)$.

If $\eta(x, y) = 1$, then above definition reduces to definition 1. If $\alpha(x, y) = 1$, then T is called an η -subadmissible mapping.

Definition 3 [11] Let (X, d) be a metric space. Let $T : X \rightarrow X$ and $\alpha, \eta : X \times X \rightarrow [0, +\infty)$ be two functions. We say that T is $\alpha - \eta$ -continuous mapping on (X, d) if for given $x \in X$, and sequence $\{x_n\}$ with

$$x_n \rightarrow x \text{ as } n \rightarrow \infty, \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \text{ for all } n \in \mathbb{N} \Rightarrow Tx_n \rightarrow Tx.$$

In 1962, Edelstein proved the following version of the Banach contraction principle.

Theorem 4 [7]. Let (X, d) be a metric space and $T : X \rightarrow X$ be a self mapping. Assume that

$$d(Tx, Ty) < d(x, y), \text{ holds for all } x, y \in X \text{ with } x \neq y.$$

Then T has a unique fixed point in X .

In 2012, Wardowski [24] introduce a new type of contractions called F -contraction and proved new fixed point theorems concerning F -contraction. He generalized the Banach contraction principle in a different way than as it was done by different investigators. Piri et al. [19] defined the F -contraction as follows.

Definition 5 [19] Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is said to be an F contraction if there exists $\tau > 0$ such that

$$\forall x, y \in X, d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)), \quad (1.1)$$

where $F : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a mapping satisfying the following conditions:

- (F1) F is strictly increasing, i.e. for all $x, y \in \mathbb{R}_+$ such that $x < y$, $F(x) < F(y)$;
- (F2) For each sequence $\{\alpha_n\}_{n=1}^\infty$ of positive numbers, $\lim_{n \rightarrow \infty} \alpha_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$;
- (F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha \rightarrow 0^+} \alpha^k F(\alpha) = 0$.

We denote by Δ_F , the set of all functions satisfying the conditions (F1)-(F3).

Example 6 [24] Let $F : \mathbb{R}_+ \rightarrow \mathbb{R}$ be given by the formula $F(\alpha) = \ln \alpha$. It is clear that F satisfied (F1)-(F2)-(F3) for any $k \in (0, 1)$. Each mapping $T : X \rightarrow X$ satisfying (1.1) is an F -contraction such that

$$d(Tx, Ty) \leq e^{-\tau} d(x, y), \text{ for all } x, y \in X, Tx \neq Ty.$$

It is clear that for $x, y \in X$ such that $Tx = Ty$ the inequality $d(Tx, Ty) \leq e^{-\tau} d(x, y)$, also holds, i.e. T is a Banach contraction.

Example 7 [24] If $F(\alpha) = \ln \alpha + \alpha$, $\alpha > 0$ then F satisfies (F1)-(F3) and the condition (1.1) is of the form

$$\frac{d(Tx, Ty)}{d(x, y)} \leq e^{d(Tx, Ty) - d(x, y)} \leq e^{-\tau}, \text{ for all } x, y \in X, Tx \neq Ty.$$

Remark 8 From (F1) and (1.1) it is easy to conclude that every F -contraction is necessarily continuous.

Wardowski [24] stated a modified version of the Banach contraction principle as follows.

Theorem 9 [24] *Let (X, d) be a complete metric space and let $T : X \rightarrow X$ be an F contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n \in \mathbb{N}}$ converges to x^* .*

Hussain et al. [11] introduced the following family of new functions.

Let Δ_G denotes the set of all functions $G : \mathbb{R}^4 \rightarrow \mathbb{R}^+$ satisfying:

(G) for all $t_1, t_2, t_3, t_4 \in \mathbb{R}^+$ with $t_1 t_2 t_3 t_4 = 0$ there exists $\tau > 0$ such that $G(t_1, t_2, t_3, t_4) = \tau$.

Definition 10 [11] *Let (X, d) be a metric space and T be a self mapping on X . Also suppose that $\alpha, \eta : X \times X \rightarrow [0, +\infty)$ be two function. We say that T is α - η -GF-contraction if for $x, y \in X$, with $\eta(x, Tx) \leq \alpha(x, y)$ and $d(Tx, Ty) > 0$ we have*

$$G(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) + F(d(Tx, Ty)) \leq F(d(x, y)),$$

where $G \in \Delta_G$ and $F \in \Delta_F$.

2 Main Result

In this section, we define a new contraction called Ćirić type α - η -GF-contraction and obtained some new fixed point theorems for such contraction in the setting of complete metric spaces. We define Ćirić type α - η -GF-contraction as follows:

Definition 11 *Let (X, d) be a metric space and T be a self mapping on X . Also suppose that $\alpha, \eta : X \times X \rightarrow [0, +\infty)$ two functions. We say that T is Ćirić type α - η -GF-contraction if for all $x, y \in X$, with $\eta(x, Tx) \leq \alpha(x, y)$ and $d(Tx, Ty) > 0$, we have*

$$G(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) + F(d(Tx, Ty)) \leq F(M(x, y)) \quad (2.1)$$

where

$$M(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\}$$

$G \in \Delta_G$ and $F \in \Delta_F$.

Now we state our main result.

Theorem 12 *Let (X, d) be a complete metric space. Let T be a Ćirić type α - η -GF-contraction satisfying the following assertions:*

- (i) T is an α -admissible mapping with respect to η ;
- (ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;
- (iii) T is $\alpha - \eta$ -continuous.

Then T has a fixed point in X . Moreover, T has a unique fixed point when $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in \text{Fix}(T)$.

Proof. Let x_0 in X such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. For $x_0 \in X$, we construct a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_1 = Tx_0$, $x_2 = Tx_1 = T^2x_0$. Continuing this process, $x_{n+1} = Tx_n = T^{n+1}x_0$, for all $n \in \mathbb{N}$. Now since, T is an α -admissible mapping with respect to η then $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0) = \eta(x_0, x_1)$. By continuing in this process we have,

$$\eta(x_{n-1}, Tx_{n-1}) = \eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n), \text{ for all } n \in \mathbb{N}. \quad (2.2)$$

If there exists $n \in \mathbb{N}$ such that $d(x_n, Tx_n) = 0$, there is nothing to prove. So, we assume that $x_n \neq x_{n+1}$ with

$$d(Tx_{n-1}, Tx_n) = d(x_n, Tx_n) > 0, \forall n \in \mathbb{N}.$$

Since T is Ćirić type α - η -GF-contraction, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} & G(d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})) \\ & + F(d(Tx_{n-1}, Tx_n)) \leq F(M(x_{n-1}, x_n)) \end{aligned}$$

which implies

$$\begin{aligned} & G(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0) \\ & + F(d(Tx_{n-1}, Tx_n)) \leq F(M(x_{n-1}, x_n)) \end{aligned} \quad (2.3)$$

Now by definition of G , $d(x_{n-1}, x_n).d(x_n, x_{n+1}).d(x_{n-1}, x_{n+1}).0 = 0$, so there exists $\tau > 0$ such that,

$$G(d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), 0) = \tau.$$

Therefore

$$F(d(x_n, x_{n+1})) = F(d(Tx_{n-1}, Tx_n)) \leq F(M(x_{n-1}, x_n)) - \tau. \quad (2.4)$$

Now

$$\begin{aligned} M(x_{n-1}, x_n) &= \max \left\{ d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)}{2} \right\} \\ &= \max \left\{ d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_{n+1})}{2} \right\} \\ &\leq \max \left\{ d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2} \right\} \\ &= \max \{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned}$$

So, we have

$$F(d(x_n, x_{n+1})) = F(d(Tx_{n-1}, Tx_n)) \leq F(\max \{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}) - \tau.$$

In this case $M(x_{n-1}, x_n) = \max \{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1})$ is impossible, because

$$F(d(x_n, x_{n+1})) = F(d(Tx_{n-1}, Tx_n)) \leq F(d(x_n, x_{n+1})) - \tau < F(d(x_n, x_{n+1})).$$

Which is a contradiction. So

$$M(x_{n-1}, x_n) = \max \{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n).$$

Thus from (2.4), we have

$$F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)) - \tau.$$

Continuing this process, we get

$$\begin{aligned}
 F(d(x_n, x_{n+1})) &\leq F(d(x_{n-1}, x_n)) - \tau \\
 &= F(d(Tx_{n-2}, Tx_{n-1})) - \tau \\
 &\leq F(d(x_{n-2}, x_{n-1})) - 2\tau \\
 &= F(d(Tx_{n-3}, Tx_{n-2})) - 2\tau \\
 &\leq F(d(x_{n-3}, x_{n-2})) - 3\tau \\
 &\vdots \\
 &\leq F(d(x_0, x_1)) - n\tau.
 \end{aligned}$$

This implies that

$$F(d(x_n, x_{n+1})) \leq F(d(x_0, x_1)) - n\tau. \quad (2.5)$$

From (2.5), we obtain $\lim_{n \rightarrow \infty} F(d(x_n, x_{n+1})) = -\infty$. Since $F \in \Delta_F$, we have

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \quad (2.6)$$

From (F3), there exists $k \in (0, 1)$ such that

$$\lim_{n \rightarrow \infty} \left((d(x_n, x_{n+1}))^k F(d(x_n, x_{n+1})) \right) = 0. \quad (2.7)$$

From (2.5), for all $n \in \mathbb{N}$, we obtain

$$(d(x_n, x_{n+1}))^k (F(d(x_n, x_{n+1})) - F(d(x_0, x_1))) \leq - (d(x_n, x_{n+1}))^k n\tau \leq 0. \quad (2.8)$$

By using (2.6), (2.7) and letting $n \rightarrow \infty$, in (2.8), we have

$$\lim_{n \rightarrow \infty} \left(n (d(x_n, x_{n+1}))^k \right) = 0. \quad (2.9)$$

We observe that from (2.9), then there exists $n_1 \in \mathbb{N}$, such that $n (d(x_n, x_{n+1}))^k \leq 1$ for all $n \geq n_1$, we get

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{\frac{1}{k}}} \text{ for all } n \geq n_1. \quad (2.10)$$

Now, $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Then, by the triangle inequality and from (2.10) we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \dots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{\infty} d(x_i, x_{i+1}) \\ &\leq \sum_{i=n}^{\infty} \frac{1}{q^{\frac{1}{k}}}. \end{aligned}$$

The series $\sum_{i=n}^{\infty} \frac{1}{q^{\frac{1}{k}}}$ is convergent. By taking limit as $n \rightarrow \infty$, in (2.11), we have $\lim_{n, m \rightarrow \infty} d(x_n, x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space there exists $x^* \in X$ such that $x_n \rightarrow x^*$ as $n \rightarrow \infty$. T is an α - η -continuous and $\eta(x_{n-1}, x_n) \leq \alpha(x_{n-1}, x_n)$, for all $n \in \mathbb{N}$ then $x_{n+1} = Tx_n \rightarrow Tx^*$ as $n \rightarrow \infty$. That is, $x^* = Tx^*$. Hence x^* is a fixed point of T . To prove uniqueness, let $x \neq y$ be any two fixed point of T , then from (2.1), we have

$$G(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) + F(d(Tx, Ty)) \leq F(M(x, y))$$

we obtain

$$\tau + F(d(x, y)) \leq F(d(x, y)).$$

which is a contradiction. Hence, $x = y$. Therefore, T has a unique fixed point.

■

Theorem 13 *Let (X, d) be a complete metric space. Let T be a self mapping satisfying the following assertions:*

- (i) T is an α -admissible mapping with respect to η ;
- (ii) T is Ćirić type α - η -GF-contraction;
- (iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;
- (iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \rightarrow x^*$ as $n \rightarrow \infty$ then either

$$\alpha(Tx_n, x) \geq \eta(Tx_n, T^2x_n) \text{ or } \alpha(T^2x_n, x) \geq \eta(T^2x_n, T^3x_n)$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point in X . Moreover, T has a unique fixed point when $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in \text{Fix}(T)$.

Proof. As similar lines of the Theorem 12, we can conclude that

$$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}) \text{ and } x_n \rightarrow x^* \text{ as } n \rightarrow \infty.$$

Since, by (iv), either

$$\alpha(Tx_n, x^*) \geq \eta(Tx_n, T^2x_n) \text{ or } \alpha(T^2x_n, x^*) \geq \eta(T^2x_n, T^3x_n),$$

holds for all $n \in \mathbb{N}$. This implies

$$\alpha(x_{n+1}, x^*) \geq \eta(x_{n+1}, x_{n+2}) \text{ or } \alpha(x_{n+2}, x^*) \geq \eta(x_{n+2}, x_{n+3}), \text{ for all } n \in \mathbb{N}.$$

Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\eta(x_{n_k}, Tx_{n_k}) = \eta(x_{n_k}, x_{n_k+1}) \leq \alpha(x_{n_k}, x^*).$$

From (2.1), we have

$$\begin{aligned} & G(d(x_{n_k}, Tx_{n_k}), d(x^*, Tx^*), d(x_{n_k}, Tx^*), d(x^*, Tx_{n_k})) + F(d(Tx_{n_k}, Tx^*)) \\ & \leq F(M(x_{n_k}, x^*)) \\ & = F\left(\max\left\{d(x_{n_k}, x^*), d(x_{n_k}, Tx_{n_k}), d(x^*, Tx^*), \frac{d(x_{n_k}, Tx^*) + d(x^*, Tx_{n_k})}{2}\right\}\right) \\ & = F\left(\max\left\{d(x_{n_k}, x^*), d(x_{n_k}, x_{n_k+1}), d(x^*, Tx^*), \frac{d(x_{n_k}, Tx^*) + d(x^*, x_{n_k+1})}{2}\right\}\right). \end{aligned}$$

Using the continuity of F and the fact that

$$\lim_{k \rightarrow \infty} d(x_{n_k}, x^*) = 0 = \lim_{k \rightarrow \infty} d(x_{n_k+1}, x^*) \quad (2.12)$$

we obtain

$$\tau + F(d(x^*, Tx^*)) \leq F(d(x^*, Tx^*)). \quad (2.13)$$

Which is a contradiction. Therefore, $d(x^*, Tx^*) = 0$, implies x^* is a fixed point of T . Uniqueness follows similar lines as in Theorem 12. ■

In the following we extend the Wardowski type fixed point theorem.

Theorem 14 *Let T be a continuous self mapping on a complete metric space X . If for $x, y \in X$ with $d(x, Tx) \leq d(x, y)$ and $d(Tx, Ty) > 0$, we have*

$$G(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) + F(d(Tx, Ty)) \leq F(M(x, y)),$$

where

$$M(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\}.$$

$G \in \Delta_G$ and $F \in \Delta_F$. Then T has a fixed point in X .

Proof. Let us define $\alpha, \eta : X \times X \rightarrow [0, +\infty)$ by

$$\alpha(x, y) = d(x, y) \text{ and } \eta(x, y) = d(x, y) \text{ for all } x, y \in X.$$

Now, $d(x, y) \leq d(x, y)$ for all $x, y \in X$, so $\alpha(x, y) \geq \eta(x, y)$ for all $x, y \in X$. That is, conditions (i) and (iii) of Theorem 12 hold true. Since T is continuous, so T is α - η -continuous. Let $\eta(x, Tx) \leq \alpha(x, y)$ and $d(Tx, Ty) > 0$, we have $d(x, Tx) \leq d(x, y)$ with $d(Tx, Ty) > 0$, then

$$G(d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)) + F(d(Tx, Ty)) \leq F(M(x, y)).$$

That is, T is Ćirić type α - η -GF-contraction mapping. Hence, all conditions of Theorem 12 satisfied and T has a fixed point. ■

Corollary 15 *Let T be a continuous selfmapping on a complete metric space X . If for $x, y \in X$ with $d(x, Tx) \leq d(x, y)$ and $d(Tx, Ty) > 0$, we have*

$$\tau + F(d(Tx, Ty)) \leq F(M(x, y)),$$

where $\tau > 0$, and $F \in \Delta_F$. Then T has a fixed point in X .

Corollary 16 *Let T be a continuous selfmapping on a complete metric space X . If for $x, y \in X$ with $d(x, Tx) \leq d(x, y)$ and $d(Tx, Ty) > 0$, we have*

$$\tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

where $\tau > 0$, and $F \in \Delta_F$. Then T has a fixed point in X .

Corollary 17 [11] *Let (X, d) be a complete metric space. Let $T : X \rightarrow X$ be a self-mapping satisfying the following assertions:*

- (i) T is an α -admissible mapping with respect to η ;
- (ii) T is an α - η -GF-contraction
- (iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;
- (iv) T is $\alpha - \eta$ -continuous.

Then T has a fixed point in X . Moreover, T has a unique fixed point when $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in \text{Fix}(T)$.

Corollary 18 [11] *Let (X, d) be a complete metric space. Let $T : X \rightarrow X$ be a self-mapping satisfying the following assertions:*

- (i) T is an α -admissible mapping with respect to η ;
- (ii) T is an α - η -GF-contraction
- (iii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$;
- (iv) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ with $x_n \rightarrow x^*$ as $n \rightarrow \infty$ then either

$$\alpha(Tx_n, x) \geq \eta(Tx_n, T^2x_n) \text{ or } \alpha(T^2x_n, x) \geq \eta(T^2x_n, T^3x_n)$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point in X . Moreover, T has a unique fixed point when $\alpha(x, y) \geq \eta(x, x)$ for all $x, y \in \text{Fix}(T)$.

Example 19 *Consider the sequence,*

$$S_1 = 1 \times 3$$

$$S_2 = 1 \times 3 + 2 \times 5$$

$$S_3 = 1 \times 3 + 2 \times 5 + 3 \times 7$$

$$S_n = 1 \times 3 + 2 \times 5 + 3 \times 7 \dots + n \times (2n + 1) = \frac{n(n+1)(4n+5)}{6}.$$

Let $X = \{S_n : n \in \mathbb{N}\}$ and $d(x, y) = |x - y|$. Then (X, d) is a complete metric space. If $F(\alpha) = \alpha + \ln \alpha$, $\alpha > 0$ and $G(t_1, t_2, t_3, t_4) = \tau$ where $\tau = 1$. Define the

mapping $T : X \rightarrow X$ by, $T(S_1) = S_1$ and $T(S_n) = S_{n-1}, n \geq 1$ and $\alpha(x, y) = 1$ if $x \in X, \eta(x, Tx) = \frac{1}{2}$ for all $x \in X$. we have

$$\lim_{n \rightarrow \infty} \frac{d(T(S_n), T(S_1))}{d(S_n, S_1)} = \lim_{n \rightarrow \infty} \frac{S_{n-1} - 3}{S_n - 3} = \frac{(n-1)n(4n+1) - 18}{n(n+1)(4n+5) - 18} = 1.$$

So we conclude the following two cases:

Case 1:

we observe that for every $m \in \mathbb{N}, m > 2, n = 1$ or $n = 1$ and $m > 1$ then $\alpha(S_m, S_n) \geq \eta(S_m, T(S_m))$, we have

$$\begin{aligned} \frac{d(T(S_m), T(S_1))}{M(S_m, S_1)} e^{d(T(S_m), T(S_1)) - M(S_m, S_1)} &= \frac{S_{m-1} - 3}{S_m - 3} e^{S_{m-1} - S_m} \\ &= \frac{(m-1)m(4m+1) - 18}{m(m+1)(4m+5) - 18} e^{-\frac{m(m+1)(4m+5)}{6}} \\ &< e^{-1}. \end{aligned}$$

Case 2:

for $m > n > 1$, then $\alpha(S_m, S_n) \geq \eta(S_m, T(S_m))$, we have

$$\begin{aligned} &\frac{d(T(S_m), T(S_n))}{M(S_m, S_n)} e^{d(T(S_m), T(S_n)) - M(S_m, S_n)} \\ &= \frac{S_{m-1} - S_{n-1}}{S_m - S_n} e^{S_n - S_{n-1} + S_{m-1} - S_m} \\ &= \frac{(m-1)m(4m+1) - (n-1)n(4n+1)}{m(m+1)(4m+5) - n(n+1)(4n+5)} e^{\frac{n(n+1)(4n+5)}{6} - \frac{m(m+1)(4m+5)}{6}} \leq e^{-1}. \end{aligned}$$

So all condition of theorems are satisfied, T has a fixed point in X .

Let (X, d, \preceq) be a partially ordered metric space. Let $T : X \rightarrow X$ is such that for $x, y \in X$, with $x \preceq y$ implies $Tx \preceq Ty$, then the mapping T is said to be non-decreasing. We derive following important result in partially ordered metric spaces.

Theorem 20 *Let (X, d, \preceq) be a complete partially ordered metric space. Assume that the following assertions hold true:*

- (i) T is nondecreasing and ordered GF -contraction;

- (ii) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;
- (iii) either for a given $x \in X$ and sequence $\{x_n\}$ in X such that $x_n \rightarrow x$ as $n \rightarrow \infty$ and $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ we have $Tx_n \rightarrow Tx$ or if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ with $x_n \rightarrow x$ as $n \rightarrow \infty$ then either

$$Tx_n \preceq x \text{ or } T^2x_n \preceq x$$

holds for all $n \in \mathbb{N}$.

Then T has a fixed point in X .

Define $F = \{\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ : \phi \text{ is a Lebesgue integral mapping which is summable, nonnegative and satisfies } \int_0^\epsilon \phi(t)dt > 0, \text{ for each } \epsilon > 0\}$.

We can easily deduce following result involving integral type inequalities.

Theorem 21 *Let T be a continuous selfmapping on a complete metric space X . If for $x, y \in X$ with*

$$\int_0^{d(x,Tx)} \phi(t)dt \leq \int_0^{d(x,y)} \phi(t)dt \text{ and } \int_0^{d(Tx,Ty)} \phi(t)dt > 0,$$

we have

$$\begin{aligned} & G\left(\int_0^{d(x,Tx)} \phi(t)dt, \int_0^{d(y,Ty)} \phi(t)dt, \int_0^{d(x,Ty)} \phi(t)dt, \int_0^{d(y,Tx)} \phi(t)dt\right) + F\left(\int_0^{d(Tx,Ty)} \phi(t)dt\right) \\ & \leq F\left(\int_0^{M(x,y)} \phi(t)dt\right), \end{aligned}$$

where

$$M(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\}.$$

$\phi \in F, G \in \Delta_G$ and $F \in \Delta_F$. Then T has a fixed point in X .

Conflict of Interests

The authors declare that they have no competing interests.

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FIXED POINT AND QUADRATIC ρ -FUNCTIONAL INEQUALITIES IN NON-ARCHIMEDEAN BANACH SPACES

JUNG RYE LEE AND DONG YUN SHIN*

ABSTRACT. In this paper, we solve the following quadratic ρ -functional inequalities

$$\begin{aligned} & \left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \right. \\ & \quad \left. - f(x) - f(y) - f(z) \right\| \\ & \leq \left\| \rho(f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) \right. \\ & \quad \left. - 4f(x) - 4f(y) - 4f(z)) \right\|, \end{aligned} \quad (0.1)$$

where ρ is a fixed non-Archimedean number with $|\rho| < \frac{1}{4!}$, and

$$\begin{aligned} & \left\| f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) \right. \\ & \quad \left. - 4f(x) - 4f(y) - 4f(z) \right\| \\ & \leq \left\| \rho \left(f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \right. \right. \\ & \quad \left. \left. - f(x) - f(y) - f(z) \right) \right\|, \end{aligned} \quad (0.2)$$

where ρ is a fixed non-Archimedean number with $|\rho| < |8|$.

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic ρ -functional inequalities (0.1) and (0.2) in non-Archimedean Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

A *valuation* is a function $|\cdot|$ from a field K into $[0, \infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r| \cdot |s|$ and the triangle inequality holds, i.e.,

$$|r+s| \leq |r| + |s|, \quad \forall r, s \in K.$$

A field K is called a *valued field* if K carries a valuation. The usual absolute values of \mathbb{R} and \mathbb{C} are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \leq \max\{|r|, |s|\}, \quad \forall r, s \in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

Definition 1.1. ([19]) Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \rightarrow [0, \infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

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- (i) $\|x\| = 0$ if and only if $x = 0$;
- (ii) $\|rx\| = |r|\|x\|$ ($r \in K, x \in X$);
- (iii) the strong triangle inequality

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}, \quad \forall x, y \in X$$

holds. Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*.

Definition 1.2. (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X . Then the sequence $\{x_n\}$ is called *Cauchy* if for a given $\varepsilon > 0$ there is a positive integer N such that

$$\|x_n - x_m\| \leq \varepsilon$$

for all $n, m \geq N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X . Then the sequence $\{x_n\}$ is called *convergent* if for a given $\varepsilon > 0$ there are a positive integer N and an $x \in X$ such that

$$\|x_n - x\| \leq \varepsilon$$

for all $n \geq N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n \rightarrow \infty} x_n = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space*.

The stability problem of functional equations originated from a question of Ulam [27] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [12] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [24] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [9] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \quad (1.1)$$

is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [26] for mappings $f : E_1 \rightarrow E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. See [7, 15, 16] for more functional equations.

The functional equation $2f\left(\frac{x+y}{2}\right) + 2\left(\frac{x-y}{2}\right) = f(x) + f(y)$ is called a *Jensen type quadratic equation*.

In [10], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\| \quad (1.2)$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

See also [25]. Gilányi [11] and Fechner [8] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [22] proved the Hyers-Ulam stability of additive functional inequalities.

Let X be a set. A function $d : X \times X \rightarrow [0, \infty]$ is called a *generalized metric* on X if d satisfies

- (1) $d(x, y) = 0$ if and only if $x = y$;
- (2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

FIXED POINT AND QUADRATIC ρ -FUNCTIONAL INEQUALITIES

Theorem 1.3. [3, 6] *Let (X, d) be a complete generalized metric space and let $J : X \rightarrow X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

- (1) $d(J^n x, J^{n+1} x) < \infty, \quad \forall n \geq n_0$;
- (2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;*
- (3) y^* *is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*
- (4) $d(y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)$ *for all $y \in Y$.*

In 1996, G. Isac and Th.M. Rassias [13] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4, 5, 14, 17, 20, 21, 23]).

In Section 2, we deal with quadratic functional equations. In Section 3, we solve the quadratic ρ -functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic ρ -functional inequality (0.1) in non-Archimedean Banach spaces. In Section 4, we solve the quadratic ρ -functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic ρ -functional inequality (0.2) in non-Archimedean Banach spaces.

Throughout this paper, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space. Let $|2| \neq 1$.

2. QUADRATIC FUNCTIONAL EQUATIONS

Theorem 2.1. *Let X and Y be vector spaces. A mapping $f : X \rightarrow Y$ satisfies*

$$f\left(\frac{x+y+z}{2} + \frac{x-y-z}{2} + \frac{y-x-z}{2} + \frac{z-x-y}{2}\right) = f(x) + f(y) + f(z) \quad (2.1)$$

if and only if the mapping $f : X \rightarrow Y$ is a quadratic mapping.

Proof. Assume that $f : X \rightarrow Y$ satisfies (2.1)

Letting $x = y = z = 0$ in (2.1), we have $4f(0) = 3f(0)$. So $f(0) = 0$.

Letting $y = z = 0$ in (2.1), we get

$$2f\left(\frac{x}{2}\right) + 2f\left(-\frac{x}{2}\right) = f(x) \quad \& \quad 2f\left(-\frac{x}{2}\right) + 2f\left(\frac{x}{2}\right) = f(-x) \quad (2.2)$$

for all $x \in X$, which imply that $f(x) = f(-x)$ for all $x \in X$.

From this and (2.2), we obtain $4f\left(\frac{x}{2}\right) = f(x)$ or $f(2x) = 4f(x)$ for all $x \in X$.

Putting $z = 0$ in (2.1), we obtain $\frac{1}{2}f(x+y) + \frac{1}{2}f(x-y) = f(x) + f(y)$ for all $x, y \in X$, which means that $f : X \rightarrow Y$ is a quadratic mapping.

The converse is obviously true. □

Corollary 2.2. *Let X and Y be vector spaces. An even mapping $f : X \rightarrow Y$ satisfies*

$$f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) = 4f(x) + 4f(y) + 4f(z) \quad (2.3)$$

for all $x, y, z \in X$. Then the mapping $f : X \rightarrow Y$ is a quadratic mapping.

Proof. Assume that $f : X \rightarrow Y$ satisfies (2.3).

Letting $x = y = z = 0$ in (2.3), we have $4f(0) = 12f(0)$. So $f(0) = 0$.

Letting $z = 0$ in (2.3), we get $2f(x+y) + 2f(x-y) = 4f(x) + 4f(y)$ and so $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in X$. □

3. QUADRATIC ρ -FUNCTIONAL INEQUALITY (0.1)

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < \frac{1}{4}$.

In this section, we solve and investigate the quadratic ρ -functional inequality (0.1) in non-Archimedean normed spaces.

Lemma 3.1. *An even mapping $f : X \rightarrow Y$ satisfies*

$$\begin{aligned} & \left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \right. \\ & \quad \left. - f(x) - f(y) - f(z) \right\| \\ & \leq \|\rho(f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) \\ & \quad - 4f(x) - 4f(y) - 4f(z))\| \end{aligned} \quad (3.1)$$

for all $x, y, z \in X$ if and only if $f : X \rightarrow Y$ is quadratic.

Proof. Assume that $f : X \rightarrow Y$ satisfies (3.1).

Letting $x = y = z = 0$ in (3.1), we get $\|f(0)\| \leq |\rho|\|8f(0)\|$. So $f(0) = 0$.

Letting $y = z = 0$ in (3.1), we get $\|4f(\frac{x}{2}) - f(x)\| \leq 0$ and so

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \quad (3.2)$$

for all $x \in X$.

By (3.1) and (3.2), we have

$$f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) = f(x) + f(y) + f(z)$$

for all $x, y, z \in X$, since $|\rho| < \frac{1}{4}$.

The converse is obviously true. \square

Now we prove the Hyers-Ulam stability of the quadratic ρ -functional inequality (3.1) in non-Archimedean Banach spaces.

Theorem 3.2. *Let $\varphi : X^3 \rightarrow [0, \infty)$ be a function with $\varphi(0, 0, 0) = 0$ such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{4}\varphi(x, y, z) \quad (3.3)$$

for all $x, y, z \in X$. Let $f : X \rightarrow Y$ be an even mapping such that

$$\begin{aligned} & \left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \right. \\ & \quad \left. - f(x) - f(y) - f(z) \right\| \\ & \leq \|\rho(f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) \\ & \quad - 4f(x) - 4f(y) - 4f(z))\| + \varphi(x, y, z) \end{aligned} \quad (3.4)$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{1-L}\varphi(x, 0, 0) \quad (3.5)$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (3.4), we get $\|f(0)\| \leq |\rho|\|8f(0)\|$. So $f(0) = 0$.

Letting $y = z = 0$ in (3.4), we get

$$\left\| 4f\left(\frac{x}{2}\right) - f(x) \right\| \leq \varphi(x, 0, 0) \quad (3.6)$$

for all $x \in X$.

Consider the set $S := \{h : X \rightarrow Y, h(0) = 0\}$ and introduce the generalized metric on S :

$$d(g, h) = \inf \{ \mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq \mu \varphi(x, 0, 0), \forall x \in X \},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [18]).

Now we consider the linear mapping $J : S \rightarrow S$ such that $Jg(x) := 4g\left(\frac{x}{2}\right)$ for all $x \in X$. Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \leq \varepsilon \varphi(x, 0, 0)$$

for all $x \in X$. Hence

$$\begin{aligned} \|Jg(x) - Jh(x)\| &= \left\| 4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right) \right\| \leq 4|\varepsilon \varphi\left(\frac{x}{2}, 0, 0\right)| \\ &\leq 4|\varepsilon| \frac{L}{|4|} \varphi(x, 0, 0) \leq L\varepsilon \varphi(x, 0, 0) \end{aligned}$$

for all $x \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that $d(Jg, Jh) \leq Ld(g, h)$ for all $g, h \in S$.

It follows from (3.6) that $d(f, Jf) \leq 1$.

By Theorem 1.3, there exists a mapping $Q : X \rightarrow Y$ satisfying the following:

(1) Q is a fixed point of J , i.e.,

$$Q(x) = 4Q\left(\frac{x}{2}\right) \quad (3.7)$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (3.7) such that there exists a $\mu \in (0, \infty)$ satisfying $\|f(x) - Q(x)\| \leq \mu \varphi(x, 0, 0)$ for all $x \in X$;

(2) $d(J^l f, Q) \rightarrow 0$ as $l \rightarrow \infty$. This implies the equality $\lim_{l \rightarrow \infty} 4^l f\left(\frac{x}{2^l}\right) = Q(x)$ for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality $d(f, Q) \leq \frac{1}{1-L}$. So $\|f(x) - Q(x)\| \leq \frac{1}{1-L} \varphi(x, 0, 0)$ for all $x \in X$.

It follows from (3.3) and (3.4) that

$$\begin{aligned} &\left\| Q\left(\frac{x+y+z}{2}\right) + Q\left(\frac{x-y-z}{2}\right) + Q\left(\frac{y-x-z}{2}\right) + Q\left(\frac{z-x-y}{2}\right) \right. \\ &\quad \left. - Q(x) - Q(y) - Q(z) \right\| \\ &= \lim_{n \rightarrow \infty} |4|^n \left\| f\left(\frac{x+y+z}{2^{n+1}}\right) + f\left(\frac{x-y-z}{2^{n+1}}\right) + f\left(\frac{y-x-z}{2^{n+1}}\right) + f\left(\frac{z-x-y}{2^{n+1}}\right) \right. \\ &\quad \left. - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right) \right\| \\ &\leq \lim_{n \rightarrow \infty} |4|^n |\rho| \left\| f\left(\frac{x+y+z}{2^n}\right) + f\left(\frac{x-y-z}{2^n}\right) + f\left(\frac{y-x-z}{2^n}\right) + f\left(\frac{z-x-y}{2^n}\right) \right. \\ &\quad \left. - 4f\left(\frac{x}{2^n}\right) - 4f\left(\frac{y}{2^n}\right) - 4f\left(\frac{z}{2^n}\right) \right\| + \lim_{n \rightarrow \infty} \frac{1}{|4|^n} \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\ &= \|\rho(Q(x+y+z) + Q(x-y-z) + Q(y-x-z) + Q(z-x-y) \\ &\quad - 4Q(x) - 4Q(y) - 4Q(z))\| \end{aligned}$$

for all $x, y, z \in X$. So

$$\begin{aligned} & \left\| Q\left(\frac{x+y+z}{2}\right) + Q\left(\frac{x-y-z}{2}\right) + Q\left(\frac{y-x-z}{2}\right) + Q\left(\frac{z-x-y}{2}\right) - Q(x) - Q(y) - Q(z) \right\| \\ & \leq \|\rho(Q(x+y+z) + Q(x-y-z) + Q(y-x-z) + Q(z-x-y) - 4Q(x) - 4Q(y) - 4Q(z))\| \end{aligned}$$

for all $x, y, z \in X$. By Lemma 3.1, the mapping $Q : X \rightarrow Y$ is quadratic.

Now, let $T : X \rightarrow Y$ be another quadratic mapping satisfying (3.5). Then we have

$$\begin{aligned} \|Q(x) - T(x)\| &= \left\| 4^q Q\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right) \right\| \\ &\leq \max \left\{ \left\| 4^q Q\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right) \right\|, \left\| 4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right) \right\| \right\} \leq \lim_{n \rightarrow \infty} \frac{1}{|4|^n} \varphi\left(\frac{x}{2^n}, 0, 0\right), \end{aligned}$$

which tends to zero as $q \rightarrow \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q . Thus the mapping $Q : X \rightarrow Y$ is a unique quadratic mapping satisfying (3.5). \square

Corollary 3.3. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be an even mapping such that*

$$\begin{aligned} & \left\| f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \right. \\ & \quad \left. - f(x) - f(y) - f(z) \right\| \\ & \leq \|\rho(f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) \\ & \quad - 4f(x) - 4f(y) - 4f(z))\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r) \end{aligned} \quad (3.8)$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{|2|^r \theta}{|2|^r - |2|^2} \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. Then we can choose $L = |2|^{2-r}$ and we get desired result. \square

Theorem 3.4. *Let $\varphi : X^3 \rightarrow [0, \infty)$ be a function with $\varphi(0, 0, 0) = 0$ such that there exists an $L < 1$ with*

$$\varphi(x, y, z) \leq |4|L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$$

for all $x, y, z \in X$. Let $f : X \rightarrow Y$ be an even mapping satisfying (3.4). Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{L}{1-L} \varphi(x, 0, 0)$$

for all $x \in X$.

Proof. It follows from (3.6) that

$$\left\| f(x) - \frac{1}{4}f(2x) \right\| \leq \frac{1}{|4|} \varphi(2x, 0, 0) \leq L\varphi(x, 0, 0) \quad (3.9)$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 3.2.

Now we consider the linear mapping $J : S \rightarrow S$ such that $Jg(x) := \frac{1}{4}g(2x)$ for all $x \in X$.

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It follows from (3.9) that $d(f, Jf) \leq L$. So $d(f, Q) \leq \frac{L}{1-L}$. So $\|f(x) - Q(x)\| \leq \frac{L}{1-L} \varphi(x, 0, 0)$ for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.2. \square

Corollary 3.5. *Let $r > 2$ and θ be positive real numbers, and let $f : X \rightarrow Y$ be an even mapping satisfying (3.8). Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that*

$$\|f(x) - Q(x)\| \leq \frac{|2|^r \theta}{|2|^2 - |2|^r} \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. Then we can choose $L = |2|^{r-2}$ and we get desired result. \square

4. QUADRATIC ρ -FUNCTIONAL INEQUALITY (0.2)

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < |8|$.

In this section, we solve and investigate the quadratic ρ -functional inequality (0.2) in non-Archimedean normed spaces.

Lemma 4.1. *An even mapping $f : X \rightarrow Y$ satisfies*

$$\begin{aligned} & \|f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) - 4f(x) - 4f(y) - 4f(z)\| \\ & \leq \left\| \rho \left(f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) \right. \right. \\ & \quad \left. \left. + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z) \right) \right\| \end{aligned} \quad (4.1)$$

for all $x, y, z \in X$ if and only if $f : X \rightarrow Y$ is quadratic.

Proof. Assume that $f : X \rightarrow Y$ satisfies (4.1).

Letting $x = y = z = 0$ in (4.1), we get $\|8f(0)\| \leq |\rho| \|f(0)\|$. So $f(0) = 0$.

Letting $x = y, z = 0$ in (4.1), we get

$$\|2f(2x) - 8f(x)\| \leq 0 \quad (4.2)$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{4}f(x)$ for all $x \in X$.

By (4.1) and (4.2), we have

$$f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) = 4f(x) + 4f(y) + 4f(z)$$

for all $x, y, z \in X$, since $|\rho| < |8| \leq |4|$.

The converse is obviously true. \square

We prove the Hyers-Ulam stability of the quadratic ρ -functional inequality (4.1) in non-Archimedean Banach spaces.

Theorem 4.2. *Let $\varphi : X^3 \rightarrow [0, \infty)$ be a function with $\varphi(0, 0, 0) = 0$ such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{|4|} \varphi(x, y, z) \quad (4.3)$$

for all $x, y, z \in X$. Let $f : X \rightarrow Y$ be an even mapping satisfying

$$\begin{aligned} & \|f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) - 4f(x) - 4f(y) - 4f(z)\| \\ & \leq \left\| \rho \left(f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) + f\left(\frac{z-x-y}{2}\right) \right. \right. \\ & \quad \left. \left. - f(x) - f(y) - f(z) \right) \right\| + \varphi(x, y, z) \end{aligned} \quad (4.4)$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{L}{|4|(1-L)}\varphi(x, x, 0) \quad (4.5)$$

for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (4.4), we get $\|8f(0)\| \leq |\rho|\|f(0)\|$. So $f(0) = 0$.

Letting $x = y, z = 0$ in (4.4), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right) \leq \frac{L}{|4|}\varphi(x, x, 0) \quad (4.6)$$

for all $x \in X$.

Consider the set $S := \{h : X \rightarrow Y, h(0) = 0\}$ and introduce the generalized metric on S :

$$d(g, h) = \inf \{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \leq \mu\varphi(x, x, 0), \forall x \in X\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [18]).

Now we consider the linear mapping $J : S \rightarrow S$ such that $Jg(x) := 4g\left(\frac{x}{2}\right)$ for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then $\|g(x) - h(x)\| \leq \varepsilon\varphi(x, x, 0)$ for all $x \in X$. Hence

$$\|Jg(x) - Jh(x)\| = \left\|4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right)\right\| \leq L\varepsilon\varphi(x, x, 0)$$

for all $a \in X$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that $d(Jg, Jh) \leq Ld(g, h)$ for all $g, h \in S$.

It follows from (4.6) that $d(f, Jf) \leq \frac{L}{|4|}$.

By Theorem 1.3, there exists a mapping $Q : X \rightarrow Y$ satisfying the following:

(1) Q is a fixed point of J , i.e.,

$$Q(x) = 4Q\left(\frac{x}{2}\right) \quad (4.7)$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set $M = \{g \in S : d(f, g) < \infty\}$. This implies that Q is a unique mapping satisfying (4.7) such that there exists a $\mu \in (0, \infty)$ satisfying $\|f(x) - Q(x)\| \leq \mu\varphi(x, x, 0)$ for all $x \in X$;

(2) $d(J^l f, Q) \rightarrow 0$ as $l \rightarrow \infty$. This implies the equality $\lim_{l \rightarrow \infty} 4^l f\left(\frac{x}{2^l}\right) = Q(x)$ for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L}d(f, Jf)$, which implies the inequality $d(f, Q) \leq \frac{L}{|4|(1-L)}$. So

$$\|f(x) - Q(x)\| \leq \frac{L}{|4|(1-L)}\varphi(x, x, 0)$$

for all $x \in X$.

It follows from (4.3) and (4.4) that

$$\begin{aligned}
 & \left\| Q\left(\frac{x+y+z}{2}\right) + Q\left(\frac{x-y-z}{2}\right) + Q\left(\frac{y-x-z}{2}\right) + Q\left(\frac{z-x-y}{2}\right) \right. \\
 & \quad \left. - Q(x) - Q(y) - Q(z) \right\| \\
 &= \lim_{n \rightarrow \infty} |4|^n \left\| f\left(\frac{x+y+z}{2^{n+1}}\right) + f\left(\frac{x-y-z}{2^{n+1}}\right) + f\left(\frac{y-x-z}{2^{n+1}}\right) + f\left(\frac{z-x-y}{2^{n+1}}\right) \right. \\
 & \quad \left. - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{z}{2^n}\right) \right\| \\
 &\leq \lim_{n \rightarrow \infty} |4|^n |\rho| \left\| f\left(\frac{x+y+z}{2^n}\right) + f\left(\frac{x-y-z}{2^n}\right) + f\left(\frac{y-x-z}{2^n}\right) + f\left(\frac{z-x-y}{2^n}\right) \right. \\
 & \quad \left. - 4f\left(\frac{x}{2^n}\right) - 4f\left(\frac{y}{2^n}\right) - 4f\left(\frac{z}{2^n}\right) \right\| + \lim_{n \rightarrow \infty} |4|^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\
 &= \|\rho(Q(x+y+z) + Q(x-y-z) + Q(y-x-z) + Q(z-x-y) \\
 & \quad - 4Q(x) - 4Q(y) - 4Q(z))\|
 \end{aligned}$$

for all $x, y, z \in X$. So

$$\begin{aligned}
 & \left\| Q\left(\frac{x+y+z}{2}\right) + Q\left(\frac{x-y-z}{2}\right) + Q\left(\frac{y-x-z}{2}\right) + Q\left(\frac{z-x-y}{2}\right) \right. \\
 & \quad \left. - Q(x) - Q(y) - Q(z) \right\| \\
 &\leq \|\rho(Q(x+y+z) + Q(x-y-z) + Q(y-x-z) + Q(z-x-y) \\
 & \quad - 4Q(x) - 4Q(y) - 4Q(z))\|
 \end{aligned}$$

for all $x, y, z \in X$. By Lemma 4.1, the mapping $Q : X \rightarrow Y$ is quadratic.

The rest of the proof is similar to the proof of Theorem 3.2. \square

Corollary 4.3. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be an even mapping such that*

$$\begin{aligned}
 & \|f(x+y+z) + f(x-y-z) + f(y-x-z) + f(z-x-y) - 4f(x) - 4f(y) - 4f(z)\| \\
 &\leq \left\| \rho \left(f\left(\frac{x+y+z}{2}\right) + f\left(\frac{x-y-z}{2}\right) + f\left(\frac{y-x-z}{2}\right) \right. \right. \\
 & \quad \left. \left. + f\left(\frac{z-x-y}{2}\right) - f(x) - f(y) - f(z) \right) \right\| + \theta(\|x\|^r + \|y\|^r + \|z\|^r)
 \end{aligned} \tag{4.8}$$

for all $x, y, z \in X$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{|2|^r - |2|^2} \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.2 by taking $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. Then we can choose $L = |2|^{2-r}$ and we get desired result. \square

Theorem 4.4. *Let $\varphi : X^3 \rightarrow [0, \infty)$ be a function with $\varphi(0, 0, 0) = 0$ such that there exists an $L < 1$ with*

$$\varphi(x, y, z) \leq |4|L\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right)$$

for all $x, y, z \in X$ Let $f : X \rightarrow Y$ be an even mapping satisfying (4.4). Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{|4|(1-L)}\varphi(x, x, 0)$$

for all $x \in X$.

Proof. It follows from (4.6) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \leq \frac{1}{|4|}\varphi(x, x, 0) \quad (4.9)$$

for all $x \in X$.

Let (S, d) be the generalized metric space defined in the proof of Theorem 4.2.

Now we consider the linear mapping $J : S \rightarrow S$ such that

$$Jg(x) := \frac{1}{4}g(2x)$$

for all $x \in X$.

It follows from (4.9) that $d(f, Jf) \leq \frac{1}{|4|}$. So $d(f, Q) \leq \frac{1}{|4|(1-L)}d(f, Jf)$, which implies the inequality

$$d(f, Q) \leq \frac{1}{1-L}.$$

So

$$\|f(x) - Q(x)\| \leq \frac{1}{|4|(1-L)}\varphi(x, x, 0)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 3.2. \square

Corollary 4.5. Let $r > 2$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be an even mapping satisfying (4.8). Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{|2|^2 - |2|^r} \|x\|^r$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.4 by taking $\varphi(x, y, z) = \theta(\|x\|^r + \|y\|^r + \|z\|^r)$ for all $x, y, z \in X$. Then we can choose $L = |2|^{r-2}$ and we get desired result. \square

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Dynamics and Global Stability of Higher Order Nonlinear Difference Equation

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ABSTRACT

In this paper, we study the behavior of the solutions of the following rational difference equation with big order

$$x_{n+1} = ax_{n-l} + bx_{n-k} + \frac{cx_{n-s} + dx_{n-t}}{ex_{n-s} + fx_{n-t}}.$$

where the parameters a, b, c, d, e and f are positive real numbers and the initial conditions $x_{-r}, x_{-r+1}, \dots, x_{-1}$ and x_0 are positive real numbers where $r = \max\{l, k, s, t\}$.

Keywords: recursive sequence, periodicity, boundedness, stability, difference equations.

Mathematics Subject Classification: 39A10

1. INTRODUCTION

Difference equations appear as natural descriptions of observed evolution phenomena because most measurements of time evolving variables are discrete and as such these equations are in their own right important mathematical models. Difference equations related to differential equations as discrete mathematics related to continuous mathematics.

In recent years nonlinear difference equations have attracted the attention of many researchers, for example: Agarwal and Elsayed [1] studied the global stability, periodicity character and gave the solution form of some special cases of the recursive sequence

$$x_{n+1} = ax_n + \frac{bx_n x_{n-3}}{cx_{n-2} + dx_{n-3}}.$$

Cinar [5] obtained the solutions of the following difference equation

$$x_{n+1} = \frac{x_{n-1}}{-1 + ax_n x_{n-1}}.$$

El-Metwally et al.[10] dealt with the following difference equation

$$y_{n+1} = \frac{y_{n-(2k+1)} + p}{y_{n-(2k+1)} + qy_{n-2l}}.$$

Elsayed [12] studied the global stability, and periodicity character of the following recursive sequence

$$x_{n+1} = ax_{n-l} + \frac{bx_{n-l}}{cx_{n-l} - dx_{n-k}}.$$

Elsayed et al. [18] investigated the behavior of the following second order rational difference equation

$$x_{n+1} = ax_n + \frac{b + cx_{n-1}}{d + ex_{n-1}}.$$

Elsayed and El-Dessoky [16] investigated the global convergence, boundedness, and periodicity of solutions of the difference equation

$$x_{n+1} = ax_{n-s} + \frac{bx_{n-l} + cx_{n-k}}{dx_{n-l} + ex_{n-k}}.$$

Karatas et al. [21] got the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2}x_{n-5}}.$$

Obaid et al. [24] studied the global attractivity and periodic character of the following fourth order difference equation

$$x_{n+1} = ax_n + \frac{bx_{n-1} + cx_{n-2} + dx_{n-3}}{\alpha x_{n-1} + \beta x_{n-2} + \gamma x_{n-3}}.$$

Yalcinkaya [29] dealt with the following difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}.$$

Zayed and El-Moeam [31], [32] studied the global asymptotic properties of the solutions of the following difference equations

$$\begin{aligned} x_{n+1} &= ax_n - \frac{bx_n}{cx_n - dx_{n-k}}, \\ x_{n+1} &= Ax_n + Bx_{n-k} + \frac{px_n + x_{n-k}}{q + x_{n-k}}. \end{aligned}$$

For some related work see [1-33].

Our goal in this article is to investigate the global stability character and the periodicity of solutions of the recursive sequence

$$x_{n+1} = ax_{n-l} + bx_{n-k} + \frac{cx_{n-s} + dx_{n-t}}{ex_{n-s} + fx_{n-t}}. \quad (1)$$

where the parameters a, b, c, d, e and f are positive real numbers and the initial conditions $x_{-r}, x_{-r+1}, \dots, x_{-1}$ and x_0 are positive real numbers where $r = \max\{l, k, s, t\}$.

2. SOME BASIC PROPERTIES AND DEFINITIONS

Here, we recall some basic definitions and some theorems that we need in the sequel.

Let I be some interval of real numbers and let $F : I^{k+1} \rightarrow I$, be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, \dots, x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots, \quad (2)$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1. (Equilibrium point) A point $\bar{x} \in I$ is called an equilibrium point of Equation (2) if

$$\bar{x} = F(\bar{x}, \bar{x}, \dots, \bar{x}).$$

That is, $x_n = \bar{x}$ for $n \geq 0$, is a solution of Equation (2), or equivalently, \bar{x} is a fixed point of F .

Definition 2. (Periodicity)

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$.

Definition 3. (Stability)

(i) The equilibrium point \bar{x} of Equation (2) is locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \dots + |x_0 - \bar{x}| < \delta,$$

we have

$$|x_n - \bar{x}| < \epsilon \text{ for all } n \geq -k.$$

(ii) The equilibrium point \bar{x} of Equation (2) is locally asymptotically stable if \bar{x} is locally stable solution of Equation (2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + \dots + |x_0 - \bar{x}| < \gamma,$$

we have

$$\lim_{n \rightarrow \infty} x_n = \bar{x}.$$

(iii) The equilibrium point \bar{x} of Equation (2) is global attractor if for all $x_{-k}, x_{-k+1}, \dots, x_{-1}, x_0 \in I$, we have

$$\lim_{n \rightarrow \infty} x_n = \bar{x}.$$

(iv) The equilibrium point \bar{x} of Equation (2) is globally asymptotically stable if \bar{x} is locally stable, and \bar{x} is also a global attractor of Equation (2).

(v) The equilibrium point \bar{x} of Equation (2) is unstable if is not locally stable.

The linearized equation of Equation (2) about the equilibrium point \bar{x} is the linear difference equation

$$y_{n+1} = \sum_{i=0}^k \frac{\partial F(\bar{x}, \bar{x}, \dots, \bar{x})}{\partial x_{n-i}} y_{n-i}. \quad (3)$$

Theorem A [22] Assume that $p_i \in R$, $i = 1, 2, \dots$ and $k \in \{0, 1, 2, \dots\}$. Then

$$\sum_{i=1}^k |p_i| < 1, \quad (4)$$

is a sufficient condition for the asymptotic stability of the difference equation

$$y_{n+k} + p_1 y_{n+k-1} + \dots + p_k y_n = 0, \quad n = 0, 1, \dots. \quad (5)$$

Theorem B [23] Let $g : [a, b]^{k+1} \rightarrow [a, b]$, be a continuous function, where k is a positive integer, and where $[a, b]$ is an interval of real numbers. Consider the difference equation

$$x_{n+1} = g(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots. \quad (6)$$

Suppose that g satisfies the following conditions.

(1) For each integer i with $1 \leq i \leq k+1$; the function $g(z_1, z_2, \dots, z_{k+1})$ is weakly monotonic in z_i for fixed $z_1, z_2, \dots, z_{i-1}, z_{i+1}, \dots, z_{k+1}$.

(2) If m, M is a solution of the system

$$m = g(m_1, m_2, \dots, m_{k+1}), \quad M = g(M_1, M_2, \dots, M_{k+1}),$$

then $m = M$, where for each $i = 1, 2, \dots, k+1$, we set

$$m_i = \begin{cases} m, & \text{if } g \text{ is non-decreasing in } z_i, \\ M, & \text{if } g \text{ is non-increasing in } z_i, \end{cases} \quad M_i = \begin{cases} M, & \text{if } g \text{ is non-decreasing in } z_i, \\ m, & \text{if } g \text{ is non-increasing in } z_i. \end{cases}$$

Then there exists exactly one equilibrium point \bar{x} of Equation (6), and every solution of Equation (6) converges to \bar{x} .

3. LOCAL STABILITY OF EQUATION (1)

In this section, we investigate the local stability character of the solutions of Equation (1). Equation (1) has a unique positive equilibrium point and is given by

$$\bar{x} = a\bar{x} + b\bar{x} + \frac{c\bar{x} + d\bar{x}}{e\bar{x} + f\bar{x}}.$$

If $(a + b) < 1$, then the unique positive equilibrium point is

$$\bar{x} = \frac{c + d}{[1 - (a + b)](e + f)}.$$

Let $f : (0, \infty)^4 \longrightarrow (0, \infty)$ be a function defined by

$$f(u_0, u_1, u_2, u_3) = au_0 + bu_1 + \frac{cu_2 + du_3}{eu_2 + fu_3}.$$

Therefore it follows that

$$\begin{aligned} \frac{\partial f(u_0, u_1, u_2, u_3)}{\partial u_0} &= a, & \frac{\partial f(u_0, u_1, u_2, u_3)}{\partial u_1} &= b, \\ \frac{\partial f(u_0, u_1, u_2, u_3)}{\partial u_2} &= \frac{(cf - de)u_3}{(eu_2 + fu_3)^2}, & \frac{\partial f(u_0, u_1, u_2, u_3)}{\partial u_3} &= \frac{(de - cf)u_2}{(eu_2 + fu_3)^2}. \end{aligned}$$

Then, we see that

$$\begin{aligned} \frac{\partial f(\bar{x}, \bar{x}, \bar{x}, \bar{x})}{\partial u_0} &= a, & \frac{\partial f(\bar{x}, \bar{x}, \bar{x}, \bar{x})}{\partial u_1} &= b, \\ \frac{\partial f(\bar{x}, \bar{x}, \bar{x}, \bar{x})}{\partial u_2} &= \frac{(cf - de)[1 - (a + b)]}{(e + f)(c + d)}, & \frac{\partial f(\bar{x}, \bar{x}, \bar{x}, \bar{x})}{\partial u_3} &= \frac{(de - cf)[1 - (a + b)]}{(e + f)(c + d)}. \end{aligned}$$

Then, the linearized equation of Equation (1) about \bar{x} is

$$y_{n+1} + ay_{n-l} + by_{n-k} + \frac{(cf - de)[1 - (a + b)]}{(e + f)(c + d)}y_{n-s} + \frac{(de - cf)[1 - (a + b)]}{(e + f)(c + d)}y_{n-p} = 0. \quad (7)$$

Theorem 1. Assume that

$$2|cf - de| < (e + f)(c + d).$$

Then the equilibrium point of Equation (1) is locally asymptotically stable.

Proof. It follows by Theorem A that Equation (7) is asymptotically stable if

$$|a| + |b| + \left| \frac{(cf - de)[1 - (a + b)]}{(e + f)(c + d)} \right| + \left| \frac{(de - cf)[1 - (a + b)]}{(e + f)(c + d)} \right| < 1,$$

or

$$2 \left| \frac{(cf - de)[1 - (a + b)]}{(e + f)(c + d)} \right| < [1 - (a + b)],$$

and so

$$2|cf - de| < (e + f)(c + d).$$

This completes the proof.

4. GLOBAL ATTRACTIVITY OF THE EQUILIBRIUM POINT OF EQUATION (1)

In this section we deals the global attractivity character of solutions of Equation (1).

Theorem 2. The equilibrium point \bar{x} is a global attractor of equation (1) if one of the following conditions holds:

$$\begin{aligned} (i) \quad cf - de &\geq 0, \quad d \geq c. \\ (ii) \quad de - cf &\geq 0, \quad c \geq d. \end{aligned}$$

Proof. Let r, s be nonnegative real numbers and assume that $f : [r, s]^4 \rightarrow [r, s]$ be a function defined by

$$f(u_0, u_1, u_2, u_3) = au_0 + bu_1 + \frac{cu_2 + du_3}{eu_2 + fu_3}.$$

Then

$$\begin{aligned} \frac{\partial f(u_0, u_1, u_2, u_3)}{\partial u_0} &= a, \quad \frac{\partial f(u_0, u_1, u_2, u_3)}{\partial u_1} = b, \\ \frac{\partial f(u_0, u_1, u_2, u_3)}{\partial u_2} &= \frac{(cf - de)u_3}{(eu_2 + fu_3)^2}, \quad \frac{\partial f(u_0, u_1, u_2, u_3)}{\partial u_3} = \frac{(de - cf)u_2}{(eu_2 + fu_3)^2}. \end{aligned}$$

We consider two cases:

Case1: Assume that $cf - de \geq 0$ is true, then we can easily see that the function $f(u_0, u_1, u_2, u_3)$ is increasing in u_0, u_1, u_2 and decreasing in u_3 . Suppose that (m, M) is a solution of the system

$$M = f(M, M, M, m) \quad \text{and} \quad m = f(m, m, m, M).$$

Then from Equation (1), we see that

$$\begin{aligned} M &= aM + bM + \frac{cM + dm}{eM + fm}, \quad m = am + bm + \frac{cm + dM}{em + fM}, \\ M[1 - (a + b)] &= \frac{cM + dm}{eM + fm}, \quad m[1 - (a + b)] = \frac{cm + dM}{em + fM}, \end{aligned}$$

then

$$\begin{aligned} M^2e[1 - (a + b)] + mMf[1 - (a + b)] &= cM + dm, \\ m^2e[1 - (a + b)] + mMf[1 - (a + b)] &= cm + dM. \end{aligned}$$

Subtracting this two equations, we obtain

$$(M - m) \{e(M + m)[1 - (a + b)] + (d - c)\} = 0,$$

under the condition $(a + b) < 1$, $d \geq c$, we see that $M = m$. It follows from Theorem B that \bar{x} is a global attractor of Equation (1).

Case 2: Similar to Case 1.

5. BOUNDEDNESS OF SOLUTIONS OF EQUATION (1)

In this section we study the boundedness nature of the solutions of Equation (1).

Theorem 3. Every solution of Equation (1) is bounded if $a + b < 1$.

Proof. Let $\{x_n\}_{n=-r}^\infty$ be a solution of Equation (1). It follows from Equation (1) that

$$x_{n+1} = ax_{n-l} + bx_{n-k} + \frac{cx_{n-s} + dx_{n-t}}{ex_{n-s} + fx_{n-t}} = ax_{n-l} + bx_{n-k} + \frac{cx_{n-s}}{ex_{n-s} + fx_{n-t}} + \frac{dx_{n-t}}{ex_{n-s} + fx_{n-t}}.$$

Then

$$x_{n+1} \leq ax_{n-l} + bx_{n-k} + \frac{cx_{n-s}}{ex_{n-s}} + \frac{dx_{n-t}}{fx_{n-t}} = ax_{n-l} + bx_{n-k} + \frac{c}{e} + \frac{d}{f} \text{ for all } n \geq 1.$$

By using a comparison, we can right hand side as follows

$$z_{n+1} = az_{n-l} + bz_{n-k} + \frac{c}{e} + \frac{d}{f}.$$

and this equation is locally asymptotically stable if $a + b < 1$, and converges to the equilibrium point $\bar{z} = \frac{cf+de}{ef[1-(a+b)]}$. Therefore

$$\lim_{n \rightarrow \infty} \sup x_n \leq \frac{cf+de}{ef[1-(a+b)]}.$$

Thus the solution is bounded.

Theorem 4. Every solution of Equation (1) is unbounded if $a > 1$ or $b > 1$.

Proof. Let $\{x_n\}_{n=-r}^{\infty}$ be a solution of Equation (1). Then from Equation (1) we see that

$$x_{n+1} = ax_{n-l} + bx_{n-k} + \frac{cx_{n-s} + dx_{n-p}}{ex_{n-s} + fx_{n-p}} > ax_{n-l} \text{ for all } n \geq 1.$$

We see that the right hand side can be written as follows $z_{n+1} = az_{n-l}$. Then

$$z_{ln+i} = a^n z_{l+i} + \text{const.}, \quad i = 0, 1, \dots, l,$$

and this equation is unstable because $a > 1$, and $\lim_{n \rightarrow \infty} z_n = \infty$. Then by using ratio test $\{x_n\}_{n=-r}^{\infty}$ is unbounded from above. When $b > 1$ is similar.

6. EXISTENCE OF PERIODIC SOLUTIONS

Here we study the existence of periodic solutions of Equation (1). The following theorem states the necessary and sufficient conditions that this equation has periodic solution of prime period two.

Theorem 5. Equation (1) has a prime period two solutions if and only if

- (i) $(d-c)(e-f)(a+b+1) > 4[cf+de(a+b)], \quad l, k, s - \text{even and } t - \text{odd}.$
- (ii) $(c-d)(f-e)(a+b+1) > 4[cf(a+b)+de], \quad l, k, t - \text{even and } s - \text{odd}.$
- (iii) $(c-d)(f-e)(1+a-b) > 4[de(1-b)+caf], \quad l, t - \text{even and } k, s - \text{odd}.$
- (iv) $(d-c)(e-f)(1+a-b) > 4[cf(1-b)+dae], \quad l, s - \text{even and } k, t - \text{odd}.$
- (v) $(c-d)(f-e) > 4de, \quad c > d, \quad f > e, \quad l, k, s - \text{odd, and } t - \text{even}.$
- (vi) $(d-c)(e-f) > 4cf, \quad d > c, \quad e > f, \quad l, k, t - \text{odd and } s - \text{even}.$
- (vii) $(d-c)(e-f)(1-a+b) > 4[cf+dbe-daf], \quad l, t - \text{odd and } k, s - \text{even}.$
- (viii) $(c-d)(f-e)(1-a+b) > 4[cbf-dae+de], \quad l, s - \text{odd and } k, t - \text{even}.$

Proof. We prove first case when l, k and s are even, and t is odd (the other cases are similar and will be left to readers). First suppose that there exists a prime period two solution $\dots p, q, p, q, \dots$, of Equation (1). We will prove that Inequality (i) holds. We see from Equation (1) when l, k, s are even, and t is odd that

$$p = aq + bq + \frac{cq + dp}{eq + fp}, \quad q = ap + bp + \frac{cp + dq}{ep + fq}.$$

Then

$$epq + fp^2 = (a+b)eq^2 + (a+b)fpq + cq + dp, \quad (8)$$

$$epq + fq^2 = (a+b)ep^2 + (a+b)fpq + cp + dq. \quad (9)$$

Subtracting (8) from (9) gives

$$\begin{aligned} f(p^2 - q^2) &= -(a+b)e(p^2 - q^2) - c(p-q) + d(p-q), \\ f(p-q)(p+q) &= -(a+b)e(p-q)(p+q) - c(p-q) + d(p-q), \end{aligned}$$

Since $p \neq q$, it follows that

$$\begin{aligned} f(p+q) &= -(a+b)e(p+q) - c + d, \\ p+q &= \frac{d-c}{f+(a+b)e}. \end{aligned} \quad (10)$$

Again, adding (8) and (9) yields

$$\begin{aligned} 2epq + f(p^2 + q^2) &= (a+b)e(p^2 + q^2) + 2(a+b)fpq + (c+d)(p+q), \\ (p^2 + q^2)[f - (a+b)e] &= (c+d)(p+q) + 2pq[(a+b)f - e], \end{aligned} \quad (11)$$

It follows by (10), (11) and the relation

$$p^2 + q^2 = (p+q)^2 - 2pq \quad \text{for all } p, q \in R,$$

that

$$\begin{aligned} [(p+q)^2 - 2pq][f - (a+b)e] &= (c+d)(p+q) + 2pq[(a+b)f - e], \\ 2pq[(a+b)f - (a+b)e + f - e] &= (p+q)^2[f - (a+b)e] - (c+d)(p+q), \\ pq &= \frac{(d-c)[cf + de(a+b)]}{(a+b+1)(e-f)[f + (a+b)e]^2}. \end{aligned} \quad (12)$$

Now it is clear from Equations (10) and (12) that p and q are the two distinct roots of the quadratic equation

$$\begin{aligned} r^2 - \left(\frac{d-c}{f+(a+b)e} \right) r + \left(\frac{(d-c)[cf + de(a+b)]}{(a+b+1)(e-f)[f + (a+b)e]^2} \right) &= 0, \\ (f + (a+b)e)r^2 - (d-c)r + \left(\frac{(d-c)[cf + de(a+b)]}{(a+b+1)(e-f)[f + (a+b)e]} \right) &= 0, \end{aligned} \quad (13)$$

and so

$$\frac{(d-c)^2}{[f + (a+b)e]^2} > \frac{4(d-c)[cf + de(a+b)]}{(a+b+1)(e-f)[f + (a+b)e]^2}.$$

Thus

$$(d-c)(e-f)(a+b+1) > 4[cf + de(a+b)].$$

Therefore Inequality (i) holds.

Second suppose that Inequality (i) is true. We will show that Equation (1) has a prime period two solution. Assume that

$$p = \frac{d-c+\xi}{2(f+Ae)}, \quad q = \frac{d-c-\xi}{2(f+Ae)},$$

where

$$\xi = \sqrt{(d-c)^2 - \frac{4(d-c)(cf + Ade)}{(A+1)(e-f)}}, \quad \text{and } A = (a+b).$$

We see from Inequality (1) that

$$(d-c)(e-f)(a+b+1) > 4[cf + de(a+b)].$$

which equivalent to

$$\frac{(d-c)^2}{[f + (a+b)e]^2} > 4 \frac{(d-c)[cf + de(a+b)]}{(a+b+1)(e-f)[f + (a+b)e]^2},$$

Therefore p and q are distinct real numbers. Set $x_{-l} = p$, $x_{-k} = p$, $x_{-s} = p$, $x_{-t} = q, \dots, x_{-2} = p$, $x_{-1} = q$, $x_0 = p$. We wish to show that

$$x_1 = x_{-1} = q \text{ and } x_2 = x_0 = p.$$

It follows from Equation (1) that

$$x_1 = A \left[\frac{d-c+\xi}{2(f+ Ae)} \right] + \frac{c \left[\frac{d-c+\xi}{2(f+ Ae)} \right] + d \left[\frac{d-c-\xi}{2(f+ Ae)} \right]}{e \left[\frac{d-c+\xi}{2(f+ Ae)} \right] + f \left[\frac{d-c-\xi}{2(f+ Ae)} \right]},$$

Dividing the denominator and numerator by $2(f+ Ae)$ gives

$$\begin{aligned} x_1 &= A \left[\frac{d-c+\xi}{2(f+ Ae)} \right] + \frac{c[d-c+\xi] + d[d-c-\xi]}{e[d-c+\xi] + f[d-c-\xi]}, \\ &= A \left[\frac{d-c+\xi}{2(f+ Ae)} \right] + \frac{d^2 - c^2 + \xi(c-d)}{e(d-c) + f(d-c) + \xi(e-f)}, \\ &= A \left[\frac{d-c+\xi}{2(f+ Ae)} \right] + \frac{(d-c)(d+c-\xi)}{(d-c)(e+f) + \xi(e-f)}, \end{aligned}$$

Multiplying the denominator and numerator of the right side by $(d-c)(e+f) - \xi(e-f)$ gives

$$\begin{aligned} x_1 &= A \left[\frac{d-c+\xi}{2(f+ Ae)} \right] + \frac{(d-c)(d+c-\xi)[(d-c)(e+f) - \xi(e-f)]}{[(d-c)(e+f) + \xi(e-f)][(d-c)(e+f) - \xi(e-f)]}, \\ &= A \left[\frac{d-c+\xi}{2(f+ Ae)} \right] + \frac{(d-c)(d+c-\xi)[(d-c)(e+f) - \xi(e-f)]}{(d-c)^2(e+f)^2 - \xi^2(e-f)^2}, \\ &= A \left[\frac{d-c+\xi}{2(f+ Ae)} \right] + \frac{(d-c)(d+c-\xi)[de + df - ce - cf - \xi e - \xi f]}{(d-c)^2(e+f)^2 - (e-f)^2 \left[(d-c)^2 - \frac{4(d-c)(cf+Ade)}{(A+1)(e-f)} \right]}, \\ &= A \left[\frac{d-c+\xi}{2(f+ Ae)} \right] + \frac{(d-c)[e(d^2 - c^2) + f(d^2 - c^2) + 2\xi(cf - de) + \xi^2(e-f)]}{(d-c)^2[e^2 + 2ef + f^2 - (e^2 - 2ef + f^2)] + \frac{4(d-c)(e-f)(cf+Ade)}{(A+1)}}, \\ &= A \left[\frac{d-c+\xi}{2(f+ Ae)} \right] + \frac{(d-c)[(d^2 - c^2)(e+f) + 2\xi(cf - de) + \xi^2(e-f)]}{\frac{4ef(A+1)(d-c)^2 + 4(d-c)(e-f)(cf+Ade)}{(A+1)}}, \\ &= A \left[\frac{d-c+\xi}{2(f+ Ae)} \right] + \frac{(d-c)[(d^2 - c^2)(e+f) + 2\xi(cf - de) + \frac{(d-c)(-3Ade - 3cf + Afc - Afd - Aec + ed - ec - fd)}{(A+1)}]}{\frac{4(d-c)[ef(Ad - Ac + d - c)] + [ecf + Ae^2d - cf^2 - Ade f]}{(A+1)}}, \\ &= A \left[\frac{d-c+\xi}{2(f+ Ae)} \right] + \frac{(d-c) \left[\frac{2(d-c)(A-1)(cf-de)}{A+1} + 2\xi(cf - de) \right]}{\left[\frac{4(d-c)[efd - Acef + Ae^2d - cf^2]}{(A+1)} \right]}, \\ &= A \left[\frac{d-c+\xi}{2(f+ Ae)} \right] + \frac{2(d-c)(cf - de) \left[\frac{(d-c)(A-1)}{A+1} + \xi \right]}{\left[\frac{4(d-c)[efd - Acef + Ae^2d - cf^2]}{(A+1)} \right]}, \\ &= A \left[\frac{d-c+\xi}{2(f+ Ae)} \right] + \frac{(cf - de)\{(d-c)(A-1) + \xi(A+1)\}}{2[efd - Acef + Ae^2d - cf^2]}, \\ &= A \left[\frac{d-c+\xi}{2(f+ Ae)} \right] + \frac{(cf - de)\{(d-c)(A-1) + \xi(A+1)\}}{2(f+ Ae)(de - cf)}, \\ &= A \left[\frac{d-c+\xi}{2(f+ Ae)} \right] + \frac{-\{(d-c)(A-1) + \xi(A+1)\}}{2(f+ Ae)}, \\ &= \frac{Ad - Ac + A\xi - Ad + d + Ac - c - A\xi - \xi}{2(f+ Ae)} = \frac{d-c-\xi}{2(f+ Ae)} = q. \end{aligned}$$

Similarly as before we can easily show that $x_2 = p$. Then it follows by induction that $x_{2n} = p$ and $x_{2n+1} = q$ for all $n \geq -1$. Thus Equation (1) has the prime period two solution \dots, p, q, p, q, \dots , where p and q are the distinct roots of the quadratic equation (13) and the proof is complete.

7. NUMERICAL EXAMPLES

For confirming the results of this article, we consider numerical examples which represent different types of solutions to Equation (1).

Example 1. We consider numerical example for the difference equation (1) when we take the constants and the initial conditions as follows: $l = 3, k = 2, s = 1, t = 3, x_{-3} = 5, x_{-2} = -12, x_{-1} = 6, x_0 = 8, a = 0.4, b = 0.3, c = 2, d = 4, e = 6, f = 8$. See Figure 1.

Example 2. See Figure (2) when we take Equation (1) with $l = 1, k = 3, s = 2, t = 3, x_{-3} = 13, x_{-2} = -9, x_{-1} = -7, x_0 = 5, a = 0.6, b = 0.4, c = 3, d = 2, e = 5, f = 8$.

Example 3. Figure (3) shows the behavior of the solution of the difference equation (1) when we put $l = 2, k = 1, s = 3, t = 3, x_{-3} = 15, x_{-2} = 11, x_{-1} = -9, x_0 = 5, a = 0.6, b = 1.4, c = 2, d = 4, e = 6, f = 9$.

Example 4. We assume $l = 2, k = 3, s = 1, t = 2, x_{-3} = 15, x_{-2} = 11, x_{-1} = -9, x_0 = 5, a = 1.5, b = 0.2, c = 2, d = 0, e = 6, f = 7$. See Figure 4.

Example 5. Figure (5) shows the period two solution of Equation (1) when $l = 0, k = 2, s = 2, t = 3, x_{-3} = p, x_{-2} = q, x_{-1} = p, x_0 = q, a = 0.06, b = 0.03, c = 1, d = 5, e = 7, f = 2$, since p and q as in the previous theorem.

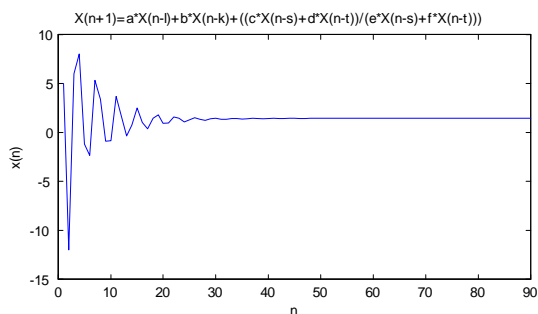


Figure 1.

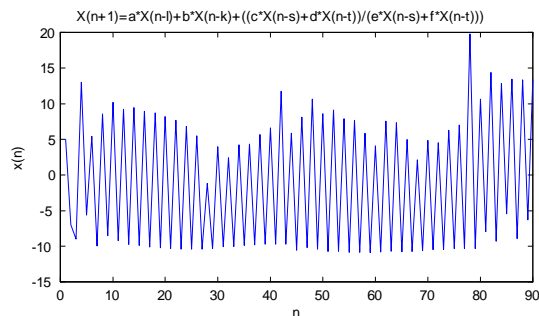


Figure 2.

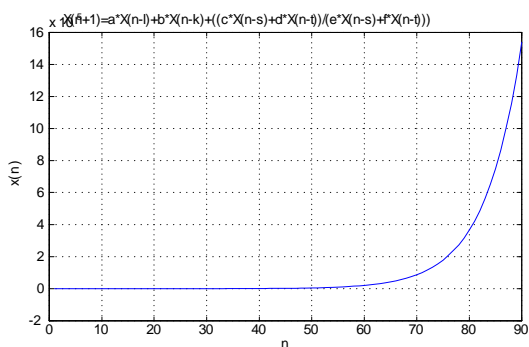


Figure 3.

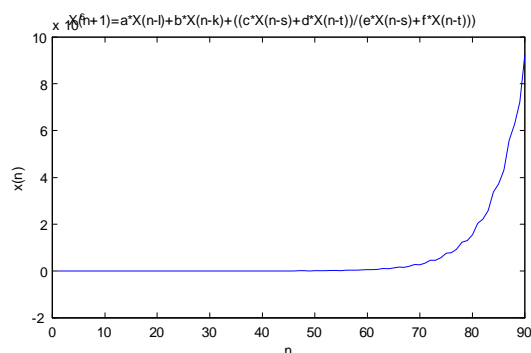


Figure 4.

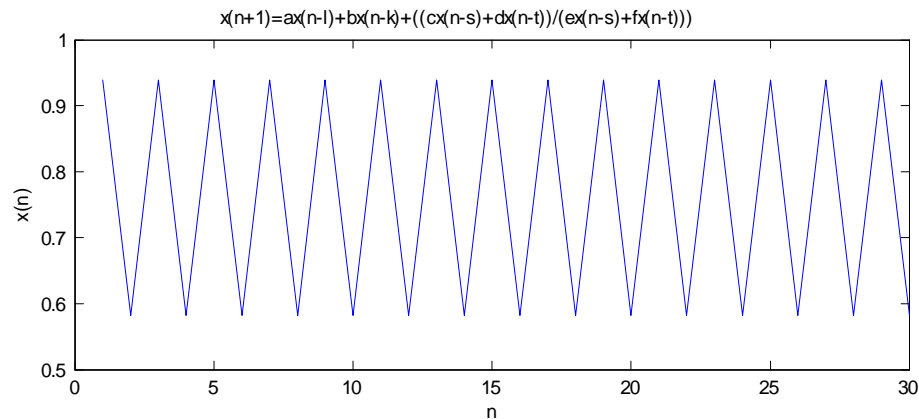


Figure 5.

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A fractional derivative inclusion problem via an integral boundary condition

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Abstract. We investigate the existence of solutions for the fractional differential inclusion ${}^c D^\alpha x(t) \in F(t, x(t))$ equipped with the boundary value problems $x(0) = 0$ and $x(1) = \int_0^\eta x(s)ds$, where $0 < \eta < 1$, $1 < \alpha \leq 2$, ${}^c D^\alpha$ is the standard Caputo differentiation and $F : [0, 1] \times \mathbb{R} \rightarrow 2^{\mathbb{R}}$ is a compact valued multifunction. An illustrative example is also discussed.

Keywords: Fixed point, Fractional differential inclusion, Integral boundary value problem.

1 Introduction

During the last decade the fractional differential equations were investigated from theoretical and applied viewpoints (see for example, [1]-[6], [8]-[15], and [32]). A special attention was given to the real world applications where the power law effect is present and where the fractional models give better results than the classical ones.

We recall that the Riemann-Liouville fractional integral of order $\alpha > 0$ of $f : (0, \infty) \rightarrow \mathbb{R}$ is given by $I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds$ provided the right side is pointwise defined on $(0, \infty)$ (see [26], [29], [31], [34] and [35]). Also, the Caputo fractional derivative of order α of f is defined by ${}^c D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}} ds$, where $n = [\alpha] + 1$ (see [26], [29], [31], [34] and [35]).

We recall that the basic theory for fractional differential inclusions is represented by the fixed point theory of multivalued mappings which was intensively investigated during last years (the reader can find more details in [18]-[25], [30] and the related references). Thus, many papers about ordinary and fractional differential inclusions were written (e.g. [1]-[2],

[7], [16], [17] and [33]).

Let (X, d) be a metric space. Let us denote by $P(X)$ and 2^X the class of all subsets and the class of all nonempty subsets of X respectively. As a result, $P_{cl}(X)$, $P_{bd}(X)$, $P_{cv}(X)$ and $P_{cp}(X)$ denote the class of all closed, bounded, convex and compact subsets of X respectively. A mapping $Q : X \rightarrow 2^X$ is called a multifunction on X and $u \in X$ is called a fixed point of Q whenever $u \in Qu$ ([24]). Also, we say that Q is convex whenever Qx is convex for all $x \in X$ ([24]). A multifunction $G : [0, 1] \rightarrow P_{cl}(\mathbb{R})$ is said to be measurable whenever the function $t \mapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$ is measurable for all $y \in \mathbb{R}$. Put $J = [0, 1]$.

The aim of this manuscript is to investigate the existence of solutions for the fractional differential inclusion

$${}^c D^\alpha x(t) \in F(t, x(t)) \quad (*)$$

via the boundary value problems $x(0) = 0$ and $x(1) = \int_0^\eta x(s)ds$, where ${}^c D^\alpha$ is the standard Caputo differentiation, $0 < \eta < 1$, $1 < \alpha \leq 2$ and $F : J \times \mathbb{R} \rightarrow 2^\mathbb{R}$ is a compact valued multifunction. We say that $F : J \times \mathbb{R} \rightarrow 2^\mathbb{R}$ is a Caratheodory multifunction whenever $t \mapsto F(t, x)$ is measurable for all $x \in \mathbb{R}$ and $x \mapsto F(t, x)$ is upper semi-continuous for almost all $t \in J$. Also, a Caratheodory multifunction $F : J \times \mathbb{R} \rightarrow 2^\mathbb{R}$ is called L^1 -Caratheodory whenever for each $\rho > 0$ there exists $\phi_\rho \in L^1(J, \mathbb{R}^+)$ such that $\|F(t, x)\| = \sup\{|v| : v \in F(t, x)\} \leq \phi_\rho(t)$ for all $\|x\|_\infty \leq \rho$ and for almost all $t \in J$. For each $x \in C(J, \mathbb{R})$, define the set of selections of F by

$$S_{F,x} := \{v \in L^1(J, \mathbb{R}) : v(t) \in F(t, x(t)) \text{ for almost all } t \in J\}.$$

Let E be a nonempty closed subset of a Banach space X and $G : E \rightarrow 2^X$ a multifunction with nonempty closed values. We say that the multifunction G is lower semi-continuous whenever the set $\{y \in E : G(y) \cap B \neq \emptyset\}$ is open for all open set B in X . It has been proved that each completely continuous multifunction is lower semi-continuous (see [24]). We shall use the following fixed point results.

Lemma 1.1. ([30]) *Let X be a Banach space, $F : J \rightarrow P_{cp,cv}(X)$ an L^1 -Caratheodory multifunction and Θ a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$. Then the operator $\Theta \circ S_F : C(J, X) \rightarrow P_{cp,cv}(C(J), X)$ defined by $(\Theta \circ S_F)(x) = \Theta(S_{F,x})$ is a closed graph operator in $C(J, X) \times C(J, X)$.*

It has been proved that if $\dim X < \infty$, then $S_F(x) \neq \emptyset$ for all $x \in C(J, X)$ ([30]).

Lemma 1.2. ([24]) *Let E be a Banach space, C a closed convex subset of E , U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \rightarrow P_{cp,cv}(C)$ is a upper semi-continuous compact map, where $P_{cp,cv}(C)$ denotes the family of nonempty, compact convex subsets of C . Then either F has a fixed point in \overline{U} or there exist $u \in \partial U$ and $\lambda \in (0, 1)$ such that $u \in \lambda F(u)$.*

Let $(X, \|\cdot\|)$ be a normed space. Define the Hausdorff metric $H_d : 2^X \times 2^X \rightarrow [0, \infty]$ by

$$H_d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\},$$

where $d(A, b) = \inf_{a \in A} d(a; b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space ([27]). A multifunction $N : X \rightarrow P_{cl}(X)$ is called a contraction whenever there exists $\gamma > 0$ such that $H_d(N(x), N(y)) \leq \gamma d(x, y)$ for all $x, y \in X$.

Lemma 1.3. ([19]) *Let (X, d) be a complete metric space. If $N : X \rightarrow P_{cl}(X)$ is a contraction, then N has a fixed point.*

Lemma 1.4. [11] *Let $0 < \eta < 1$. Then x is a solution for the differential equation ${}^c D^\alpha x(t) = v(t)$ ($t \in J$ and $1 < \alpha \leq 2$) via the boundary value conditions $x(0) = 0$ and $x(1) = \int_0^\eta x(s)ds$ if and only if x is a solution of the integral equation*

$$\begin{aligned} x(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s) ds \\ & + \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta \left(\int_0^s (s-m)^{\alpha-1} v(m) dm \right) ds \quad (t \in J). \end{aligned}$$

2 Main results

Here, we give our results about the existence of solutions for the inclusion problem (*).

Theorem 2.1. *Suppose that $F : J \times \mathbb{R} \rightarrow 2^\mathbb{R}$ is a Caratheodory multifunction with compact and convex values and there exist a bounded continuous non-decreasing map $\psi : [0, \infty) \rightarrow (0, \infty)$ and a continuous function $p : J \rightarrow (0, \infty)$ such that*

$$\|F(t, x(t))\| = \sup\{|v| : v \in F(t, x(t))\} \leq p(t)\psi(\|x\|_\infty)$$

for all $t \in J$ and $x \in C(J, \mathbb{R})$. Then the problem (*) has at least one solution.

Proof. By using Lemma 1.4, we know that the existence of solution for the problem (*) is equivalent to the existence of solution for the integral equation

$$\begin{aligned} x(t) \in & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s) ds \\ & + \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta \left(\int_0^s (s-m)^{\alpha-1} v(m) dm \right) ds \quad (t \in J). \end{aligned}$$

Put $E = C(J, \mathbb{R})$. Define the operator $N : E \rightarrow 2^E$ by

$$\begin{aligned} N(x) = \{h \in E : h(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s) ds \\ & + \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta \left(\int_0^s (s-m)^{\alpha-1} v(m) dm \right) ds, \text{ for some } v \in S_{F,x}\}. \end{aligned}$$

We show that the operator N satisfies the assumptions of Lemma 1.2. First, we show that $N(x)$ is convex for all $x \in C(J, \mathbb{R})$. Let $h_1, h_2 \in N(x)$. Choose $v_1, v_2 \in S_{F,x}$ such that

$$\begin{aligned} h_i(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_i(s) ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v_i(s) ds \\ & + \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta \left(\int_0^s (s-m)^{\alpha-1} v_i(m) dm \right) ds \end{aligned}$$

for all $t \in J$ and $i = 1, 2$. Let $0 \leq w \leq 1$. Then, we have

$$\begin{aligned} [wh_1 + (1-w)h_2](t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [wv_1(s) + (1-w)v_2(s)] ds \\ & - \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} [wv_1(s) + (1-w)v_2(s)] ds \\ & + \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta \left(\int_0^s (s-m)^{\alpha-1} [wv_1(m) + (1-w)v_2(m)] dm \right) ds \end{aligned}$$

for all $t \in J$. Since $S_{F,x}$ is convex (because F has convex values), $wh_1 + (1-w)h_2 \in N(x)$. Now, we show that $N(x)$ maps bounded sets of $C(J, \mathbb{R})$ into bounded sets. Let $r > 0$ and $B_r = \{x \in C(J, \mathbb{R}) : \|x\|_\infty \leq r\}$. For each $h \in N(x)$ and $x \in B_r$ choose $v \in S_{F,x}$ such that

$$\begin{aligned} h(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s) ds \\ & + \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta \left(\int_0^s (s-m)^{\alpha-1} v(m) dm \right) ds \end{aligned}$$

and

$$\begin{aligned} |h(t)| \leq & \sup_{t \in J} \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s) ds \right. \\ & \left. + \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta \left(\int_0^s (s-m)^{\alpha-1} v(m) dm \right) ds \right| \\ \leq & \sup_{t \in J} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |v(s)| ds + \sup_{t \in [0,1]} \left| \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \right| \int_0^1 (1-s)^{\alpha-1} |v(s)| ds \\ & + \sup_{t \in J} \left| \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \right| \int_0^\eta \left(\int_0^s (s-m)^{\alpha-1} |v(m)| dm \right) ds \leq \|p\|_\infty \psi(\|x\|_\infty) A \end{aligned}$$

for all $t \in J$, where $\|p\|_\infty = \sup_{t \in J} p(t)$ and $A = \frac{(\alpha+1)(2-\eta^2)+2(\alpha+1)+2\eta^{\alpha+1}}{(2-\eta^2)\Gamma(\alpha+2)}$. Thus,

$$\|h(t)\|_\infty = \sup_{t \in J} |h(t)| \leq A\|p\|_\infty \psi(\|x\|_\infty).$$

Now, we show that N maps bounded sets into equi-continuous sets of $C(J, \mathbb{R})$. Let $t_1, t_2 \in J$ with $t_1 < t_2$ and $x \in B_r$. Then,

$$\begin{aligned} |h(t_2) - h(t_1)| = & \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} v(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} v(s) ds \right. \\ & - \frac{2t_2}{(2 - \eta^2)\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} v(s) ds + \frac{2t_1}{(2 - \eta^2)\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} v(s) ds \\ & + \frac{2t_2}{(2 - \eta^2)\Gamma(\alpha)} \int_0^\eta \left(\int_0^s (s - m)^{\alpha-1} v(m) dm \right) ds \\ & \left. - \frac{2t_1}{(2 - \eta^2)\Gamma(\alpha)} \int_0^\eta \left(\int_0^s (s - m)^{\alpha-1} v(m) dm \right) ds \right| \\ \leq & \|p\|_\infty \psi(\|x\|_\infty) \left[\frac{(2 - \eta^2)(t_2^\alpha - t_1^\alpha) + 2(t_1 - t_2)}{(2 - \eta^2)\Gamma(\alpha + 1)} + \frac{2(t_2 - t_1)\eta^{\alpha+1}}{(2 - \eta^2)\Gamma(\alpha + 2)} \right] \end{aligned}$$

For all $h \in N(x)$. Thus, $\lim_{t_2 \rightarrow t_1} |h(t_2) - h(t_1)| = 0$ for all $x \in B_r$. Hence by using the Arzela-Ascoli theorem, N is completely continuous. Here, we show that N has a closed graph. Let $x_n \rightarrow x_0$, $h_n \in N(x_n)$ for all n and $h_n \rightarrow h_0$. We have to show that $h_0 \in N(x_0)$. For each n choose $v_n \in S_{F, x_n}$ such that

$$\begin{aligned} h_n(t) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} v_n(s) ds - \frac{2t}{(2 - \eta^2)\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} v_n(s) ds \\ & + \frac{2t}{(2 - \eta^2)\Gamma(\alpha)} \int_0^\eta \left(\int_0^s (s - m)^{\alpha-1} v_n(m) dm \right) ds \end{aligned}$$

for all $t \in J$. Define the continuous linear operator $\theta : L^1(J, \mathbb{R}) \rightarrow C(J, \mathbb{R})$ by

$$\begin{aligned} \theta(v) = & \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} v(s) ds - \frac{2t}{(2 - \eta^2)\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} v(s) ds \\ & + \frac{2t}{(2 - \eta^2)\Gamma(\alpha)} \int_0^\eta \left(\int_0^s (s - m)^{\alpha-1} v(m) dm \right) ds. \end{aligned}$$

Note that,

$$\begin{aligned} \|h_n(t) - h_0(t)\| = & \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} (v_n(s) - v_0(s)) ds \right. \\ & - \frac{2t}{(2 - \eta^2)\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} (v_n(s) - v_0(s)) ds \\ & \left. + \frac{2t}{(2 - \eta^2)\Gamma(\alpha)} \int_0^\eta \left(\int_0^s (s - m)^{\alpha-1} (v_n(m) - v_0(m)) dm \right) ds \right\| \end{aligned}$$

for all n and so $\lim_{n \rightarrow \infty} \|h_n(t) - h_0(t)\| = 0$. By using Lemma 1.1, $\theta o S_F$ is a closed graph operator. Since $h_n(t) \in \theta(S_{F,x_n})$ for all n and $x_n \rightarrow x_0$, there exists $v_0 \in S_{F,x_0}$ such that

$$h_0(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_0(s) ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v_0(s) ds \\ + \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta \left(\int_0^s (s-m)^{\alpha-1} v_0(m) dm \right) ds.$$

Thus, N has a closed graph. If there exists $\lambda \in (0, 1)$ such that $x \in \lambda N(x)$, then there exists $v \in S_{F,x}$ such that

$$x(t) = \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds - \frac{2\lambda t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s) ds \\ + \frac{2\lambda t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta \left(\int_0^s (s-m)^{\alpha-1} v(m) dm \right) ds$$

for all $t \in J$. Now, choose $M > 0$ such that $\frac{\|p\|_\infty \psi(\|x\|_\infty) ([\alpha+1](2-\eta^2) + 2(\alpha+1) + 2\eta^{\alpha+1})}{(2-\eta^2)\Gamma(\alpha+2)} < M$ for all $x \in E$. This is possible because ψ is bounded. Thus,

$$\|x\|_\infty \leq \frac{\|p\|_\infty \psi(\|x\|_\infty) ([\alpha+1](2-\eta^2) + 2(\alpha+1) + 2\eta^{\alpha+1})}{(2-\eta^2)\Gamma(\alpha+2)} < M.$$

Now, put $U = \{x \in C(J, \mathbb{R}) : \|x\|_\infty < M+1\}$. Thus, there are not $x \in \partial U$ and $\lambda \in (0, 1)$ such that $x \in \lambda N(x)$. Note that, the operator $N : \overline{U} \rightarrow P_{cp,cv}(\overline{U})$ is upper semi-continuous because it is completely continuous. Now by using Lemma 1.2, N has a fixed point in \overline{U} which is a solution of the problem (*). This completes the proof. \square

Now, we present our next result about the existence of solutions for the problem (*) with non-convex valued assumption.

Theorem 2.2. *Let $m \in C(J, \mathbb{R}^+)$ be such that $\|m\|_\infty \left(\frac{4-\eta^2}{(2-\eta^2)\Gamma(\alpha+1)} + \frac{2\eta^{\alpha+1}}{(2-\eta^2)\Gamma(\alpha+2)} \right) < 1$. Suppose that $F : J \times \mathbb{R} \rightarrow P_{cp}(\mathbb{R})$ is a multifunction such that $H_d(F(t, x), F(t, y)) \leq m(t)|x - y|$ and $d(x, F(t, x)) \leq m(t)$ for almost all $t \in J$ and $x, y \in \mathbb{R}$. Then the boundary value inclusion problem (*) has a solution.*

Proof. Note that, $S_{F,x}$ is nonempty for all $x \in C(J, \mathbb{R})$. By using Theorem III.6 in [18], we get F has a measurable selection. Now, similar to the proof of Theorem 2.1, consider the operator $N : E \rightarrow 2^E$ by

$$N(x) = \{h \in E : h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s) ds \\ + \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta \left(\int_0^s (s-m)^{\alpha-1} v(m) dm \right) ds, \text{ for some } v \in S_{F,x}\},$$

where $E = C(J, \mathbb{R})$. First, we show that $N(x)$ is a closed subset of E for all $x \in E$. Let $x \in E$ and $\{u_n\}_{n \geq 1}$ be a sequence in $N(x)$ with $u_n \rightarrow u$. For each n , choose $v_n \in S_{F,x}$ such that

$$u_n(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_n(s) ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v_n(s) ds \\ + \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta \left(\int_0^s (s-m)^{\alpha-1} v_n(m) dm \right) ds$$

for all $t \in J$. Since F has compact values, $\{v_n\}_{n \geq 1}$ has a subsequence which converges to some $v \in L^1(J, \mathbb{R})$. We denote this subsequence again by $\{v_n\}_{n \geq 1}$. It is easy to check that $v \in S_{F,x}$ and

$$u_n(t) \rightarrow u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s) ds \\ + \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta \left(\int_0^s (s-m)^{\alpha-1} v(m) dm \right) ds$$

for all $t \in J$. This implies that $u \in N(x)$ and so the multifunction N has closed values. Now, we show that N is a contractive multifunction with constant

$$\gamma = \|m\|_\infty \left(\frac{4-\eta^2}{(2-\eta^2)\Gamma(\alpha+1)} + \frac{2\eta^{\alpha+1}}{(2-\eta^2)\Gamma(\alpha+2)} \right) < 1.$$

Let $x, y \in E$ and $h_1 \in N(x)$. Choose $v_1 \in S_{F,x}$ such that

$$h_1(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_1(s) ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v_1(s) ds \\ + \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta \left(\int_0^s (s-m)^{\alpha-1} v_1(m) dm \right) ds$$

for all $t \in J$. Since $H_d(F(t, x), F(t, y)) \leq m(t)|x(t) - y(t)|$ for almost all $t \in J$, there exists $w_0 \in F(t, y(t))$ such that $|v_1 - w_0| \leq m(t)|x(t) - y(t)|$ for almost all $t \in [0, 1]$. Define the multifunction $U : J \rightarrow 2^{\mathbb{R}}$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \leq m(t)|x(t) - y(t)|\} \text{ for almost all } t \in J\}.$$

By using Proposition III.4 in [18], we get the multifunction $U(t) \cap F(t, y(t))$ is measurable. It is easy to see that there exists $v_2 \in S_{F,y}$ such that $|v_1(t) - v_2(t)| \leq m(t)|x(t) - y(t)|$ For all $t \in J$. Now, define

$$h_2(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_2(s) ds - \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v_2(s) ds \\ + \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \int_0^\eta \left(\int_0^s (s-m)^{\alpha-1} v_2(m) dm \right) ds$$

for all $t \in J$. Thus,

$$\begin{aligned} |h_1(t) - h_2(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |v_1(s) - v_2(s)| ds \\ &\quad + \left| \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \right| \int_0^1 (1-s)^{\alpha-1} |v_1(s) - v_2(s)| ds \\ &\quad + \left| \frac{2t}{(2-\eta^2)\Gamma(\alpha)} \right| \int_0^\eta \left(\int_0^s (s-m)^{\alpha-1} |v_1(m) - v_2(m)| dm \right) ds \\ &\leq \|m\|_\infty \left(\frac{4-\eta^2}{(2-\eta^2)\Gamma(\alpha+1)} + \frac{2\eta^{\alpha+1}}{(2-\eta^2)\Gamma(\alpha+2)} \right) \|x-y\|_\infty = \gamma \|x-y\|_\infty. \end{aligned}$$

Therefore, the multifunction N is a contraction with closed values. By using Lemma 1.3, N has a fixed point which is a solution of the inclusion problem (*). \square

3 Application

Consider the problem

$${}^c D^{3/2} x(t) \in F(t, x(t)) \quad (t \in [0, 1])$$

via the boundary value conditions $x(0) = 0$ and $x(1) = \int_0^{3/4} x(s) ds$, where $F : [0, 1] \times \mathbb{R} \rightarrow 2^{\mathbb{R}}$ is the multifunction defined by

$$F(t, x) = \left[\frac{x^5}{4(x^5+3)} + \frac{t+1}{8}, \frac{1}{4} \sin x + \frac{1}{4}(t+1) \right].$$

Since $\max \left[\frac{x^5}{4(x^5+3)} + \frac{t+1}{8}, \frac{1}{4} \sin x + \frac{1}{4}(t+1) \right] \leq \frac{3}{4}$, it is easy to check that

$$\sup \{ |\gamma| : \gamma \in F(t, x) \} \leq p(t) \psi(\|x\|_\infty)$$

for all $x \in C([0, 1], \mathbb{R})$, where $p(t) = 1$ and $\psi(t) = \frac{3}{4}$ for all $t \in [0, 1]$. Thus by using Theorem 2.1, this inclusion problem has at least one solution.

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Stability and hyperstability of generalized orthogonally quadratic ternary homomorphisms in non-Archimedean ternary Banach algebras: a fixed point approach

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Abstract: Using the fixed point method, we prove the stability and the hyperstability of generalized orthogonally quadratic ternary homomorphisms in non-Archimedean ternary Banach algebras.

1. Introduction and preliminaries

The stability problem of functional equations had been first raised by Ulam [29]. This problem solved by Hyers [16] in the framework of Banach spaces. For more details about the result concerning such problems, we refer the reader to ([1, 3, 11, 17, 22, 25, 26, 27, 28, 31, 32]). The stability of homomorphisms and derivations in Banach algebras, Banach ternary algebras, C^* -algebras, Lie C^* -algebras, C^* -ternary algebras has been studied by many authors (see [9, 25, 26, 27, 28]).

Let \mathcal{A}, \mathcal{B} be two ternary algebras. A mapping $f : \mathcal{A} \rightarrow \mathcal{B}$ is called a quadratic ternary homomorphism if f is a quadratic mapping (i.e. $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in \mathcal{A}$) and satisfies

$$f([a, b, c]) = [f(a), f(b), f(c)]$$

for all $a, b, c \in \mathcal{A}$.

A mapping $g : \mathcal{A} \rightarrow \mathcal{B}$ is called a generalized quadratic ternary homomorphism if there exists a quadratic ternary homomorphism $f : \mathcal{A} \rightarrow \mathcal{B}$ such that

$$g([a, b, c]) = [g(a), f(b), f(c)]$$

for all $a, b, c \in \mathcal{A}$.

In 2003, Cădariu and Radu applied the fixed point methods to the investigation of Jensen functional equations [4] (see also [5, 6, 12, 21, 24]).

Arriola and Beyer [2] initiated the stability of functional equations in non-Archimedean spaces. In fact they established the stability of the Cauchy functional equation over p -adic fields. After their results some papers (see, for instance, ([7, 8, 9, 10]) on the stability of other equations in such spaces have been published.

In 1897, Hensel [15] discovered the p -adic numbers as a number theoretical analogue of power series in complex analysis. During the last three decades p -adic numbers have gained the interest in of physicists for their research, in particular, in the problems coming from quantum physics, p -adic strings and hyperstrings [18, 19]. A key property of p -adic numbers is that they do not satisfy the Archimedean axiom: For any $x, y > 0$, there exists $n \in \mathbb{N}$ such that $x < ny$ (see [13, 30]).

Let \mathbb{K} denote a field and function (valuation absolute) $|\cdot|$ from \mathbb{K} into $[0, \infty)$. A non-Archimedean valuation is a function $|\cdot|$ that satisfies the strong triangle inequality; namely $|x+y| \leq \max\{|x|, |y|\} \leq |x| + |y|$ for all $x, y \in \mathbb{K}$. The associated field \mathbb{K} is referred to as a non-Archimedean field. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \geq 1$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except 0 into 1 and $|0| = 0$. We always assume in addition that $|\cdot|$ is non trivial, i.e., there is a $z \in \mathbb{K}$ such that $|z| \neq 0, 1$.

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Let X be a linear space over a field \mathbb{K} with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \rightarrow [0, \infty)$ is said to be a non-Archimedean norm if it is a norm over \mathbb{K} with the strong triangle inequality (ultrametric); namely, $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in \mathbb{K}$. Then $(X, \|\cdot\|)$ is called a non-Archimedean space. In any such a space a sequence $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence if and only if $\{x_{n+1}, x_n\}_{n \in \mathbb{N}}$ converges to zero. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent. A non-Archimedean ternary Banach algebra is a complete non-Archimedean space \mathcal{A} equipped with a ternary product $(x, y, z) \rightarrow [x, y, z]$ of \mathcal{A}^3 into \mathcal{A} which is \mathcal{K} -linear in each variables and associative in the sense that

$$[x, y, [z, w, v]] = [x, [w, z, y], v] = [[x, y, z], w, v]$$

and satisfies the following:

$$\|[x, y, z]\| \leq \|x\| \cdot \|y\| \cdot \|z\|$$

(see [14]).

Let X be a nonempty set and let $d : X \times X \rightarrow [0, \infty]$ satisfy: $d(x, y) = 0$ if and only if $x = y$, $d(x, y) = d(y, x)$ and $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ (strong triangle inequality), for all $x, y, z \in X$. Then (X, d) is called a non-Archimedean generalized metric space. (X, d) is called complete if every d -Cauchy sequence in X is d -convergent.

Suppose that X is a real vector space (or an algebra) with $\dim X \geq 2$ and \perp is a binary relation on X with the following properties:

- (O₁) totality of \perp for zero: $x \perp 0, 0 \perp x$ for all $x \in X$;
- (O₂) independence: if $x, y \in X - \{0\}$, $x \perp y$, then x, y are linearly independent;
- (O₃) homogeneity: if $x, y \in X$, $x \perp y$, then $\alpha x \perp \beta y$ for all $\alpha, \beta \in \mathbb{R}$;
- (O₄) the Thalesian property: if P is a 2-dimensional subspace (subalgebra) of X , $x \in P$ and $\lambda \in \mathbb{R}_+$, then there exists $u_x \in P$ such that $x \perp u_x$ and $x + u_x \perp \lambda x - u_x$.

The pair (X, \perp) is called an orthogonality space (algebra). By an orthogonality normed space (normed algebra) we mean an orthogonality space (algebra) having a normed structure (see [23]).

2. Main results

Using the strong triangle inequality in the proof of the main result of [20], we get to the following result:

Theorem 2.1. (Non-Archimedean Alternative Contraction Principle) Suppose that (Ω, d) is a non-Archimedean generalized complete metric space and $T : \Omega \rightarrow \Omega$ is a strictly contractive mapping with the Lipschitz constant L . Let $x \in \Omega$. If either

- (i) $d(T^m(x), T^{m+1}(x)) = \infty$ for all $m \geq 0$, or
- (ii) there exists some m_0 such that $d(T^m(x), T^{m+1}(x)) < \infty$ for all $m \geq m_0$, then the sequence $\{T^m(x)\}$ is convergent to a fixed point x^* of T ; x^* is the unique fixed point of T in the set

$$\Lambda = \{y \in \Omega : d(T^{m_0}(x), y) < \infty\};$$

and $d(y, x^*) \leq d(y, T(y))$ for all y in this set.

In this section, we suppose that \mathcal{A} is a non-Archimedean ternary Banach algebra with $\perp := \bigcup \{(x, \alpha x) : x \in \mathcal{A}, \alpha \in \mathbb{R}\}$, where $\perp \cup$ is an orthogonality on \mathcal{A} , and \mathcal{B} is a non-Archimedean ternary Banach algebra and $l \in \{1, -1\}$ is fixed. Also, let $|4| < 1$ and we assume that $4 \neq 0$ in \mathbb{K} (i.e., the characteristic of \mathbb{K} is not 4).

Theorem 2.2. Let $g, f : \mathcal{A} \rightarrow \mathcal{B}$ be two mappings with $g(0) = f(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^8 \rightarrow [0, \infty]$ such that

$$\begin{aligned} & \|\eta(ax + by) + \eta(ax - by) - 2a^2\eta(x) - 2b^2\eta(y)\| + \|f([u, v, w]) - [f(u), f(v), f(w)]\| \\ & + \|g([r, s, t]) - [g(r), f(s), f(t)]\| \leq \varphi(x, y, u, v, w, r, s, t) \end{aligned} \quad (2.1)$$

for all $\eta \in \{f, g\}$, $x, y \in \mathcal{A}$ with $x \perp y$ and for all $u, v, w, r, s, t \in \mathcal{A}$, that are mutually orthogonal and nonzero fixed integers a, b . Suppose that there exists $L < 1$ such that

$$\varphi(x, y, u, v, w, r, s, t) \leq |4|^{l(l+2)} L \varphi\left(\frac{x}{2^l}, \frac{y}{2^l}, \frac{u}{2^l}, \frac{v}{2^l}, \frac{w}{2^l}, \frac{r}{2^l}, \frac{s}{2^l}, \frac{t}{2^l}\right) \quad (2.2)$$

for all $x, y \in \mathcal{A}$ with $x \perp y$ and for all $u, v, w, r, s, t \in \mathcal{A}$, that are mutually orthogonal. Then there exist a unique orthogonally quadratic ternary homomorphism $h : \mathcal{A} \rightarrow \mathcal{B}$ and a unique generalized orthogonally quadratic ternary homomorphism $H : \mathcal{A} \rightarrow \mathcal{B}$ (respect to h) such that

$$\max\{\|g(x) - H(x)\|, \|f(x) - h(x)\|\} \leq \frac{L^{\frac{1-l}{2}}}{|4|} \psi(x) \quad (2.3)$$

for all $x \in \mathcal{A}$, where

$$\begin{aligned} \psi(x) := & \max\{\varphi(\frac{x}{a}, \frac{x}{b}, 0, 0, 0, 0, 0, 0), \varphi(\frac{x}{a}, 0, 0, 0, 0, 0, 0, 0), \frac{1}{|2b^2|} \varphi(x, x, 0, 0, 0, 0, 0, 0), \\ & \frac{1}{|2b^2|} \varphi(x, -x, 0, 0, 0, 0, 0, 0), \varphi(0, \frac{x}{b}, 0, 0, 0, 0, 0, 0)\}. \end{aligned}$$

for all $x \in \mathcal{A}$.

Proof. By (2.2), one can show that

$$\lim_{n \rightarrow \infty} \frac{1}{|4|^{l(l+2)n}} \varphi(2^{ln}x, 2^{ln}y, 2^{ln}u, 2^{ln}v, 2^{ln}w, 2^{ln}r, 2^{ln}s, 2^{ln}t) = 0 \quad (2.4)$$

for all $x, y \in \mathcal{A}$ with $x \perp y$ and for all $u, v, w, r, s, t \in \mathcal{A}$, that are mutually orthogonal. Putting $\eta = g$ in (2.1) and $u = v = w = r = s = t = 0$ in (2.1), we get

$$\|g(ax + by) + g(ax - by) - 2a^2g(x) - 2b^2g(y)\| \leq \varphi(x, y, 0, 0, 0, 0, 0, 0) \quad (2.5)$$

for all $x, y \in \mathcal{A}$ with $x \perp y$. Putting $y = 0$ in (2.5). Since $x \perp 0$, we get

$$\|2g(ax) - 2a^2g(x)\| \leq \varphi(x, 0, 0, 0, 0, 0, 0, 0) \quad (2.6)$$

for all $x \in \mathcal{A}$. Setting $y = -y$ in (2.5), by the definition of \perp , we get

$$\|g(ax - by) + g(ax + by) - 2a^2g(x) - 2b^2g(-y)\| \leq \varphi(x, -y, 0, 0, 0, 0, 0, 0) \quad (2.7)$$

for all $x, y \in \mathcal{A}$ with $x \perp y$. It follows from (2.5) and (2.7) that

$$\|2b^2g(y) - 2b^2g(-y)\| \leq \max\{\varphi(x, y, 0, 0, 0, 0, 0, 0), \varphi(x, -y, 0, 0, 0, 0, 0, 0)\} \quad (2.8)$$

for all $x, y \in \mathcal{A}$ with $x \perp y$. Putting $y = by$ in (2.8), by the definition of \perp , we get

$$\|g(by) - g(-by)\| \leq \max\{\frac{1}{|2b^2|} \varphi(x, by, 0, 0, 0, 0, 0, 0), \frac{1}{|2b^2|} \varphi(x, -by, 0, 0, 0, 0, 0, 0)\} \quad (2.9)$$

for all $x, y \in \mathcal{A}$ with $x \perp y$. Let $x = 0$ in (2.5). Since $0 \perp x$, we get

$$\|g(by) + g(-by) - 2b^2g(y)\| \leq \varphi(0, y, 0, 0, 0, 0, 0, 0) \quad (2.10)$$

for all $y \in \mathcal{A}$. It follows from (2.9) and (2.10) that

$$\|2g(by) - 2b^2g(y)\| \leq \max\{\frac{1}{|2b^2|} \varphi(x, by, 0, 0, 0, 0, 0, 0), \frac{1}{|2b^2|} \varphi(x, -by, 0, 0, 0, 0, 0, 0), \varphi(0, y, 0, 0, 0, 0, 0, 0)\} \quad (2.11)$$

for all $x, y \in \mathcal{A}$ with $x \perp y$. Replacing x and y by $\frac{x}{a}$ and $\frac{y}{b}$ in (2.5), respectively, and by the definition of \perp , we get

$$\|g(2x) - 2a^2g(\frac{x}{a}) - 2b^2g(\frac{x}{b})\| \leq \varphi(\frac{x}{a}, \frac{x}{b}, 0, 0, 0, 0, 0, 0) \quad (2.12)$$

for all $x \in \mathcal{A}$. Setting $x = \frac{x}{a}$ in (2.6), by the definition of \perp , we get

$$\|2a^2g(\frac{x}{a}) - 2g(x)\| \leq \varphi(\frac{x}{a}, 0, 0, 0, 0, 0, 0, 0) \quad (2.13)$$

for all $x \in \mathcal{A}$. Putting $y = \frac{x}{b}$ in (2.9), by the definition of \perp , we get

$$\|2b^2g(\frac{x}{b}) - 2g(x)\| \leq \max\{\frac{1}{|2b^2|} \varphi(x, x, 0, 0, 0, 0, 0, 0), \frac{1}{|2b^2|} \varphi(x, -x, 0, 0, 0, 0, 0, 0), \varphi(0, \frac{x}{b}, 0, 0, 0, 0, 0, 0)\} \quad (2.14)$$

for all $x \in \mathcal{A}$. It follows from (2.12), (2.13) and (2.14) that

$$\|g(2x) - 4g(x)\| \leq \psi(x) \quad (2.15)$$

for all $x \in \mathcal{A}$. Consider the set

$$X := \{\acute{g} : \acute{g} : \mathcal{A} \rightarrow \mathcal{B} \mid \acute{g}(0) = 0\}.$$

For every $\dot{g}, \dot{h} \in X$, define

$$d(\dot{g}, \dot{h}) := \inf\{K \in (0, \infty) : \|\dot{g}(x) - \dot{h}(x)\| \leq K\psi(x), \forall x \in \mathcal{A}\}.$$

It is easy to show that (X, d) is a complete generalized non-Archimedean metric space. Now, we consider the $\mathcal{J} : X \rightarrow X$ such that

$$\mathcal{J}(\dot{g})(x) := \frac{1}{4^l} \dot{g}(2^l x)$$

for all $x \in \mathcal{A}$. For any $\dot{g}, \dot{h} \in X$, it follows that for all $x \in \mathcal{A}$

$$\begin{aligned} d(\dot{g}, \dot{h}) < K &\Rightarrow \|\dot{g}(x) - \dot{h}(x)\| \leq K\psi(x) \\ &\Rightarrow \left\| \frac{\dot{g}(2^l x)}{4^l} - \frac{\dot{h}(2^l x)}{4^l} \right\| \leq K \frac{\psi(2^l x)}{|4|^l} \\ &\Rightarrow \|\mathcal{J}\dot{g}(x) - \mathcal{J}\dot{h}(x)\| \leq LK\psi(x). \end{aligned}$$

Hence we have

$$d(\mathcal{J}(\dot{g}), \mathcal{J}(\dot{h})) \leq Ld(\dot{g}, \dot{h}).$$

By applying the inequality (2.15), we see that $d(\mathcal{J}(f), f) \leq \frac{L \cdot \frac{1-l}{2}}{|4|}$. It follows from Theorem 2.1 that \mathcal{J} has a unique fixed point $H : \mathcal{A} \rightarrow \mathcal{B}$ in the set $\Lambda : \{\dot{g} \in X : d(\dot{g}, g) < \infty\}$, where H is defined by

$$H(x) = \lim_{n \rightarrow \infty} \mathcal{J}^n g(x) = \lim_{n \rightarrow \infty} \frac{1}{4^{ln}} g(2^{ln} x) \quad (2.16)$$

for all $x \in \mathcal{A}$. It follows from (2.4), (2.5) and (2.16) that

$$\begin{aligned} &\|H(ax + by) + H(ax - by) - 2a^2 H(x) - 2b^2 H(y)\| \\ &= \lim_{n \rightarrow \infty} \frac{1}{|4|^{ln}} \|g(2^{ln} ax + 2^{ln} by) + g(2^{ln} ax - 2^{ln} by) - 2a^2 g(2^{ln} x) - 2b^2 g(2^{ln} y)\| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{|4|^{ln}} \varphi(2^{ln} x, 2^{ln} y, 0, 0, 0, 0, 0) \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{|4|^{ln(l+2)}} \varphi(2^{ln} x, 2^{ln} y, 0, 0, 0, 0, 0) = 0 \end{aligned}$$

for all $x, y \in \mathcal{A}$ with $x \perp y$. This shows that H is an orthogonally quadratic.

Putting $\eta = f$, $u = v = w = r = s = t = 0$ in (2.1), we get

$$\|f(ax + by) + f(ax - by) - 2a^2 f(x) - 2b^2 f(y)\| \leq \varphi(x, y, 0, 0, 0, 0, 0)$$

for all $x, y \in \mathcal{A}$ with $x \perp y$. By the same reasoning as above, we can show that the limit

$$h(x) =: \lim_{n \rightarrow \infty} \frac{1}{4^{ln}} f(2^{ln} x)$$

exists for all $x \in \mathcal{A}$. Moreover, we can show that h is an orthogonally quadratic mapping on \mathcal{A} satisfying (2.3). On the other hand, we have

$$\begin{aligned} \|h([u, v, w]) - [h(u), h(v), h(w)]\| &= \lim_{n \rightarrow \infty} \frac{1}{|4|^{2ln}} \|f(4^{ln} [u, v, w]) - [f(2^{ln} u), f(2^{ln} v), f(2^{ln} w)]\| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{|4|^{2ln}} \varphi(0, 0, 2^{ln} u, 2^{ln} v, 2^{ln} w, 0, 0) \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{|4|^{l(l+2)n}} \varphi(0, 0, 2^{ln} u, 2^{ln} v, 2^{ln} w, 0, 0) = 0 \end{aligned}$$

for all $u, v, w \in \mathcal{A}$, that are mutually orthogonal. Therefore, h is an orthogonally quadratic ternary homomorphism on \mathcal{A} . Also, we have

$$\begin{aligned} \|H([r, s, t]) - [H(r), H(s), H(t)]\| &= \lim_{n \rightarrow \infty} \frac{1}{|4|^{2ln}} \|g(4^{ln} [r, s, t]) - [g(2^{ln} r), g(2^{ln} s), g(2^{ln} t)]\| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{|4|^{2ln}} \varphi(0, 0, 0, 0, 0, 2^{ln} r, 2^{ln} s, 2^{ln} t) \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{|4|^{l(l+2)n}} \varphi(0, 0, 0, 0, 0, 2^{ln} r, 2^{ln} s, 2^{ln} t) = 0 \end{aligned}$$

for all $r, s, t \in \mathcal{A}$, that are mutually orthogonal. It follows that H is a generalized orthogonally quadratic ternary homomorphism (respect to h) on \mathcal{A} . This completes the proof. \square

From now on, we use the following abbreviation for any mappings $g, f : \mathcal{A} \rightarrow \mathcal{B}$:

$$\begin{aligned} \Delta(g, f)(z_1, \dots, z_8) := & \|f(az_1 + bz_2) + f(az_1 - bz_2) - 2a^2f(z_1) - 2b^2f(z_2)\| \\ & + \|g(az_1 + bz_2) + g(az_1 - bz_2) - 2a^2g(z_1) - 2b^2g(z_2)\| \\ & + \|f([z_3, z_4, z_5]) - [f(z_3), f(z_4), f(z_5)]\| \\ & + \|g([z_6, z_7, z_8]) - [g(z_6), f(z_7), f(z_8)]\|. \end{aligned}$$

Corollary 2.3. Let $\mathbb{K} = \mathbb{Q}_2$ be the 2-adic number field. Let \mathcal{A} be a non-Archimedean ternary Banach algebra on \mathbb{K} with $\perp = \bigcup \{(x, \alpha x) : x \in X, \alpha \in \mathbb{R}\}$ and \mathcal{B} be a non-Archimedean ternary Banach algebra on \mathbb{K} . Let ϵ be a nonnegative real number and let p be a real number such that $p > 6$ if $l = 1$ and $0 < p < 2$ if $l = -1$. Suppose that mappings $g, f : \mathcal{A} \rightarrow \mathcal{B}$ satisfy $f(0) = g(0) = 0$ and

$$\Delta(g, f)(z_1, \dots, z_8) \leq \epsilon \max\{\|z_i\|^p : 1 \leq i \leq 8\}$$

for all $z_1, z_2 \in \mathcal{A}$ with $z_1 \perp z_2$ and for all $z_3, \dots, z_8 \in \mathcal{A}$, that are mutually orthogonal. Then there exist a unique orthogonally quadratic ternary homomorphism $h : \mathcal{A} \rightarrow \mathcal{B}$ and a unique generalized orthogonally quadratic ternary homomorphism $H : \mathcal{A} \rightarrow \mathcal{B}$ (respect to h) such that

$$\max\{\|g(z) - H(z)\|, \|f(z) - h(z)\|\} \leq |2|^{\frac{l(4-p)+p}{2}} \epsilon \|z\|^p \begin{cases} 2, & \gcd(a, 2) = \gcd(b, 2) = 1; \\ \max\{2^{ip}, 2\}, & a = k2^i, \gcd(b, 2) = 1; \\ \max\{2^{jp}, 2^{2j+1}\}, & \gcd(a, 2) = 1, b = m2^j \vee a = k2^i, b = m2^j (j \geq i); \\ \max\{2^{jp}, 2^{2j+1}\}, & a = k2^i, b = m2^j (i \geq j) \end{cases}$$

for all $x \in \mathcal{A}$, where $i, j, k, m \geq 1$ are integers and $\gcd(k, 2) = \gcd(m, 2) = 1$.

Now, we have the following result on hyperstability of generalized orthogonally quadratic ternary homomorphisms in non-Archimedean ternary Banach algebras.

Corollary 2.4. Let $p > 0$ be a nonnegative real number such that $|2|^{(2l+4)p} \geq 1$ and let $j \in \{3, 4, \dots, 8\}$ be fixed. Suppose that mappings $g, f : \mathcal{A} \rightarrow \mathcal{B}$ satisfy $f(0) = g(0) = 0$ and

$$\Delta(g, f)(z_1, \dots, z_8) \leq \left(\sum_{i=1}^8 \|z_i\|^p\right) \|z_j\|^p$$

for all $z_1, z_2 \in \mathcal{A}$ with $z_1 \perp z_2$ and for all $z_3, \dots, z_8 \in \mathcal{A}$, that are mutually orthogonal, where a, b are positive fixed integers. Then f is an orthogonally quadratic ternary homomorphism and g is a generalized orthogonally quadratic ternary homomorphism related to f .

Proof. It follows from Theorem 2.2 by taking

$$\varphi(z_1, \dots, z_8) = \left(\sum_{i=1}^8 \|z_i\|^p\right) \|z_j\|^p$$

for all $z_1, z_2 \in \mathcal{A}$ with $z_1 \perp z_2$ and for all $z_3, \dots, z_8 \in \mathcal{A}$, that are mutually orthogonal and putting $L = |2|^{-(2l+4)p}$. \square

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SYMMETRY IDENTITIES OF HIGHER-ORDER q -EULER POLYNOMIALS UNDER THE SYMMETRIC GROUP OF DEGREE FOUR

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ABSTRACT. In this paper, we give some new identities of symmetry for the higher-order q -Euler polynomials under the symmetric group of degree four which are derived from the fermionic p -adic q -integrals on \mathbb{Z}_p .

1. INTRODUCTION

Let p be a fixed odd prime number. Throughout this paper, \mathbb{Z}_p , \mathbb{Q}_p , and \mathbb{C}_p will denote the ring of p -adic integers, the field of p -adic rational numbers, and the completion of the algebraic closure of \mathbb{Q}_p . Let ν_p be the normalized exponential valuation of \mathbb{C}_p with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$. Let us assume that q is an indeterminate in \mathbb{C}_p such that $|1 - q|_p < p^{-\frac{1}{p-1}}$. The q -number of x is defined as $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q \rightarrow 1} [x]_q = x$. Let $C(\mathbb{Z}_p)$ be the space of all \mathbb{C}_p -valued continuous functions on \mathbb{Z}_p . For $f \in C(\mathbb{Z}_p)$, the fermionic p -adic q -integral on \mathbb{Z}_p is defined by Kim as

$$\begin{aligned} (1.1) \quad I_{-q}(f) &= \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) \\ &= \lim_{N \rightarrow \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x) (-q)^x, \quad (\text{see [9, 10, 12, 13]}). \end{aligned}$$

Thus, by (1.1), we get

$$(1.2) \quad qI_{-q}(f_1) + I_{-q}(f) = [2]_q f(0), \quad (\text{see [9]}),$$

where $f_1(x) = f(x+1)$. The Carlitz-type q -Euler numbers are defined by

$$(1.3) \quad q(E_q + 1)^n + E_{n,q} = [2]_q \delta_{0,n}, \quad E_{0,q} = 1, \quad (\text{see [9, 10]}),$$

with the usual convention about replacing E_q^n by $E_{n,q}$.

The q -Euler polynomials are given by

$$(1.4) \quad E_{n,q}(x) = \sum_{l=0}^n \binom{n}{l} [x]_q^{n-l} q^{lx} E_{l,q}, \quad (\text{see [9]}).$$

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From (1.1) and (1.4), we have

$$(1.5) \quad \int_{\mathbb{Z}_p} [x+y]_q^n d\mu_{-q}(y) = E_{n,q}(x), \quad (n \geq 0), \quad (\text{see [9, 10, 12]}).$$

For $r \in \mathbb{N}$, we consider the higher-order q -Euler polynomials as follows:

$$(1.6) \quad \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{t[x_1+\cdots+x_r+x]_q} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) = \sum_{n=0}^{\infty} E_{n,q}^{(r)}(x) \frac{t^n}{n!}.$$

Thus, by (1.3), we get

$$(1.7) \quad E_{n,q}^{(r)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_r + x]_q^n d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r), \quad (\text{see [9]}).$$

When $x = 0$, $E_{n,q}^{(r)} = E_{n,q}^{(r)}(0)$ are called the higher-order q -Euler numbers.

In this paper, we give some new identities of symmetry for the higher-order q -Euler polynomials under the symmetric group S_4 of degree four.

Recently, several authors have studied q -extensions of Euler numbers and polynomials in the several different areas (see [1–23]).

2. SYMMETRY IDENTITIES OF $E_{n,q}^{(r)}(x)$ UNDER S_4

Let $w_1, w_2, w_3, w_4 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod{2}$, $w_2 \equiv 1 \pmod{2}$, $w_3 \equiv 1 \pmod{2}$, $w_4 \equiv 1 \pmod{2}$. Then we have

$$(2.1) \quad \begin{aligned} & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{[w_1 w_2 w_3 \sum_{l=1}^r x_l + w_1 w_2 w_3 w_4 x + w_4 w_2 w_3 \sum_{l=1}^r i_l + w_4 w_1 w_3 \sum_{l=1}^r j_l + w_4 w_1 w_2 \sum_{l=1}^r k_l]_q} t \\ & \times d\mu_{-q^{w_1 w_2 w_3}}(x_1) \cdots d\mu_{-q^{w_1 w_2 w_3}}(x_r) \\ & = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_{-q^{w_1 w_2 w_3}}^r} \sum_{x_1, \dots, x_r=0}^{p^N-1} (-q^{w_1 w_2 w_3})^{\sum_{l=1}^r x_l} \\ & \times e^{[w_1 w_2 w_3 \sum_{l=1}^r x_l + w_1 w_2 w_3 w_4 x + w_4 w_2 w_3 \sum_{l=1}^r i_l + w_4 w_1 w_3 \sum_{l=1}^r j_l + w_4 w_1 w_2 \sum_{l=1}^r k_l]_q} t \\ & = \lim_{N \rightarrow \infty} \frac{1}{[w_4 p^N]_{-q^{w_1 w_2 w_3}}^r} \sum_{l_1, \dots, l_r=0}^{w_4-1} \sum_{x_1, \dots, x_r=0}^{p^N-1} (-1)^{\sum_{i=1}^r l_i} q^{w_1 w_2 w_3 \sum_{i=1}^r (l_i + w_4 x_i)} (-1)^{x_1 + \cdots + x_r} \\ & \times e^{[w_1 w_2 w_3 \sum_{i=1}^r (l_i + w_4 x_i) + w_1 w_2 w_3 w_4 x + w_4 w_2 w_3 \sum_{l=1}^r i_l + w_4 w_1 w_3 \sum_{l=1}^r j_l + w_4 w_1 w_2 \sum_{l=1}^r k_l]_q} t. \end{aligned}$$

Now, we observe that

$$(2.2) \quad \begin{aligned} & \frac{1}{[2]_{q^{w_1 w_2 w_3}}^r} \sum_{i_1, \dots, i_r=0}^{w_1-1} \sum_{j_1, \dots, j_r=0}^{w_2-1} \sum_{k_1, \dots, k_r=0}^{w_3-1} (-1)^{\sum_{l=1}^r (i_l + j_l + k_l)} \\ & \times q^{w_4 w_2 w_3 \sum_{l=1}^r i_l + w_4 w_1 w_3 \sum_{l=1}^r j_l + w_4 w_1 w_2 \sum_{l=1}^r k_l} \\ & \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{[w_1 w_2 w_3 \sum_{l=1}^r x_l + w_1 w_2 w_3 w_4 x + w_4 w_2 w_3 \sum_{l=1}^r i_l + w_4 w_1 w_3 \sum_{l=1}^r j_l + w_4 w_1 w_2 \sum_{l=1}^r k_l]_q} t \\ & \times d\mu_{-q^{w_1 w_2 w_3}}(x_1) \cdots d\mu_{-q^{w_1 w_2 w_3}}(x_r) \end{aligned}$$

$$\begin{aligned}
&= \lim_{N \rightarrow \infty} \left(\frac{1}{1 + q^{w_1 w_2 w_3 w_4 p^N}} \right)^r \sum_{i_1, \dots, i_r=0}^{w_1-1} \sum_{j_1, \dots, j_r=0}^{w_2-1} \sum_{k_1, \dots, k_r=0}^{w_3-1} \sum_{l_1, \dots, l_r=0}^{w_4-1} (-1)^{\sum_{n=1}^r (l_n + j_n + i_n + k_n)} \\
&\quad \times q^{w_4 w_2 w_3 \sum_{i=1}^r i_l + w_4 w_1 w_3 \sum_{j=1}^r j_l + w_4 w_1 w_2 \sum_{l=1}^r k_l + w_1 w_2 w_3 \sum_{i=1}^r l_i} \\
&\quad \times \sum_{x_1, \dots, x_r=0}^{p^N-1} q^{w_1 w_2 w_3 \sum_{i=1}^r x_i} (-1)^{x_1 + \dots + x_r} \\
&\quad \times e^{\left[w_1 w_2 w_3 \sum_{i=1}^r (l_i + x_i w_4) + w_1 w_2 w_3 w_4 x + w_4 w_2 w_3 \sum_{i=1}^r i_l + w_4 w_1 w_3 \sum_{i=1}^r j_l + w_4 w_1 w_2 \sum_{i=1}^r k_l \right]_q t}.
\end{aligned}$$

As this expression is invariant under S_4 , we have the following theorem.

Theorem 2.1. For $w_1, w_2, w_3, w_4 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod{2}$, $w_2 \equiv 1 \pmod{2}$, $w_3 \equiv 1 \pmod{2}$, $w_4 \equiv 1 \pmod{2}$, the following expression

$$\begin{aligned}
&\frac{1}{[2]_q^{w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)}}} \sum_{i_1, \dots, i_r=0}^{w_{\sigma(1)}-1} \sum_{j_1, \dots, j_r=0}^{w_{\sigma(2)}-1} \sum_{k_1, \dots, k_r=0}^{w_{\sigma(3)}-1} (-1)^{\sum_{i=1}^r (i_l + j_l + k_l)} \\
&\quad \times q^{w_{\sigma(4)} w_{\sigma(2)} w_{\sigma(3)} \sum_{i=1}^r i_l + w_{\sigma(4)} w_{\sigma(1)} w_{\sigma(3)} \sum_{i=1}^r j_l + w_{\sigma(4)} w_{\sigma(1)} w_{\sigma(2)} \sum_{i=1}^r k_l} \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} e^{[A]_q t} \\
&\quad \times d\mu_{-q^{w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)}}} (x_1) \dots d\mu_{-q^{w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)}}} (x_r)
\end{aligned}$$

are the same for any $\sigma \in S_4$,

where $A = w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)} \sum_{l=1}^r i_l + w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)} w_{\sigma(4)} x + w_{\sigma(4)} w_{\sigma(2)} w_{\sigma(3)} \sum_{l=1}^r i_l + w_{\sigma(4)} w_{\sigma(1)} w_{\sigma(3)} \sum_{l=1}^r j_l + w_{\sigma(4)} w_{\sigma(1)} w_{\sigma(2)} \sum_{l=1}^r k_l$.

From (1.7), we have

$$\begin{aligned}
(2.3) \quad &\int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} e^{\left[w_1 w_2 w_3 \sum_{i=1}^r x_l + w_1 w_2 w_3 w_4 x + w_4 w_2 w_3 \sum_{i=1}^r i_l + w_4 w_1 w_3 \sum_{i=1}^r j_l + w_4 w_1 w_2 \sum_{i=1}^r k_l \right]_q t} \\
&\quad \times d\mu_{-q^{w_1 w_2 w_3}} (x_1) \dots d\mu_{-q^{w_1 w_2 w_3}} (x_r) \\
&= \sum_{n=0}^{\infty} [w_1 w_2 w_3]_q^n \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} \left[\sum_{l=1}^r x_l + w_4 x + \frac{w_4}{w_1} \sum_{l=1}^r i_l + \frac{w_4}{w_2} \sum_{l=1}^r j_l + \frac{w_4}{w_3} \sum_{l=1}^r k_l \right]_q^n \\
&\quad \times d\mu_{-q^{w_1 w_2 w_3}} (x_1) \dots d\mu_{-q^{w_1 w_2 w_3}} (x_r) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} [w_1 w_2 w_3]_q^n E_{n,q^{w_1 w_2 w_3}}^{(r)} \left(w_4 x + \frac{w_4}{w_1} \sum_{l=1}^r i_l + \frac{w_4}{w_2} \sum_{l=1}^r j_l + \frac{w_4}{w_3} \sum_{l=1}^r k_l \right) \frac{t^n}{n!}.
\end{aligned}$$

Thus, by (2.3), we get

$$\begin{aligned}
(2.4) \quad &\int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} \left[w_1 w_2 w_3 \sum_{l=1}^r x_l + w_4 w_2 w_3 \sum_{l=1}^r i_l + w_4 w_1 w_3 \sum_{l=1}^r j_l + w_4 w_1 w_2 \sum_{l=1}^r k_l \right]_q^n \\
&\quad \times d\mu_{-q^{w_1 w_2 w_3}} (x_1) \dots d\mu_{-q^{w_1 w_2 w_3}} (x_r) \\
&= [w_1 w_2 w_3]_q^n E_{n,q^{w_1 w_2 w_3}}^{(r)} \left(w_4 x + \frac{w_4}{w_1} \sum_{l=1}^r i_l + \frac{w_4}{w_2} \sum_{l=1}^r j_l + \frac{w_4}{w_3} \sum_{l=1}^r k_l \right).
\end{aligned}$$

Therefore, by (2.4) and Theorem 2.1, we obtain the following theorem.

Theorem 2.2. For $n \geq 0$, $w_1, w_2, w_3, w_4 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod{2}$, $w_2 \equiv 1 \pmod{2}$, $w_3 \equiv 1 \pmod{2}$, $w_4 \equiv 1 \pmod{2}$, the following expression

$$\begin{aligned} & \frac{[w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}]_q^n}{[2]_q^{r w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}} \sum_{i_1, \dots, i_r=0}^{w_{\sigma(1)}-1} \sum_{j_1, \dots, j_r=0}^{w_{\sigma(2)}-1} \sum_{k_1, \dots, k_r=0}^{w_{\sigma(3)}-1} (-1)^{\sum_{l=1}^r (i_l + j_l + k_l)} \\ & \times q^{w_{\sigma(4)}w_{\sigma(2)}w_{\sigma(3)} \sum_{l=1}^r i_l + w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(3)} \sum_{l=1}^r j_l + w_{\sigma(4)}w_{\sigma(1)}w_{\sigma(2)} \sum_{l=1}^r k_l} \\ & \times E_{n,q}^{(r)}{}_{w_{\sigma(1)}w_{\sigma(2)}w_{\sigma(3)}} \left(w_{\sigma(4)}x + \frac{w_{\sigma(4)}}{w_{\sigma(1)}} \sum_{l=1}^r i_l + \frac{w_{\sigma(4)}}{w_{\sigma(2)}} \sum_{l=1}^r j_l + \frac{w_{\sigma(4)}}{w_{\sigma(3)}} \sum_{l=1}^r k_l \right) \end{aligned}$$

are the same for any $\sigma \in S_4$.

Now, we observe that

$$\begin{aligned} (2.5) \quad & \left[\sum_{l=1}^r x_l + w_4x + \frac{w_4}{w_1} \sum_{l=1}^r i_l + \frac{w_4}{w_2} \sum_{l=1}^r j_l + \frac{w_4}{w_3} \sum_{l=1}^r k_l \right]_{q^{w_1w_2w_3}} \\ & = \frac{[w_4]_q}{[w_1w_2w_3]_q} \left[w_2w_3 \sum_{l=1}^r i_l + w_1w_3 \sum_{l=1}^r j_l + w_1w_2 \sum_{l=1}^r k_l \right]_{q^{w_4}} \\ & \quad + q^{w_2w_3w_4 \sum_{l=1}^r i_l + w_1w_3w_4 \sum_{l=1}^r j_l + w_1w_2w_4 \sum_{l=1}^r k_l}. \end{aligned}$$

By (2.5), we get

$$\begin{aligned} (2.6) \quad & \left[\sum_{l=1}^r x_l + w_4x + \frac{w_4}{w_1} \sum_{l=1}^r i_l + \frac{w_4}{w_2} \sum_{l=1}^r j_l + \frac{w_4}{w_3} \sum_{l=1}^r k_l \right]_{q^{w_1w_2w_3}}^n \\ & = \sum_{m=0}^n \binom{n}{m} \left(\frac{[w_4]_q}{[w_1w_2w_3]_q} \right)^{n-m} \left[w_2w_3 \sum_{l=1}^r i_l + w_1w_3 \sum_{l=1}^r j_l + w_1w_2 \sum_{l=1}^r k_l \right]_{q^{w_4}}^{n-m} \\ & \quad \times q^{m(w_2w_3w_4 \sum_{l=1}^r i_l + w_1w_3w_4 \sum_{l=1}^r j_l + w_1w_2w_4 \sum_{l=1}^r k_l)} \left[\sum_{l=1}^r x_l + w_4x \right]_{q^{w_1w_2w_3}}^m. \end{aligned}$$

From (2.6), we can derive the following equation:

$$\begin{aligned} (2.7) \quad & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[\sum_{l=1}^r x_l + w_4x + \frac{w_4}{w_1} \sum_{l=1}^r i_l + \frac{w_4}{w_2} \sum_{l=1}^r j_l + \frac{w_4}{w_3} \sum_{l=1}^r k_l \right]_{q^{w_1w_2w_3}}^n \\ & \times d\mu_{-q^{w_1w_2w_3}}(x_1) \cdots d\mu_{-q^{w_1w_2w_3}}(x_r) \\ & = \sum_{m=0}^n \binom{n}{m} \left(\frac{[w_4]_q}{[w_1w_2w_3]_q} \right)^{n-m} \left[w_2w_3 \sum_{l=1}^r i_l + w_1w_3 \sum_{l=1}^r j_l + w_1w_2 \sum_{l=1}^r k_l \right]_{q^{w_4}}^{n-m} \\ & \quad \times E_{m,q}^{(r)}{}_{w_1w_2w_3}(w_4x) \\ & \quad \times q^{m(w_2w_3w_4 \sum_{l=1}^r i_l + w_1w_3w_4 \sum_{l=1}^r j_l + w_1w_2w_4 \sum_{l=1}^r k_l)}. \end{aligned}$$

Thus, by (2.7), we get

$$(2.8) \quad \frac{[w_1w_2w_3]_q^n}{[2]_q^{r w_1w_2w_3}} \sum_{i_1, \dots, i_r=0}^{w_1-1} \sum_{j_1, \dots, j_r=0}^{w_2-1} \sum_{k_1, \dots, k_r=0}^{w_3-1} (-1)^{\sum_{l=1}^r (i_l + j_l + k_l)} q^{w_2w_3w_4 \sum_{l=1}^r i_l + w_4w_1w_3 \sum_{l=1}^r j_l}$$

$$\begin{aligned}
& \times q^{w_1 w_2 w_4 \sum_{i=1}^r k_i} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[\sum_{l=1}^r x_l + w_4 x + \frac{w_4}{w_1} \sum_{l=1}^r i_l + \frac{w_4}{w_2} \sum_{l=1}^r j_l + \frac{w_4}{w_3} \sum_{l=1}^r k_l \right]_{q^{w_1 w_2 w_3}}^n \\
& \times d\mu_{-q^{w_1 w_2 w_3}}(x_1) \cdots d\mu_{-q^{w_1 w_2 w_3}}(x_r) \\
& = \sum_{m=0}^n \binom{n}{m} \frac{[w_1 w_2 w_3]_q^m}{[2]_{q^{w_1 w_2 w_3}}^r} [w_4]_q^{n-m} E_{m,q^{w_1 w_2 w_3}}^{(r)}(w_4 x) \sum_{i_1, \dots, i_r=0}^{w_1-1} \sum_{j_1, \dots, j_r=0}^{w_2-1} \sum_{k_1, \dots, k_r=0}^{w_3-1} \\
& \times (-1)^{\sum_{l=1}^r (i_l + j_l + k_l)} q^{(m+1)(w_2 w_3 w_4 \sum_{l=1}^r i_l + w_4 w_1 w_3 \sum_{l=1}^r j_l + w_4 w_1 w_2 \sum_{l=1}^r k_l)} \\
& \times \left[w_2 w_3 \sum_{l=1}^r i_l + w_1 w_3 \sum_{l=1}^r j_l + w_1 w_2 \sum_{l=1}^r k_l \right]_{q^{w_4}}^{n-m} \\
& = \sum_{m=0}^n \binom{n}{m} \frac{[w_1 w_2 w_3]_q^m}{[2]_{q^{w_1 w_2 w_3}}^r} [w_4]_q^{n-m} E_{m,q^{w_1 w_2 w_3}}^{(r)}(w_4 x) T_{n,q^{w_4}}^{(r)}(w_1, w_2, w_3 \mid m),
\end{aligned}$$

where

$$\begin{aligned}
(2.9) \quad & T_{n,q}^{(r)}(w_1, w_2, w_3 \mid m) \\
& = \sum_{i_1, \dots, i_r=0}^{w_1-1} \sum_{j_1, \dots, j_r=0}^{w_2-1} \sum_{k_1, \dots, k_r=0}^{w_3-1} (-1)^{\sum_{l=1}^r (i_l + j_l + k_l)} \\
& \times q^{(m+1)(w_2 w_3 \sum_{l=1}^r j_l + w_1 w_3 \sum_{l=1}^r i_l + w_1 w_2 \sum_{l=1}^r k_l)} \\
& \times \left[w_2 w_3 \sum_{l=1}^r i_l + w_1 w_3 \sum_{l=1}^r j_l + w_1 w_2 \sum_{l=1}^r k_l \right]_q^{n-m}.
\end{aligned}$$

As this expression is invariant under S_4 , we have the following theorem.

Theorem 2.3. For $n \geq 0$, $w_1, w_2, w_3, w_4 \in \mathbb{N}$ with $w_1 \equiv 1 \pmod{2}$, $w_2 \equiv 1 \pmod{2}$, $w_3 \equiv 1 \pmod{2}$, $w_4 \equiv 1 \pmod{2}$, the following expression

$$\begin{aligned}
& \sum_{m=0}^n \binom{n}{m} \frac{[w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)}]_q^m}{[2]_{q^{w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)}}^r} [w_{\sigma(4)}]_q^{n-m} \\
& \times E_{m,q^{w_{\sigma(1)} w_{\sigma(2)} w_{\sigma(3)}}}^{(r)}(w_{\sigma(4)} x) T_{n,q^{w_{\sigma(4)}}}^{(r)}(w_{\sigma(1)}, w_{\sigma(2)}, w_{\sigma(3)} \mid m)
\end{aligned}$$

are the same for any $\sigma \in S_4$.

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Soft saturated and dried values with applications in BCK/BCI -algebras

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Abstract

The notions of soft saturated values and soft dried values are introduced, and their applications in BCK/BCI -algebras are discussed. Using these notions, properties of energetic subsets are investigated. Using the concepts of intersectional (union) ideals, properties of right vanished (stable) subsets are explored.

Keywords:

Energetic subset, Right vanished subset, Right stable subset, Saturated value, Dried value.

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1 Introduction

The real world is inherently uncertain, imprecise and vague. Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [32]. In response to this situation Zadeh [33] introduced *fuzzy set theory* as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a

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more general framework, the approach to uncertainty is outlined by Zadeh [34]. To solve complicated problem in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [29]. Maji et al. [26] and Molodtsov [29] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [29] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. Worldwide, there has been a rapid growth in interest in soft set theory and its applications in recent years. Evidence of this can be found in the increasing number of high-quality articles on soft sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences in recent years. Maji et al. [26] described the application of soft set theory to a decision making problem. Maji et al. [25] also studied several operations on the theory of soft sets. Aktaş and Çağman [2] studied the basic concepts of soft set theory, and compared soft sets to fuzzy and rough sets, providing examples to clarify their differences. They also discussed the notion of soft groups.

BCK and BCI-algebras are two classes of logical algebras which are introduced by Imai and Iséki (see [9, 10]). This notion originated from two different ways:

- (1) set theory, and
- (2) classical and non-classical propositional calculi.

In set theory, we have the following simple relations: $(A - B) - (A - C) \subseteq C - B$ and $A - (A - B) \subseteq B$. Several properties on BCK/BCI-algebras are investigated in the papers [11, 12, 13, 14] and [27]. There is a deep relation between BCK/BCI-algebras and posets. Today BCK/BCI-algebras have been studied by many authors and they have been applied to many branches of mathematics, such as group, functional analysis, probability theory, topology, fuzzy set theory, and so on. Jun and Park [24] studied applications of soft sets in ideal theory of BCK/BCI-algebras. Jun et al. [20, 22] introduced the notion of

intersectional soft sets, and considered its applications to BCK/BCI -algebras. Also, Jun [16] discussed the union soft sets with applications in BCK/BCI -algebras. We refer the reader to the papers [1, 3, 5, 6, 7, 15, 18, 19, 21, 23, 30, 31, 35] for further information regarding algebraic structures/properties of soft set theory.

In this paper, we introduce the notions of soft saturated values and soft dried values, and discuss their applications in BCK/BCI -algebras. Using these notions, we investigate several properties of energetic subsets. Using the concepts of intersectional (union) ideals, we explore some properties of right vanished (stable) subsets.

2 Preliminaries

A BCK/BCI -algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a BCI -algebra if it satisfies the following conditions:

- (I) $(\forall x, y, z \in X) (((x * y) * (x * z)) * (z * y) = 0),$
- (II) $(\forall x, y \in X) ((x * (x * y)) * y = 0),$
- (III) $(\forall x \in X) (x * x = 0),$
- (IV) $(\forall x, y \in X) (x * y = 0, y * x = 0 \Rightarrow x = y).$

If a BCI -algebra X satisfies the following identity:

- (V) $(\forall x \in X) (0 * x = 0),$

then X is called a BCK -algebra. Any BCK/BCI -algebra X satisfies the following axioms:

$$(\forall x \in X) (x * 0 = x), \quad (2.1)$$

$$(\forall x, y, z \in X) (x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x), \quad (2.2)$$

$$(\forall x, y, z \in X) ((x * y) * z = (x * z) * y), \quad (2.3)$$

$$(\forall x, y, z \in X) ((x * z) * (y * z) \leq x * y) \quad (2.4)$$

where $x \leq y$ if and only if $x * y = 0$. A nonempty subset S of a BCK/BCI -algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI -algebra

X is called an *ideal* of X if it satisfies:

$$0 \in I, \quad (2.5)$$

$$(\forall x \in X) (\forall y \in I) (x * y \in I \Rightarrow x \in I). \quad (2.6)$$

We refer the reader to the books [8, 28] for further information regarding *BCK/BCI*-algebras.

A soft set theory is introduced by Molodtsov [29], and Çağman et al. [4] provided new definitions and various results on soft set theory.

In what follows, let U be an initial universe set and E be a set of parameters. We say that the pair (U, E) is a *soft universe*. Let $\mathcal{P}(U)$ denote the power set of U and $A, B, C, \dots \subseteq E$.

Definition 2.1 ([4, 29]). A *soft set* (f, A) over U is defined to be the set of ordered pairs

$$(f, A) := \{(x, f(x)) : x \in E, f(x) \in \mathcal{P}(U)\},$$

where $f : E \rightarrow \mathcal{P}(U)$ such that $f(x) = \emptyset$ if $x \notin A$.

The function f is called an approximate function of the soft set (f, A) . The subscript A in the notation f indicates that f is the approximate function of (f, A) .

Definition 2.2 ([16]). Let $(U, E) = (U, X)$ where X is a *BCK/BCI*-algebra. A soft set (f, X) over U is called a *union soft subalgebra* over U if the following condition holds:

$$(\forall x, y \in X) (f(x * y) \subseteq f(x) \cup f(y)). \quad (2.7)$$

Definition 2.3 ([16]). Let $(U, E) = (U, X)$ where X is a *BCK/BCI*-algebra. A soft set (f, X) over U is called a *union soft ideal* over U if it satisfies:

$$(\forall x, y \in X) (f(0) \subseteq f(x) \subseteq f(x * y) \cup f(y)). \quad (2.8)$$

Proposition 2.4 ([16]). Let $(U, E) = (U, X)$ where X is a *BCK/BCI*-algebra. Every union soft ideal (f, X) over U satisfies the following condition:

$$(\forall x, y \in X) (x \leq y \Rightarrow f(x) \subseteq f(y)). \quad (2.9)$$

3 Energetic subsets and soft saturated (dried) values

In what follows, let $\left\{ \begin{array}{c} \mathcal{Q}(U) \\ \mathcal{R}(U) \end{array} \right\}$ be the class of all subsets of U such that

$$(\forall A, B, C \in \mathcal{P}(U)) \left\{ \begin{array}{l} A \cap B \subseteq C \Rightarrow A \subseteq C \text{ or } B \subseteq C \\ A \subseteq B \cup C \Rightarrow A \subseteq B \text{ or } A \subseteq C \end{array} \right\},$$

and let $(U, E) = (U, X)$ where X is a BCK/BCI -algebra unless otherwise specified.

Definition 3.1 ([17]). A non-empty subset G of X is said to be S -energetic if it satisfies:

$$(\forall a, b \in X) (a * b \in G \Rightarrow \{a, b\} \cap G \neq \emptyset). \quad (3.1)$$

Example 3.2 ([17]). Let $X = \{0, a, b, c, d\}$ be a BCK -algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	0	a
c	c	b	a	0	b
d	d	a	a	a	0

The set $G := \{a, b, c\}$ is an S -energetic subset of X , but $H := \{a, b\}$ is not an S -energetic subset of X since $d * c = a \in H$ but $\{d, c\} \cap H = \emptyset$.

Definition 3.3 ([22]). A soft set (f, X) over U is called an *int-soft subalgebra* over U if it satisfies:

$$(\forall x, y \in X) (f(x * y) \supseteq f(x) \cap f(y)). \quad (3.2)$$

Definition 3.4 ([22]). A soft set (f, X) over U is called an *int-soft ideal* over U if it satisfies:

$$(\forall x \in X) (f(x) \subseteq f(0)), \quad (3.3)$$

$$(\forall x, y \in X) (f(x * y) \cap f(y) \subseteq f(x)). \quad (3.4)$$

Lemma 3.5 ([22]). Every *int-soft ideal* (f, X) over U satisfies the following conditions:

$$(1) (\forall x, y \in X) (x \leq y \Rightarrow f(y) \subseteq f(x)).$$

$$(2) (\forall x, y, z \in X) (x * y \leq z \Rightarrow f(y) \cap f(z) \subseteq f(x)).$$

Given a soft set (f, X) over U and $\alpha \in \mathcal{P}(U)$, we define useful subsets of X .

$$\begin{aligned} f_{\alpha}^{\subseteq} &:= \{x \in X \mid f(x) \subseteq \alpha\}, \quad f_{\alpha}^{\subset} := \{x \in X \mid f(x) \subset \alpha\}, \\ f_{\alpha}^{\supseteq} &:= \{x \in X \mid f(x) \supseteq \alpha\}, \quad f_{\alpha}^{\supset} := \{x \in X \mid f(x) \supset \alpha\}. \end{aligned}$$

Proposition 3.6. *If (f, X) is an int-soft subalgebra over U with $f : X \rightarrow \mathcal{Q}(U)$, then*

$$(\forall \alpha \in \mathcal{Q}(U)) (f_{\alpha}^{\subseteq} \neq \emptyset \Rightarrow f_{\alpha}^{\subseteq} \text{ is an S-energetic subset of } X).$$

Proof. Let $x, y \in X$ be such that $x * y \in f_{\alpha}^{\subseteq}$. Then

$$f(x) \cap f(y) \subseteq f(x * y) \subseteq \alpha,$$

and so $f(x) \subseteq \alpha$ or $f(y) \subseteq \alpha$, that is, $x \in f_{\alpha}^{\subseteq}$ or $y \in f_{\alpha}^{\subseteq}$. Hence $\{x, y\} \cap f_{\alpha}^{\subseteq} \neq \emptyset$. Therefore f_{α}^{\subseteq} is an S-energetic subset of X . \square

Corollary 3.7. *If (f, X) is an int-soft subalgebra over U with $f : X \rightarrow \mathcal{Q}(U)$, then*

$$(\forall \alpha \in \mathcal{Q}(U)) (f_{\alpha}^{\subseteq} \neq \emptyset \Rightarrow f_{\alpha}^{\subseteq} \text{ is an S-energetic subset of } X).$$

Proof. Straightforward. \square

The following example shows that the converse of Proposition 3.6 is not true.

Example 3.8. Let $(U, E) = (U, X)$ where $X = \{0, a, b, c, d\}$ is a BCK-algebra as in Example 3.2. Let (f, X) be a soft set over U in which f is given as follows:

$$f : X \rightarrow \mathcal{Q}(U), \quad x \mapsto \begin{cases} \gamma_2 & \text{if } x = 0, \\ \gamma_3 & \text{if } x = d, \\ \gamma_1 & \text{if } x \in \{a, b, c\}, \end{cases}$$

where $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{Q}(U)$ with $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3$. For any $\alpha \in \mathcal{Q}(U)$, if $\gamma_1 \subseteq \alpha \subsetneq \gamma_2$ then $f_{\alpha}^{\subseteq} = \{a, b, c\}$ is an S-energetic subset of X . But (f, X) is not an int-soft subalgebra over U since

$$f(d * d) = f(0) = \gamma_2 \not\supseteq \gamma_3 = f(d) \cap f(d).$$

Let $(U, E) = (U, X)$ where X is a BCK-algebra. Then every int-soft ideal over U is an int-soft subalgebra over U (see [22]). Hence we have the following corollary.

Corollary 3.9. Let $(U, E) = (U, X)$ where X is a BCK -algebra. If (f, X) is an int-soft ideal over U with $f : X \rightarrow \mathcal{Q}(U)$, then

$$(\forall \alpha \in \mathcal{Q}(U)) (f_\alpha^\subseteq \neq \emptyset \Rightarrow f_\alpha^\subseteq \text{ is an } S\text{-energetic subset of } X).$$

The following example shows that the converse of Corollary 3.9 is not true.

Example 3.10. Consider the soft set (f, X) over U as in Example 3.8. For any $\alpha \in \mathcal{Q}(U)$, if $\gamma_1 \subseteq \alpha \subsetneq \gamma_2$ then $f_\alpha^\subseteq = \{a, b, c\}$ is an S-energetic subset of X . But (f, X) is not an int-soft ideal over U since $f(d) = \gamma_3 \not\subseteq \gamma_2 = f(0)$.

Definition 3.11. Let (f, X) be a soft set over U and $\alpha \in \mathcal{P}(U)$ with $f_\alpha^\supseteq \neq \emptyset$. Then α is called a *soft saturated S-value* for (f, X) if the following assertion is valid:

$$(\forall a, b \in X) (f(a * b) \supseteq \alpha \Rightarrow f(a) \cup f(b) \supseteq \alpha). \quad (3.5)$$

Example 3.12. Let $(U, E) = (U, X)$ where $X = \{0, 1, 2, 3\}$ is a BCK -algebra with the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	2
3	3	1	3	0

Consider a soft set (f, X) over U in which f is given as follows:

$$f : X \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \gamma_1 & \text{if } x = 0, \\ \gamma_2 & \text{if } x = 1, \\ \gamma_3 & \text{if } x \in \{2, 3\}, \end{cases}$$

where γ_1, γ_2 and γ_3 are subsets of U with $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3$. Take $\alpha \in \mathcal{P}(U)$ with $\gamma_2 \subsetneq \alpha \subseteq \gamma_3$. Then $f_\alpha^\supseteq = \{2, 3\}$, and it is easy to check that α is a soft saturated S-value for (f, X) .

Example 3.13. Let $(U, E) = (\mathbb{N}, X)$ where \mathbb{N} is the set of all natural numbers and $X = \{0, 1, 2, a, b\}$ is a BCI -algebra with the following Cayley table:

$*$	0	1	2	a	b
0	0	0	0	b	a
1	1	0	1	b	a
2	2	2	0	b	a
a	a	a	a	0	b
b	b	b	b	a	0

Consider a soft set (f, X) over U in which f is given as follows:

$$f : X \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \mathbb{N} & \text{if } x = 0, \\ 2\mathbb{N} & \text{if } x \in \{1, a\}, \\ 2\mathbb{N} - \{2, 4, 6\} & \text{if } x = 2, \\ 2\mathbb{N} - \{4, 6, 8\} & \text{if } x = b. \end{cases}$$

If $\alpha = 2\mathbb{N} - \{4\}$, then $f_\alpha^\supseteq = \{0, 1, a\} \neq \emptyset$, $f(2 * b) = f(a) = 2\mathbb{N} \supseteq \alpha$, and $f(2) \cup f(b) = 2\mathbb{N} - \{4, 6\} \not\supseteq \alpha$. Hence α is not a soft saturated S-value for (f, X) .

Proposition 3.14. *Let (f, X) be an int-soft subalgebra over U with $f : X \rightarrow \mathcal{R}(U)$. If $\alpha \in \mathcal{R}(U)$ is a soft saturated S-value for (f, X) , then*

$$f_\alpha^\supseteq \neq \emptyset \Rightarrow f_\alpha^\supseteq \text{ is an S-energetic subset of } X.$$

Proof. Let $a, b \in X$ be such that $a * b \in f_\alpha^\supseteq$. Then $f(a * b) \supseteq \alpha$, which implies from (3.5) that $f(a) \cup f(b) \supseteq \alpha$. Thus $f(a) \supseteq \alpha$ or $f(b) \supseteq \alpha$, that is, $a \in f_\alpha^\supseteq$ or $b \in f_\alpha^\supseteq$. Hence $\{a, b\} \cap f_\alpha^\supseteq \neq \emptyset$. Therefore f_α^\supseteq is an S-energetic subset of X . \square

Theorem 3.15. *Let (f, X) be a soft set over U and $\alpha \in \mathcal{P}(U)$ be such that $f_\alpha^\supseteq \neq \emptyset$. If (f, X) is a union soft subalgebra over U , then α is a soft saturated S-value for (f, X) .*

Proof. Let $x, y \in X$ be such that $f(x * y) \supseteq \alpha$. Then

$$\alpha \subseteq f(x * y) \subseteq f(x) \cup f(y),$$

and so α is a soft saturated S-value for (f, X) . \square

Corollary 3.16. *Let $(U, E) = (U, X)$ where X is a BCK-algebra. Let (f, X) be a soft set over U and let $\alpha \in \mathcal{P}(U)$ be such that $f_\alpha^\supseteq \neq \emptyset$. If (f, X) is a union soft ideal over U , then α is a soft saturated S-value for (f, X) .*

Definition 3.17. Let (f, X) be a soft set over U and $\alpha \in \mathcal{P}(U)$ with $f_\alpha^\subseteq \neq \emptyset$. Then α is called a *soft dried S-value* for (f, X) if the following assertion is valid:

$$(\forall a, b \in X) (f(a * b) \subseteq \alpha \Rightarrow f(a) \cap f(b) \subseteq \alpha). \quad (3.6)$$

Example 3.18. Let $(U, E) = (U, X)$ where $X = \{0, 1, 2, 3\}$ is a BCK-algebra as in Example 3.12. Consider a soft set (f, X) over U in which f is given as follows:

$$f : X \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \gamma_2 & \text{if } x = 0, \\ \gamma_1 & \text{if } x = 1, \\ \gamma_3 & \text{if } x \in \{2, 3\}, \end{cases}$$

where γ_1, γ_2 and γ_3 are subsets of U with $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3$. For any $\alpha \in \mathcal{P}(U)$ with $\gamma_1 \subseteq \alpha \subsetneq \gamma_2$, $f_\alpha^\subseteq = \{1\}$ and α is a soft dried S-value for (f, X) .

Theorem 3.19. Let (f, X) be a union soft subalgebra over U with $f : X \rightarrow \mathcal{Q}(U)$. For any soft dried S-value $\alpha \in \mathcal{Q}(U)$ for (f, X) , we have

$$f_\alpha^\subseteq \neq \emptyset \Rightarrow f_\alpha^\subseteq \text{ is an S-energetic subset of } X.$$

Proof. Let $a, b \in X$ be such that $a * b \in f_\alpha^\subseteq$. Then $f(a * b) \subseteq \alpha$, and so $f(a) \cap f(b) \subseteq \alpha$ by (3.6). Thus $f(a) \subseteq \alpha$ or $f(b) \subseteq \alpha$, i.e., $a \in f_\alpha^\subseteq$ or $b \in f_\alpha^\subseteq$. Hence $\{a, b\} \cap f_\alpha^\subseteq \neq \emptyset$. Therefore f_α^\subseteq is an S-energetic subset of X . \square

Corollary 3.20. Let $(U, E) = (U, X)$ where X is a BCK-algebra. Let (f, X) be a union soft ideal over U with $f : X \rightarrow \mathcal{Q}(U)$. For any soft dried S-value $\alpha \in \mathcal{Q}(U)$ for (f, X) , we have

$$f_\alpha^\subseteq \neq \emptyset \Rightarrow f_\alpha^\subseteq \text{ is an S-energetic subset of } X.$$

Theorem 3.21. Let (f, X) be an int-soft subalgebra over U and let $\alpha \in \mathcal{P}(U)$ be such that $f_\alpha^\subseteq \neq \emptyset$. Then α is a soft dried S-value for (f, X) .

Proof. Let $a, b \in X$ be such that $f(a * b) \subseteq \alpha$. Then $\alpha \supseteq f(a * b) \supseteq f(a) \cap f(b)$, which shows that α is a soft dried S-value for (f, X) . \square

Corollary 3.22. Let $(U, E) = (U, X)$ where X is a BCK-algebra. Let (f, X) be an int-soft ideal over U and let $\alpha \in \mathcal{P}(U)$ be such that $f_\alpha^\subseteq \neq \emptyset$. Then α is a soft dried S-value for (f, X) .

Definition 3.23 ([17]). Let X be a BCK/BCI-algebra. A non-empty subset G of X is said to be *I-energetic* if it satisfies:

$$(\forall x, y \in X) (y \in G \Rightarrow \{x, y * x\} \cap G \neq \emptyset). \quad (3.7)$$

Example 3.24 ([17]). Let $X = \{0, 1, 2, a, b\}$ be a BCI-algebra with the following Cayley table:

$*$	0	1	2	a	b
0	0	0	0	b	a
1	1	0	1	b	a
2	2	2	0	b	a
a	a	a	a	0	b
b	b	b	b	a	0

It is routine to verify that $G := \{a, b\}$ is an I-energetic subset of X .

Example 3.25 ([17]). Let $X = \{0, 1, 2, 3, 4\}$ be a BCK -algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	1
2	2	1	0	1	2
3	3	1	1	0	3
4	4	4	4	4	0

It is routine to verify that $G := \{0, 1, 4\}$ is an I-energetic subset of X .

The notion of I-energetic subsets is independent to the notion of S-energetic subsets. In fact, the S-energetic subset $G := \{a, b, c\}$ in Example 3.2 is not an I-energetic subset of X since $\{d, a * d\} \cap G = \emptyset$. Also, in Example 3.25, the I-energetic subset $G := \{0, 1, 4\}$ is not an S-energetic subset of X since $3 * 2 = 1 \in G$ and $\{3, 2\} \cap G = \emptyset$ (see [17]).

Definition 3.26. Let (f, X) be a soft set over U and $\alpha \in \mathcal{P}(U)$ with $f_\alpha^\supseteq \neq \emptyset$. Then α is called a *soft saturated I-value* for (f, X) if the following assertion is valid:

$$(\forall x, y \in X) (f(y) \supseteq \alpha \Rightarrow f(y * x) \cup f(x) \supseteq \alpha). \quad (3.8)$$

Example 3.27. Let $(U, E) = (U, X)$ where $X = \{0, 1, 2, 3\}$ is a BCK -algebra as in Example 3.12. Consider a soft set (f, X) over U in which f is given as follows:

$$f : X \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \gamma_3 & \text{if } x = 0, \\ \gamma_2 & \text{if } x \in \{1, 3\}, \\ \gamma_1 & \text{if } x = 2, \end{cases}$$

where γ_1, γ_2 and γ_3 are subsets of U with $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3$. Put $\alpha \in \mathcal{P}(U)$ with $\gamma_1 \subsetneq \alpha \subseteq \gamma_2$. Then $f_\alpha^\supseteq = \{0, 1, 3\}$. It is easy to check that α is a soft saturated I-value for (f, X) .

Theorem 3.28. Let $(U, E) = (U, X)$ where X is a BCK -algebra. If (f, X) is a union soft subalgebra over U , then every soft saturated I-value for (f, X) is a soft saturated S-value for (f, X) .

Proof. Since (f, X) is a union soft subalgebra over U , $f(0) \subseteq f(x)$ for all $x \in X$. Let $\alpha \in \mathcal{P}(U)$ be a soft saturated I-value for (f, X) . Assume that $f(a * b) \supseteq \alpha$ for all $a, b \in X$. Using (3.8), (2.3), (III) and (V), we have

$$\begin{aligned} \alpha &\subseteq f((a * b) * a) \cup f(a) = f((a * a) * b) \cup f(a) \\ &= f(0 * b) \cup f(a) = f(0) \cup f(a) = f(a). \end{aligned}$$

Thus $f(a) \cup f(b) \supseteq f(a) \supseteq \alpha$ and therefore α is a soft saturated S-value for (f, X) . \square

Corollary 3.29. *Let $(U, E) = (U, X)$ where X is a BCK-algebra. If (f, X) is a union soft ideal over U , then every soft saturated I-value for (f, X) is a soft saturated S-value for (f, X) .*

Proof. Straightforward. □

The converse of Theorem 3.28 is not true as seen in the following example.

Example 3.30. Let $(U, E) = (U, X)$ where $X = \{0, a, b, c\}$ is a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Consider a soft set (f, X) over U in which f is given as follows:

$$f : X \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \gamma_1 & \text{if } x = 0, \\ \gamma_2 & \text{if } x = a, \\ \gamma_3 & \text{if } x \in \{b, c\}, \end{cases}$$

where γ_1, γ_2 and γ_3 are subsets of U with $\gamma_1 \subsetneq \gamma_2 \subsetneq \gamma_3$. Take $\alpha \in \mathcal{P}(U)$ with $\gamma_2 \subsetneq \alpha \subseteq \gamma_3$. Then $f_\alpha^\supseteq = \{b, c\}$. It is easy to check that α is a soft saturated S-value for (f, X) , but not a soft saturated I-value for (f, X) since $f(b) \supseteq \gamma_3$ and $f(b * a) \cup f(a) = f(a) = \gamma_2 \not\supseteq \gamma_3$.

Theorem 3.31. *Let (f, X) be an int-soft ideal over U with $f : X \rightarrow \mathcal{Q}(U)$. Then*

$$(\forall \alpha \in \mathcal{Q}(U)) (f_\alpha^\subseteq \neq \emptyset \Rightarrow f_\alpha^\subseteq \text{ is an I-energetic subset of } X).$$

Proof. Let $x, y \in X$ be such that $y \in f_\alpha^\subseteq$. Then $f(y) \subseteq \alpha$. It follows from (3.4) that

$$\alpha \supseteq f(y) \supseteq f(y * x) \cap f(x).$$

Thus $f(y * x) \subseteq \alpha$ or $f(x) \subseteq \alpha$, i.e., $y * x \in f_\alpha^\subseteq$ or $x \in f_\alpha^\subseteq$. Hence $\{x, y * x\} \cap f_\alpha^\subseteq \neq \emptyset$, and so f_α^\subseteq is an I-energetic subset of X . □

Theorem 3.32. *Let (f, X) be an int-soft ideal over U with $f : X \rightarrow \mathcal{R}(U)$. If $\alpha \in \mathcal{R}(U)$ is a soft saturated I-value for (f, X) , then*

$$f_\alpha^\supseteq \neq \emptyset \Rightarrow f_\alpha^\supseteq \text{ is an I-energetic subset of } X.$$

Proof. Let $x, y \in X$ be such that $y \in f_{\alpha}^{\supseteq}$. Then $f(y) \supseteq \alpha$, which implies from (3.8) that $f(y * x) \cup f(x) \supseteq \alpha$. Hence $f(y * x) \supseteq \alpha$ or $f(x) \supseteq \alpha$, that is, $y * x \in f_{\alpha}^{\supseteq}$ or $x \in f_{\alpha}^{\supseteq}$. Thus $\{x, y * x\} \cap f_{\alpha}^{\supseteq} \neq \emptyset$, and therefore f_{α}^{\supseteq} is an I-energetic subset of X . \square

Theorem 3.33. *Let $\alpha \in \mathcal{P}(U)$ be such that $f_{\alpha}^{\supseteq} \neq \emptyset$. If (f, X) is a union soft ideal over U , then α is a soft saturated I-value for (f, X) .*

Proof. Let $x, y \in X$ be such that $f(y) \supseteq \alpha$. Then $\alpha \subseteq f(y) \subseteq f(y * x) \cup f(x)$ by (2.8). Hence α is a soft saturated I-value for (f, X) . \square

Theorem 3.34. *If (f, X) is a union soft ideal over U with $f : X \rightarrow \mathcal{R}(U)$, then*

$$(\forall \alpha \in \mathcal{R}(U)) (f_{\alpha}^{\supseteq} \neq \emptyset \Rightarrow f_{\alpha}^{\supseteq} \text{ is an I-energetic subset of } X).$$

Proof. Let $x, y \in X$ be such that $y \in f_{\alpha}^{\supseteq}$. Then $f(y) \supseteq \alpha$, and so

$$\alpha \subseteq f(y) \subseteq f(y * x) \cup f(x)$$

by (2.8). Thus $f(y * x) \supseteq \alpha$ or $f(x) \supseteq \alpha$, i.e., $y * x \in f_{\alpha}^{\supseteq}$ or $x \in f_{\alpha}^{\supseteq}$. Hence $\{x, y * x\} \cap f_{\alpha}^{\supseteq} \neq \emptyset$, and so f_{α}^{\supseteq} is an I-energetic subset of X . \square

Definition 3.35. Let (f, X) be a soft set over U and $\alpha \in \mathcal{P}(U)$ with $f_{\alpha}^{\subseteq} \neq \emptyset$. Then α is called a *soft dried I-value* for (f, X) if the following assertion is valid:

$$(\forall x, y \in X) (f(y) \subseteq \alpha \Rightarrow f(y * x) \cap f(x) \subseteq \alpha). \quad (3.9)$$

Example 3.36. Let $(U, E) = (U, X)$ where $X = \{0, a, b, c\}$ is a BCK-algebra as in Example 3.30. Consider the soft set (f, X) over U in Example 3.30. Take $\alpha \in \mathcal{P}(U)$ with $\gamma_2 \subseteq \alpha \subsetneq \gamma_3$. Then $f_{\alpha}^{\subseteq} = \{0, a\}$. It is easy to check that α is a soft dried I-value for (f, X) .

Theorem 3.37. *Let $(U, E) = (U, X)$ where X is a BCK-algebra. If (f, X) is an int-soft subalgebra over U , then every soft dried I-value for (f, X) is a soft dried S-value for (f, X) .*

Proof. Since (f, X) is an int-soft subalgebra over U , $f(0) \supseteq f(x)$ for all $x \in X$. Let $\alpha \in \mathcal{P}(U)$ be a soft dried I-value for (f, X) . Assume that $f(a * b) \subseteq \alpha$ for all $a, b \in X$. Using (3.9), (2.3), (III) and (V), we have

$$\begin{aligned} \alpha &\supseteq f((a * b) * a) \cap f(a) = f((a * a) * b) \cap f(a) \\ &= f(0 * b) \cap f(a) = f(0) \cap f(a) = f(a). \end{aligned}$$

Thus $f(a) \cap f(b) \subseteq f(a) \subseteq \alpha$ and therefore α is a soft dried S-value for (f, X) . \square

Theorem 3.38. *If (f, X) is a union soft ideal over U with $f : X \rightarrow \mathcal{R}(U)$, then*

$$(\forall \alpha \in \mathcal{R}(U)) (f_\alpha^\supset \neq \emptyset \Rightarrow f_\alpha^\supset \text{ is an I-energetic subset of } X).$$

Proof. Let $x, y \in X$ be such that $y \in f_\alpha^\supset$. Then $f(y) \supset \alpha$. It follows from (2.8) that

$$\alpha \subseteq f(y) \subseteq f(y * x) \cup f(x).$$

Thus $f(y * x) \supseteq \alpha$ or $f(x) \supseteq \alpha$, i.e., $y * x \in f_\alpha^\supset$ or $x \in f_\alpha^\supset$. Hence $\{x, y * x\} \cap f_\alpha^\supset \neq \emptyset$, and so f_α^\supset is an I-energetic subset of X . \square

Theorem 3.39. *Let (f, X) be a union soft ideal over U with $f : X \rightarrow \mathcal{Q}(U)$. If $\alpha \in \mathcal{Q}(U)$ is a soft dried I-value for (f, X) , then*

$$f_\alpha^\subseteq \neq \emptyset \Rightarrow f_\alpha^\subseteq \text{ is an I-energetic subset of } X.$$

Proof. Let $x, y \in X$ be such that $y \in f_\alpha^\subseteq$. Then $f(y) \subseteq \alpha$, which implies from (3.9) that $f(y * x) \cap f(x) \subseteq \alpha$. Hence $f(y * x) \subseteq \alpha$ or $f(x) \subseteq \alpha$, that is, $y * x \in f_\alpha^\subseteq$ or $x \in f_\alpha^\subseteq$. Thus $\{x, y * x\} \cap f_\alpha^\subseteq \neq \emptyset$, and therefore f_α^\subseteq is an I-energetic subset of X . \square

Definition 3.40 ([17]). Let Q be a non-empty subset of a BCK/BCI -algebra X . Then Q is said to be *right vanished* if it satisfies:

$$(\forall a, b \in X) (a * b \in Q \Rightarrow a \in Q). \quad (3.10)$$

Q is said to be *right stable* if $Q * X := \{a * x \mid a \in Q, x \in X\} \subseteq Q$.

Theorem 3.41. *Let $(U, E) = (U, X)$ where X is a BCK -algebra and let (f, X) be an int-soft ideal over U . Then f_α^\supset and f_α^\supset are right stable subsets of X for any $\alpha \in \mathcal{P}(U)$ with $f_\alpha^\supset \neq \emptyset \neq f_\alpha^\supset$.*

Proof. Let $x \in X$ and $a \in f_\alpha^\supset$. Then $f(a) \supseteq \alpha$. Since $a * x \leq a$ and (f, X) is an int-soft ideal over U , it follows from Lemma 3.5(1) that $f(a * x) \supseteq f(a) \supseteq \alpha$, i.e., $a * x \in f_\alpha^\supset$. Hence f_α^\supset is a right stable subset of X . Similarly, f_α^\supset is a right stable subset of X . \square

Theorem 3.42. *Let $(U, E) = (U, X)$ where X is a BCK -algebra. If (f, X) is a union soft ideal over U , then f_α^\subseteq and f_α^\subseteq are right stable subsets of X for any $\alpha \in \mathcal{P}(U)$ with $f_\alpha^\subseteq \neq \emptyset \neq f_\alpha^\subseteq$.*

Proof. Let $\alpha \in \mathcal{P}(U)$ and $x, a \in X$ be such that $a \in f_\alpha^\subseteq$. Then $f(a) \subseteq \alpha$. Note that $a * x \leq a$, i.e., $(a * x) * a = 0$. Since (f, X) is a union soft ideal of X , it follows that

$$f(a * x) \subseteq f((a * x) * a) \cup f(a) = f(0) \cup f(a) = f(a) \subseteq \alpha.$$

Hence $a * x \in f_\alpha^\subseteq$, and so f_α^\subseteq is a right stable subset of X . Similarly, f_α^\supseteq is a right stable subset of X . \square

Theorem 3.43. *Let $(U, E) = (U, X)$ where X is a BCK-algebra. If (f, X) is a union soft ideal over U , then f_α^\supseteq and f_α^\supseteq are right vanished subsets of X for any $\alpha \in \mathcal{P}(U)$ with $f_\alpha^\supseteq \neq \emptyset \neq f_\alpha^\supseteq$.*

Proof. Let $\alpha \in \mathcal{P}(U)$ and $a, b \in X$ be such that $a * b \in f_\alpha^\supseteq$. Then $f(a * b) \supseteq \alpha$. Note that $a * b \leq a$, i.e., $(a * b) * a = 0$. Since (f, X) is a union soft ideal of X , it follows from (2.8), (2.3), (III) and (V) that

$$\begin{aligned} \alpha &\subseteq f(a * b) \subseteq f((a * b) * a) \cup f(a) \\ &= f((a * a) * b) \cup f(a) = f(0 * b) \cup f(a) \\ &= f(0) \cup f(a) = f(a), \end{aligned}$$

and so $a \in f_\alpha^\supseteq$. Therefore f_α^\supseteq is a right vanished subset of X . Similarly, f_α^\supseteq is a right vanished subset of X . \square

4 Conclusions

We have introduced the notions of soft saturated values and soft dried values, and discussed their applications in BCK/BCI-algebras. Using these notions, we have investigated several properties of energetic subsets. Using the concepts of int-soft ideals (union ideals), we have explored some properties of right vanished (stable) subsets. Work is on going. Some important issues for further work are:

1. To develop strategies for obtaining more valuable results,
2. To apply these notions and results for studying related notions in other (soft) algebraic structures such as soft (semi-, near-, Γ -) rings, soft lattices, soft BL-algebras, soft R_0 -algebras, soft MV-algebras and soft MTL-algebras, etc.,
3. To study (fuzzy) rough set theoretical aspects based on this article.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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Boundedness from Below of Composition Followed by Differentiation on Bloch-type Spaces

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Abstract. Let φ be an analytic self-map of the unit disk \mathbb{D} . The composition followed by differentiation operator, denoted by DC_φ , is defined by

$$DC_\varphi f(z) = (f \circ \varphi)' = f'(\varphi)\varphi', \quad f \in H(\mathbb{D}).$$

In this paper, under some assumption conditions, we give a necessary and sufficient condition for the operator $DC_\varphi : \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ to be bounded below.

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Keywords: Bloch-type space, composition operator, differentiation operator, bounded below.

1 Introduction

Let \mathbb{D} denote the open unit disk in the complex plane \mathbb{C} and $\partial\mathbb{D}$ be its boundary. Let $H(\mathbb{D})$ be the space of analytic functions on \mathbb{D} . For $0 < \alpha < \infty$, an $f \in H(\mathbb{D})$ is said to belong to Bloch-type space (or α -Bloch space), denoted by $\mathcal{B}^\alpha = \mathcal{B}^\alpha(\mathbb{D})$, if

$$\|f\|_\alpha = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

It is easy to check that \mathcal{B}^α is a Banach space with the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \|f\|_\alpha$. When $\alpha = 1$, $\mathcal{B}^1 = \mathcal{B}$ is the well-known Bloch space.

Throughout the paper, $S(\mathbb{D})$ denotes the set of all analytic self-maps of \mathbb{D} . Associated with $\varphi \in S(\mathbb{D})$ is the composition operator C_φ defined by $C_\varphi f = f \circ \varphi$ for $f \in H(\mathbb{D})$. The main subject in the study of composition operators is to describe operator theoretic properties of C_φ in terms of function theoretic properties of φ . See [4] and the references therein for the study of the composition operator. See [7, 8, 9, 10, 11, 12, 13, 14, 15] for the study of composition operators on Bloch-type spaces.

Let D be the differentiation operator and $\varphi \in S(\mathbb{D})$. The composition followed by differentiation operator, denoted by DC_φ , is defined as follows.

$$DC_\varphi f(z) = (f \circ \varphi)' = f'(\varphi)\varphi', \quad f \in H(\mathbb{D}).$$

In [7], the authors studied the boundedness and compactness of DC_φ between Bloch-type spaces. For example, they obtained the following results:

Theorem A. [7] *Let $\alpha, \beta > 0$ and $\varphi \in S(\mathbb{D})$. Then $DC_\varphi: \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded if and only if*

$$M_1 := \sup_{z \in \mathbb{D}} \frac{|\varphi'(z)|^2(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}} < \infty, \quad M_2 := \sup_{z \in \mathbb{D}} \frac{|\varphi''(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} < \infty.$$

Theorem B. [7] *Let $\alpha, \beta > 0$, $\varphi \in S(\mathbb{D})$ such that $DC_\varphi: \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded. Then $DC_\varphi: \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is compact if and only if*

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{|\varphi'(z)|^2(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}} = 0 \quad (1)$$

and

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{|\varphi''(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} = 0. \quad (2)$$

Recall that the operator $DC_\varphi: \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is said to be bounded, if there exists a $C > 0$, such that $\|DC_\varphi f\|_{\mathcal{B}^\beta} \leq C\|f\|_{\mathcal{B}^\alpha}$ for all $f \in \mathcal{B}^\alpha$. A bounded operator $DC_\varphi: \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is said to be bounded below, if there exists a $\delta > 0$, such that

$$\|DC_\varphi f\|_{\mathcal{B}^\beta} \geq \delta\|f\|_{\mathcal{B}^\alpha}$$

for all $f \in \mathcal{B}^\alpha$. We notice that $DC_\varphi: \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded below if and only if DC_φ has closed range. The boundedness from below of composition operator C_φ on \mathcal{B} was studied by Gathage, Zheng and Zorboska in terms of sampling sets, see [6]. More precisely, they proved that C_φ is bounded below on \mathcal{B} if and only if there exists $\varepsilon > 0$, such that $G_\varepsilon = \varphi(\Omega_\varepsilon)$ is a sampling set for \mathcal{B} , where

$$\Omega_\varepsilon = \{z \in \mathbb{D} : \frac{(1-|z|^2)|\varphi'(z)|}{1-|\varphi(z)|^2} \geq \varepsilon\}.$$

See [1, 2, 5, 6] for other characterizations of the boundedness from below of composition operator on \mathcal{B} . The boundedness from below of multiplication operator on Bloch-type spaces was studied in [3].

In this paper, we give a necessary and sufficient condition for the boundedness from below of the operator $DC_\varphi: \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$, i.e., we obtain the following results.

Theorem 1. *Let $0 < \alpha, \beta < \infty$. Let $\varphi \in S(\mathbb{D})$ such that $\varphi'(z) \not\equiv 0$ and (2) holds. Suppose that DC_φ is bounded from \mathcal{B}^α to \mathcal{B}^β and $\lim_{\varphi(z) \rightarrow \partial\mathbb{D}} \frac{|\varphi'(z)|^2(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}}$ exists. Then $DC_\varphi: \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded below if and only if*

$$\lim_{\varphi(z) \rightarrow \partial\mathbb{D}} \frac{|\varphi'(z)|^2(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}} > 0. \quad (3)$$

Theorem 2. Let $0 < \alpha, \beta < \infty$. Let $\varphi \in S(\mathbb{D})$ such that $\varphi'(z) \not\equiv 0$ and (1) holds. Suppose that DC_φ is bounded from \mathcal{B}^α to \mathcal{B}^β and $\lim_{\varphi(z) \rightarrow \partial\mathbb{D}} \frac{|\varphi''(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}$ exists. Then $DC_\varphi: \mathcal{B}^\alpha \rightarrow \mathcal{B}^\beta$ is bounded below if and only if

$$\lim_{\varphi(z) \rightarrow \partial\mathbb{D}} \frac{|\varphi''(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha} > 0. \quad (4)$$

Throughout the paper, we denote by C a positive constant which may differ from one occurrence to the next. We say that $P \preceq Q$ if there exists a constant C such that $P \leq CQ$. The symbol $P \approx Q$ means that $P \preceq Q \preceq P$.

2 Proof of main results

In this section, we prove the main results in this paper. For this purpose, we need the following lemma.

Lemma 1. Let $\varphi \in S(\mathbb{D})$ such that $\varphi'(z) \not\equiv 0$. Suppose that $\beta > 0$ and $f_n \in H(\mathbb{D})$ for $n = 1, 2, \dots$. If $\|DC_\varphi f_n\|_{\mathcal{B}^\beta} \rightarrow 0$ as $n \rightarrow \infty$, then $f'_n \circ \varphi \rightarrow 0$ as $n \rightarrow \infty$, locally uniformly in \mathbb{D} .

Proof. The proof is similar to the proof of Lemma 2.9 in [3]. For the convenience of the readers, we give the detail of the proof. Since $\varphi'(z) \not\equiv 0$, then for any $r_0 \in (0, 1)$, there exists an r' such that $r_0 < r' < 1$ and $\varphi'(z) \neq 0$ for $|z| = r'$. By Lemma 2.2 of [3],

$$|\varphi'(z)f'_n(\varphi(z))| \leq C_{\beta, r'} \|\varphi' f'_n \circ \varphi\|_{\mathcal{B}^\beta}$$

for $n = 1, 2, \dots$, and $|z| = r'$. Let $\delta = \min_{|z|=r'} |\varphi'(z)| > 0$. Then we have $|f'_n(\varphi(z))| \leq (C_{\beta, r'}/\delta) \|\varphi' f'_n \circ \varphi\|_{\mathcal{B}^\beta}$, for $n = 1, 2, \dots$, and $|z| = r'$. By Maximum principle and the assumption that $\|\varphi' f'_n \circ \varphi\|_{\mathcal{B}^\beta} \rightarrow 0$ as $n \rightarrow \infty$, we have $f'_n \circ \varphi \rightarrow 0$ as $n \rightarrow \infty$, uniformly for $|z| \leq r'$. The proof of the lemma is finished.

Lemma 2. [16] Let m be a positive integer and $\alpha > 0$. Then $f \in \mathcal{B}^\alpha$ if and only if

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^{m+\alpha-1} |f^{(m)}(z)| < \infty.$$

Moreover,

$$\|f\|_{\mathcal{B}^\alpha} \approx \sum_{j=0}^{m-1} |f^{(j)}(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^{m+\alpha-1} |f^{(m)}(z)|.$$

Proof of Theorem 1. Necessity. By the assumption that DC_φ is bounded from \mathcal{B}^α to \mathcal{B}^β , from Theorem A, we have

$$M_1 := \sup_{z \in \mathbb{D}} \frac{|\varphi'(z)|^2 (1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^{\alpha+1}} < \infty.$$

Hence,

$$M_3 := \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |\varphi'(z)|^2 < \sup_{z \in \mathbb{D}} \frac{|\varphi'(z)|^2 (1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^{\alpha+1}} < \infty.$$

Assume that (3) does not hold, i.e., for any $\eta_1 > 0$, there exists a $\delta_1 > 0$ such that

$$\frac{|\varphi'(z)|^2 (1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^{\alpha+1}} < \eta_1 \quad (5)$$

for $|\varphi(z)| > \delta_1$. Let $a_n \in \mathbb{D}$ such that $\varphi(a_n) \rightarrow \partial\mathbb{D}$ as $n \rightarrow \infty$ ($n = 1, 2, \dots$). Set

$$f_n(z) = \frac{1}{\alpha \varphi(a_n)} \frac{1 - |\varphi(a_n)|^2}{(1 - \overline{\varphi(a_n)}z)^\alpha}.$$

It is easy to check that $1 \leq \|f_n\|_\alpha \leq 2^{\alpha+1}$. Since DC_φ from \mathcal{B}^α to \mathcal{B}^β is bounded below, then, there exists a $\delta > 0$ such that

$$\|DC_\varphi f_n\|_{\mathcal{B}^\beta} \geq \delta \|f_n\|_{\mathcal{B}^\alpha} \geq \delta \|f_n\|_\alpha \geq \delta. \quad (6)$$

On the other hand, we obtain

$$\begin{aligned} \|DC_\varphi f_n\|_{\mathcal{B}^\beta} &= |\varphi'(0)f'_n(\varphi(0))| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |(\varphi' \cdot f'_n \circ \varphi)'(z)| \\ &= |\varphi'(0)f'_n(\varphi(0))| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |\varphi'^2(z)f''_n(\varphi(z)) + \varphi''(z)f'_n(\varphi(z))| \\ &\leq |\varphi'(0)f'_n(\varphi(0))| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |\varphi'^2(z)f''_n(\varphi(z))| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |\varphi''(z)f'_n(\varphi(z))| \\ &= |\varphi'(0)f'_n(\varphi(0))| + E_1 + E_2, \end{aligned}$$

where

$$E_1 = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |\varphi'^2(z)f''_n(\varphi(z))| \text{ and } E_2 = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |\varphi''(z)f'_n(\varphi(z))|.$$

First we estimate E_1 . For any $z \in \mathbb{D}$ such that $|\varphi(z)| > \delta_1$, by (5), we have

$$\begin{aligned} &(1 - |z|^2)^\beta |\varphi'^2(z)f''_n(\varphi(z))| \\ &= (\alpha + 1)(1 - |z|^2)^\beta |\varphi'(z)|^2 |\varphi(a_n)| \frac{1 - |\varphi(a_n)|^2}{|1 - \overline{\varphi(a_n)}\varphi(z)|^{\alpha+2}} \\ &\leq (\alpha + 1) \frac{|\varphi'(z)|^2 (1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^{\alpha+1}} \frac{(1 - |\varphi(a_n)|^2)(1 - |\varphi(z)|^2)^{\alpha+1}}{|1 - \overline{\varphi(a_n)}\varphi(z)|^{\alpha+2}} \\ &\leq (\alpha + 1) \frac{|\varphi'(z)|^2 (1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^{\alpha+1}} (1 + |\varphi(a_n)|)(1 + |\varphi(z)|)^{\alpha+1} \\ &\leq (\alpha + 1) 2^{\alpha+2} \eta_1. \end{aligned} \quad (7)$$

For any $\eta_2 > 0$, there exists a positive integer N , $1 - |\varphi(a_n)|^2 < \eta_2$ holds for all $n > N$. For any $z \in \mathbb{D}$ such that $|\varphi(z)| \leq \delta_1$ and $n > N$, we deduce

$$\begin{aligned}
& (1 - |z|^2)^\beta |\varphi'^2(z) f_n''(\varphi(z))| \\
&= (\alpha + 1)(1 - |z|^2)^\beta |\varphi'(z)|^2 |\varphi(a_n)| \frac{1 - |\varphi(a_n)|^2}{|1 - \overline{\varphi(a_n)}\varphi(z)|^{\alpha+2}} \\
&\leq (\alpha + 1)(1 - |z|^2)^\beta |\varphi'(z)|^2 \frac{1 - |\varphi(a_n)|^2}{(1 - |\varphi(z)|)^{\alpha+2}} \\
&\leq (\alpha + 1)(1 - |z|^2)^\beta |\varphi'(z)|^2 \frac{\eta_2}{(1 - \delta_1)^{\alpha+2}} \\
&\leq (\alpha + 1) \frac{M_3}{(1 - \delta_1)^{\alpha+2}} \eta_2.
\end{aligned} \tag{8}$$

From (7) and (8), we have

$$\begin{aligned}
E_1 &\leq \sup_{|\varphi(z)| > \delta_1} (1 - |z|^2) |\varphi'(z)|^2 |f_n''(\varphi(z))| \\
&\quad + \sup_{|\varphi(z)| \leq \delta_1} (1 - |z|^2) |\varphi'(z)|^2 |f_n''(\varphi(z))| \\
&< (\alpha + 1) 2^{\alpha+2} \eta_1 + (\alpha + 1) \frac{M_3}{(1 - \delta_1)^{\alpha+2}} \eta_2, \quad \text{as } n > N.
\end{aligned}$$

By the arbitrary of η_1 and η_2 , we see that $E_1 \rightarrow 0$ as $n \rightarrow \infty$.

Next we estimate E_2 . From (2), for any $\eta_3 > 0$, there exists a $\delta_2 > 0$ such that

$$\frac{|\varphi''(z)|(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^\alpha} < \eta_3, \quad \text{when } |\varphi(z)| > \delta_2.$$

For any $z \in \mathbb{D}$ such that $|\varphi(z)| > \delta_2$,

$$\begin{aligned}
(1 - |z|^2)^\beta |\varphi''(z) f_n'(\varphi(z))| &= (1 - |z|^2)^\beta |\varphi''(z)| \frac{1 - |\varphi(a_n)|^2}{|1 - \overline{\varphi(a_n)}\varphi(z)|^{\alpha+1}} \\
&\leq 2^\alpha (1 + |\varphi(a_n)|) \frac{|\varphi''(z)|(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^\alpha} \\
&< 2^{\alpha+1} \eta_3.
\end{aligned} \tag{9}$$

For any $z \in \mathbb{D}$ such that $|\varphi(z)| \leq \delta_2$ and $n > N$, we have

$$\begin{aligned}
(1 - |z|^2)^\beta |\varphi''(z) f_n'(\varphi(z))| &= (1 - |z|^2)^\beta |\varphi''(z)| \frac{1 - |\varphi(a_n)|^2}{|1 - \overline{\varphi(a_n)}\varphi(z)|^{\alpha+1}} \\
&\leq 2^\alpha \frac{|\varphi''(z)|(1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^\alpha} \frac{1 - |\varphi(a_n)|^2}{1 - |\varphi(z)|} \\
&\leq 2^\alpha \frac{M_2 \eta_2}{1 - \delta_2}.
\end{aligned} \tag{10}$$

Then,

$$\begin{aligned} E_2 &\leq \sup_{|\varphi(z)| > \delta_2} (1 - |z|^2) |\varphi''(z) f'_n(\varphi(z))| + \sup_{|\varphi(z)| \leq \delta_2} (1 - |z|^2) |\varphi''(z) f'_n(\varphi(z))| \\ &\leq 2^{\alpha+1} \eta_3 + 2^\alpha \frac{M_2}{1 - \delta_2} \eta_2, \quad \text{as } n > N. \end{aligned}$$

Since η_2 and η_3 are arbitrary, then $E_2 \rightarrow 0$ as $n \rightarrow \infty$. In addition, $|\varphi'(0) f'_n(\varphi(0))| \rightarrow 0$ as $n \rightarrow \infty$. Therefore, $\|DC_\varphi f_n\|_{\mathcal{B}^\beta} \rightarrow 0$ as $n \rightarrow \infty$, which contradicts (6). Therefore (3) holds.

Sufficiency. Now assume that (3) holds. Denoted

$$\epsilon = \lim_{\varphi(z) \rightarrow \partial \mathbb{D}} \frac{|\varphi'(z)|^2 (1 - |z|^2)^\beta}{(1 - |\varphi(z)|^2)^{\alpha+1}} > 0. \quad (11)$$

Suppose on the contrary that DC_φ is not bounded below from \mathcal{B}^α to \mathcal{B}^β . Then, there exists a sequence $\{f_n\} \subset \mathcal{B}^\alpha$ such that $\|f_n\|_{\mathcal{B}^\alpha} = 1$ for $n = 1, 2, \dots$, and

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |(f'_n(\varphi) \varphi')'(z)| \leq \|DC_\varphi f_n\|_{\mathcal{B}^\beta} \rightarrow 0$$

as $n \rightarrow \infty$. By Lemma 1, $f'_n \circ \varphi \rightarrow 0$ and hence $f'_n \rightarrow 0$ as $n \rightarrow \infty$, locally uniformly in \mathbb{D} . By Cauchy's estimate we see that $f''_n \rightarrow 0$ as $n \rightarrow \infty$, locally uniformly in \mathbb{D} . Let $z_n \in \mathbb{D}$ be a sequence such that

$$(1 - |\varphi(z_n)|^2)^{\alpha+1} |f''_n(\varphi(z_n))| \geq \frac{1}{2}. \quad (12)$$

Since for every $n = 1, 2, \dots$, $\|f_n\|_{\mathcal{B}^\alpha} = 1$, we see that the above $\{z_n\}$ exist by Lemma 2. Then $\varphi(z_n) \rightarrow \partial \mathbb{D}$ as $n \rightarrow \infty$. Hence by (2) and (11), we get

$$\frac{|\varphi''(z_n)| (1 - |z_n|^2)^\beta}{(1 - |\varphi(z_n)|^2)^\alpha} \rightarrow 0 \quad (13)$$

and

$$\frac{|\varphi'(z_n)|^2 (1 - |z_n|^2)^\beta}{(1 - |\varphi(z_n)|^2)^{\alpha+1}} \geq \epsilon/2 \quad (14)$$

for sufficiently large n , respectively. Therefore, by (12), (13), (14) and Lemma 2, we obtain

$$\begin{aligned} \|DC_\varphi f_n\|_{\mathcal{B}^\beta} &\geq (1 - |z|^2)^\beta |(\varphi' \cdot f'_n \circ \varphi)'(z)| \\ &\geq \frac{|\varphi'(z_n)|^2 (1 - |z_n|^2)^\beta}{(1 - |\varphi(z_n)|^2)^{\alpha+1}} (1 - |\varphi(z_n)|^2)^{\alpha+1} |f''_n(\varphi(z_n))| \\ &\quad - \frac{|\varphi''(z_n)| (1 - |z_n|^2)^\beta}{(1 - |\varphi(z_n)|^2)^\alpha} (1 - |\varphi(z_n)|^2)^\alpha |f'_n(\varphi(z_n))| \\ &\geq \frac{\epsilon}{4}, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

We arrive at a contradiction. Therefore DC_φ is bounded below from \mathcal{B}^α to \mathcal{B}^β . This completes the proof of this theorem.

Proof of Theorem 2. The proof of Theorem 2 is similar to the proof of Theorem 1. Hence we omit the details.

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Uni-soft filters of BE -algebras

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Abstract

Further properties of uni-soft filters in a BE -algebra are investigated. The problem of classifying uni-soft filters by their τ -exclusive filter is solved. New uni-soft filter from old one is established.

Keywords: (Self distributive) BE -algebra, Filter, Uni-soft filter,

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1 Introduction

To solve complicated problem in economics, engineering, and environment, we can't successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which we can consider as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties can't be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [8]. Maji et al. [7] and Molodtsov [8] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [8] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from

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the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. [7] described the application of soft set theory to a decision making problem. Maji et al. The algebraic structure of set theories dealing with uncertainties has been studied by some authors. Çağman et al. [3] introduced fuzzy parameterized (FP) soft sets and their related properties. They proposed a decision making method based on FP-soft set theory, and provided an example which shows that the method can be successfully applied to the problems that contain uncertainties. Feng [4] considered the application of soft rough approximations in multicriteria group decision making problems. Aktaş and Çağman [2] studied the basic concepts of soft set theory, and compared soft sets to fuzzy and rough sets, providing examples to clarify their differences. They also discussed the notion of soft groups. As a generalization of a BCK-algebra, Kim and Kim [6] introduced the notion of a *BE*-algebra, and investigated several properties. In [1], Ahn and So introduced the notion of ideals in *BE*-algebras. They gave several descriptions of ideals in *BE*-algebras. Jun et al. [5] introduced the notion of uni-soft filter of a *BE*-algebra, and investigated their properties. They considered characterizations of a uni-soft filter, and provided conditions for a soft set to be a uni-soft filter.

In this paper, we investigate further properties of a uni-soft filter. We solve the problem of classifying uni-soft filters by their τ -exclusive filters. We make a new uni-soft filter from old one.

2 Preliminaries

Let $K(\tau)$ be the class of all algebras of type $\tau = (2, 0)$. By a *BE*-algebra (see [6]) we mean a system $(X; *, 1) \in K(\tau)$ in which the following axioms hold:

$$(\forall x \in X) (x * x = 1), \quad (2.1)$$

$$(\forall x \in X) (x * 1 = 1), \quad (2.2)$$

$$(\forall x \in X) (1 * x = x), \quad (2.3)$$

$$(\forall x, y, z \in X) (x * (y * z) = y * (x * z)). \quad (\text{exchange}) \quad (2.4)$$

A relation " \leq " on a *BE*-algebra X is defined by

$$(\forall x, y \in X) (x \leq y \iff x * y = 1). \quad (2.5)$$

A *BE*-algebra $(X; *, 1)$ is said to be *transitive* (see [1]) if it satisfies:

$$(\forall x, y, z \in X) (y * z \leq (x * y) * (x * z)). \quad (2.6)$$

A BE -algebra $(X; *, 1)$ is said to be *self distributive* (see [6]) if it satisfies:

$$(\forall x, y, z \in X) (x * (y * z) = (x * y) * (x * z)). \quad (2.7)$$

Every self distributive BE -algebra $(X; *, 1)$ satisfies the following properties:

$$(\forall x, y, z \in X) (x \leq y \Rightarrow z * x \leq z * y, y * z \leq x * z). \quad (2.8)$$

$$(\forall x, y \in X) (x * (x * y) = x * y), \quad (2.9)$$

$$(\forall x, y, z \in X) (x * y \leq (z * x) * (z * y)), \quad (2.10)$$

$$(\forall x, y, z \in X) ((x * y) * (x * z) \leq x * (y * z)). \quad (2.11)$$

Note that every self distributive BE -algebra is transitive, but the converse is not true in general (see [1]).

Let $(X; *, 1)$ be a BE -algebra and let F be a non-empty subset of X . Then F is a *filter* of X (see [6]) if

$$(F1) \quad 1 \in F;$$

$$(F2) \quad (\forall x, y \in X) (x * y, x \in F \Rightarrow y \in F).$$

A soft set theory is introduced by Molodtsov [8]. In what follows, let U be an initial universe set and X be a set of parameters. Let $\mathcal{P}(U)$ denotes the power set of U and $A, B, C, \dots \subseteq X$.

A *soft set* $(\tilde{\mathcal{F}}, A)$ of X over U is defined to be the set of ordered pairs

$$(\tilde{\mathcal{F}}, A) := \left\{ (x, \tilde{\mathcal{F}}(x)) : x \in X, \tilde{\mathcal{F}}(x) \in \mathcal{P}(U) \right\},$$

where $\tilde{\mathcal{F}} : X \rightarrow \mathcal{P}(U)$ such that $\tilde{\mathcal{F}}(x) = \emptyset$ if $x \notin A$.

For a soft set $(\tilde{\mathcal{F}}, A)$ of X and a subset τ of U , the τ -*exclusive set* of $(\tilde{\mathcal{F}}, A)$, denoted by $e_A(\tilde{\mathcal{F}}; \tau)$, is defined to be the set

$$e_A(\tilde{\mathcal{F}}; \tau) := \left\{ x \in A \mid \tau \supseteq \tilde{\mathcal{F}}(x) \right\}.$$

For any soft sets $(\tilde{\mathcal{F}}, X)$ and $(\tilde{\mathcal{G}}, X)$ of X , we call $(\tilde{\mathcal{F}}, X)$ a *soft subset* of $(\tilde{\mathcal{G}}, X)$, denoted by $(\tilde{\mathcal{F}}, X) \subseteq (\tilde{\mathcal{G}}, X)$, if $\tilde{\mathcal{F}}(x) \subseteq \tilde{\mathcal{G}}(x)$ for all $x \in X$. The *soft union* of $(\tilde{\mathcal{F}}, X)$ and $(\tilde{\mathcal{G}}, X)$, denoted by $(\tilde{\mathcal{F}}, X) \tilde{\cup} (\tilde{\mathcal{G}}, X)$, is defined to be the soft set $(\tilde{\mathcal{F}} \tilde{\cup} \tilde{\mathcal{G}}, X)$ of X over U in which $\tilde{\mathcal{F}} \tilde{\cup} \tilde{\mathcal{G}}$ is defined by

$$(\tilde{\mathcal{F}} \tilde{\cup} \tilde{\mathcal{G}})(x) = \tilde{\mathcal{F}}(x) \cup \tilde{\mathcal{G}}(x) \text{ for all } x \in M.$$

The *soft intersection* of $(\tilde{\mathcal{F}}, X)$ and $(\tilde{\mathcal{G}}, X)$, denoted by $(\tilde{\mathcal{F}}, X) \tilde{\cap} (\tilde{\mathcal{G}}, X)$, is defined to be the soft set $(\tilde{\mathcal{F}} \tilde{\cap} \tilde{\mathcal{G}}, M)$ of X over U in which $\tilde{\mathcal{F}} \tilde{\cap} \tilde{\mathcal{G}}$ is defined by

$$(\tilde{\mathcal{F}} \tilde{\cap} \tilde{\mathcal{G}})(x) = \tilde{\mathcal{F}}(x) \cap \tilde{\mathcal{G}}(x) \text{ for all } x \in S.$$

3 Uni-soft filters

In what follows, we take a *BE*-algebra X , as a set of parameters unless otherwise specified.

Definition 3.1 ([5]). A soft set $(\tilde{\mathcal{F}}, X)$ of X over U is called a *uni-soft filter* of X if it satisfies:

$$(\forall x \in X) (\tilde{\mathcal{F}}(1) \subseteq \tilde{\mathcal{F}}(x)), \quad (3.1)$$

$$(\forall x, y \in X) (\tilde{\mathcal{F}}(y) \subseteq \tilde{\mathcal{F}}(x * y) \cup \tilde{\mathcal{F}}(x)). \quad (3.2)$$

We make a new uni-soft filter from old one.

Lemma 3.2 ([5]). For a soft set $(\tilde{\mathcal{F}}, X)$ over U , the following are equivalent.

- (i) $(\tilde{\mathcal{F}}, X)$ is a uni-soft filter of X over U .
- (ii) The τ -exclusive set $e_X(\tilde{\mathcal{F}}; \tau)$ is a filter of X for all $\tau \in \mathcal{P}(U)$ with $e_X(\tilde{\mathcal{F}}; \tau) \neq \emptyset$.

Theorem 3.3. For a soft set $(\tilde{\mathcal{F}}, X)$ over U , define a soft set $(\tilde{\mathcal{F}}^*, X)$ over U by

$$\tilde{\mathcal{F}}^* : X \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \tilde{\mathcal{F}}(x) & \text{if } x \in e_X(\tilde{\mathcal{F}}; \tau), \\ U & \text{otherwise} \end{cases}$$

where τ is a nonempty subset of U . If $(\tilde{\mathcal{F}}, X)$ is a uni-soft filter of X over U , then so is $(\tilde{\mathcal{F}}^*, X)$.

Proof. Assume that $(\tilde{\mathcal{F}}, X)$ is a uni-soft filter of X over U . Then $e_X(\tilde{\mathcal{F}}; \tau) (\neq \emptyset)$ is a filter of X over U for all $\tau \subseteq U$ by Lemma 3.2. Hence $1 \in e_X(\tilde{\mathcal{F}}; \tau)$, and so $\tilde{\mathcal{F}}^*(1) = \tilde{\mathcal{F}}(1) \subseteq \tilde{\mathcal{F}}(x) \subseteq \tilde{\mathcal{F}}^*(x)$ for all $x \in X$. Let $x, y \in X$. If $x * y \in e_X(\tilde{\mathcal{F}}; \tau)$ and $x \in e_X(\tilde{\mathcal{F}}; \tau)$, then $y \in e_X(\tilde{\mathcal{F}}; \tau)$. Hence

$$\tilde{\mathcal{F}}^*(y) = \tilde{\mathcal{F}}(y) \subseteq \tilde{\mathcal{F}}(x * y) \cup \tilde{\mathcal{F}}(x) = \tilde{\mathcal{F}}^*(x * y) \cup \tilde{\mathcal{F}}^*(x).$$

If $x * y \notin e_X(\tilde{\mathcal{F}}; \tau)$ or $x \notin e_X(\tilde{\mathcal{F}}; \tau)$, then $\tilde{\mathcal{F}}^*(x * y) = U$ or $\tilde{\mathcal{F}}^*(x) = U$. Thus

$$\tilde{\mathcal{F}}^*(y) \subseteq U = \tilde{\mathcal{F}}^*(x * y) \cup \tilde{\mathcal{F}}^*(x).$$

Therefore $(\tilde{\mathcal{F}}^*, X)$ is a uni-soft filter of X over U . \square

Theorem 3.4. *If $(\tilde{\mathcal{F}}, X)$ and $(\tilde{\mathcal{G}}, X)$ are uni-soft filters of X over U , then the soft union $(\tilde{\mathcal{F}}, X) \tilde{\cup} (\tilde{\mathcal{G}}, X)$ of $(\tilde{\mathcal{F}}, X)$ and $(\tilde{\mathcal{G}}, X)$ is a uni-soft filter of X over U .*

Proof. For any $x \in X$, we have

$$(\tilde{\mathcal{F}} \tilde{\cup} \tilde{\mathcal{G}})(1) = \tilde{\mathcal{F}}(1) \cup \tilde{\mathcal{G}}(1) \subseteq \tilde{\mathcal{F}}(x) \cup \tilde{\mathcal{G}}(x) = (\tilde{\mathcal{F}} \tilde{\cup} \tilde{\mathcal{G}})(x).$$

Let $x, y \in X$. Then

$$\begin{aligned} (\tilde{\mathcal{F}} \tilde{\cup} \tilde{\mathcal{G}})(y) &= \tilde{\mathcal{F}}(y) \cup \tilde{\mathcal{G}}(y) \\ &\subseteq (\tilde{\mathcal{F}}(x * y) \cup \tilde{\mathcal{F}}(x)) \cup (\tilde{\mathcal{G}}(x * y) \cup \tilde{\mathcal{G}}(x)) \\ &= (\tilde{\mathcal{F}}(x * y) \cup \tilde{\mathcal{G}}(x * y)) \cup (\tilde{\mathcal{F}}(x) \cup \tilde{\mathcal{G}}(x)) \\ &= (\tilde{\mathcal{F}} \tilde{\cup} \tilde{\mathcal{G}})(x * y) \cup (\tilde{\mathcal{F}} \tilde{\cup} \tilde{\mathcal{G}})(x). \end{aligned}$$

Hence $(\tilde{\mathcal{F}}, X) \tilde{\cup} (\tilde{\mathcal{G}}, X)$ is a uni-soft filter of X over U . \square

The following example shows that the soft intersection of uni-soft filters of X over U may not be a uni-soft filter of X over U

Example 3.5. Consider a BE -algebra $X = \{1, a, b, c, d, 0\}$ with the Cayley table which is given in Table 1 (see [1]).

Let $E = X$ be the set of parameters and $U = \mathbb{Z}$ be the initial universe set. Define two soft sets $(\tilde{\mathcal{F}}, X)$ and $(\tilde{\mathcal{G}}, X)$ over U as follows:

$$\tilde{\mathcal{F}} : X \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} 4\mathbb{N} & \text{if } x \in \{1, c\} \\ 2\mathbb{Z} & \text{if } x \in \{a, b, d, 0\} \end{cases}$$

and

$$\tilde{\mathcal{G}} : X \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} 8\mathbb{N} & \text{if } x \in \{1, a, b\} \\ 4\mathbb{Z} & \text{if } x \in \{c, d, 0\} \end{cases}$$

Table 1: Cayley table for the “*”-operation

*	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

respectively. It is easy to check that $(\tilde{\mathcal{F}}, X)$ and $(\tilde{\mathcal{G}}, X)$ are uni-soft filters of X over U . But $(\tilde{\mathcal{F}}, X) \tilde{\cap} (\tilde{\mathcal{G}}, X) = (\tilde{\mathcal{F}} \tilde{\cap} \tilde{\mathcal{G}}, X)$ is not a uni-soft filter of X over U , since

$$\begin{aligned}
 (\tilde{\mathcal{F}} \tilde{\cap} \tilde{\mathcal{G}})(d) &= \tilde{\mathcal{F}}(d) \cap \tilde{\mathcal{G}}(d) = 2\mathbb{Z} \cap 4\mathbb{Z} \\
 &= 4\mathbb{Z} \not\subseteq 4\mathbb{N} = 8\mathbb{N} \cup 4\mathbb{N} \\
 &= (\tilde{\mathcal{F}}(a) \cap \tilde{\mathcal{G}}(a)) \cup (\tilde{\mathcal{F}}(c) \cap \tilde{\mathcal{G}}(c)) \\
 &= (\tilde{\mathcal{F}} \tilde{\cap} \tilde{\mathcal{G}})(a) \cup (\tilde{\mathcal{F}} \tilde{\cap} \tilde{\mathcal{G}})(c) \\
 &= (\tilde{\mathcal{F}} \tilde{\cap} \tilde{\mathcal{G}})(c * d) \cup (\tilde{\mathcal{F}} \tilde{\cap} \tilde{\mathcal{G}})(c).
 \end{aligned}$$

Theorem 3.6. Let $(\tilde{\mathcal{F}}, X)$ be a uni-soft filter of X . Let τ_1 and τ_2 be subsets of U such that $\tau_1 \supsetneq \tau_2$. If the τ_1 -exclusive set of $(\tilde{\mathcal{F}}, X)$ is equal to the τ_2 -exclusive set of $(\tilde{\mathcal{F}}, X)$, then there is no $x \in X$ such that $\tau_1 \supsetneq \tilde{\mathcal{F}}(x) \supsetneq \tau_2$.

Proof. Straightforward. □

The converse of Theorem 3.6 is not true in general as seen in the following example.

Example 3.7. Consider a BE -algebra $X = \{1, a, b, c\}$ with the Cayley table which is given in Table 2.

Given $U = X$, consider a soft set $(\tilde{\mathcal{F}}, X)$ of X over U which is given by

$$\tilde{\mathcal{F}} : X \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \emptyset & \text{if } x = 1, \\ \{1, a, c\} & \text{if } x \in \{a, b\}, \\ \{1, a\} & \text{if } x = c. \end{cases}$$

Table 2: Cayley table for the “*”-operation

*	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	a	a	1

Then $(\tilde{\mathcal{F}}, X)$ is a uni-soft filter of X . The τ -exclusive sets of $(\tilde{\mathcal{F}}, X)$ are described as follows:

$$e_X(\tilde{\mathcal{F}}; \tau) = \begin{cases} X & \text{if } \tau \in \{X, \{1, a, c\}\} \\ \{1, c\} & \text{if } \{1, a\} \subseteq \tau \subsetneq \{1, a, c\}, \\ \{1\} & \text{otherwise.} \end{cases}$$

If we take $\tau_1 = X$ and $\tau_2 = \{1, b\}$, then $\tau_1 \supsetneq \tau_2$ and there is no $x \in X$ such that $\tau_1 \supsetneq \tilde{\mathcal{F}}(x) \supsetneq \tau_2$. But $e_X(\tilde{\mathcal{F}}; \tau_1) = X \neq \{1\} = e_X(\tilde{\mathcal{F}}; \tau_2)$.

Theorem 3.8. *Let $(\tilde{\mathcal{F}}, X)$ be a uni-soft filter of X . Let τ_1 and τ_2 be subsets of U such that $\tau_1 \supsetneq \tau_2$ and $\{\tau_1, \tau_2, \tilde{\mathcal{F}}(x)\}$ is totally ordered by set inclusion for all $x \in X$. If there is no $x \in X$ such that $\tau_1 \supsetneq \tilde{\mathcal{F}}(x) \supsetneq \tau_2$, then the τ_1 -exclusive set of $(\tilde{\mathcal{F}}, X)$ is equal to the τ_2 -exclusive set of $(\tilde{\mathcal{F}}, X)$.*

Proof. Since $\tau_1 \supsetneq \tau_2$, we have $e_X(\tilde{\mathcal{F}}; \tau_2) \subseteq e_X(\tilde{\mathcal{F}}; \tau_1)$. If $x \in e_X(\tilde{\mathcal{F}}; \tau_1)$, then $\tau_1 \supsetneq \tilde{\mathcal{F}}(x)$. Since $\{\tau_1, \tau_2, \tilde{\mathcal{F}}(x) \mid x \in X\}$ is totally ordered by set inclusion and there is no $x \in X$ such that $\tau_1 \supsetneq \tilde{\mathcal{F}}(x) \supsetneq \tau_2$, it follows that $\tau_2 \supsetneq \tilde{\mathcal{F}}(x)$, that is, $x \in e_X(\tilde{\mathcal{F}}; \tau_2)$. Therefore the τ_1 -exclusive set of $(\tilde{\mathcal{F}}, X)$ is equal to the τ_2 -exclusive set of $(\tilde{\mathcal{F}}, X)$. \square

We have the following question.

Question. *Given a uni-soft filter $(\tilde{\mathcal{F}}, X)$ of X , does any filter can be represented as a τ -exclusive set of $(\tilde{\mathcal{F}}, X)$?*

The following example shows that the answer to the question above is false.

Example 3.9. Let $X = \{1, a, b, c\}$ be the BE -algebra as in Example 3.7. Given $U = X$, consider a soft set $(\tilde{\mathcal{F}}, X)$ of X over U which is given by

$$\tilde{\mathcal{F}} : X \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \{c\} & \text{if } x = 1, \\ \{1, c\} & \text{if } x \in \{a, b, c\}. \end{cases}$$

Then $(\tilde{\mathcal{F}}, X)$ is a uni-soft filter of X . The τ -exclusive sets of $(\tilde{\mathcal{F}}, X)$ are described as follows:

$$e_X(\tilde{\mathcal{F}}; \tau) = \begin{cases} X & \text{if } \tau \supseteq \{1, c\}, \\ \{1\} & \text{if } \{c\} \subseteq \tau \subsetneq \{1, c\}, \\ \emptyset & \text{otherwise.} \end{cases}$$

The filter $\{1, b\}$ cannot be a τ -exclusive set $e_X(\tilde{\mathcal{F}}; \tau)$, since there is no $\tau \subseteq U$ such that $e_X(\tilde{\mathcal{F}}; \tau) = \{1, b\}$.

However, we have the following theorem.

Theorem 3.10. *Every filter of a BE -algebra can be represented as a τ -exclusive set of a uni-soft filter, that is, given a filter F of a BE -algebra X , there exists a uni-soft filter $(\tilde{\mathcal{F}}, X)$ of X over U such that F is the τ -exclusive set of $(\tilde{\mathcal{F}}, X)$ for a nonempty subset τ of U .*

Proof. Let F be a filter of a BE -algebra X . For a nonempty subset τ of U , define a soft set $(\tilde{\mathcal{F}}, X)$ over U by

$$\tilde{\mathcal{F}} : X \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \tau & \text{if } x \in F, \\ U & \text{if } x \notin F. \end{cases}$$

Obviously, $F = e_X(\tilde{\mathcal{F}}; \tau)$. We now prove that $(\tilde{\mathcal{F}}, X)$ is a uni-soft filter of X . Since $1 \in F = e_X(\tilde{\mathcal{F}}; \tau)$, we have $\tilde{\mathcal{F}}(1) \subseteq \tau \subseteq \tilde{\mathcal{F}}(x)$ for all $x \in X$. Let $x, y \in X$. If $x * y \in F$ and $x \in F$, then $y \in F$ because F is a filter of X . Hence $\tilde{\mathcal{F}}(x * y) = \tilde{\mathcal{F}}(x) = \tilde{\mathcal{F}}(y) = \tau$, and so $\tilde{\mathcal{F}}(y) \subseteq \tilde{\mathcal{F}}(x * y) \cup \tilde{\mathcal{F}}(x)$. If $x * y \notin F$ or $x \notin F$, then $\tilde{\mathcal{F}}(x * y) = U$ or $\tilde{\mathcal{F}}(x) = U$. Hence $\tilde{\mathcal{F}}(y) \subseteq U = \tilde{\mathcal{F}}(x * y) \cup \tilde{\mathcal{F}}(x)$. Therefore $(\tilde{\mathcal{F}}, X)$ is a uni-soft filter of X . \square

Note that if $E = X$ is a finite BE -algebra, then the number of filters of X is finite whereas the number of τ -exclusive sets of a uni-soft filter of X over $U = \mathbb{Z}$ appears to be infinite. But, since every τ -exclusive set is indeed a filter of X , not all these τ -exclusive sets are distinct. The next theorem characterizes this aspect.

Theorem 3.11. Let $(\tilde{\mathcal{F}}, X)$ be a uni-soft filter of X over $U = \mathbb{Z}$ and let $\tau_1 \subsetneq \tau_2 \subseteq U$ be such that $\{\tau_1, \tau_2, \tilde{\mathcal{F}}(x)\}$ is a chain for all $x \in X$. Two τ -exclusive sets $e_X(\tilde{\mathcal{F}}; \tau_1)$ and $e_X(\tilde{\mathcal{F}}; \tau_2)$ are equal if and only if there is no $x \in X$ such that $\tau_1 \subsetneq \tilde{\mathcal{F}}(x) \subsetneq \tau_2$.

Proof. Let τ_1 and τ_2 be subsets of U such that $e_X(\tilde{\mathcal{F}}; \tau_1) = e_X(\tilde{\mathcal{F}}; \tau_2)$. Assume that there exists $x \in X$ such that $\tau_1 \subsetneq \tilde{\mathcal{F}}(x) \subsetneq \tau_2$. Then $e_X(\tilde{\mathcal{F}}; \tau_2)$ is a proper superset of $e_X(\tilde{\mathcal{F}}; \tau_1)$, which contradicts the hypothesis.

Conversely, suppose that there is no $x \in X$ such that $\tau_1 \subsetneq \tilde{\mathcal{F}}(x) \subsetneq \tau_2$. Obviously, $e_X(\tilde{\mathcal{F}}; \tau_2) \supseteq e_X(\tilde{\mathcal{F}}; \tau_1)$. If $x \in e_X(\tilde{\mathcal{F}}; \tau_2)$, then $\tau_2 \supseteq \tilde{\mathcal{F}}(x)$. It follows from the hypothesis that $\tau_1 \supseteq \tilde{\mathcal{F}}(x)$, i.e., $x \in e_X(\tilde{\mathcal{F}}; \tau_1)$. Therefore $e_X(\tilde{\mathcal{F}}; \tau_1) = e_X(\tilde{\mathcal{F}}; \tau_2)$. \square

Let $(\tilde{\mathcal{F}}, X)$ be a soft set of X over U . For any $a, b \in X$ and $k \in \mathbb{N}$, consider the set

$$\tilde{\mathcal{F}}[a^k; b] := \left\{ x \in X \mid \tilde{\mathcal{F}}(a^k * (b * x)) = \tilde{\mathcal{F}}(1) \right\}$$

where $\tilde{\mathcal{F}}(a^k * x) = \tilde{\mathcal{F}}(a * (a * (\cdots * (a * (a * x)) \cdots)))$ in which a appears k -times. Note that $a, b, 1 \in \tilde{\mathcal{F}}[a^k; b]$ for all $a, b \in X$ and $k \in \mathbb{N}$.

Proposition 3.12. Let $(\tilde{\mathcal{F}}, X)$ be a soft set of X over U satisfying the condition (3.1) and $\tilde{\mathcal{F}}(x * y) = \tilde{\mathcal{F}}(x) \cap \tilde{\mathcal{F}}(y)$ for all $x, y \in X$. For any $a, b \in X$ and $k \in \mathbb{N}$, if $x \in \tilde{\mathcal{F}}[a^k; b]$, then $y * x \in \tilde{\mathcal{F}}[a^k; b]$ for all $y \in X$.

Proof. Assume that $x \in \tilde{\mathcal{F}}[a^k; b]$. Then $\tilde{\mathcal{F}}(a^k * (b * x)) = \tilde{\mathcal{F}}(1)$, and so

$$\begin{aligned} \tilde{\mathcal{F}}(a^k * (b * (y * x))) &= \tilde{\mathcal{F}}(a^k * (y * (b * x))) \\ &= \tilde{\mathcal{F}}(y * (a^k * (b * x))) \\ &= \tilde{\mathcal{F}}(y) \cap \tilde{\mathcal{F}}(a^k * (b * x)) \\ &= \tilde{\mathcal{F}}(y) \cap \tilde{\mathcal{F}}(1) = \tilde{\mathcal{F}}(1) \end{aligned}$$

for all $y \in X$ by the exchange property of the operation $*$ and (3.1). Hence $y * x \in \tilde{\mathcal{F}}[a^k; b]$ for all $y \in X$. \square

Proposition 3.13. For any soft set $(\tilde{\mathcal{F}}, X)$ of X over U , if an element $a \in X$ satisfies $a * x = 1$ for all $x \in X$ then $\tilde{\mathcal{F}}[a^k; b] = X = \tilde{\mathcal{F}}[b^k; a]$ for all $b \in X$ and $k \in \mathbb{N}$.

Proof. For any $x \in X$, we have

$$\begin{aligned} \tilde{\mathcal{F}}(a^k * (b * x)) &= \tilde{\mathcal{F}}(a^{k-1} * (a * (b * x))) \\ &= \tilde{\mathcal{F}}(a^{k-1} * (b * (a * x))) \\ &= \tilde{\mathcal{F}}(a^{k-1} * (b * 1)) \\ &= \tilde{\mathcal{F}}(1), \end{aligned}$$

and so $x \in \tilde{\mathcal{F}}[a^k; b]$. Similarly, $x \in \tilde{\mathcal{F}}[b^k; a]$. \square

Proposition 3.14. *Let X be a self distributive BE -algebra and let $(\tilde{\mathcal{F}}, X)$ be an order-reversing soft set of X over U with the property (3.1). If $b \leq c$ in X , then $\tilde{\mathcal{F}}[a^k; c] \subseteq \tilde{\mathcal{F}}[a^k; b]$ for all $a \in X$ and $k \in \mathbb{N}$.*

Proof. Let $a, b, c, \in X$ be such that $b \leq c$. For any $k \in \mathbb{N}$, if $x \in \tilde{\mathcal{F}}[a^k; c]$, then

$$\begin{aligned}\tilde{\mathcal{F}}(1) &= \tilde{\mathcal{F}}(a^k * (c * x)) = \tilde{\mathcal{F}}(c * (a^k * x)) \\ &\supseteq \tilde{\mathcal{F}}(b * (a^k * x)) = \tilde{\mathcal{F}}(a^k * (b * x))\end{aligned}$$

by (2.4) and (2.8), and so $\tilde{\mathcal{F}}(a^k * (b * x)) = \tilde{\mathcal{F}}(1)$. Thus $x \in \tilde{\mathcal{F}}[a^k; b]$, which completes the proof. \square

The following example shows that there exists a soft set $(\tilde{\mathcal{F}}, X)$ of X over U , $a, b \in X$ and $k \in \mathbb{N}$ such that $\tilde{\mathcal{F}}[a^k; b]$ is not a filter of X .

Example 3.15. Consider the BE -algebra $X = \{1, a, b, c\}$ in Example 3.7. Let $(\tilde{\mathcal{F}}, X)$ be a soft set of X over $U = \mathbb{N}$ which is given by

$$\tilde{\mathcal{F}} : X \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} 6\mathbb{N} & \text{if } x = 1, \\ 3\mathbb{N} & \text{if } x \in \{a, b, c\}. \end{cases}$$

Then it is a soft set of X over U . But $\tilde{\mathcal{F}}[c; b] = \{x \in X \mid \tilde{\mathcal{F}}(c * (b * x)) = \tilde{\mathcal{F}}(1)\} = \{1, a, b\}$ is not a filter, since $a * c = a \in \tilde{\mathcal{F}}[c; b]$ and $c \notin \tilde{\mathcal{F}}[c; b]$.

We provide conditions for a set $\tilde{\mathcal{F}}[a^k; b]$ to be a filter.

Theorem 3.16. *Let $(\tilde{\mathcal{F}}, X)$ be a soft set over X . If X is a self distributive BE -algebra and $\tilde{\mathcal{F}}$ is injective, then $\tilde{\mathcal{F}}[a^k; b]$ is a filter of X for all $a, b \in X$ and $k \in \mathbb{N}$.*

Proof. Assume that X is a self distributive BE -algebra and $\tilde{\mathcal{F}}$ is injective. Obviously, $1 \in \tilde{\mathcal{F}}[a^k; b]$. Let $a, b, x, y \in X$ and $k \in \mathbb{N}$ be such that $x * y \in \tilde{\mathcal{F}}[a^k; b]$ and $x \in \tilde{\mathcal{F}}[a^k; b]$. Then $\tilde{\mathcal{F}}(a^k * (b * x)) = \tilde{\mathcal{F}}(1)$ which implies that $a^k * (b * x) = 1$ since $\tilde{\mathcal{F}}$ is injective.

Using (2.7), we have

$$\begin{aligned}
 \tilde{\mathcal{F}}(1) &= \tilde{\mathcal{F}}(a^k * (b * (x * y))) \\
 &= \tilde{\mathcal{F}}(a^{k-1} * (a * (b * (x * y)))) \\
 &= \tilde{\mathcal{F}}(a^{k-1} * (a * ((b * x) * (b * y)))) \\
 &= \dots \\
 &= \tilde{\mathcal{F}}((a^k * (b * x)) * (a^k * (b * y))) \\
 &= \tilde{\mathcal{F}}(1 * (a^k * (b * y))) \\
 &= \tilde{\mathcal{F}}(a^k * (b * y)),
 \end{aligned}$$

which implies that $y \in \tilde{\mathcal{F}}[a^k; b]$. Therefore $\tilde{\mathcal{F}}[a^k; b]$ is a filter of X for all $a, b \in X$ and $k \in \mathbb{N}$. \square

Theorem 3.17. *Let X be a self distributive BE-algebra. Let $(\tilde{\mathcal{F}}, X)$ be a soft set of X over U satisfying the condition (3.1) and*

$$(\forall x, y \in X) \left(\tilde{\mathcal{F}}(x * y) = \tilde{\mathcal{F}}(x) \cup \tilde{\mathcal{F}}(y) \right). \quad (3.3)$$

Then $\tilde{\mathcal{F}}[a^k; b]$ is a filter of X for all $a, b \in X$ and $k \in \mathbb{N}$.

Proof. Let $a, b \in X$ and $k \in \mathbb{N}$. Obviously, $1 \in \tilde{\mathcal{F}}[a^k; b]$. Let $x, y \in X$ be such that $x * y \in \tilde{\mathcal{F}}[a^k; b]$ and $x \in \tilde{\mathcal{F}}[a^k; b]$. Then $\tilde{\mathcal{F}}(a^k * (b * x)) = \tilde{\mathcal{F}}(1)$, which implies from (3.3) and (3.1) that

$$\begin{aligned}
 \tilde{\mathcal{F}}(1) &= \tilde{\mathcal{F}}(a^k * (b * (x * y))) \\
 &= \tilde{\mathcal{F}}(a^{k-1} * (a * (b * (x * y)))) \\
 &= \tilde{\mathcal{F}}(a^{k-1} * (a * ((b * x) * (b * y)))) \\
 &= \dots \\
 &= \tilde{\mathcal{F}}((a^k * (b * x)) * (a^k * (b * y))) \\
 &= \tilde{\mathcal{F}}(a^k * (b * x)) \cup \tilde{\mathcal{F}}(a^k * (b * y)) \\
 &= \tilde{\mathcal{F}}(1) \cup \tilde{\mathcal{F}}(a^k * (b * y)) \\
 &= \tilde{\mathcal{F}}(a^k * (b * y)).
 \end{aligned}$$

Hence $y \in \tilde{\mathcal{F}}[a^k; b]$ and therefore $\tilde{\mathcal{F}}[a^k; b]$ is a filter of X for all $a, b \in X$ and $k \in \mathbb{N}$. \square

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On q -analogue of Stancu-Schurer-Kantorovich operators based on q -Riemann integral

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Abstract

In the present paper we introduce the Kantorovich type generalization of Stancu-Schurer operators based on q -Riemann integral. A convergence theorem using the well known Bohman-Korovkin criterion is proven and the rate of convergence involving the modulus of continuity is established. Also, we obtain a Voronovskaja type theorem for these operators.

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1 Introduction

In recent years, the applications of q -calculus have played an important role in the area of approximation theory, generalizations of many well-known linear and positive operators based on the q -integers were studied by numbers of authors ([2, 5, 7, 10–12, 14, 16, 18–20]). In 1987, Lupaş [9] introduced and studied q -analogue of Bernstein operators and in 1996 another generalization of these operators were introduced by Philips [17]. More results on q -Bernstein polynomials were obtained by Ostrowska in [15]. In [1], Agratini introduced a new class of q -Bernstein type operators which fix certain polynomials and studied their approximation properties. Very recently, Muraru [14] proposed the q -Bernstein-Schurer

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operators. Agrawal et al. [3] introduced the Stancu type generalization of Bernstein-Schurer operators. Aral et al. [4] also presented many results on convergence of different q -operators recently in their book.

First, we present some basic definitions and notations from q -calculus. Let $q > 0$. For each nonnegative integer k , the q -integer $[k]_q$ and q -factorial $[k]_q!$ are respectively defined by

$$[k]_q := \begin{cases} \frac{1-q^k}{1-q}, & q \neq 1, \\ k, & q = 1, \end{cases} \quad [k]_q! := \begin{cases} [k]_q[k-1]_q \cdots [1]_q, & k \geq 1, \\ 1, & k = 0. \end{cases}$$

For the integers n, k satisfying $n \geq k \geq 0$, the q -binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

We denote $(a+b)_q^k = \prod_{j=0}^{k-1} (a+bq^j)$.

The q -Jackson integral on the interval $[0, b]$ is defined as

$$\int_0^b f(t) d_q t = (1-q)b \sum_{j=0}^{\infty} f(q^j b) q^j, \quad 0 < q < 1,$$

provided that sums converge absolutely. Suppose $0 < a < b$. The q -Jackson integral on the interval $[a, b]$ is defined as

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t, \quad 0 < q < 1.$$

Dalmanoğlu [5], Mahmudov and Sabancigil [12] defined some q -type generalizations of Bernstein-Kantorovich operators using q -Jackson integral. In [18], Ren and Zeng were introduced two kinds of Kantorovich-type q -Bernstein-Stancu operators. The first version is defined using q -Jackson integral as follows

$$S_{n,q}^{(\alpha,\beta)}(f, x) = ([n+1]_q + \beta) \sum_{k=0}^n q^{-k} p_{n,k}(q; x) \int_{\frac{[k]_q + \alpha}{[n+1]_q + \beta}}^{\frac{[k+1]_q + \alpha}{[n+1]_q + \beta}} f(t) d_q t, \quad (1.1)$$

where $0 \leq \alpha \leq \beta$, $f \in C[0, 1]$ and $p_{n,k}(q; x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (1-x)_q^{n-k}$.

To guarantee the positivity of q -Bernstein-Stancu-Kantorovich operators, in [18] is considered the Riemann-type q -integral (see [13]) defined by

$$\int_a^b f(t) d_q^R t = (1-q)(b-a) \sum_{j=0}^{\infty} f(a + (b-a)q^j) q^j. \quad (1.2)$$

Ren and Zeng [18] redefine $S_{n,q}^{(\alpha,\beta)}$ by putting the Riemann-type q -integral into the operators instead of the q -Jackson integral as

$$\tilde{S}_{n,q}^{(\alpha,\beta)}(f, x) = ([n+1]_q + \beta) \sum_{k=0}^n q^{-k} p_{n,k}(q; x) \int_{\frac{[k]_q + \alpha}{[n+1]_q + \beta}}^{\frac{[k+1]_q + \alpha}{[n+1]_q + \beta}} f(t) d_q^R t. \quad (1.3)$$

Very recently, the q -Kantorovich extension of the Bernstein-Schurer operators have been considered by Kumar et al. [8] as follows:

$$K_{n,p}(f, q; x) = [n+1]_q \sum_{k=0}^{n+p} b_{n,k}(q; x) q^{-k} \int_{[k]_q/[n+1]_q}^{[k+1]_q/[n+1]_q} f(t) d_q^R t, \quad (1.4)$$

where $x \in [0, 1]$, $f \in C[0, 1+p]$, $p \in \mathbb{N}^0 = \mathbb{N} \cup \{0\}$ and $b_{n,k}(q; x) = \begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k (1-x)_q^{n+p-k}$.

In the present paper, inspired by the new Kantorovich type generalization of the q -Bernstein-Schurer operators we introduce the Kantorovich type of Stancu-Schurer operators involving the Riemann-type q -integral.

2 Construction of the operators

In this section we construct the Stancu-Schurer-Kantorovich operators based on q -integers. Let $\alpha, \beta \in R$ be such that $0 \leq \alpha \leq \beta$ and $p \in \mathbb{N}^0 = \mathbb{N} \cup \{0\}$, then for any $f \in C[0, 1+p]$, $q \in (0, 1)$, the Stancu-Schurer-Kantorovich operators are defined using q -Riemann integral as follows

$$K_{n,p}^{(\alpha,\beta)}(f, q; x) = ([n+1]_q + \beta) \sum_{k=0}^{n+p} q^{-k} b_{n,k}(q; x) \int_{\frac{[k]_q + \alpha}{[n+1]_q + \beta}}^{\frac{[k+1]_q + \alpha}{[n+1]_q + \beta}} f(t) d_q^R t. \quad (2.1)$$

Lemma 2.1. Let $K_{n,p}^{(\alpha,\beta)}(f, q; x)$ be given by (2.1). Then the following equalities hold:

- (i) $K_{n,p}^{(\alpha,\beta)}(1, q; x) = 1$;
- (ii) $K_{n,p}^{(\alpha,\beta)}(t, q; x) = \frac{1}{[n+1]_q + \beta} \left\{ \frac{1}{[2]_q} + \alpha + \frac{2q}{[2]_q} [n+p]_q x \right\}$;
- (iii) $K_{n,p}^{(\alpha,\beta)}(t^2, q; x) = \frac{1}{([n+1]_q + \beta)^2} \left\{ q \left(1 + \frac{2(q-1)}{[2]_q} + \frac{(q-1)^2}{[3]_q} \right) [n+p]_q [n+p-1]_q x^2 + \left(\frac{3q+1}{[2]_q} + \frac{4\alpha q}{[2]_q} + \frac{q^2-1}{[3]_q} \right) [n+p]_q x + \frac{1}{[3]_q} + \frac{2\alpha}{[2]_q} + \alpha^2 \right\}$.

Proof. By definition of q -Riemann integral (1.2), we have

$$\int_{\frac{[k]_q + \alpha}{[n+1]_q + \beta}}^{\frac{[k+1]_q + \alpha}{[n+1]_q + \beta}} d_q^R t = \frac{q^k}{[n+1]_q + \beta}; \quad (2.2)$$

$$\int_{\frac{[k]_q + \alpha}{[n+1]_q + \beta}}^{\frac{[k+1]_q + \alpha}{[n+1]_q + \beta}} t d_q^R t = \frac{1}{([n+1]_q + \beta)^2} \left\{ q^k ([k]_q + \alpha) + \frac{q^{2k}}{[2]_q} \right\}; \quad (2.3)$$

$$\int_{\frac{[k]_q + \alpha}{[n+1]_q + \beta}}^{\frac{[k+1]_q + \alpha}{[n+1]_q + \beta}} t^2 d_q^R t = \frac{1}{([n+1]_q + \beta)^3} \left\{ q^k ([k]_q + \alpha)^2 + \frac{2q^{2k}}{[2]_q} ([k]_q + \alpha) + \frac{q^{3k}}{[3]_q} \right\}. \quad (2.4)$$

The following identities hold

$$\sum_{k=0}^{n+p} b_{n,k}(q; x) q^k = 1 - (1-q)[n+p]_q x; \quad (2.5)$$

$$\sum_{k=0}^{n+p} b_{n,k}(q; x) q^{2k} = 1 - (1-q^2)[n+p]_q x + q(1-q)^2 [n+p]_q [n+p-1]_q x^2. \quad (2.6)$$

Hence, by using equality $\sum_{k=0}^{n+p} b_{n,k}(q; x) = 1$ and equation (2.2), we get

$$K_{n,p}^{(\alpha,\beta)}(1, q; x) = 1.$$

By using relations (2.3) and (2.5) we have

$$\begin{aligned} K_{n,p}^{(\alpha,\beta)}(t, q; x) &= \frac{1}{[n+1]_q + \beta} \sum_{k=0}^{n+p} q^{-k} b_{n,k}(q; x) \left\{ q^k ([k]_q + \alpha) + \frac{q^{2k}}{[2]_q} \right\} \\ &= \frac{1}{[n+1]_q + \beta} \left\{ \frac{1}{[2]_q} + \alpha + \frac{2q}{[2]_q} [n+p]_q x \right\}. \end{aligned} \quad (2.7)$$

Now, from the equations (2.4) and (2.6), we get

$$\begin{aligned} &K_{n,p}^{(\alpha,\beta)}(t^2, q; x) \\ &= \frac{1}{([n+1]_q + \beta)^2} \sum_{k=0}^{n+p} q^{-k} b_{n,k}(q; x) \left\{ q^k ([k]_q + \alpha)^2 + \frac{2q^{2k}}{[2]_q} ([k]_q + \alpha) + \frac{q^{3k}}{[3]_q} \right\} \\ &= \frac{1}{([n+1]_q + \beta)^2} \sum_{k=0}^{n+p} b_{n,k}(q; x) \left\{ \left(\frac{1}{1-q} + \alpha \right)^2 \right. \\ &\quad \left. + 2q^k \left[\frac{1}{[2]_q} \left(\frac{1}{1-q} + \alpha \right) - \left(\frac{1}{1-q} + \alpha \right) \frac{1}{1-q} \right] + q^{2k} \left(\frac{1}{(1-q)^2} - \frac{2}{(1-q)[2]_q} + \frac{1}{[3]_q} \right) \right\} \\ &= \frac{1}{([n+1]_q + \beta)^2} \left\{ q \left(1 + \frac{2(q-1)}{[2]_q} + \frac{(q-1)^2}{[3]_q} \right) [n+p]_q [n+p-1]_q x^2 \right. \\ &\quad \left. + \left(\frac{3q+1}{[2]_q} + \frac{4\alpha q}{[2]_q} + \frac{q^2-1}{[3]_q} \right) [n+p]_q x + \frac{1}{[3]_q} + \frac{2\alpha}{[2]_q} + \alpha^2 \right\}. \end{aligned}$$

□

Remark 2.2. From Lemma 2.1, we get

$$K_{n,p}^{(\alpha,\beta)}(t-x, q; x) = \frac{1 + [2]_q \alpha}{[2]_q ([n+1]_q + \beta)} + \left(\frac{2q}{[2]_q} \frac{[n+p]_q}{[n+1]_q + \beta} - 1 \right) x;$$

$$\begin{aligned} &K_{n,p}^{(\alpha,\beta)}((t-x)^2, q; x) \\ &= K_{n,p}^{(\alpha,\beta)}(t^2; x) - 2x K_{n,p}^{(\alpha,\beta)}(t; x) + x^2 K_{n,p}^{(\alpha,\beta)}(1; x) \\ &= \frac{1}{([n+1]_q + \beta)^2} \left\{ \frac{q^2(4q^2 + q + 1)}{[2]_q [3]_q} [n+p]_q [n+p-1]_q x^2 \right. \\ &\quad \left. + \frac{q(4q^2 + 5q + 3) + 4\alpha q(q^2 + q + 1)}{[2]_q [3]_q} [n+p]_q x + \frac{1}{[3]_q} + \frac{2\alpha}{[2]_q} + \alpha^2 \right\} \\ &\quad - \frac{2x}{[n+1]_q + \beta} \left\{ \frac{1}{[2]_q} + \alpha + \frac{2q}{[2]_q} [n+p]_q x \right\} + x^2 \end{aligned}$$

$$\begin{aligned}
&= \left[\frac{q^2(4q^2 + q + 1)}{[2]_q[3]_q} \frac{[n+p]_q[n+p-1]_q}{([n+1]_q + \beta)^2} - \frac{4q}{[2]_q} \frac{[n+p]_q}{[n+1]_q + \beta} + 1 \right] x^2 \\
&+ \left[\frac{q(4q^2 + 5q + 3) + 4\alpha q(q^2 + q + 1)}{[2]_q[3]_q} \frac{[n+p]_q}{([n+1]_q + \beta)^2} - \frac{2(1 + [2]_q\alpha)}{[2]_q([n+1]_q + \beta)} \right] x \\
&+ \frac{1}{([n+1]_q + \beta)^2} \left(\frac{1}{[3]_q} + \frac{2\alpha}{[2]_q} + \alpha^2 \right).
\end{aligned}$$

Lemma 2.3. For $f \in C[0, p+1]$, we have

$$\|K_{n,p}^{(\alpha,\beta)}(f, q; \cdot)\|_{C[0,1]} \leq \|f\|_{C[0,p+1]},$$

where $\|\cdot\|_{C[0,p+1]}$ is the sup-norm on $[0, p+1]$.

Proof. We have

$$\begin{aligned}
|K_{n,p}^{(\alpha,\beta)}(f, q; \cdot)| &\leq ([n+1]_q + \beta) \sum_{k=0}^{n+p} q^{-k} b_{n,k}(q; x) \int_{\frac{[k]_q + \alpha}{[n+1]_q + \beta}}^{\frac{[k+1]_q + \alpha}{[n+1]_q + \beta}} |f(t)| d_q^R t \\
&\leq \|f\|_{C[0,p+1]} K_{n,p}^{(\alpha,\beta)}(1, q; x) = \|f\|_{C[0,p+1]}.
\end{aligned}$$

□

Lemma 2.4. For each $x \in [0, 1]$ and $0 < q < 1$, we have

$$K_{n,p}^{(\alpha,\beta)}((t-x)^2, q; x) \leq 4C \frac{\varphi^2(x)}{[n+p]_q} + \frac{8(\alpha^2 + 3\beta^2 + 3[p]_q^2 + 4)}{([n+1]_q + \beta)^2}, \quad (2.8)$$

$$K_{n,p}^{(\alpha,\beta)}((t-x)^4, q; x) \leq 64\tilde{C} \frac{\varphi^2(x)}{[n+p]_q^2} + \frac{8^3(\alpha^4 + 27[p]_q^4 + 27\beta^4 + 28)}{([n+1]_q + \beta)^4}, \quad (2.9)$$

where $\varphi^2(x) = x(1-x)$ and C, \tilde{C} are some constants.

Proof. We have

$$\begin{aligned}
&K_{n,p}^{(\alpha,\beta)}((t-x)^2, q; x) \\
&= ([n+1]_q + \beta) \sum_{k=0}^{n+p} q^{-k} b_{n,k}(q; x) \int_{\frac{[k]_q + \alpha}{[n+1]_q + \beta}}^{\frac{[k+1]_q + \alpha}{[n+1]_q + \beta}} (t-x)^2 d_q^R t \\
&= (1-q) \sum_{k=0}^{n+p} b_{n,k}(q; x) \sum_{j=0}^{\infty} \left(\frac{[k]_q + \alpha}{[n+1]_q + \beta} + \frac{q^k}{[n+1]_q + \beta} q^j - x \right)^2 q^j \\
&\leq 2(1-q) \sum_{k=0}^{n+p} b_{n,k}(q; x) \left[\sum_{j=0}^{\infty} \left(\frac{[k]_q + \alpha}{[n+1]_q + \beta} - x \right)^2 q^j + \sum_{j=0}^{\infty} \frac{q^{2k} q^{3j}}{([n+1]_q + \beta)^2} \right] \\
&\leq 2 \sum_{k=0}^{n+p} b_{n,k}(q; x) \left[\frac{[k]_q + \alpha}{[n+1]_q + \beta} - \frac{[k]_q}{[n+p]_q} - \left(x - \frac{[k]_q}{[n+p]_q} \right) \right]^2 \\
&\quad + \frac{2}{[3]_q} \sum_{k=0}^{n+p} b_{n,k}(q; x) \frac{q^{2k}}{([n+1]_q + \beta)^2}
\end{aligned}$$

$$\begin{aligned}
&\leq 4 \sum_{k=0}^{n+p} b_{n,k}(q; x) \left(\frac{[k]_q + \alpha}{[n+1]_q + \beta} - \frac{[k]_q}{[n+p]_q} \right)^2 + 4 \sum_{k=0}^{n+p} b_{n,k}(q; x) \left(x - \frac{[k]_q}{[n+p]_q} \right)^2 \\
&\quad + \frac{2}{[3]_q([n+1]_q + \beta)^2} \\
&\leq 8 \sum_{k=0}^{n+p} b_{n,k}(q; x) \left(\frac{[k]_q + \alpha}{[n+1]_q + \beta} - \frac{[k]_q}{[n+1]_q + \beta} \right)^2 \\
&\quad + 8 \sum_{k=0}^{n+p} b_{n,k}(q; x) \left(\frac{[k]_q}{[n+1]_q + \beta} - \frac{[k]_q}{[n+p]_q} \right)^2 \\
&\quad + 4B_{n+p}((t-x)^2, q; x) + \frac{2}{[3]_q} \frac{1}{([n+1]_q + \beta)^2},
\end{aligned}$$

where $B_n(f, q; x) = \sum_{k=0}^n \binom{n}{k}_q x^k (1-x)^{n-k} f([k]_q/[n]_q)$ is the q -Bernstein operators. On the other hand by [10], we have

$$|B_n((t-x)^m, q; x)| \leq K_m \frac{x(1-x)}{[n]_q^{\lfloor (m+1)/2 \rfloor}},$$

for some constant $K_m > 0$, where $\lfloor a \rfloor$ is the integer part of $a \geq 0$. We find that

$$\begin{aligned}
&K_{n,p}^{(\alpha,\beta)}((t-x)^2, q; x) \\
&\leq \frac{8\alpha^2}{([n+1]_q + \beta)^2} + 8 \sum_{k=0}^{n+p} b_{n,k}(q; x) [k]_q^2 \frac{(q^{n+1}[p]_q - q^{n+p} - \beta)^2}{[n+p]_q^2([n+1]_q + \beta)^2} \\
&\quad + 4C \frac{\varphi^2(x)}{[n+p]_q} + \frac{2}{[3]_q([n+1]_q + \beta)^2} \\
&\leq \frac{8\alpha^2}{([n+1]_q + \beta)^2} + \frac{24([p]^2 + 1 + \beta^2)}{([n+1]_q + \beta)^2} + 4C \frac{\varphi^2(x)}{[n+p]_q} + \frac{2}{[3]_q([n+1]_q + \beta)^2} \\
&\leq 4C \frac{\varphi^2(x)}{[n+p]_q} + \frac{8(\alpha^2 + 3\beta^2 + 3[p]_q^2 + 4)}{([n+1]_q + \beta)^2}.
\end{aligned}$$

Also, we obtain

$$\begin{aligned}
&K_{n,p}^{(\alpha,\beta)}((t-x)^4, q; x) \\
&= ([n+1]_q + \beta) \sum_{k=0}^{n+p} q^{-k} b_{n,k}(q; x) \int_{\frac{[k]_q + \alpha}{[n+1]_q + \beta}}^{\frac{[k+1]_q + \alpha}{[n+1]_q + \beta}} (t-x)^4 d_q^R t \\
&= (1-q) \sum_{k=0}^{n+p} b_{n,k}(q; x) \sum_{j=0}^{\infty} \left(\frac{[k]_q + \alpha}{[n+1]_q + \beta} + \frac{q^k}{[n+1]_q + \beta} q^j - x \right)^4 q^j \\
&\leq 8(1-q) \sum_{k=0}^{n+p} b_{n,k}(q; x) \sum_{j=0}^{\infty} \left(\frac{[k]_q + \alpha}{[n+1]_q + \beta} - x \right)^4 q^j \\
&\quad + 8(1-q) \sum_{k=0}^{n+p} b_{n,k}(q; x) \sum_{j=0}^{\infty} \left(\frac{q^k}{[n+1]_q + \beta} \right)^4 q^{5j}
\end{aligned}$$

$$\begin{aligned}
&= 8 \sum_{k=0}^{n+p} b_{n,k}(q; x) \left(\frac{[k]_q + \alpha}{[n+1]_q + \beta} - x \right)^4 + \frac{8}{[5]_q} \sum_{k=0}^{n+p} b_{n,k}(q; x) \left(\frac{q^k}{[n+1]_q + \beta} \right)^4 \\
&= 8 \sum_{k=0}^{n+p} b_{n,k}(q; x) \left[\frac{[k]_q + \alpha}{[n+1]_q + \beta} - \frac{[k]_q}{[n+p]_q} - \left(x - \frac{[k]_q}{[n+p]_q} \right) \right]^4 \\
&\quad + \frac{8}{[5]_q} \sum_{k=0}^{n+p} b_{n,k}(q; k) \left(\frac{q^k}{[n+1]_q + \beta} \right)^4 \\
&\leq 64B_{n+p}((t-x)^4, q; x) + 64 \sum_{k=0}^{n+p} b_{n,k}(q; x) \left[\frac{[k]_q + \alpha}{[n+1]_q + \beta} - \frac{[k]_q}{[n+p]_q} \right]^4 \\
&\quad + \frac{8}{[5]_q([n+1]_q + \beta)^4} \\
&\leq 64B_{n+p}((t-x)^4, q; x) + 8^3 \sum_{k=0}^{n+p} b_{n,k}(q; x) \left(\frac{[k]_q + \alpha}{[n+1]_q + \beta} - \frac{[k]_q}{[n+1]_q + \beta} \right)^4 \\
&\quad + 8^3 \sum_{k=0}^{n+p} b_{n,k}(q; x) \left(\frac{[k]_q}{[n+1]_q + \beta} - \frac{[k]_q}{[n+p]_q} \right)^4 + \frac{8}{[5]_q([n+1]_q + \beta)^4} \\
&\leq 64\tilde{C} \frac{\varphi^2(x)}{[n+p]_q^2} + \frac{8^3\alpha^4}{([n+1]_q + \beta)^4} + 8^3 \sum_{k=0}^{n+p} b_{n,k}(q; x) [k]_q^4 \frac{(q^{n+1}[p]_q - q^{n+p} - \beta)^4}{[n+p]_q^4 ([n+1]_q + \beta)^4} \\
&\quad + \frac{8}{[5]_q([n+1]_q + \beta)^4} \\
&\leq 64\tilde{C} \frac{\varphi^2(x)}{[n+p]_q^2} + \frac{8^3\alpha^4}{([n+1]_q + \beta)^4} + 24^3 \frac{[p]_q^4 + 1 + \beta^4}{([n+1]_q + \beta)^4} + \frac{8}{[5]_q([n+1]_q + \beta)^4} \\
&\leq 64\tilde{C} \frac{\varphi^2(x)}{[n+p]_q^2} + \frac{8^3(\alpha^4 + 27[p]_q^4 + 27\beta^4 + 28)}{([n+1]_q + \beta)^4}.
\end{aligned}$$

□

3 Direct theorems

In this section we propose to study some approximation properties of the Stancu-Schurer-Kantorovich operators defined in (2.1). First, we prove the basic convergence theorem of $K_{n,p}^{(\alpha,\beta)}$ and then obtain the rate convergence of these operators in term of the modulus of continuity. Further, we study local direct results for the q -analogue of Stancu-Schurer-Kantorovich operators.

Theorem 3.1. *Let $(q_n)_n$, $0 < q_n < 1$ be a sequence satisfying the following conditions*

$$\lim_{n \rightarrow \infty} q_n = 1, \quad \lim_{n \rightarrow \infty} q_n^n = a, \quad a \in [0, 1). \quad (3.1)$$

Then for any $f \in C[0, p+1]$, the sequence $K_{n,p}^{(\alpha,\beta)}(f, q_n; x)$ converges to f uniformly on $[0, 1]$.

Proof. From (3.1) we obtain $[n+1]_{q_n} \rightarrow \infty$ as $n \rightarrow \infty$. Further $\frac{[n+p]_{q_n}}{[n+1]_{q_n}} \rightarrow 1$, hence $K_{n,p}^{(\alpha,\beta)}(1, q_n; x) \rightarrow 1$, $K_{n,p}^{(\alpha,\beta)}(t, q_n; x) \rightarrow x$ and $K_{n,p}^{(\alpha,\beta)}(t^2, q_n; x) \rightarrow x^2$ uniformly on $[0, 1]$ as $n \rightarrow \infty$. Therefore, using the Bohman-Korovkin theorem implies that $K_{n,p}(f, q_n; \cdot)$ converges to f uniformly on $[0, 1]$. \square

Let us consider the following K -functional

$$K_2(f, \delta) := \inf \{ \|f - g\| + \delta \|g''\|, g \in C^2[0, p+1] \}, \quad \text{where } \delta \geq 0. \quad (3.2)$$

It is known (see [6]) there exist an absolute constant $C > 0$ such that

$$K_2(f, \delta) \leq C \omega_2(f, \sqrt{\delta}), \quad (3.3)$$

where the second order modulus of smoothness for $f \in C[0, p+1]$ is defined as

$$\omega_2(f, \delta) := \sup_{0 < h < \delta; x, x+2h \in [0, p+1]} |f(x+2h) - 2f(x+h) + f(x)|, \quad \text{where } \delta > 0.$$

The usual modulus of continuity for $f \in C[0, p+1]$ is defined as

$$\omega(f, \delta) := \sup_{0 < h < \delta; x, x+h \in [0, p+1]} |f(x+h) - f(x)|.$$

Theorem 3.2. Let $(q_n)_n$ be a sequence satisfying conditions (3.1) and $f \in C[0, 1+p]$. Then

$$|K_{n,p}^{(\alpha,\beta)}(f, q_n; x) - f(x)| \leq 2\omega(f, \delta_n),$$

where

$$\delta_n = \sqrt{\frac{C}{[n+p]_{q_n}} + \frac{8(\alpha^2 + 3\beta^2 + 3[p]_{q_n}^2 + 4)}{([n+1]_{q_n} + \beta)^2}},$$

and C is a constant.

Proof. For any $t, x \in [a, b]$, it is known that

$$|f(t) - f(x)| \leq \omega(f, \delta) \left(\frac{(t-x)^2}{\delta^2} + 1 \right).$$

Therefore, we obtain

$$|K_{n,p}^{(\alpha,\beta)}(f, q_n; x) - f(x)| \leq \omega(f, \delta) \left(1 + \frac{1}{\delta^2} K_{n,p}^{(\alpha,\beta)}((t-x)^2, q_n; x) \right).$$

By using the relation (2.8), we obtain the required result. \square

In what follows, we give estimate of the rate of convergence by means of the Lipschitz function for the operators defined in (2.1). Let

$$Lip_M(\gamma) = \{f \in C[0, p+1], |f(t) - f(x)| \leq M|t-x|^\gamma\}, \quad 0 < \gamma \leq 1,$$

be the Lipschitz class.

Theorem 3.3. Let $(q_n)_n$ be a sequence satisfying conditions (3.1) and $f \in Lip_M(\gamma)$. Then

$$|K_{n,p}^{(\alpha,\beta)}(f, q_n; x) - f(x)| \leq M(\delta_n(x))^{\gamma/2},$$

where $\delta_n(x) = K_{n,p}^{(\alpha,\beta)}((t-x)^2, q_n; x)$.

Proof. Since $K_{n,p}^{(\alpha,\beta)}(e_0, q_n; \cdot) = e_0$ and $f \in Lip_M(\gamma)$, we have

$$|K_{n,p}^{(\alpha,\beta)}(f, q_n; x) - f(x)| \leq K_{n,p}^{(\alpha,\beta)}(|f(t) - f(x)|, q_n; x) \leq MK_{n,p}^{(\alpha,\beta)}(|t-x|^\gamma, q_n; x).$$

Applying the Hölder's inequality with $p = \frac{2}{\gamma}$ and $q = \frac{2}{2-\gamma}$, we get

$$|K_{n,p}^{(\alpha,\beta)}(f, q_n; x) - f(x)| \leq MK_{n,p}^{(\alpha,\beta)}(|t-x|^2, q_n; x)^{\frac{\gamma}{2}} = M(\delta_n(x))^{\frac{\gamma}{2}}.$$

□

Theorem 3.4. Let $(q_n)_n$ be a sequence satisfying conditions (3.1) and $f \in C[0, p+1]$. Then, for every $x \in [0, 1]$ we have

$$|K_{n,p}^{(\alpha,\beta)}(f, q_n; x) - f(x)| \leq 4K_2(f, \psi_{n,p}(q_n; x)) + \omega(f, \gamma_{n,p}),$$

where C is a constant and

$$\begin{aligned} \psi_{n,p}(q_n; x) &= \frac{C\varphi^2(x)}{[n+p]_{q_n}} + \frac{12[p]_{q_n}^2 + 7\beta^2 + 3(\alpha+1)^2 + 10}{([n+1]_{q_n} + \beta)^2}, \\ \gamma_{n,p} &= \frac{\alpha + \beta + 2 + 2[p]_{q_n}}{[n+1]_{q_n} + \beta}, \quad \varphi^2(x) = x(1-x). \end{aligned}$$

Proof. We define the auxiliary operators

$$K_{n,p}^{*(\alpha,\beta)}(f, q_n; x) = K_{n,p}^{(\alpha,\beta)}(f, q_n; x) + f(x) - f(a_n x + b_n), \quad (3.4)$$

where

$$a_n = \frac{2q_n}{[2]_{q_n}} \frac{[n+p]_{q_n}}{[n+1]_{q_n} + \beta}, \quad b_n = \frac{1}{[n+1]_{q_n} + \beta} \left(\frac{1}{[2]_{q_n}} + \alpha \right).$$

From Lemma 2.1 we obtain

$$K_{n,p}^{*(\alpha,\beta)}(1, q_n; x) = 1 \text{ and } K_{n,p}^{*(\alpha,\beta)}(t, q_n; x) = x.$$

Using Taylor's formula,

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-s)g''(s)ds, \quad g \in C^2[0, p+1],$$

we get

$$\begin{aligned} &K_{n,p}^{*(\alpha,\beta)}(g, q_n; x) - g(x) \\ &= g'(x)K_{n,p}^{*(\alpha,\beta)}(t-x, q_n; x) + K_{n,p}^{*(\alpha,\beta)}\left(\int_x^t (t-s)g''(s)ds, q_n; x\right) \\ &= K_{n,p}^{*(\alpha,\beta)}\left(\int_x^t (t-s)g''(s)ds, q_n; x\right) \\ &= K_{n,p}^{*(\alpha,\beta)}\left(\int_x^t (t-s)g''(s)ds, q_n; x\right) - \int_x^{a_n x + b_n} (a_n x + b_n - s)g''(s)ds. \end{aligned}$$

Therefore, we have

$$\begin{aligned}
 & |K_{n,p}^{*(\alpha,\beta)}(g, q_n; x) - g(x)| \\
 & \leq K_{n,p}^{(\alpha,\beta)} \left(\left| \int_x^t (t-s)g''(s)ds \right|, q_n; x \right) + \left| \int_x^{a_n x + b_n} (a_n x + b_n - s)g''(s)ds \right| \\
 & \leq K_{n,p}^{(\alpha,\beta)} ((t-x)^2, q_n; x) \|g''\|_{C[0,p+1]} + (a_n x + b_n - x)^2 \|g''\|_{C[0,p+1]}. \quad (3.5)
 \end{aligned}$$

Using the relation (3.4) we obtain

$$\begin{aligned}
 |K_{n,p}^{(\alpha,\beta)}(f, q_n; x) - f(x)| & \leq |K_{n,p}^{*(\alpha,\beta)}(f - g, q_n; x)| + |K_{n,p}^{*(\alpha,\beta)}(g, q_n; x) - g(x)| \\
 & \quad + |f(x) - g(x)| + |f(a_n x + b_n) - f(x)|.
 \end{aligned}$$

Since $\|K_{n,p}^{*(\alpha,\beta)}\|_{C[0,1]} \leq 3\|f\|_{C[0,p+1]}$, and using (3.5) we have

$$\begin{aligned}
 & |K_{n,p}^{(\alpha,\beta)}(f, q_n; x) - f(x)| \\
 & \leq 4\|f - g\|_{C[0,p+1]} + [K_{n,p}^{(\alpha,\beta)}((t-x)^2, q_n; x) + (a_n x + b_n - x)^2] \|g''\|_{C[0,p+1]} \\
 & \quad + \omega(f, |(a_n - 1)x + b_n|).
 \end{aligned}$$

Since

$$\begin{aligned}
 (a_n x + b_n - x)^2 & = \left[\frac{2q_n}{[2]_{q_n}} \frac{[n+p]_{q_n}}{[n+1]_{q_n} + \beta} x + \frac{1}{[n+1]_{q_n} + \beta} \left(\alpha + \frac{1}{[2]_{q_n}} \right) - x \right]^2 \\
 & = \frac{1}{([n+1]_{q_n} + \beta)^2} \left\{ \left(\frac{2q_n}{[2]_{q_n}} [n+p]_{q_n} - [n+1]_{q_n} \right) x - \beta x + \alpha + \frac{1}{[2]_{q_n}} \right\}^2 \\
 & \leq \frac{3}{([n+1]_{q_n} + \beta)^2} \left\{ \left(\frac{-1 + 2q_n^{n+1}[p]_{q_n} - q_n^{n+1}}{1 + q_n} \right)^2 + \beta^2 + \left(\alpha + \frac{1}{[2]_{q_n}} \right)^2 \right\} \\
 & \leq \frac{6 + 24[p]_{q_n}^2 + 3\beta^2 + 3(\alpha + 1)^2}{([n+1]_{q_n} + \beta)^2},
 \end{aligned}$$

we have

$$\begin{aligned}
 K_{n,p}^{(\alpha,\beta)}((t-x)^2, q_n; x) + (a_n x + b_n - x)^2 & \leq \frac{4C\varphi^2(x)}{[n+p]_{q_n}^2} + \frac{4(12[p]_{q_n}^2 + 7\beta^2 + 3(\alpha+1)^2) + 10}{([n+1]_{q_n} + \beta)^2} \\
 & \leq 4\psi_{n,p}(q_n; x).
 \end{aligned}$$

Also $|(a_n - 1)x + b_n| \leq \gamma_{n,p}$. Therefore

$$|K_{n,p}^{(\alpha,\beta)}(f, q_n; x) - f(x)| \leq 4\|f - g\|_{C[0,p+1]} + 4\psi_{n,p}(q_n; x) \|g''\|_{C[0,p+1]} + \omega(f, \gamma_{n,p}).$$

Taking the infimum over all $g \in C^2[0, p+1]$ and using (3.2), the proof of the theorem is completed. \square

4 A Voronovskaya theorem for q -Stancu-Schurer-Kantorovich operators

In this section we shall establish a Voronovskaja type theorem for q -Stancu-Schurer-Kantorovich operators. First, we need the auxiliary results contained in the following lemmas.

Lemma 4.1. *Let $(q_n)_n$ be a sequence satisfying conditions (3.1). Then we have*

$$\begin{aligned}\lim_{n \rightarrow \infty} [n]_{q_n} K_{n,p}^{(\alpha,\beta)}(t-x, q_n; x) &= -\frac{1+a-2ap+2\beta}{2}x + \alpha + \frac{1}{2}, \\ \lim_{n \rightarrow \infty} [n]_{q_n} K_{n,q_n}^{(\alpha,\beta)}((t-x)^2, q_n; x) &= -x^2 + x.\end{aligned}$$

Proof. Using the formulas for $K_{n,p}^{(\alpha,\beta)}(t^i, q_n; x)$, $i = 0, 1, 2$ given in Lemma 2.1, we get

$$\begin{aligned}& \lim_{n \rightarrow \infty} [n]_{q_n} K_{n,p}^{(\alpha,\beta)}(t-x, q_n; x) \\ &= \lim_{n \rightarrow \infty} [n]_{q_n} \left\{ \left(\frac{2q_n}{[2]_{q_n}} \frac{[n+p]_{q_n}}{[n+1]_{q_n} + \beta} - 1 \right) x + \frac{1}{[n+1]_{q_n} + \beta} \left(\frac{1}{[2]_{q_n}} + \alpha \right) \right\} \\ &= \lim_{n \rightarrow \infty} \left\{ \frac{[n]_{q_n}}{[2]_{q_n}([n+1]_{q_n} + \beta)} (-1 - q_n^{n+p+1} + q_n^{n+1}(1+q_n)[p]_{q_n} - [2]_{q_n}\beta)x \right. \\ &\quad \left. + \frac{\alpha[n]_{q_n}}{[n+1]_{q_n} + \beta} + \frac{[n]_{q_n}}{[2]_{q_n}([n+1]_{q_n} + \beta)} \right\} \\ &= -\frac{1+a-2ap+2\beta}{2}x + \alpha + \frac{1}{2},\end{aligned}$$

and

$$\begin{aligned}& \lim_{n \rightarrow \infty} [n]_{q_n} K_{n,p}^{(\alpha,\beta)}((t-x)^2, q_n; x) \\ &= \lim_{n \rightarrow \infty} [n]_{q_n} \left\{ K_{n,p}^{(\alpha,\beta)}(t^2, q_n; x) - x^2 - 2xK_{n,p}^{(\alpha,\beta)}(t-x, q_n; x) \right\} \\ &= \lim_{n \rightarrow \infty} [n]_{q_n} \left(\frac{4q_n^4 + q_n^3 + q_n^2}{[2]_{q_n}[3]_{q_n}} \cdot \frac{[n+p-1]_{q_n}[n+p]_{q_n}}{([n+1]_{q_n} + \beta)^2} - 1 \right) x^2 \\ &\quad + \lim_{n \rightarrow \infty} \frac{q_n(4q_n^2 + 5q_n + 3) + 4\alpha q_n(q_n^2 + q_n + 1)}{[2]_{q_n}[3]_{q_n}([n+1]_{q_n} + \beta)^2} [n]_{q_n}[n+p]_{q_n}x \\ &\quad + \lim_{n \rightarrow \infty} \frac{[n]_{q_n}}{([n+1]_{q_n} + \beta)^2} \left(\alpha^2 + \frac{2\alpha}{[2]_{q_n}} + \frac{1}{[3]_{q_n}} \right) - \lim_{n \rightarrow \infty} 2x[n]_{q_n} K_{n,p}^{(\alpha,\beta)}(t-x, q_n; x) \\ &= -x^2 + x.\end{aligned}$$

□

Lemma 4.2. *Let $(q_n)_n$ be a sequence satisfying conditions (3.1). Then for each $x \in [0, 1]$ we have*

$$K_{n,p}^{(\alpha,\beta)}((t-x)^2, q_n; x) = O\left(\frac{1}{[n]_{q_n}}\right); \quad K_{n,p}^{(\alpha,\beta)}((t-x)^4, q_n; x) = O\left(\frac{1}{[n]_{q_n}^2}\right).$$

Proof. This result follows from Lemma 2.4. \square

The main result of this section is the following Voronovskaja type theorem:

Theorem 4.3. *Let $(q_n)_n$ be a sequence satisfying conditions (3.1) and $f'' \in C[0, p+1]$. Then we have*

$$\begin{aligned} & \lim_{n \rightarrow \infty} [n]_{q_n} (K_{n,p}^{(\alpha,\beta)}(f, q; x) - f(x)) \\ &= \left(-\frac{1+a-2ap+2\beta}{2}x + \alpha + \frac{1}{2} \right) f'(x) + \frac{1}{2} (-x^2 + x) f''(x). \end{aligned}$$

Proof. From the Taylor's theorem, we have

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2}(t-x)^2 f''(x) + \xi(t, x)(t-x)^2, \quad (4.1)$$

where the function $\xi(\cdot, x)$ is the Peano form of remainder, $\xi(\cdot, x) \in C[0, p+1]$ and $\xi(t, x) \rightarrow 0$ as $t \rightarrow x$.

Applying $K_{n,p}^{(\alpha,\beta)}$ on both side of (4.1), we obtain

$$\begin{aligned} & [n]_{q_n} (K_{n,p}^{(\alpha,\beta)}(f, q_n; x) - f(x)) \\ &= [n]_{q_n} f'(x) K_{n,p}^{(\alpha,\beta)}(t-x, q_n; x) + \frac{1}{2} [n]_{q_n} f''(x) K_{n,p}^{(\alpha,\beta)}((t-x)^2, q_n; x) \\ & \quad + [n]_{q_n} K_{n,p}^{(\alpha,\beta)}(\xi(t, x)(t-x)^2, q_n; x). \end{aligned} \quad (4.2)$$

Now, we shall show that

$$\lim_{n \rightarrow \infty} [n]_{q_n} K_{n,p}^{(\alpha,\beta)}(\xi(t, x)(t-x)^2, q_n; x) = 0. \quad (4.3)$$

From the Cauchy-Schwarz inequality, we have

$$K_{n,p}^{(\alpha,\beta)}(\xi(t, x)(t-x)^2, q_n; x) \leq \sqrt{K_{n,p}^{(\alpha,\beta)}(\xi^2(t, x), q_n; x)} \sqrt{K_{n,p}^{(\alpha,\beta)}((t-x)^4, q_n; x)}.$$

From Theorem 3.1, we have

$$\lim_{n \rightarrow \infty} K_{n,p}^{(\alpha,\beta)}(\xi^2(t, x), q_n; x) = \xi^2(x, x) = 0$$

uniformly with respect to $x \in [0, 1]$. Since $K_{n,p}^{(\alpha,\beta)}((t-x)^4, q_n; x) = O\left(\frac{1}{[n]_{q_n}^2}\right)$ (see Lemma 2.4), it follows (4.3). In view of Lemma 4.1 the theorem is proved. \square

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Mathematical analysis of a cell mediated immunity in a virus dynamics model with nonlinear infection rate and removal

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Abstract

In this paper, we investigate the dynamical behavior of a nonlinear model for viral infection with Cytotoxic T Lymphocyte (CTL) immune response. The model is a generalization of several models presented in the literature by considering more general functions for the: (i) intrinsic growth rate of uninfected cells; (ii) incidence rate of infection; (iii) natural death rate of infected cells; (iv) rate at which the infected cells are killed by CTL cells; (v) production and removal rates of viruses; (vi) activation and natural death rates of CTLs. We derive two threshold parameters R_0 (the basic infection reproduction number) and R_1 (the CTL immune response activation number) and establish a set of conditions on the general functions which are sufficient to determine the global dynamics of the model. By using suitable Lyapunov functions and LaSalle's invariance principle, we prove the global asymptotic stability of all equilibria of the model.

Keywords: Viral infection; global stability; CTL immune response; Lyapunov functional.

1 Introduction

During the last decades, several mathematical models have been proposed to describe the interaction of the virus with target cells (see e.g. [1]-[15]). The immune response is universal and necessary to eliminate or control the disease after viral infection. The Cytotoxic T Lymphocyte (CTL) cells are responsible to attack and kill the infected cells. Several viral infection models have been introduced in the literature to model the CTL immune response to several diseases [16]-[20]. The basic virus dynamics model with CTL immune response has four state variables: x , the population of uninfected target cells; y , the population of infected cells; v , the population of free virus particles in the blood; and z , the population of CTL cells. The model equations are as follows [1]:

$$\dot{x} = s - dx - \beta xv, \quad (1)$$

$$\dot{y} = \beta xv - ay - qyz, \quad (2)$$

$$\dot{v} = ky - cv, \quad (3)$$

$$\dot{z} = ryz - \mu z. \quad (4)$$

Parameters s , k and r represent, respectively, the rate at which new healthy cells are generated from the source within the body, the generation rate constant of free viruses produced from the infected cells and the rate at which the CTL cells are produced. Parameters d , a , c and μ are the natural death rate constants of the uninfected target cells, infected cells, free virus particles and CTL cells, respectively. Parameter β is the infection rate constant and q is the removal rate constant of the infected cells due to the CTLs. All the parameters given in model (1)-(4) are positive. Our aim in this paper is to propose and analyze a nonlinear viral infection model which generalizes model (1)-(4) and several models presented in the literature. We consider the

following nonlinear viral infection model with CTL immune response.

$$\dot{x} = n(x) - \psi(x, y, v), \quad (5)$$

$$\dot{y} = \psi(x, y, v) - a\varphi_1(y) - q\varphi_3(z)\varphi_1(y), \quad (6)$$

$$\dot{v} = k\varphi_1(y) - c\varphi_2(v), \quad (7)$$

$$\dot{z} = r\varphi_3(z)\varphi_1(y) - \mu\varphi_3(z), \quad (8)$$

where $n(x)$ represents the intrinsic growth rate of uninfected cells accounting for both production and natural mortality; $\psi(x, y, v)$ denotes the incidence rate of infection; $a\varphi_1(y)$ refers to the natural death rate of infected cells; $q\varphi_3(z)\varphi_1(y)$ represents the rate at which the infected cells are killed by the CTL cells; $k\varphi_1(y)$ denotes the generation rate of free virus particles; $c\varphi_2(v)$ accounts for the removal rate of free virus particles; $r\varphi_3(z)\varphi_1(y)$ and $\mu\varphi_3(z)$ refer to the activation and natural death rates of CTLs, respectively. Functions $n, \psi, \varphi_i, i = 1, 2, 3$ are continuously differentiable and satisfy the following assumptions:

Assumption A1. (i) there exists x_0 such that $n(x_0) = 0, n(x) > 0$ for $x \in [0, x_0]$,

(ii) $n'(x) < 0$ for all $x > 0$,

(iii) there are $s, \bar{s} > 0$ such that $n(x) \leq s - \bar{s}x$ for $x \geq 0$.

Assumption A2. (i) $\psi(x, y, v) > 0$ and $\psi(0, y, v) = \psi(x, y, 0) = 0$ for all $x > 0, y \geq 0, v > 0$,

(ii) $\frac{\partial \psi(x, y, v)}{\partial x} > 0, \frac{\partial \psi(x, y, v)}{\partial y} < 0, \frac{\partial \psi(x, y, v)}{\partial v} > 0$ and $\frac{\partial \psi(x, 0, 0)}{\partial v} > 0$ for all $x > 0, y \geq 0, v > 0$,

(iii) $\frac{d}{dx} \left(\frac{\partial \psi(x, 0, 0)}{\partial v} \right) > 0$ for all $x > 0$.

Assumption A3. (i) $\varphi_j(u) > 0$ for all $u > 0, \varphi_j(0) = 0, j = 1, 2, 3$,

(ii) $\varphi'_j(u) > 0$, for all $u > 0, j = 1, 3, \varphi'_2(u) > 0$, for all $u \geq 0$,

(iii) there are $\alpha_j \geq 0, j = 1, 2, 3$ such that $\varphi_j(u) \geq \alpha_j u$, for all $u \geq 0$.

Assumption A4. $\frac{\psi(x, y, v)}{\varphi_2(v)}$ is decreasing with respect to v for all $v > 0$.

1.1 Properties of solutions

In this subsection, we study some properties of the solution of the model such as the non-negativity and boundedness of solutions.

Proposition. Assume that Assumptions A1-A3 are satisfied. Then there exist positive numbers $L_i, i = 1, 2, 3$, such that the compact set

$$\Gamma = \{(x, y, v, z) \in \mathbb{R}_{\geq 0}^4 : 0 \leq x, y \leq L_1, 0 \leq v \leq L_2, 0 \leq z \leq L_3\}$$

is positively invariant.

Proof. We have

$$\dot{x} \big|_{x=0} = n(0) > 0, \quad \dot{y} \big|_{y=0} = \psi(x, 0, v) \geq 0 \text{ for all } x \geq 0, v \geq 0,$$

$$\dot{v} \big|_{v=0} = k\varphi_1(y) \geq 0 \text{ for all } y \geq 0, \quad \dot{z} \big|_{z=0} = 0.$$

Hence, the orthant $\mathbb{R}_{\geq 0}^4$ is positively invariant for system (5)-(8). Next, we show that the solutions of the system are bounded. Let $T(t) = x(t) + y(t) + \frac{a}{2k}v(t) + \frac{q}{r}z(t)$, then

$$\begin{aligned} \dot{T}(t) &= n(x) - \frac{a}{2}\varphi_1(y) - \frac{ac}{2k}\varphi_2(v) - \frac{q\mu}{r}\varphi_3(z) \leq s - \bar{s}x - \frac{a}{2}\alpha_1y - \frac{ac}{2k}\alpha_2v - \frac{q\mu}{r}\alpha_3z \\ &\leq s - \sigma \left(x + y + \frac{a}{2k}v + \frac{q}{r}z \right) = s - \sigma T(t), \end{aligned}$$

where $\sigma = \min\{\bar{s}, \frac{a}{2}\alpha_1, c\alpha_2, \mu\alpha_3\}$. Then,

$$T(t) \leq T(0)e^{-\sigma t} + \frac{s}{\sigma} (1 - e^{-\sigma t}) = e^{-\sigma t} \left(T(0) - \frac{s}{\sigma} \right) + \frac{s}{\sigma}.$$

Hence, $0 \leq T(t) \leq L_1$ if $T(0) \leq L_1$ for $t \geq 0$ where $L_1 = \frac{s}{\sigma}$. It follows that, $0 \leq x(t), y(t) \leq L_1, 0 \leq v(t) \leq L_2$ and $0 \leq z(t) \leq L_3$ for all $t \geq 0$ if $x(0) + y(0) + \frac{a}{2k}v(0) + \frac{q}{r}z(0) \leq L_1$, where $L_2 = \frac{2kL_1}{a}$ and $L_3 = \frac{rL_1}{q}$. Therefore, $x(t), y(t), v(t)$ and $z(t)$ are all bounded. \square

1.2 The equilibria and threshold parameters

In this subsection we calculate the equilibria of the model and derive two threshold parameters.

Lemma. Assume that Assumptions A1-A4 are satisfied, then there exist two threshold parameters $R_0 > 0$ and $R_1 > 0$ with $R_1 < R_0$ such that

- (i) if $R_0 \leq 1$, then there exists only one positive equilibrium $E_0 \in \Gamma$,
- (ii) if $R_1 \leq 1 < R_0$, then there exist only two positive equilibria $E_0 \in \Gamma$ and $E_1 \in \Gamma$, and
- (iii) if $R_1 > 1$, then there exist three positive equilibria $E_0 \in \Gamma$, $E_1 \in \Gamma$ and $E_2 \in \overset{\circ}{\Gamma}$.

Proof. At any equilibrium we have

$$n(x) - \psi(x, y, v) = 0, \quad (9)$$

$$\psi(x, y, v) - a\varphi_1(y) - q\varphi_1(y)\varphi_3(z) = 0, \quad (10)$$

$$k\varphi_1(y) - c\varphi_2(v) = 0, \quad (11)$$

$$(r\varphi_1(y) - \mu)\varphi_3(z) = 0. \quad (12)$$

From Eq. (12), either $\varphi_3(z) = 0$ or $\varphi_3(z) \neq 0$. If $\varphi_3(z) = 0$, then from Assumption A3 we get, $z = 0$ and from Eqs. (9)-(11) we have

$$n(x) = \psi(x, y, v) = a\varphi_1(y) = \frac{ac\varphi_2(v)}{k}. \quad (13)$$

From Eq. (13), we have $\varphi_1(y) = \frac{n(x)}{a}$, $\varphi_2(v) = \frac{kn(x)}{ac}$. Since φ_1, φ_2 are continuous and strictly increasing functions with $\varphi_1(0) = \varphi_2(0) = 0$, then $\varphi_1^{-1}, \varphi_2^{-1}$ exist and they are also continuous and strictly increasing [21]. Let $\varkappa_1(x) = \varphi_1^{-1}\left(\frac{n(x)}{a}\right)$ and $\varkappa_2(x) = \varphi_2^{-1}\left(\frac{kn(x)}{ac}\right)$, then

$$y = \varkappa_1(x), \quad v = \varkappa_2(x). \quad (14)$$

Obviously from Assumption A1, $\varkappa_1(x), \varkappa_2(x) > 0$ for $x \in [0, x_0]$ and $\varkappa_1(x_0) = \varkappa_2(x_0) = 0$. Substituting from Eq. (14) into Eq. (13) we get

$$\psi(x, \varkappa_1(x), \varkappa_2(x)) - \frac{ac}{k}\varphi_2(\varkappa_2(x)) = 0. \quad (15)$$

We note that, $x = x_0$ is a solution of Eq. (15). Then, from Eq. (14) we have $y = v = 0$, and this leads to the infection-free equilibrium $E_0 = (x_0, 0, 0, 0)$. Let

$$\Phi_1(x) = \psi(x, \varkappa_1(x), \varkappa_2(x)) - \frac{ac}{k}\varphi_2(\varkappa_2(x)) = 0.$$

Then from Assumptions A1-A3, we have

$$\Phi_1(0) = -\frac{ac}{k}\varphi_2(\varkappa_2(0)) < 0,$$

$$\Phi_1(x_0) = \psi(x_0, 0, 0) - \frac{ac}{k}\varphi_2(0) = 0.$$

Moreover,

$$\Phi_1'(x_0) = \frac{\partial\psi(x_0, 0, 0)}{\partial x} + \varkappa_1'(x_0)\frac{\partial\psi(x_0, 0, 0)}{\partial y} + \varkappa_2'(x_0)\frac{\partial\psi(x_0, 0, 0)}{\partial v} - \frac{ac}{k}\varphi_2'(0)\varkappa_2'(x_0).$$

Assumption A2 implies that $\frac{\partial\psi(x_0, 0, 0)}{\partial x} = \frac{\partial\psi(x_0, 0, 0)}{\partial y} = 0$. Also, from Assumption A3, we have $\varphi_2'(0) > 0$, then

$$\Phi_1'(x_0) = \frac{ac}{k}\varkappa_2'(x_0)\varphi_2'(0)\left(\frac{k}{ac\varphi_2'(0)}\frac{\partial\psi(x_0, 0, 0)}{\partial v} - 1\right).$$

From Eq. (14), we get

$$\Phi_1'(x_0) = n'(x_0)\left(\frac{k}{ac\varphi_2'(0)}\frac{\partial\psi(x_0, 0, 0)}{\partial v} - 1\right).$$

From Assumption A1, we have $n'(x_0) < 0$. Therefore, if $\frac{k}{ac\varphi_2'(0)}\frac{\partial\psi(x_0, 0, 0)}{\partial v} > 1$, then $\Phi_1'(x_0) < 0$ and there exists a $x_1 \in (0, x_0)$ such that $\Phi_1(x_1) = 0$. From Eq. (14), we have $y_1 = \varkappa_1(x_1) > 0$ and $v_1 = \varkappa_2(x_1) > 0$.

It follows that, a chronic-infection equilibrium without CTL immune response $E_1 = (x_1, y_1, v_1, 0)$ exists when $\frac{k}{ac\varphi'_2(0)} \frac{\partial\psi(x_0, 0, 0)}{\partial v} > 1$. Let us define

$$R_0 = \frac{k}{ac\varphi'_2(0)} \frac{\partial\psi(x_0, 0, 0)}{\partial v},$$

which describes the average number of newly infected cells generated from one infected cell at the beginning of the infectious process. The other possibility of Eq. (12) is $\varphi_3(z) \neq 0$ which leads to $y_2 = \varphi_1^{-1}\left(\frac{\mu}{r}\right) > 0$ and $v_2 = \varphi_2^{-1}\left(\frac{k\mu}{cr}\right) > 0$. Substituting $y = y_2$ and $v = v_2$ in Eq. (9), we letting

$$\Phi_2(x) = n(x) - \psi(x, y_2, v_2) = 0.$$

Clearly,

$$\Phi_2(0) = n(0) > 0 \text{ and } \Phi_2(x_0) = -\psi(x_0, y_2, v_2) < 0.$$

According to Assumptions A1 and A2, $\Phi_2(x)$ is a strictly decreasing function of x . Thus, there exists a unique $x_2 \in (0, x_0)$ such that $\Phi_2(x_2) = 0$. Now from Eq. (10) we have

$$z_2 = \varphi_3^{-1}\left(\frac{a}{q}\left(\frac{k\psi(x_2, y_2, v_2)}{ac\varphi_2(v_2)} - 1\right)\right).$$

From Assumption A3, we have if $\frac{k\psi(x_2, y_2, v_2)}{ac\varphi_2(v_2)} > 1$, then $z_2 > 0$. Now we define

$$R_1 = \frac{k\psi(x_2, y_2, v_2)}{ac\varphi_2(v_2)},$$

which represents the immune response reproduction ratio which expresses the CTL load during the lifespan of a CTL cell. Hence, z_2 can be rewritten as $z_2 = \varphi_3^{-1}\left(\frac{a}{q}(R_1 - 1)\right)$. It follows that, there is a chronic-infection equilibrium with CTL immune response $E_2 = (x_2, y_2, v_2, z_2)$ if $R_1 > 1$.

Now we show that $E_0, E_1 \in \Gamma$ and $E_2 \in \overset{\circ}{\Gamma}$. Clearly, $E_0 \in \Gamma$. We have $x_1 < x_0$, then from Assumption A1

$$0 = n(x_0) < n(x_1) \leq s - \bar{s}x_1.$$

It follows that

$$0 < x_1 < \frac{s}{\bar{s}} \leq \frac{s}{\sigma} = L_1.$$

From Eqs. (9)-(10), we get

$$a\alpha_1 y_1 \leq a\varphi_1(y_1) = n(x_1) < n(0) \leq s \Rightarrow 0 < y_1 < \frac{s}{a\alpha_1} < \frac{s}{\frac{a}{2}\alpha_1} \leq L_1.$$

Eq. (13) implies that,

$$c\alpha_2 v_1 \leq c\varphi_2(v_1) = k\varphi_1(y_1) = \frac{k}{a}n(x_1) < \frac{k}{a}n(0) \leq \frac{ks}{a} \Rightarrow 0 < v_1 < \frac{ks}{ac\alpha_2} < \frac{2ks}{ac\alpha_2} \leq L_2.$$

Moreover, $z_1 = 0$ and then, $E_1 \in \Gamma$. Let $R_1 > 1$, then one can show that $0 < x_2 < L_1$ and $0 < v_2 < L_2$. From Eq. (10), we have

$$a\varphi_1(y_2) + q\varphi_1(y_2)\varphi_3(z_2) = n(x_2).$$

Then

$$a\alpha_1 y_2 \leq a\varphi_1(y_2) \leq n(x_2) \Rightarrow a\alpha_1 y_2 \leq n(x_2) < n(0) \leq s \Rightarrow 0 < y_2 \leq \frac{s}{a\alpha_1} \leq L_1.$$

and

$$\frac{q\mu\alpha_3}{r} z_2 \leq q\varphi_1(y_2)\varphi_3(z_2) \leq n(x_2) < n(0) \leq s \Rightarrow 0 < z_2 \leq \frac{sr}{q\mu\alpha_3} \leq L_3.$$

Then, $E_2 \in \overset{\circ}{\Gamma}$. Clearly from Assumptions A2 and A4, we obtain

$$\begin{aligned} R_1 &= \frac{k\psi(x_2, y_2, v_2)}{ac\varphi_2(v_2)} < \frac{k\psi(x_2, 0, v_2)}{ac\varphi_2(v_2)} \leq \frac{k}{ac} \lim_{v \rightarrow 0^+} \frac{\psi(x_2, 0, v)}{\varphi_2(v)} \\ &= \frac{k}{ac\varphi'_2(0)} \frac{\partial\psi(x_2, 0, 0)}{\partial v} < \frac{k}{ac\varphi'_2(0)} \frac{\partial\psi(x_0, 0, 0)}{\partial v} = R_0. \quad \square \end{aligned}$$

2 Global stability analysis

In this section, we investigate the global asymptotic stability of the infection-free, chronic-infection without CTL immune response and chronic-infection with CTL immune response equilibria of model (5)-(8) by using direct Lyapunov method and applying LaSalle's invariance principle. Throughout the paper, we define the function $H : (0, \infty) \rightarrow [0, \infty)$ as: $H(w) = w - 1 - \ln w$, where $H(w) \geq 0$ for any $w > 0$ and H has the global minimum $H(1) = 0$.

Theorem 1. Let Assumptions A1-A4 hold true and $R_0 \leq 1$, then the infection-free equilibrium E_0 is globally asymptotically stable (GAS) in Γ .

Proof. We construct a Lyapunov functional as:

$$U_0(x, y, v, z) = x - x_0 - \int_{x_0}^x \lim_{v \rightarrow 0^+} \frac{\psi(x_0, 0, v)}{\psi(\eta, 0, v)} d\eta + y + \frac{a}{k}v + \frac{aq}{rk}z. \quad (16)$$

It is seen that, $U_0(x, y, v, z) > 0$ for all $x, y, v, z > 0$ while $U_0(x, y, v, z)$ reaches its global minimum at E_0 . We calculate $\frac{dU_0}{dt}$ along the solutions of model (5)-(8) as:

$$\begin{aligned} \frac{dU_0}{dt} &= \left(1 - \lim_{v \rightarrow 0^+} \frac{\psi(x_0, 0, v)}{\psi(x, 0, v)}\right) (n(x) - \psi(x, y, v)) + \psi(x, y, v) - a\varphi_1(y) \\ &\quad + \frac{a}{k} (k\varphi_1(y) - c\varphi_2(v) - q\varphi_2(v)\varphi_3(z)) + \frac{aq}{rk} (r\varphi_3(z)\varphi_2(v) - \mu\varphi_3(z)) \\ &= n(x) \left(1 - \lim_{v \rightarrow 0^+} \frac{\psi(x_0, 0, v)}{\psi(x, 0, v)}\right) + \psi(x, y, v) \lim_{v \rightarrow 0^+} \frac{\psi(x_0, 0, v)}{\psi(x, 0, v)} - \frac{ac}{k} \varphi_2(v) - \frac{aq\mu}{rk} \varphi_3(z). \end{aligned} \quad (17)$$

Since $n(x_0) = 0$, then we get

$$\frac{dU_0}{dt} = (n(x) - n(x_0)) \left(1 - \lim_{v \rightarrow 0^+} \frac{\psi(x_0, 0, v)}{\psi(x, 0, v)}\right) + \frac{ac}{k} \left(\frac{k}{ac} \frac{\psi(x, y, v)}{\varphi_2(v)} \lim_{v \rightarrow 0^+} \frac{\psi(x_0, 0, v)}{\psi(x, 0, v)} - 1\right) \varphi_2(v) - \frac{aq\mu}{rk} \varphi_3(z). \quad (18)$$

From Assumptions A2 and A4 we have

$$\frac{\psi(x, y, v)}{\varphi_2(v)} < \frac{\psi(x, 0, v)}{\varphi_2(v)} \leq \lim_{v \rightarrow 0^+} \frac{\psi(x, 0, v)}{\varphi_2(v)} = \frac{1}{\varphi_2'(0)} \frac{\partial \psi(x, 0, 0)}{\partial v}.$$

Then,

$$\begin{aligned} \frac{dU_0}{dt} &\leq (n(x) - n(x_0)) \left(1 - \frac{(\partial \psi(x_0, 0, 0)/\partial v)}{(\partial \psi(x, 0, 0)/\partial v)}\right) + \frac{ac}{k} \left(\frac{k}{ac\varphi_2'(0)} \frac{\partial \psi(x_0, 0, 0)}{\partial v} - 1\right) \varphi_2(v) - \frac{aq\mu}{rk} \varphi_3(z) \\ &= (n(x) - n(x_0)) \left(1 - \frac{(\partial \psi(x_0, 0, 0)/\partial v)}{(\partial \psi(x, 0, 0)/\partial v)}\right) + \frac{ac}{k} (R_0 - 1) \varphi_2(v) - \frac{aq\mu}{rk} \varphi_3(z). \end{aligned} \quad (19)$$

From Assumptions A1 and A2, we have

$$(n(x) - n(x_0)) \left(1 - \frac{(\partial \psi(x_0, 0, 0)/\partial v)}{(\partial \psi(x, 0, 0)/\partial v)}\right) \leq 0.$$

Therefore, if $R_0 \leq 1$, then $\frac{dU_0}{dt} \leq 0$ for all $x, v, z > 0$. We note that solutions of system (5)-(8) limited to Υ , the largest invariant subset of $\{\frac{dU_0}{dt} = 0\}$ [22]. We see that, $\frac{dU_0}{dt} = 0$ if and only if $x(t) = x_0$, $v(t) = 0$ and $z(t) = 0$ for all t . By the above discussion, each element of Υ satisfies $v(t) = 0$ and $z(t) = 0$. Then from Eq. (7) we get

$$\dot{v}(t) = 0 = k\varphi_1(y(t)).$$

It follows from Assumption A3 that, $y(t) = 0$ for all t . Using LaSalle's invariance principle, we derive that E_0 is GAS. \square

To prove the global stability of the two equilibria E_1 and E_2 , we need the following condition on the incidence rate function.

Assumption A5.

$$\left(\frac{\psi(x, y, v)}{\psi(x, y_i, v_i)} - \frac{\varphi_2(v)}{\varphi_2(v_i)} \right) \left(1 - \frac{\psi(x, y_i, v_i)}{\psi(x, y, v)} \right) \leq 0, \quad x, y, v > 0, \quad i = 1, 2$$

Theorem 2. Assume that Assumptions A1-A5 are satisfied and $R_1 \leq 1 < R_0$, then the chronic-infection equilibrium without CTL immune response E_1 is GAS in Γ .

Proof. We define the following Lyapunov functional

$$\begin{aligned} U_1(x, y, v, z) = & x - x_1 - \int_{x_1}^x \frac{\psi(x_1, y_1, v_1)}{\psi(\eta, y_1, v_1)} d\eta + y - y_1 - \int_{y_1}^y \frac{\varphi_1(y_1)}{\varphi_1(\eta)} d\eta \\ & + \frac{a}{k} \left(v - v_1 - \int_{v_1}^v \frac{\varphi_2(v_1)}{\varphi_2(\eta)} d\eta \right) + \frac{q}{r} z. \end{aligned} \quad (20)$$

It is seen that, $U_1(x, y, v, z) > 0$ for all $x, y, v, z > 0$ while $U_1(x, y, v, z)$ reaches its global minimum at E_1 . The time derivative of U_1 along the trajectories of (5)-(8) is given by

$$\begin{aligned} \frac{dU_1}{dt} = & \left(1 - \frac{\psi(x_1, y_1, v_1)}{\psi(x, y_1, v_1)} \right) (n(x) - \psi(x, y, v)) + \left(1 - \frac{\varphi_1(y_1)}{\varphi_1(y)} \right) (\psi(x, y, v) - a\varphi_1(y) - q\varphi_1(y)\varphi_3(z)) \\ & + \frac{a}{k} \left(1 - \frac{\varphi_2(v_1)}{\varphi_2(v)} \right) (k\varphi_1(y) - c\varphi_2(v)) + \frac{q}{r} (r\varphi_1(y)\varphi_3(z) - \mu\varphi_3(z)) \\ = & \left(1 - \frac{\psi(x_1, y_1, v_1)}{\psi(x, y_1, v_1)} \right) n(x) + \psi(x_1, y_1, v_1) \frac{\psi(x, y, v)}{\psi(x, y_1, v_1)} - \frac{\varphi_1(y_1)}{\varphi_1(y)} \psi(x, y, v) \\ & + a\varphi_1(y_1) + q\varphi_1(y_1)\varphi_3(z) - \frac{ac}{k} \varphi_2(v) - a\varphi_1(y) \frac{\varphi_2(v_1)}{\varphi_2(v)} + \frac{ac}{k} \varphi_2(v_1) - \frac{q\mu}{r} \varphi_3(z). \end{aligned} \quad (21)$$

Using the equilibrium conditions for E_1 :

$$n(x_1) = \psi(x_1, y_1, v_1) = a\varphi_1(y_1) = \frac{ac}{k} \varphi_2(v_1),$$

we obtain

$$\begin{aligned} \frac{dU_1}{dt} = & (n(x) - n(x_1)) \left(1 - \frac{\psi(x_1, y_1, v_1)}{\psi(x, y_1, v_1)} \right) + 3a\varphi_1(y_1) - a\varphi_1(y_1) \frac{\psi(x_1, y_1, v_1)}{\psi(x, y_1, v_1)} + a\varphi_1(y_1) \frac{\psi(x, y, v)}{\psi(x, y_1, v_1)} \\ & - a\varphi_1(y_1) \frac{\varphi_1(y_1)\psi(x, y, v)}{\varphi_1(y)\psi(x_1, y_1, v_1)} - a\varphi_1(y_1) \frac{\varphi_2(v)}{\varphi_2(v_1)} - a\varphi_1(y_1) \frac{\varphi_2(v_1)\varphi_1(y)}{\varphi_2(v)\varphi_1(y_1)} + q \left(\varphi_1(y_1) - \frac{\mu}{r} \right) \varphi_3(z). \end{aligned} \quad (22)$$

Collecting terms to get

$$\begin{aligned} \frac{dU_1}{dt} = & (n(x) - n(x_1)) \left(1 - \frac{\psi(x_1, y_1, v_1)}{\psi(x, y_1, v_1)} \right) + a\varphi_1(y_1) \left(\frac{\psi(x, y, v)}{\psi(x, y_1, v_1)} - \frac{\varphi_2(v)}{\varphi_2(v_1)} - 1 + \frac{\varphi_2(v)\psi(x, y_1, v_1)}{\varphi_2(v_1)\psi(x, y, v)} \right) \\ & + a\varphi_1(y_1) \left[4 - \frac{\psi(x_1, y_1, v_1)}{\psi(x, y_1, v_1)} - \frac{\varphi_1(y_1)\psi(x, y, v)}{\varphi_1(y)\psi(x_1, y_1, v_1)} - \frac{\varphi_2(v_1)\varphi_1(y)}{\varphi_2(v)\varphi_1(y_1)} - \frac{\varphi_2(v)\psi(x, y_1, v_1)}{\varphi_2(v_1)\psi(x, y, v)} \right] \\ & + q(\varphi_1(y_1) - \varphi_1(y_2)) \varphi_3(z). \end{aligned} \quad (23)$$

Eq. (23) can be rewritten as:

$$\begin{aligned} \frac{dU_1}{dt} = & (n(x) - n(x_1)) \left(1 - \frac{\psi(x_1, y_1, v_1)}{\psi(x, y_1, v_1)} \right) + a\varphi_1(y_1) \left(\frac{\psi(x, y, v)}{\psi(x, y_1, v_1)} - \frac{\varphi_2(v)}{\varphi_2(v_1)} \right) \left(1 - \frac{\psi(x, y_1, v_1)}{\psi(x, y, v)} \right) \\ & + a\varphi_1(y_1) \left[4 - \frac{\psi(x_1, y_1, v_1)}{\psi(x, y_1, v_1)} - \frac{\varphi_1(y_1)\psi(x, y, v)}{\varphi_1(y)\psi(x_1, y_1, v_1)} - \frac{\varphi_2(v_1)\varphi_1(y)}{\varphi_2(v)\varphi_1(y_1)} - \frac{\varphi_2(v)\psi(x, y_1, v_1)}{\varphi_2(v_1)\psi(x, y, v)} \right] \\ & + q(\varphi_1(y_1) - \varphi_1(y_2)) \varphi_3(z). \end{aligned} \quad (24)$$

From Assumptions A1-A5, we get that the first and second terms of Eq. (24) are less than or equal zero. Since the geometrical mean is less than or equal to the arithmetical mean, the third term of Eq. (24) is also

less than or equal zero. Now, if $R_1 \leq 1$, then E_2 does not exist since $z_2 = \frac{a}{q}(R_1 - 1) \leq 0$. It follows that, $\dot{z}(t) = (r\varphi_1(y) - \mu)\varphi_3(z) \leq 0$, i.e. $\varphi_1(y_1) \leq \varphi_1(y_2)$. It follows from above that, $\frac{dU_1}{dt} \leq 0$ for all $x, y, v, z > 0$. The solutions of system (5)-(8) limited to Υ , the largest invariant subset of $\{(x, y, v, z) : \frac{dU_1}{dt} = 0\}$ [22]. We have $\frac{dU_1}{dt} = 0$ if and only if $x(t) = x_1$, $y(t) = y_1$, $v(t) = v_1$ and $z(t) = 0$. So, Υ contains a unique point, that is E_1 . Thus, the global asymptotic stability of the chronic-infection equilibrium without CTL immune response E_1 follows from LaSalle's invariance principle. \square

Theorem 3. Let Assumptions A1-A5 hold true and $R_1 > 1$, then the chronic-infection equilibrium with CTL immune response E_2 is GAS in $\bar{\Gamma}$.

Proof. We construct a Lyapunov functional as follows:

$$U_2(x, y, v, z) = x - x_2 - \int_{x_2}^x \frac{\psi(x_2, y_2, v_2)}{\psi(\eta, y_2, v_2)} d\eta + y - y_2 - \int_{y_2}^y \frac{\varphi_1(y_2)}{\varphi_1(\eta)} d\eta \\ + \left(\frac{a + q\varphi_3(z_2)}{k} \right) \left(v - v_2 - \int_{v_2}^v \frac{\varphi_2(v_2)}{\varphi_2(\eta)} d\eta \right) + \frac{q}{r} \left(z - z_2 - \int_{z_2}^z \frac{\varphi_3(z_2)}{\varphi_3(\eta)} d\eta \right). \quad (25)$$

We have $U_2(x, y, v, z) > 0$ for all $x, y, v, z > 0$ and $U_2(x_2, y_2, v_2, z_2) = 0$. Calculating the derivative of U_2 along positive solutions of (5)-(8) gives us the following

$$\frac{dU_2}{dt} = \left(1 - \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)} \right) (n(x) - \psi(x, y, v)) + \left(1 - \frac{\varphi_1(y_2)}{\varphi_1(y)} \right) (\psi(x, y, v) - a\varphi_1(y) - q\varphi_1(y)\varphi_3(z)) \\ + \left(\frac{a + q\varphi_3(z_2)}{k} \right) \left(1 - \frac{\varphi_2(v_2)}{\varphi_2(v)} \right) (k\varphi_1(y) - c\varphi_2(v)) + \frac{q}{r} \left(1 - \frac{\varphi_3(z_2)}{\varphi_3(z)} \right) (r\varphi_1(y)\varphi_3(z) - \mu\varphi_3(z)). \quad (26)$$

Collecting terms of Eq. (26) and using $n(x_2) = \psi(x_2, y_2, v_2)$ we obtain

$$\frac{dU_2}{dt} = (n(x) - n(x_2)) \left(1 - \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)} \right) + \psi(x_2, y_2, v_2) - \psi(x_2, y_2, v_2) \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)} \\ + \psi(x, y, v) \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)} - \frac{\varphi_1(y_2)\psi(x, y, v)}{\varphi_1(y)} + a\varphi_1(y_2) + q\varphi_1(y_2)\varphi_3(z) - \frac{ac}{k}\varphi_2(v) - a\varphi_1(y) \frac{\varphi_2(v_2)}{\varphi_2(v)} \\ + \frac{ac}{k}\varphi_2(v_2) - \frac{qc}{k}\varphi_3(z_2)\varphi_2(v) - q\varphi_3(z_2)\varphi_1(y) \frac{\varphi_2(v_2)}{\varphi_2(v)} + \frac{qc}{k}\varphi_3(z_2)\varphi_2(v_2) - \frac{q\mu}{r}\varphi_3(z) + \frac{q\mu}{r}\varphi_3(z_2). \quad (27)$$

By using the equilibrium conditions of E_2

$$\psi(x_2, y_2, v_2) = a\varphi_1(y_2) + q\varphi_1(y_2)\varphi_3(z_2) = \frac{ac}{k}\varphi_2(v_2) + \frac{qc}{k}\varphi_3(z_2)\varphi_2(v_2), \quad \varphi_1(y_2) = \frac{\mu}{r},$$

we obtain

$$\frac{dU_2}{dt} = (n(x) - n(x_2)) \left(1 - \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)} \right) + 3\psi(x_2, y_2, v_2) - \psi(x_2, y_2, v_2) \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)} \\ + \psi(x_2, y_2, v_2) \frac{\psi(x, y, v)}{\psi(x, y_2, v_2)} - \psi(x_2, y_2, v_2) \frac{\varphi_1(y_2)\psi(x, y, v)}{\varphi_1(y)\psi(x_2, y_2, v_2)} \\ - \psi(x_2, y_2, v_2) \frac{\varphi_2(v)}{\varphi_2(v_2)} - a\varphi_1(y_2) \frac{\varphi_2(v_2)\varphi_1(y)}{\varphi_2(v)\varphi_1(y_2)} - q\varphi_1(y_2)\varphi_3(z_2) \frac{\varphi_2(v_2)\varphi_1(y)}{\varphi_2(v)\varphi_1(y_2)}. \quad (28)$$

Collecting terms of Eq. (28), we get

$$\frac{dU_2}{dt} = (n(x) - n(x_2)) \left(1 - \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)} \right) + \psi(x_2, y_2, v_2) \left(\frac{\psi(x, y, v)}{\psi(x, y_2, v_2)} - \frac{\varphi_2(v)}{\varphi_2(v_2)} - 1 + \frac{\varphi_2(v)\psi(x, y_2, v_2)}{\varphi_2(v_2)\psi(x, y, v)} \right) \\ + \psi(x_2, y_2, v_2) \left[4 - \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)} - \frac{\varphi_1(y_2)\psi(x, y, v)}{\varphi_1(y)\psi(x_2, y_2, v_2)} - \frac{\varphi_2(v_2)\varphi_1(y)}{\varphi_2(v)\varphi_1(y_2)} - \frac{\varphi_2(v)\psi(x, y_2, v_2)}{\varphi_2(v_2)\psi(x, y, v)} \right]. \quad (29)$$

We can rewrite (29) as

$$\frac{dU_2}{dt} = (n(x) - n(x_2)) \left(1 - \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)} \right) + \psi(x_2, y_2, v_2) \left(\frac{\psi(x, y, v)}{\psi(x, y_2, v_2)} - \frac{\varphi_2(v)}{\varphi_2(v_2)} \right) \left(1 - \frac{\psi(x, y_2, v_2)}{\psi(x, y, v)} \right) \\ + \psi(x_2, y_2, v_2) \left[4 - \frac{\psi(x_2, y_2, v_2)}{\psi(x, y_2, v_2)} - \frac{\varphi_1(y_2)\psi(x, y, v)}{\varphi_1(y)\psi(x_2, y_2, v_2)} - \frac{\varphi_2(v_2)\varphi_1(y)}{\varphi_2(v)\varphi_1(y_2)} - \frac{\varphi_2(v)\psi(x, y_2, v_2)}{\varphi_2(v_2)\psi(x, y, v)} \right]. \quad (30)$$

We note that from Assumptions A1-A5 and the relationship between the arithmetical and geometrical means, we have $\frac{dU_2}{dt} \leq 0$ for all $x, y, v, z > 0$. The solutions of model (5)-(8) limited to Υ , the largest invariant subset of $\{(x, y, v, z) : \frac{dU_2}{dt} = 0\}$ [22]. We have $\frac{dU_2}{dt} = 0$ if and only if $x(t) = x_2$, $y(t) = y_2$ and $v(t) = v_2$ for all t . Therefore, if $v(t) = v_2$ and $y(t) = y_2$, then from Eq. (6), $\psi(x_2, y_2, v_2) - a\varphi_1(y_2) - q\varphi_1(y_2)\varphi_3(z(t)) = 0$, which gives $z(t) = z_2$ for all t . Thus, $\frac{dU_2}{dt} = 0$ occurs at E_2 . The global asymptotic stability of the chronic-infection equilibrium with CTL immune response E_2 follows from LaSalle's invariance principle. \square

3 Conclusion

In this paper, we have proposed and analyzed a nonlinear viral infection model with CTL immune response. We have considered more general nonlinear functions for the: (i) intrinsic growth rate of uninfected cells; (ii) incidence rate of infection; (iii) natural death rate of infected cells; (iv) rate at which the infected cells are killed by CTL cells; (v) production and removal rates of viruses; (vi) activation and natural death rates of CTLs. We have derived a set of conditions on these general functions and have determined two threshold parameters to prove the existence and the global stability of the model's equilibria. The global asymptotic stability of the three equilibria, infection-free, chronic-infection without CTL immune response and chronic-infection with CTL immune response has been proven by using direct Lyapunov method and LaSalle's invariance principle.

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ON THE STABILITY OF CUBIC LIE *-DERIVATIONS

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ABSTRACT. We will show the general solution of the functional equation $f(sx + y) + f(x - sy) - s^2f(x + y) - sf(x - y) = (s - 1)(s^2 - 1)f(x) - (s + 1)(s^2 - 1)f(y)$ and investigate the stability of cubic Lie *-derivations associated with the given functional equation.

1. INTRODUCTION

The concept of stability problem of a functional equation was first posed by Ulam [14] concerning the stability of group homomorphisms as follows: *Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot, \cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \rightarrow G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$ then there is a homomorphism $H : G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?* In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism then there exists a true homomorphism near it. By the problem raised by Ulam, several stability problems of a large variety of functional equations have been extensively studied and generalized by a number of authors and many interesting results have been obtained for the last nearly fifty years. For further information about the topic, we refer the reader to [9], [5], [1] and [2].

Recall that a Banach *-algebra is a Banach algebra (complete normed algebra) which has an isometric involution. Jang and Park [6] investigated the stability of *-derivations and of quadratic *-derivations with Cauchy functional equation and the Jensen functional equation on Banach *-algebra. The stability of *-derivations on Banach *-algebra by using fixed point alternative was proved by Park and Bodaghi and also Yang et al.; see [12] and [15], respectively. Also, the stability of cubic Lie derivations was introduced by Fošner and Fošner; see [4].

Jun and Kim [8] introduced the following cubic functional equation:

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x)$$

and established a general solution. Najati [11] investigated the following generalized cubic functional equation:

$$(1.1) \quad f(sx + y) + f(sx - y) = sf(x + y) + sf(x - y) + 2(s^3 - s)f(x)$$

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for a positive integer $s \geq 2$. Also, Jun and Kim [7] proved the Hyers-Ulam-Rassias stability of a Euler-Lagrange type cubic mapping as follows:

$$(1.2) \quad \begin{aligned} f(sx + y) + f(x + sy) \\ = (s + 1)(s - 1)^2[f(x) + f(y)] + s(s + 1)f(x + y), \end{aligned}$$

where $s \in \mathbb{Z} (s \neq 0, \pm 1)$.

In this paper, we deal with the following functional equation:

$$(1.3) \quad \begin{aligned} f(sx + y) + f(x - sy) - s^2f(x + y) - sf(x - y) \\ = (s - 1)(s^2 - 1)f(x) - (s + 1)(s^2 - 1)f(y) \end{aligned}$$

for all $s \in \mathbb{Z} (s \neq 0, \pm 1)$. We will show the general solution of the functional equation (1.3) and investigate the stability of cubic Lie $*$ -derivations associated with the given functional equation on normed algebras.

2. CUBIC FUNCTIONAL EQUATIONS

In this section let X and Y be vector spaces and we investigate the general solution of the functional equation (1.3).

Theorem 2.1. *A function $f : X \rightarrow Y$ satisfies the functional equation (1.3) if and only if it satisfies the functional equation (1.1).*

Proof. Suppose that f satisfies the equation (1.3). It is easy to show that $f(0) = 0$, $f(sx) = s^3f(x)$ for all $x \in X$ and all $s \in \mathbb{Z} (s \neq 0, \pm 1)$. Letting $x = -x$ in the equation (1.3), we have

$$(2.1) \quad \begin{aligned} -f(sx - y) - f(x + sy) + (s + 1)(s^2 - 1)f(y) \\ = -s^2f(x - y) - sf(x + y) - (s - 1)(s^2 - 1)f(x) \end{aligned}$$

for all $x, y \in X$. Replacing x and y in the equation (2.1), we get

$$(2.2) \quad \begin{aligned} f(x - sy) - f(sx + y) + (s + 1)(s^2 - 1)f(x) \\ = s^2f(x - y) - sf(x + y) - (s - 1)(s^2 - 1)f(y) \end{aligned}$$

for all $x, y \in X$. Subtracting the equation (2.2) from the equation (1.3), we obtain

$$(2.3) \quad \begin{aligned} 2f(sx + y) + 2(s^2 - 1)f(y) \\ = (s^2 + s)f(x + y) + (s - s^2)f(x - y) + 2s(s^2 - 1)f(x) \end{aligned}$$

for all $x, y \in X$. Now, letting $y = -y$ in the equation (2.3)

$$(2.4) \quad \begin{aligned} 2f(sx - y) - 2(s^2 - 1)f(y) \\ = (s^2 + s)f(x - y) + (s - s^2)f(x + y) + 2s(s^2 - 1)f(x) \end{aligned}$$

for all $x, y \in X$. Adding two equations (2.3) and (2.4), we have

$$(2.5) \quad 2f(sx + y) + 2f(sx - y) = 2sf(x + y) + 2sf(x - y) + 4s(s^2 - 1)f(x)$$

for all $x, y \in X$. Thus we have the equation (1.1). Conversely, suppose that f satisfies the equation (1.1). It is easy to see that $f(0) = 0$, $f(sx) = s^3f(x)$

for all $x \in X$ and all $s \in \mathbb{Z}(s \neq 0)$. Letting $y = sy$ in the equation (1.1), we have

$$(2.6) \quad f(x + sy) + f(x - sy) = s^2 f(x + y) + s^2 f(x - y) - 2(s^2 - 1)f(x)$$

for all $x, y \in X$. Replacing x and y in the equation (2.6), we get

$$(2.7) \quad f(sx + y) - f(sx - y) = s^2 f(x + y) - s^2 f(x - y) - 2(s^2 - 1)f(y)$$

for all $x, y \in X$. By adding two equations (1.1) and (2.7), we obtain

$$(2.8) \quad 2f(sx + y) = (s^2 + s)f(x + y) + (s - s^2)f(x - y) + 2s(s^2 - 1)f(x) - 2(s^2 - 1)f(y)$$

for all $x, y \in X$. Now, letting $y = sy$ in the equation (2.7), we have

$$(2.9) \quad f(x + sy) - f(x - sy) = sf(x + y) - sf(x - y) + 2s(s^2 - 1)f(y)$$

for all $x, y \in X$. Subtracting the equation (2.9) from the equation (2.6), we know that

$$(2.10) \quad 2f(x - sy) = (s^2 - s)f(x + y) + (s^2 + s)f(x - y) - 2(s^2 - 1)f(x) - 2s(s^2 - 1)f(y)$$

for all $x, y \in X$. By adding two equations (2.8) and (2.10), we have the desired equation (1.3). \square

3. CUBIC LIE *-DERIVATIONS

Throughout this section, we assume that A is a complex normed $*$ -algebra and M is a Banach A -bimodule. We will use the same symbol $\|\cdot\|$ as norms on a normed algebra A and a normed A -bimodule M . A mapping $f : A \rightarrow M$ is a *cubic homogeneous mapping* if $f(\mu a) = \mu^3 f(a)$, for all $a \in A$ and $\mu \in \mathbb{C}$. A cubic homogeneous mapping $f : A \rightarrow M$ is called a *cubic derivation* if

$$f(xy) = f(x)y^3 + x^3 f(y)$$

holds for all $x, y \in A$. For all $x, y \in A$, the symbol $[x, y]$ will denote the commutator $xy - yx$. We say that a cubic homogeneous mapping $f : A \rightarrow M$ is a cubic Lie derivation if

$$f([x, y]) = [f(x), y^3] + [x^3, f(y)]$$

for all $x, y \in A$. In addition, if f satisfies in condition $f(x^*) = f(x)^*$ for all $x \in A$, then it is called the *cubic Lie $*$ -derivation*.

Example 3.1. Let $A = \mathbb{C}$ be a complex field endowed with the map $z \mapsto z^* = \bar{z}$ (where \bar{z} is the complex conjugate of z). We define $f : A \rightarrow A$ by $f(a) = a^3$ for all $a \in A$. Then f is cubic and

$$f([a, b]) = [f(a), b^3] + [a^3, f(b)] = 0$$

for all $a \in A$. Also,

$$f(a^*) = f(\bar{a}) = \bar{a}^3 = \overline{a^3} = f(a)^*$$

for all $a \in A$. Thus f is a cubic Lie $*$ -derivation.

In the following, \mathbb{T}^1 will stand for the set of all complex units, that is,

$$\mathbb{T}^1 = \{\mu \in \mathbb{C} \mid |\mu| = 1\}.$$

For the given mapping $f : A \rightarrow M$, we consider

$$(3.1) \quad \begin{aligned} \Delta_\mu f(x, y) &:= f(s\mu x + \mu y) + f(\mu x - s\mu y) - s^2\mu^3 f(x+y) - s\mu^3 f(x-y) \\ &\quad - \mu^3(s-1)(s^2-1)f(x) + \mu^3(s+1)(s^2-1)f(y), \\ \Delta f(x, y) &:= f([x, y]) - [f(x), y^3] - [x^3, f(y)] \end{aligned}$$

for all $x, y \in A$, $\mu \in \mathbb{C}$ and $s \in \mathbb{Z} (s \neq 0, \pm 1)$.

Theorem 3.2. Suppose that $f : A \rightarrow M$ is a mapping with $f(0) = 0$ for which there exists a function $\phi : A^5 \rightarrow [0, \infty)$ such that

$$(3.2) \quad \tilde{\phi}(a, b, x, y, z) := \sum_{j=0}^{\infty} \frac{1}{|s|^{3j}} \phi(s^j a, s^j b, s^j x, s^j y, s^j z) < \infty$$

$$(3.3) \quad \|\Delta_\mu f(a, b)\| \leq \phi(a, b, 0, 0, 0)$$

$$(3.4) \quad \|\Delta f(x, y) + f(z^*) - f(z)^*\| \leq \phi(0, 0, x, y, z)$$

for all $\mu \in \mathbb{T}^1_{\frac{1}{n_0}} = \{e^{i\theta} \mid 0 \leq \theta \leq \frac{2\pi}{n_0}\}$ and all $a, b, x, y, z \in A$ in which $n_0 \in \mathbb{N}$. Also, if for each fixed $a \in A$ the mapping $r \mapsto f(ra)$ from \mathbb{R} to M is continuous, then there exists a unique cubic Lie $*$ -derivation $L : A \rightarrow M$ satisfying

$$(3.5) \quad \|f(a) - L(a)\| \leq \frac{1}{|s|^3} \tilde{\phi}(a, 0, 0, 0, 0).$$

Proof. Let $b = 0$ and $\mu = 1$ in the inequality (3.3), we have

$$(3.6) \quad \|f(a) - \frac{1}{s^3} f(sa)\| \leq \frac{1}{|s|^3} \phi(a, 0, 0, 0, 0)$$

for all $a \in A$. Using the induction, it is easy to show that

$$(3.7) \quad \left\| \frac{1}{s^{3t}} f(s^t a) - \frac{1}{s^{3k}} f(s^k a) \right\| \leq \frac{1}{|s|^3} \sum_{j=k}^{t-1} \frac{\phi(s^j a, 0, 0, 0, 0)}{|s|^{3j}}$$

for $t > k \geq 0$ and $a \in A$. The inequalities (3.2) and (3.7) imply that the sequence $\{\frac{1}{s^{3n}} f(s^n a)\}_{n=0}^{\infty}$ is a Cauchy sequence. Since M is complete, the sequence is convergent. Hence we can define a mapping $L : A \rightarrow M$ as

$$(3.8) \quad L(a) = \lim_{n \rightarrow \infty} \frac{1}{s^{3n}} f(s^n a)$$

for $a \in A$. By letting $t = n$ and $k = 0$ in the inequality (3.7), we have

$$(3.9) \quad \left\| \frac{1}{s^{3n}} f(s^n a) - f(a) \right\| \leq \frac{1}{|s|^3} \sum_{j=0}^{n-1} \frac{\phi(s^j a, 0, 0, 0, 0)}{|s|^{3j}}$$

for $n > 0$ and $a \in A$. By taking $n \rightarrow \infty$ in the inequality (3.9), the inequalities (3.2) implies that the inequality (3.5) holds.

Now, we will show that the mapping L is a unique cubic Lie *-derivation such that the inequality (3.5) holds for all $a \in A$. We note that

$$(3.10) \quad \begin{aligned} \|\Delta_\mu L(a, b)\| &= \lim_{n \rightarrow \infty} \frac{1}{|s|^{3n}} \|\Delta_\mu f(s^n a, s^n b)\| \\ &\leq \lim_{n \rightarrow \infty} \frac{\phi(s^n a, s^n b, 0, 0, 0)}{|s|^{3n}} = 0, \end{aligned}$$

for all $a, b \in A$ and $\mu \in \mathbb{T}_{\frac{1}{n_0}}^1$. By taking $\mu = 1$ in the inequality (3.10), it follows that the mapping L is a Euler-Lagrange cubic mapping. Also, the inequality (3.10) implies that $\Delta_\mu L(a, 0) = 0$. Hence

$$L(\mu a) = \mu^3 L(a)$$

for all $a \in A$ and $\mu \in \mathbb{T}_{\frac{1}{n_0}}^1$. Let $\mu \in \mathbb{T}^1 = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$. Then $\mu = e^{i\theta}$, where $0 \leq \theta \leq 2\pi$. Let $\mu_1 = \mu^{\frac{1}{n_0}} = e^{\frac{i\theta}{n_0}}$. Hence we have $\mu_1 \in \mathbb{T}_{\frac{1}{n_0}}^1$. Then

$$L(\mu a) = L(\mu_1^{n_0} a) = \mu_1^{3n_0} L(a) = \mu^3 L(a)$$

for all $\mu \in \mathbb{T}^1$ and $a \in A$. Suppose that ρ is any continuous linear functional on A and a is a fixed element in A . Then we can define a function $g : \mathbb{R} \rightarrow \mathbb{R}$ by

$$g(r) = \rho(L(ra))$$

for all $r \in \mathbb{R}$. It is easy to check that g is cubic. Let

$$g_k(r) = \rho\left(\frac{f(s^k ra)}{s^{3k}}\right)$$

for all $k \in \mathbb{N}$ and $r \in \mathbb{R}$.

Note that g as the pointwise limit of the sequence of measurable functions g_k is measurable. Hence g as a measurable cubic function is continuous (see [3]) and

$$g(r) = r^3 g(1)$$

for all $r \in \mathbb{R}$. Thus

$$\rho(L(ra)) = g(r) = r^3 g(1) = r^3 \rho(L(a)) = \rho(r^3 L(a))$$

for all $r \in \mathbb{R}$. Since ρ was an arbitrary continuous linear functional on A we may conclude that

$$L(ra) = r^3 L(a)$$

for all $r \in \mathbb{R}$. Let $\mu \in \mathbb{C} (\mu \neq 0)$. Then $\frac{\mu}{|\mu|} \in \mathbb{T}^1$. Hence

$$L(\mu a) = L\left(\frac{\mu}{|\mu|} |\mu| a\right) = \left(\frac{\mu}{|\mu|}\right)^3 L(|\mu| a) = \left(\frac{\mu}{|\mu|}\right)^3 |\mu|^3 L(a) = \mu^3 L(a)$$

for all $a \in A$ and $\mu \in \mathbb{C} (\mu \neq 0)$. Since a was an arbitrary element in A , we may conclude that L is cubic homogeneous.

Next, replacing x, y by $s^k x, s^k y$, respectively, and $z = 0$ in the inequality (3.4), we have

$$\begin{aligned} \|\Delta L(x, y)\| &= \lim_{n \rightarrow \infty} \left\| \frac{\Delta f(s^n x, s^n y)}{s^{3n}} \right\| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{|s|^{3n}} \phi(0, 0, s^n x, s^n y, 0) = 0 \end{aligned}$$

for all $x, y \in A$. Hence we have $\Delta L(x, y) = 0$ for all $x, y \in A$. That is, L is a cubic Lie derivation. Letting $x = y = 0$ and replacing z by $s^k z$ in the inequality (3.4), we get

$$(3.11) \quad \left\| \frac{f(s^n z^*)}{s^{3n}} - \frac{f(s^n z)^*}{s^{3n}} \right\| \leq \frac{\phi(0, 0, 0, 0, s^n z)}{|s|^{3n}}$$

for all $z \in A$. As $n \rightarrow \infty$ in the inequality (3.11), we have

$$L(z^*) = L(z)^*$$

for all $z \in A$. This means that L is a cubic Lie $*$ -derivation. Now, assume $L' : A \rightarrow A$ is another cubic $*$ -derivation satisfying the inequality (3.5). Then

$$\begin{aligned} \|L(a) - L'(a)\| &= \frac{1}{|s|^{3n}} \|L(s^n a) - L'(s^n a)\| \\ &\leq \frac{1}{|s|^{3n}} \left(\|L(s^n a) - f(s^n a)\| + \|f(s^n a) - L'(s^n a)\| \right) \\ &\leq \frac{1}{|s|^{3n+1}} \sum_{j=0}^{\infty} \frac{1}{|s|^{3j}} \phi(s^{j+n} a, 0, 0, 0, 0) \\ &= \frac{1}{|s|^3} \sum_{j=n}^{\infty} \frac{1}{|s|^{3j}} \phi(s^j a, 0, 0, 0, 0), \end{aligned}$$

which tends to zero as $k \rightarrow \infty$, for all $a \in A$. Thus $L(a) = L'(a)$ for all $a \in A$. This proves the uniqueness of L . \square

Corollary 3.3. *Let θ, r be positive real numbers with $r < 3$ and let $f : A \rightarrow M$ be a mapping with $f(0) = 0$ such that*

$$\|\Delta_\mu f(a, b)\| \leq \theta(\|a\|^r + \|b\|^r)$$

$$\|\Delta f(x, y) + f(z^*) - f(z)^*\| \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r)$$

for all $\mu \in \mathbb{T}_{n_0}^1$ and $a, b, x, y, z \in A$. Then there exists a unique cubic Lie $*$ -derivation $L : A \rightarrow M$ satisfying

$$\|f(a) - L(a)\| \leq \frac{\theta\|a\|^r}{|s|^3 - |s|^r}$$

for all $a \in A$.

Proof. The proof follows from Theorem 3.2 by taking $\phi(a, b, x, y, z) = \theta(\|a\|^r + \|b\|^r + \|x\|^r + \|y\|^r + \|z\|^r)$ for all $a, b, x, y, z \in A$. \square

Now, we will investigate the stability of the given functional equation (3.1) using the alternative fixed point method. Before proceeding the proof, we will state the theorem, the alternative of fixed point; see [10] and [13].

Definition 3.4. Let X be a set. A function $d : X \times X \rightarrow [0, \infty]$ is called a *generalized metric* on X if d satisfies

- (1) $d(x, y) = 0$ if and only if $x = y$;
- (2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 3.5 (The alternative of fixed point [10], [13]). *Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \rightarrow \Omega$ with Lipschitz constant l . Then for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1} x) = \infty \text{ for all } n \geq 0,$$

or there exists a natural number n_0 such that

- (1) $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$;
- (2) *The sequence $(T^n x)$ is convergent to a fixed point y^* of T ;*
- (3) y^* *is the unique fixed point of T in the set*

$$\Delta = \{y \in \Omega | d(T^{n_0} x, y) < \infty\};$$

- (4) $d(y, y^*) \leq \frac{1}{1-l} d(y, Ty)$ for all $y \in \Delta$.

Theorem 3.6. *Let $f : A \rightarrow M$ be a continuous mapping with $f(0) = 0$ and let $\phi : A^5 \rightarrow [0, \infty)$ be a continuous mapping such that*

$$(3.12) \quad \|\Delta_\mu f(a, b)\| \leq \phi(a, b, 0, 0, 0)$$

$$(3.13) \quad \|\Delta f(x, y) + f(z^*) - f(z)^*\| \leq \phi(0, 0, x, y, z)$$

for all $\mu \in \mathbb{T}_{\frac{1}{n_0}}^1$ and $a, b, x, y, z \in A$. If there exists a constant $l \in (0, 1)$ such that

$$(3.14) \quad \phi(sa, sb, sx, sy, sz) \leq |s|^3 l \phi(a, b, x, y, z)$$

*for all $a, b, x, y, z \in A$, then there exists a cubic Lie *-derivation $L : A \rightarrow M$ satisfying*

$$(3.15) \quad \|f(a) - L(a)\| \leq \frac{1}{|s|^3(1-l)} \phi(a, 0, 0, 0, 0)$$

for all $a \in A$.

Proof. Consider the set

$$\Omega = \{g | g : A \rightarrow A, g(0) = 0\}$$

and introduce the generalized metric on Ω ,

$$d(g, h) = \inf \{c \in (0, \infty) | \|g(a) - h(a)\| \leq c\phi(a, 0, 0, 0, 0), \text{ for all } a \in A\}.$$

It is easy to show that (Ω, d) is complete. Now we define a function $T : \Omega \rightarrow \Omega$ by

$$(3.16) \quad T(g)(a) = \frac{1}{s^3} g(sa)$$

for all $a \in A$. Note that for all $g, h \in \Omega$, let $c \in (0, \infty)$ be an arbitrary constant with $d(g, h) \leq c$. Then

$$(3.17) \quad \|g(a) - h(a)\| \leq c\phi(a, 0, 0, 0, 0)$$

for all $a \in A$. Letting $a = sa$ in the inequality (3.17) and using (3.14) and (3.16), we have

$$\begin{aligned} \|T(g)(a) - T(h)(a)\| &= \frac{1}{|s|^3} \|g(sa) - h(sa)\| \\ &\leq \frac{1}{|s|^3} c\phi(sa, 0, 0, 0, 0) \leq cl\phi(a, 0, 0, 0, 0), \end{aligned}$$

that is,

$$d(Tg, Th) \leq cl.$$

Hence we have that

$$d(Tg, Th) \leq l d(g, h),$$

for all $g, h \in \Omega$, that is, T is a strictly self-mapping of Ω with the Lipschitz constant l . Letting $\mu = 1, b = 0$ in the inequality (3.12), we get

$$\|\frac{1}{s^3} f(sa) - f(a)\| \leq \frac{1}{|s|^3} \phi(a, 0, 0, 0, 0)$$

for all $a \in A$. This means that

$$d(Tf, f) \leq \frac{1}{|s|^3}.$$

We can apply the alternative of fixed point and since $\lim_{n \rightarrow \infty} d(T^n f, L) = 0$, there exists a fixed point L of T in Ω such that

$$(3.18) \quad L(a) = \lim_{n \rightarrow \infty} \frac{f(s^n a)}{s^{3n}},$$

for all $a \in A$. Hence

$$d(f, L) \leq \frac{1}{1-l} d(Tf, f) \leq \frac{1}{|s|^3} \frac{1}{1-l}.$$

This implies that the inequality (3.15) holds for all $a \in A$. Since $l \in (0, 1)$, the inequality (3.14) shows that

$$(3.19) \quad \lim_{n \rightarrow \infty} \frac{\phi(s^n a, s^n b, s^n x, s^n y, s^n z)}{|s|^{3n}} = 0.$$

Replacing a, b by $s^n a, s^n b$, respectively, in the inequality (3.12), we have

$$\frac{1}{|s|^{3n}} \|\Delta_\mu f(s^n a, s^n b)\| \leq \frac{\phi(s^n a, s^n b, 0, 0, 0)}{|s|^{3n}}.$$

Taking the limit as k tend to infinity, we have $\Delta_\mu f(a, b) = 0$ for all $a, b \in A$ and all $\mu \in \mathbb{T}_{n_0}^1$. The remains are similar to the proof of Theorem 3.2. \square

Corollary 3.7. *Let θ, r be positive real numbers with $r < 3$ and let $f : A \rightarrow M$ be a mapping with $f(0) = 0$ such that*

$$\|\Delta_\mu f(a, b)\| \leq \theta(\|a\|^r + \|b\|^r)$$

$$\|\Delta f(x, y) + f(z^*) - f(z)^*\| \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r)$$

*for all $\mu \in \mathbb{T}_{n_0}^1$ and $a, b, x, y, z \in A$. Then there exists a unique cubic Lie *-derivation $L : A \rightarrow M$ satisfying*

$$\|f(a) - L(a)\| \leq \frac{\theta\|a\|^r}{|s|^3(1-l)}$$

for all $a \in A$.

Proof. The proof follows from Theorem 3.6 by taking $\phi(a, b, x, y, z) = \theta(\|a\|^r + \|b\|^r + \|x\|^r + \|y\|^r + \|z\|^r)$ for all $a, b, x, y, z \in A$. \square

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Fuzzy share functions for cooperative fuzzy games[†]

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Abstract In this paper, the concept of fuzzy share functions of cooperative fuzzy games with fuzzy characteristic functions is proposed. Players in the proposed cooperative fuzzy game do not need to know precise information about the payoff value. We generalize the axiom of additivity by introducing a positive fuzzy value function $\tilde{\mu}$ on the class of cooperative fuzzy games in fuzzy characteristic function form. The so-called axiom of $\tilde{\mu}$ -additivity generalizes the classical axiom of additivity by putting the weight $\tilde{\mu}(\tilde{v})$ on the value of the game \tilde{v} . We show that any additive function $\tilde{\mu}$ determines a unique fuzzy share function satisfying the axioms of efficient shares, null player property, symmetry and $\tilde{\mu}$ -additivity on the subclass of games on which $\tilde{\mu}$ is positive and which contains all positively scaled unanimity games. Finally, we introduce the fuzzy Shapley share functions and fuzzy Banzhaf share functions for the cooperative fuzzy games with fuzzy characteristic functions.

Keywords: Cooperative fuzzy game; Fuzzy share functions; Characteristic functions; Fuzzy numbers.

1. Introduction

A cooperative game with transferable utility, or simply a TU-game, is a finite set of players N and for any subset (coalition) of players a worth representing the total payoff that the coalition can obtain by cooperating. A value function for TU-games is a function that assigns to every TU-game with n players an n -dimensional vector representing a distribution of payoffs among the players. A value function is efficient if for every game it distributes exactly the worth of the 'grand coalition', N , over all players. The most famous efficient value function is the Shapley value [16]. An example of a value function that is not efficient is the Banzhaf value [3, 8, 14]. Since the Banzhaf value is not efficient, it is not adequate in allocating the worth $v(N)$. In order to allocate $v(N)$ and according to the Banzhaf value, Van der Laan et al. in [18] characterize the normalized Banzhaf value, which distributes the worth $v(N)$ proportional to the Banzhaf values of the players.

A different approach to efficiently allocate the worth $v(N)$ is described in [19], who introduce share functions as an alternative type of solution for TU-games. A share vector for an n -player game is an n -dimensional real vector whose components add up to one. The i th component is player i 's share in the total payoff that is to be distributed among the players. A share function assigns such a share vector to every game. The share function corresponding to the Shapley value is the Shapley share function, which is obtained by dividing the Shapley value of each player by $v(N)$, i.e., by the sum of the Shapley values of all players. Similarly, the Banzhaf share function is obtained by dividing the Banzhaf-value or normalized Banzhaf-value by the corresponding sum of payoffs over all players. One advantage of share functions over value functions is that share functions avoid the "efficiency issue", i.e., they avoid the question of what is the final worth to be distributed over the players. This yields some major simplifications. For example, although the Banzhaf and normalized Banzhaf value are very different value functions (e.g. the Banzhaf value satisfies linearity and the dummy player property which are not satisfied by the normalized Banzhaf value), they correspond to the same Banzhaf share function. Another main advantage of share functions has been discovered by [15], who shows that on a ratio scale meaningful statements can be made for a certain class of share functions, whereas all statements with respect to value functions are meaningless. Besides the advantages of share functions for general TU-games, in [2, 20] they study share

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functions for so-called games in coalition structure, for which an extra advantage is that they provide a natural method to define solutions for such games. Share functions, when multiplied by the worth of the grand coalition N , yield a distribution of the worth of the grand coalition reflecting the individual bargaining position of the players.

Mares and Vlach [12, 13] were concerned about the uncertainty in the value of the characteristic function associated with a game. In their models, the domain of the characteristic function of a game remains to be the class of crisp (deterministic) coalitions but the values assigned to them are fuzzy quantities. However, the implicit assumption that all players and coalitions know the expected payoffs even before the negotiation process, is evidently unrealistic. In fact, during the process of negotiation and coalition forming, the players can have only vague idea about the real outcome of the situation, and this vague expectation can be modeled by mathematical tools (see [12]).

In this paper, we consider the fuzzy share functions of a cooperative fuzzy game with fuzzy characteristic function. The paper will be organized as follows. In Section 2, we introduce the concepts of fuzzy numbers and the Hukuhara difference on fuzzy numbers. Then, the model of cooperative fuzzy games is introduced. Moreover, some basic concepts of crisp games will be discussed. In Section 3, the fuzzy share functions of cooperative fuzzy games with fuzzy characteristic function is proposed, we generalize the axiom of additivity by introducing a positive fuzzy valued function $\tilde{\mu}$ on the class of cooperative fuzzy games in fuzzy characteristic function form. The so-called axiom of $\tilde{\mu}$ -additivity generalizes the classical axiom of additivity by putting the weight $\tilde{\mu}(\tilde{v})$ on the value of the game \tilde{v} . We show that any additive function $\tilde{\mu}$ determines a unique fuzzy share function satisfying the axioms of efficient shares, null player property, symmetry and $\tilde{\mu}$ -additivity on the subclass of games on which $\tilde{\mu}$ is positive and which contains all positively scaled unanimity games. In Section 4, we introduce fuzzy Shapley share functions and fuzzy Banzhaf share functions, furthermore, an applicable example is given. Finally, some conclusions will be discussed in Section 5.

2. Preliminaries

In this section, we first recall the concept of fuzzy number, and then introduce some basic concepts and notations in cooperative games with fuzzy characteristic functions.

2.1 A review of fuzzy numbers

Let us start by recalling the most general definition of a fuzzy number. Let \mathbb{R} be $(-\infty, +\infty)$, i.e., the set of all real numbers.

Definition 2.1. A fuzzy number, denoted by \tilde{a} , is a fuzzy subset of \mathbb{R} with membership function $u_{\tilde{a}} : \mathbb{R} \rightarrow [0, 1]$ satisfying the following conditions:

- (1) there exists at least one number $a_0 \in \mathbb{R}$ such that $u_{\tilde{a}}(a_0) = 1$;
- (2) $u_{\tilde{a}}(x)$ is nondecreasing on $(-\infty, a_0)$ and nonincreasing on $(a_0, +\infty)$;
- (3) $u_{\tilde{a}}(x)$ is upper semi-continuous, i.e., $\lim_{x \rightarrow x_0^+} u_{\tilde{a}}(x) = u_{\tilde{a}}(x_0)$ if $x_0 < a_0$; and $\lim_{x \rightarrow x_0^-} u_{\tilde{a}}(x) = u_{\tilde{a}}(x_0)$ if $x_0 > a_0$;
- (4) $\text{Supp}(u_{\tilde{a}})$, the support set of \tilde{a} , is compact, where $\text{Supp}(u_{\tilde{a}}) = \text{cl}\{x \in (R) | u_{\tilde{a}}(x) > 0\}$.

We denote the set of all fuzzy numbers by \mathfrak{R} . An important type of fuzzy numbers in common use is the triangular fuzzy number [9], whose membership function has the form

$$u_{\tilde{a}}(x) = \begin{cases} \frac{x - a_1}{a_2 - a_1}, & x \in [a_1, a_2], \\ \frac{a_3 - x}{a_3 - a_2}, & x \in [a_2, a_3], \\ 0, & \text{otherwise.} \end{cases}$$

where $a_1, a_2, a_3 \in \mathbb{R}$ with $a_1 \leq a_2 \leq a_3$.

For a fuzzy number $\tilde{a} \in \mathfrak{R}$, the level set is defined as $\tilde{a}_\lambda = \{x \in \mathbb{R} | u_{\tilde{a}}(x) \geq \lambda\}$, $u_{\tilde{a}}(x) \in [0, 1]$. It follows from the properties of the membership function of a fuzzy number \tilde{a} that each of its λ -cuts \tilde{a}_λ

is an interval number, denoted by $\tilde{a}_\lambda = [\tilde{a}_\lambda^L, \tilde{a}_\lambda^R]$, $\lambda \in (0, 1]$, where \tilde{a}_λ^L and \tilde{a}_λ^R mean the lower and upper bounds of \tilde{a}_λ .

Let $\tilde{a}, \tilde{b} \in \mathfrak{R}$, and let $*$ be a binary operation on \mathbb{R} . The $*$ operation can be extended to fuzzy numbers by means of Zadeh's extension principle [22] in the following way:

$$u_{\tilde{a}*\tilde{b}}(z) = \sup_{x*y} \min\{u_{\tilde{a}}(x), u_{\tilde{b}}(y)\}, z \in \mathbb{R}, \quad (2.1)$$

where $\tilde{a} * \tilde{b}$ is a fuzzy number with the membership function $u_{\tilde{a}*\tilde{b}}$.

It is not easy to apply Eq.(2.1) in calculation directly. However, calculating λ -cuts of the fuzzy number $\tilde{a} * \tilde{b}$ is an easy task in each case because

$$\begin{aligned} (\tilde{a} + \tilde{b})_\lambda &= \tilde{a}_\lambda + \tilde{b}_\lambda = [\tilde{a}_\lambda^L + \tilde{b}_\lambda^L, \tilde{a}_\lambda^R + \tilde{b}_\lambda^R], \\ (\tilde{a} - \tilde{b})_\lambda &= \tilde{a}_\lambda - \tilde{b}_\lambda = [\tilde{a}_\lambda^L - \tilde{b}_\lambda^R, \tilde{a}_\lambda^R - \tilde{b}_\lambda^L], \\ (m\tilde{a})_\lambda &= m\tilde{a}_\lambda = [m\tilde{a}_\lambda^L, m\tilde{a}_\lambda^R], \forall m \in \mathbb{R}, m > 0, \\ (\tilde{a}\tilde{b})_\lambda &= [\min\{\tilde{a}_\lambda^L\tilde{b}_\lambda^L, \tilde{a}_\lambda^L\tilde{b}_\lambda^R, \tilde{a}_\lambda^R\tilde{b}_\lambda^L, \tilde{a}_\lambda^R\tilde{b}_\lambda^R\}, \max\{\tilde{a}_\lambda^L\tilde{b}_\lambda^L, \tilde{a}_\lambda^L\tilde{b}_\lambda^R, \tilde{a}_\lambda^R\tilde{b}_\lambda^L, \tilde{a}_\lambda^R\tilde{b}_\lambda^R\}]. \end{aligned}$$

Definition 2.2. For any two fuzzy numbers $\tilde{a}, \tilde{b} \in \mathfrak{R}$, we write

- (1) $\tilde{a} \geq \tilde{b}$ if and only if $\tilde{a}_\lambda^L \geq \tilde{b}_\lambda^L$ and $\tilde{a}_\lambda^R \geq \tilde{b}_\lambda^R$, $\forall \lambda \in (0, 1]$;
- (2) $\tilde{a} = \tilde{b}$ if and only if $\tilde{a} \geq \tilde{b}$ and $\tilde{b} \geq \tilde{a}$;
- (3) $\tilde{a} \subseteq \tilde{b}$ if and only if $\tilde{a}_\lambda^L \geq \tilde{b}_\lambda^L$ and $\tilde{a}_\lambda^R \leq \tilde{b}_\lambda^R$, $\forall \lambda \in (0, 1]$.

Remark 2.1. The ordering " \geq " between fuzzy numbers in Definition 2.2 has been defined in [9], which is the extension of the max operator to fuzzy numbers with Zadeh's extension principle, i.e.,

$$\tilde{a} \geq \tilde{b} \text{ if and only if } \max\{\tilde{a}, \tilde{b}\} = \tilde{a}, \forall \tilde{a}, \tilde{b} \in \mathfrak{R}.$$

In this paper, we will use the Hukuhara difference between fuzzy numbers [4,10] as follows.

Definition 2.3. Let $\tilde{a}, \tilde{b} \in \mathfrak{R}$. If there exists $\tilde{c} \in \mathfrak{R}$ such that $\tilde{a} = \tilde{b} + \tilde{c}$, then \tilde{c} is called the Hukuhara difference, and denoted by $\tilde{c} = \tilde{a} -_H \tilde{b}$.

Remark 2.2. The Hukuhara difference is defined as an inverse calculation of the "+" operator defined based on Zadeh's extension principle. But the Hukuhara difference between two fuzzy numbers does not always exists. Regarding the existence of the Hukuhara difference, there is an extensive literature described in [9].

Theorem 2.1. Let $\tilde{a}, \tilde{b} \in \mathfrak{R}$. The Hukuhara difference $\tilde{c} = \tilde{a} -_H \tilde{b}$ exists if and only if

$$\tilde{a}_\lambda^L - \tilde{b}_\lambda^L \leq \tilde{a}_\beta^L - \tilde{b}_\beta^L \leq \tilde{a}_\beta^R - \tilde{b}_\beta^R \leq \tilde{a}_\lambda^R - \tilde{b}_\lambda^R, \forall \lambda, \beta \in (0, 1], \beta > \lambda.$$

Lemma 2.1. Let $\tilde{a}, \tilde{b} \in \mathfrak{R}$. If $\tilde{a} -_H \tilde{b}$ exists, then for any $\lambda \in (0, 1]$,

$$(\tilde{a} -_H \tilde{b})_\lambda = \tilde{a}_\lambda -_H \tilde{b}_\lambda = [\tilde{a}_\lambda^L - \tilde{b}_\lambda^L, \tilde{a}_\lambda^R - \tilde{b}_\lambda^R].$$

Lemma 2.2. Let $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \mathfrak{R}$. If $\tilde{a} -_H \tilde{b}$ and $\tilde{c} -_H \tilde{d}$ exists, then

$$(\tilde{a} + \tilde{c}) -_H (\tilde{b} + \tilde{d}) = (\tilde{a} -_H \tilde{b}) + (\tilde{c} -_H \tilde{d}).$$

2.2 Cooperative games with fuzzy characteristic functions

We consider cooperative games with the set of players $N = \{1, 2, \dots, n\}$. A cooperative crisp game is defined by (N, v) , in which N is the set of players and the characteristic function $v : 2^N \rightarrow \mathbb{R}_+ = \{r \in \mathbb{R} | r \geq 0\}$ satisfies the condition that $v(\emptyset) = 0$.

In a cooperative crisp game, a characteristic function v describes a cooperative game and associates a crisp coalition S with the worth $v(S)$, which is interpreted as the payoff that the coalition S can acquire only through the action of S . The cooperative crisp game is based on the assumption that all players and coalitions know the payoff value v before the cooperation begins. As Borkotokey [5] says, this assumption is not realistic because there are many uncertain factors during negotiation and coalition formation. In many situations, the players can have only vague ideas about the real payoff value. Taking into account the imprecision of information in decision making problems, we incorporate a fuzzy characteristic function, which is represented by fuzzy numbers $\tilde{v}(S)$. Therefore, the characteristic

function of such a game associates a crisp coalition $S \in \mathcal{P}(N)$ with a fuzzy number $\tilde{v}(S)$. Assessing such fuzzy numbers for any crisp coalition $S \in \mathcal{P}(N)$, we define a cooperative game with fuzzy characteristic values by a pair (N, \tilde{v}) , where the fuzzy characteristic function $\tilde{v} : \mathcal{P}(N) \rightarrow \mathfrak{R}_+$ is such that $\tilde{v}(\emptyset) = 0$. Obviously, games with fuzzy characteristic functions are a kind of cooperative fuzzy games. Hereinafter, a cooperative game with fuzzy characteristic function will be called a "cooperative fuzzy game" for short.

Along the paper we use the $|\cdot|$ operator to denote the cardinality of a finite set, i.e., $|S|$ is the number of players in S , for any $S \subseteq N$. Alternatively, sometimes we use lowercase letters to denote cardinalities, and thus $s = |S|$ for any $S \subseteq N$. A fuzzy game (N, \tilde{v}) is called *monotone* if for every $S, T \subseteq N$ with $T \subseteq S$, it holds that $\tilde{v}_\lambda^L(T) \leq \tilde{v}_\lambda^L(S)$ and $\tilde{v}_\lambda^R(T) \leq \tilde{v}_\lambda^R(S)$. That is, monotone fuzzy games are those in which the cooperation among players is never pernicious. Since the whole paper deals with monotone games, henceforth we will simply say game instead of monotone game.

For each $S \subseteq N$ and $i \in N$, we will write $S \cup i$ instead of $S \cup \{i\}$ and $S \setminus i$ instead of $S \setminus \{i\}$. For a pair of fuzzy games $(N, \tilde{w}), (N, \tilde{v}) \in \mathcal{FG}$, the game (N, \tilde{z}) is defined by $\tilde{z}(S) = \tilde{w}(S) + \tilde{v}(S)$ for all $S \subseteq N$. Further, given $S \in \mathcal{P}(N)$, the unanimity game with carrier S , $(N, u_T(S))$, is defined by $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise. Notice that $(N, u_T(S)) \in \mathcal{FG}$ for every $S \in \mathcal{P}(N)$.

Given $(N, \tilde{v}) \in \mathcal{FG}$, a player $i \in N$ is a *dummy* if $\tilde{v}(S \cup i) = \tilde{v}(S) + \tilde{v}(i)$ for all $S \subseteq N \setminus i$, that is, if all her marginal contributions are equal to $\tilde{v}(i)$. A player $i \in N$ is called a *null player* if she is a dummy and $\tilde{v}(i) = 0$. Two players $i, j \in N$ are *symmetric* if $\tilde{v}(S \cup i) = \tilde{v}(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$, that is, if their marginal contributions to each coalition coincide.

3. Fuzzy Share Functions

In this section we extend the share function introduced by Van der Laan et al. in [18] to a fuzzy environment. We consider a class of fuzzy share functions for n -person, the basic concept of fuzzy share functions is that it assigns to each player his fuzzy share in the payoff $\tilde{v}(N)$ of the grand coalition N , i.e., a fuzzy share function on a class \mathcal{FC} of games is a function $\tilde{\rho} : \mathcal{FC} \rightarrow \mathfrak{R}^n$ giving player i the share $\tilde{\rho}_i(\tilde{v})$ in the value $\tilde{v}(N)$, where \mathcal{FC} is the subset of \mathcal{FG} that is $\mathcal{FC} \subset \mathcal{FG}$. So, for any game \tilde{v} , a fuzzy share function $\tilde{\rho}$ gives a fuzzy payoff $\tilde{\rho}_i(\tilde{v})\tilde{v}(N)$ to player $i, i = 1, 2, \dots, n$. Observe that we do not require a priori that the share is nonnegative, although for monotone games this seems to be reasonable. We return to this point at the end of this section. Of course the total payoff equals $\tilde{v}(N)$ if and only if $\sum_{i=1}^n \tilde{\rho}_i(\tilde{v}) = 1$. Therefore, for a share function $\tilde{\rho}$ on \mathcal{FC} , we redefine the axiom of efficiency as follows.

AXIOM 3.1. For any $\tilde{v} \in \mathcal{FC}$, $\sum_{i=1}^n \tilde{\rho}_i(\tilde{v}) = 1$.

Now, let $\tilde{\mu} : \mathcal{FC} \rightarrow \mathfrak{R}$ be a fuzzy valued function on the class \mathcal{FC} of games. Then we have the following definition.

Definition 3.1.

(1) A fuzzy valued function $\tilde{\mu} : \mathcal{FC} \rightarrow \mathfrak{R}$ is called *additive* on the class \mathcal{FC} of fuzzy games if for any pair \tilde{w}, \tilde{v} on the class \mathcal{FC} such that $\tilde{w} + \tilde{v} \in \mathcal{FC}$, it holds that $\tilde{\mu}_\lambda^L(\tilde{w} + \tilde{v}) = \tilde{\mu}_\lambda^L(\tilde{w}) + \tilde{\mu}_\lambda^L(\tilde{v})$, $\tilde{\mu}_\lambda^R(\tilde{w} + \tilde{v}) = \tilde{\mu}_\lambda^R(\tilde{w}) + \tilde{\mu}_\lambda^R(\tilde{v})$.

(2) A fuzzy valued function $\tilde{\mu} : \mathcal{FC} \rightarrow \mathfrak{R}$ is called *linear* on the class \mathcal{FC} of fuzzy games if it is additive and if for any \tilde{v} on \mathcal{FC} it holds that $\tilde{\mu}(\alpha\tilde{v}) = \alpha\tilde{\mu}(\tilde{v})$.

(3) A fuzzy valued function $\tilde{\mu} : \mathcal{FC} \rightarrow \mathfrak{R}$ is called *positive* on the class \mathcal{FC} of fuzzy games if $\tilde{\mu}(\tilde{v}) > 0$ for all $\tilde{v} \in \mathcal{FC}$.

For instance the function $\tilde{\mu}$ defined by $\tilde{\mu}(\tilde{v}) = \sum_{T \subseteq N} \tilde{v}(T)$ is additive function. For a given function $\tilde{\mu}$, we generalize the axioms of additivity and linearity to the concepts of $\tilde{\mu}$ -additivity and $\tilde{\mu}$ -linearity of a share function $\tilde{\rho}$ on a class \mathcal{FC} .

AXIOM 3.2.($\tilde{\mu}$ -additivity) Let $\tilde{\mu} : \mathcal{FC} \rightarrow \mathfrak{R}$ be given. Then for any pair \tilde{w} and \tilde{v} of games in \mathcal{FC} such that $\tilde{w} + \tilde{v} \in \mathcal{FC}$ it holds that $\tilde{\mu}_\lambda(\tilde{w} + \tilde{v})\tilde{\rho}_\lambda(\tilde{w} + \tilde{v}) = [\tilde{\mu}(\tilde{w})\tilde{\rho}(\tilde{w})]_\lambda + [\tilde{\mu}(\tilde{v})\tilde{\rho}(\tilde{v})]_\lambda$.

AXIOM 3.3.($\tilde{\mu}$ -linearity) Let $\tilde{\mu} : \mathcal{FC} \rightarrow \mathfrak{R}$ be given. Then for any pair \tilde{w} and \tilde{v} of games in \mathcal{FC} such that $\tilde{\mu}_\lambda(a\tilde{w} + b\tilde{v})\tilde{\rho}_\lambda(a\tilde{w} + b\tilde{v}) = a[\tilde{\mu}(\tilde{w})\tilde{\rho}(\tilde{w})]_\lambda + b[\tilde{\mu}(\tilde{v})\tilde{\rho}(\tilde{v})]_\lambda$ for any pair of real numbers a and b such that $a\tilde{w} + b\tilde{v} \in \mathcal{FC}$.

Definition 3.2[7]. Let $I : \mathfrak{R} \rightarrow \mathbb{R}$, this function is defined by

$$I(m) = \frac{\int_0^1 xm(x)dx}{\int_0^1 m(x)dx}.$$

$I(m)$ denotes the center of gravity of m .

If $m = (a_1, a_2, a_3)$ be a triangular fuzzy number, the center of gravity of m is defined by

$$I_t(m) = \frac{\int_0^1 xm(x)dx}{\int_0^1 m(x)dx} = \frac{a_1 + a_2 + a_3}{3}.$$

Since the center of gravity could approximately denote the value of fuzzy number, so we use the center of gravity $I(\tilde{v})$ to denote the value $\tilde{v}(N)$.

We are now able to define a class of fuzzy share functions. Therefore, let \mathcal{FG} be the collection of all subclasses of games such that for any subclass $\mathcal{FC} \in \mathcal{FG}$ holds that $\alpha u_T \in \mathcal{FC}$ for any $T \subset N$ and any real number $\alpha > 0$, i.e., \mathcal{FG} is the collection of all subclasses containing all positively scaled unanimity games. As is known, see [10, 11, 21], every $\tilde{v} \in \mathcal{FG}$ can be expressed as $\tilde{v}(S) = \sum_{T \subset P(I): T \neq \emptyset} u_T(S) c_T(\tilde{v})$ with the so-called *dividends* $c_T(\tilde{v})$. So, any game $\tilde{v} \in \mathcal{FG}$ can be written as the sum of scaled unanimity games with the dividend $c_T(\tilde{v})$ as the scale of u_T , $T \subset N$. In [21] we know that

$$\tilde{v}(S) = \sum_{T \in P(I): T \neq \emptyset} u_T(S) \tilde{c}_T(\tilde{v}) = \sum_{\substack{T \in P(I): T \neq \emptyset \\ \tilde{c}_T(\tilde{v}) \geq 0}} u_T(S) \tilde{c}_T(\tilde{v}) - \sum_{\substack{T \in P(I): T \neq \emptyset \\ \tilde{c}_T(\tilde{v}) < 0}} u_T(S) (-\tilde{c}_T(\tilde{v}))$$

where $\tilde{c}_T(\tilde{v}) = \text{Sup}\{\lambda \in [0, 1] | x \in \tilde{c}_T^\lambda(\tilde{v})\}$, $\tilde{c}_T^\lambda(\tilde{v}) = [c_T(\tilde{v}_\lambda^L), c_T(\tilde{v}_\lambda^R)]$ and

$$c_T(\tilde{v}_\lambda^L) = \sum_{T \subset N} (-1)^{(|T|-|S|)} \tilde{v}_\lambda^L(S), \quad c_T(\tilde{v}_\lambda^R) = \sum_{T \subset N} (-1)^{(|T|-|S|)} \tilde{v}_\lambda^R(S).$$

Especially, for any $S \in \mathcal{P}(N)$, we let $[\tilde{v}_0^L(S), \tilde{v}_0^R(S)] = cl\{x \in R | \tilde{v}(S)(x) > 0\}$, where cl denotes the closure of sets, and let

$$c_T(\tilde{v}_0^L) = \sum_{T \subset N} (-1)^{(|T|-|S|)} \tilde{v}_0^L(S), \quad c_T(\tilde{v}_0^R) = \sum_{T \subset N} (-1)^{(|T|-|S|)} \tilde{v}_0^R(S),$$

for $T \in \mathcal{P}(N) \setminus \emptyset$.

Proposition 3.1[21]. Let $v \in G_H(I)$ satisfy the following three conditions:

- (i) $c_T(v_\lambda^R) \geq c_T(v_\lambda^L)$, $\forall \lambda \in (0, 1]$, $\forall T \in P(I)$;
- (ii) $c_T(v_0^R) \leq 0$ or $c_T(v_0^L) \geq 0$, $\forall T \in P(I)$;
- (iii) $[c_T(v_\beta^L), c_T(v_\beta^R)] \subseteq [c_T(v_\lambda^L), c_T(v_\lambda^R)]$, $\forall T \in P(I)$, $\forall \lambda, \beta \in (0, 1]$, $\lambda < \beta$.

Then the Hukuhara-Shapley function is the unique Shapley value for game v .

Theorem 3.1. For some subclass of games $\mathcal{FC} \in \mathcal{FG}$, let $\tilde{\mu} : \mathcal{FC} \rightarrow \mathfrak{R}$ be a positive fuzzy value function on \mathcal{FC} . Then on the subclass \mathcal{FC} there exists a unique fuzzy share function $\tilde{\rho} : \mathcal{FC} \rightarrow \mathfrak{R}^n$ satisfying the axioms of efficient shares, null player property, symmetry and $\tilde{\mu}$ -additivity if and only if $\tilde{\mu}$ is additive on \mathcal{FC} .

Proof. Firstly, we suppose $\tilde{\rho}$ satisfies efficiency and $\tilde{\mu}$ -additivity. From the $\tilde{\mu}$ -additivity it follows that

$$\begin{aligned} & \tilde{\mu}_\lambda(\tilde{w} + \tilde{v}) \sum_{i=1}^n \rho_i(\tilde{w} + \tilde{v}) \\ &= \tilde{\mu}_\lambda(\tilde{w} + \tilde{v}) [(\rho_1(\tilde{w} + \tilde{v}))_\lambda + \dots + (\rho_n(\tilde{w} + \tilde{v}))_\lambda] \\ &= \tilde{\mu}_\lambda(\tilde{w} + \tilde{v}) [\rho_1(\tilde{w} + \tilde{v})]_\lambda + \dots + \tilde{\mu}_\lambda(\tilde{w} + \tilde{v}) [\rho_n(\tilde{w} + \tilde{v})]_\lambda \\ &= [\tilde{\mu}_\lambda(\tilde{w}) (\rho_1(\tilde{w}))_\lambda + \tilde{\mu}_\lambda(\tilde{v}) (\rho_1(\tilde{v}))_\lambda] + \dots + [\tilde{\mu}_\lambda(\tilde{w}) (\rho_n(\tilde{w}))_\lambda + \tilde{\mu}_\lambda(\tilde{v}) (\rho_n(\tilde{v}))_\lambda] \\ &= \tilde{\mu}_\lambda(\tilde{w}) [(\rho_1(\tilde{w}))_\lambda + \dots + (\rho_n(\tilde{w}))_\lambda] + \tilde{\mu}_\lambda(\tilde{v}) [(\rho_1(\tilde{v}))_\lambda + \dots + (\rho_n(\tilde{v}))_\lambda] \\ &= \tilde{\mu}_\lambda(\tilde{w}) \sum_{i=1}^n \rho_i(\tilde{w}) + \tilde{\mu}_\lambda(\tilde{v}) \sum_{i=1}^n \rho_i(\tilde{v}) \end{aligned}$$

and for any $\tilde{w}, \tilde{v} \in \mathcal{FC}$ such that $\tilde{w} + \tilde{v} \in \mathcal{FC}$. Efficiency then implies that $\tilde{\mu}(\tilde{w} + \tilde{v}) = \tilde{\mu}(\tilde{w}) + \tilde{\mu}(\tilde{v})$. Hence $\tilde{\mu}$ must be additive.

Secondly, we assume that $\tilde{\mu}$ is additive. We shall show that there can be at most one share function $\tilde{\rho} : \mathcal{FC} \rightarrow \mathbb{R}^n$ satisfying the four axioms. Therefore, let $\tilde{\rho} : \mathcal{FC} \rightarrow \mathbb{R}^n$ be a function satisfying the axioms. Recall that any positively scaled unanimity game belongs to the subclass \mathcal{FC} . For a unanimity game u_T , we have that two players i and j are symmetric if they are both in T , whereas a player not in T is a null player. Hence from the symmetry, null player property and efficient shares axioms it follows that for any positively scaled unanimity game αu_T , $\alpha > 0$, it holds that

$$\tilde{\rho}_i(\alpha u_T) = \frac{1}{|T|}, \quad \text{when } i \in T, \quad (3.1)$$

$$\tilde{\rho}_i(\alpha u_T) = 0, \quad \text{when } i \notin T, \quad (3.2)$$

Now, with $\tilde{c}_T(\tilde{v})$ the dividends of the game \tilde{v} , we can rewrite $\tilde{c}_T(\tilde{v})$ as the difference of two sums of positively scaled unanimity games by

$$\tilde{v}(S) = \sum_{T \in P(I): T \neq \emptyset} u_T(S) \tilde{c}_T(\tilde{v}) = \sum_{\substack{T \in P(I): T \neq \emptyset \\ \tilde{c}_T(\tilde{v}) \geq 0}} u_T(S) \tilde{c}_T(\tilde{v}) - \sum_{\substack{T \in P(I): T \neq \emptyset \\ \tilde{c}_T(\tilde{v}) < 0}} u_T(S) (-\tilde{c}_T(\tilde{v})).$$

Since \tilde{v} is a positive function on \mathcal{FC} and \mathcal{FC} contains \tilde{v} and all positively scaled unanimity games, it follows by applying the axiom of \tilde{u} -additivity repeatedly that $\tilde{\rho}(\tilde{v})$ is uniquely defined by

$$\tilde{\mu}(\tilde{v}) \tilde{\rho}(\tilde{v}) = \sum_{\tilde{c}_T(\tilde{v}) \geq 0} \tilde{\mu}(\tilde{c}_T(\tilde{v}) u_T) \tilde{\rho}(\tilde{c}_T(\tilde{v}) u_T) - \sum_{\tilde{c}_T(\tilde{v}) < 0} \tilde{\mu}(-\tilde{c}_T(\tilde{v}) u_T) \tilde{\rho}(-\tilde{c}_T(\tilde{v}) u_T). \quad (3.3)$$

It only need to prove that $\tilde{\rho}$ indeed satisfies the axioms.

First, because of the additivity of $\tilde{\mu}$ it holds that

$$\tilde{\mu}(\tilde{v}) = \sum_{\tilde{c}_T(\tilde{v}) \geq 0} \tilde{\mu}(\tilde{c}_T(\tilde{v}) u_T) - \sum_{\tilde{c}_T(\tilde{v}) < 0} \tilde{\mu}(-\tilde{c}_T(\tilde{v}) u_T). \quad (3.4)$$

Hence it follows from equation (3.1), (3.2), (3.3) and (3.4) that $\sum_{j=1}^n \tilde{\rho}_j(\tilde{v}) = 1$ and therefore the axiom of efficient shares is satisfied. Second, observe that a null player in \tilde{v} is a null player in any u_T with nonzero dividend $\tilde{c}_T(\tilde{v})$. Hence, by Equations (3.1) and (3.3) and the positiveness of \tilde{v} it follows that $\tilde{\rho}$ satisfies the null player property. Third, if i and j are two symmetric players in \tilde{v} , then $\tilde{c}_T(\tilde{v})_i = \tilde{c}_T(\tilde{v})_j$, whereas for each other $T \subset N$ with nonzero weight $\tilde{c}_T(\tilde{v})$, i and j are either both in T or both not in T . Hence by Equations (3.1), (3.2) and (3.3) and the positiveness of $\tilde{\mu}$ it follows that $\tilde{\rho}$ satisfies the symmetry property. Finally, for any two games $\tilde{v}, \tilde{w} \in \mathcal{FC}$ we have that $\tilde{v} + \tilde{w} = \sum_{T \subset N} (\tilde{c}_T(\tilde{v}) + \tilde{c}_T(\tilde{w})) u_T$. Together with Equation (3.3) and the additivity of \tilde{u} this implies that $\tilde{\mu}(\tilde{v} + \tilde{w}) \tilde{\rho}(\tilde{v} + \tilde{w}) = \tilde{\mu}(\tilde{v}) \tilde{\rho}(\tilde{v}) + \tilde{\mu}(\tilde{w}) \tilde{\rho}(\tilde{w})$ and hence $\tilde{\rho}$ is $\tilde{\mu}$ -additive.

Theorem 3.2. For given positive numbers ω_t , $t = 1, \dots, n$, let the function $\tilde{\mu}^\omega$ be defined by

$$\tilde{\mu}^\omega(\tilde{v}) = \sum_{i \in N} \sum_{\{T | i \in T\}} \omega_t m_T^i = \sum_{i \in N} \sum_{\{T | i \in T\}} \omega_t [\tilde{v}(T \cup i) -_H \tilde{v}(T)] = I(\tilde{v}),$$

where $t = |T|$. Then the share function $\tilde{\rho}^\omega$ defined by

$$\tilde{\rho}_i^\omega(\tilde{v}) = \frac{\sum_{\{T | i \in T\}} \omega_t m_T^i}{I(\tilde{v})} = \frac{\sum_{\{T | i \in T\}} \omega_t [\tilde{v}(T \cup i) -_H \tilde{v}(T)]}{I(\tilde{v})}, \quad i \in N, \quad (3.5)$$

is the unique share function satisfying the axioms of efficient shares, null player property, symmetry and $\tilde{\mu}^\omega$ -additive on the subclass \mathcal{FC} of \mathcal{FG} on which $\tilde{\mu}^\omega$ is positive.

Proof. By definition of $\tilde{\mu}^\omega$, all positively scaled unanimity games αu_T are $\tilde{\mu}^\omega$ positively and hence $\mathcal{FC} \in \mathcal{FG}$. Moreover, $\tilde{\mu}^\omega$ is additive. Hence, it follows from Theorem 3.1 that there exists a unique share function that satisfies the four axioms with respect to $\tilde{\mu}^\omega$ on the class \mathcal{FC} of $\tilde{\mu}^\omega$ -positive games.

It remains to show that $\tilde{\rho}^\omega$ indeed satisfies the four axioms. First, by definition we have that $\tilde{\rho}^\omega$ satisfies the efficient shares axiom. Second, since $m_T^i(\tilde{v}) = 0$ for all $T \subset N$ if i is a null player, the null player property is satisfied. Third, if i and j are symmetric we have that $m_T^i(\tilde{v}) = m_T^j(\tilde{v})$ for all $T \subset N$ containing both i and j , $m_{T \cup \{i\}}^i(\tilde{v}) = m_{T \cup \{j\}}^j(\tilde{v})$ for all $T \subset N$ such that both $i, j \notin T$ and $m_T^i(\tilde{v}) = m_{T \cup \{j\} \setminus \{i\}}^j(\tilde{v})$ for all $T \subset N$ such that $i \in T$ and $j \notin T$. Since the weights ω_t only depend on t this implies that the symmetry axiom holds. Finally, observe that

$$\tilde{\mu}^\omega(\tilde{v})\rho_i^\omega(\tilde{v}) = \sum_{\{T|i \in T\}} \omega_t m_T^i(\tilde{v}), \quad i = 1, \dots, n.$$

Since for all i and T it holds that $m_T^i(a\tilde{v} + b\tilde{w}) = am_T^i(\tilde{v}) + bm_T^i(\tilde{w})$, it follows that $\tilde{\rho}^\omega$ is $\tilde{\mu}^\omega$ -linear and hence also $\tilde{\mu}^\omega$ -additive.

Using the same method in [21], we could get the following Lemma 3.1.

Lemma 3.1[21]. Let $S \in P(N)$ and $i \in S$. Then we have

$$[\tilde{\rho}(\tilde{v})(S)]_\lambda = \tilde{\rho}(\tilde{v})_\lambda(S), \quad \forall \lambda \in (0, 1],$$

where the function is defined by Eq.(3.5).

4. Fuzzy Share Functions: Examples

In this section, we will introduce fuzzy Shapley share functions and fuzzy Banzhaf share functions, furthermore, an applicable example is given.

Definition 4.1[21]. The Shapley value, assigns to any game $(N, v) \in \mathcal{G}$ a vector in \mathbb{R}^n defined as

$$\phi_i^S(\tilde{v}) = \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!} [\tilde{v}(S \cup i) - {}_H \tilde{v}(S)], \quad i \in N. \quad (4.1)$$

Definition 4.2[21]. The Banzhaf value, assigns to any game $(N, v) \in G$ a vector in \mathbb{R}^n defined as

$$\phi_i^B(\tilde{v}) = \sum_{S \subseteq N \setminus i} \frac{1}{2^{n-1}} [\tilde{v}(S \cup i) - {}_H \tilde{v}(S)], \quad i \in N. \quad (4.2)$$

Definition 4.3[2].

(i) Given a game $(N, v) \in G$, the Shapley share function, ρ^S , assigns to any game $(N, v) \in \mathcal{G}$ a vector in \mathbb{R}^n defined as $\rho_i^S(N, v) = \frac{\phi_i^S(v)}{v(N)}$, $i \in N$, if $v \neq v_0$, and $\rho_i^S(N, v) = \frac{1}{|N|}$, $i \in N$.

(ii) Given a game $(N, v) \in G$, the Banzhaf share function, ρ^B , assigns to any game $(N, v) \in \mathcal{G}$ a vector in \mathbb{R}^n defined as $\rho_i^B(N, v) = \frac{\phi_i^B(v)}{\sum_{j \in N} \phi_j^B(v)}$, $i \in N$, if $v \neq v_0$, and $\rho_i^B(N, v) = \frac{1}{|N|}$, $i \in N$.

Definition 4.4 Given a game $(N, v) \in \mathcal{FG}$, the fuzzy Shapley share function, $\tilde{\rho}^S$, assigns to any game $(N, v) \in \mathcal{FG}$ a vector in \mathbb{R}^n defined as $\tilde{\rho}_i^S(N, v) = \frac{\tilde{\phi}_i^S(N, v)}{I(\tilde{v})}$, $i \in N$, if $v \neq v_0$, and $\tilde{\rho}_i^S(N, v) = \frac{1}{|N|}$, $i \in N$.

Definition 4.5 Given a game $(N, v) \in \mathcal{FG}$, the fuzzy Banzhaf share function, $\tilde{\rho}^B$, assigns to any game $(N, v) \in \mathcal{FG}$ a vector in \mathbb{R}^n defined as $\tilde{\rho}_i^B(N, v) = \frac{\tilde{\phi}_i^B(N, v)}{I(\tilde{v})}$, $i \in N$, if $v \neq v_0$, and $\tilde{\rho}_i^B(N, v) = \frac{1}{|N|}$, $i \in N$.

In Theorem 3.1 the class \mathcal{FC} is restricted by the condition that the function $\tilde{\mu}$ must satisfy $\tilde{\mu}(\tilde{v}) > 0$ for all $\tilde{v} \in \mathcal{FC}$. So, the restrictions on the class \mathcal{FC} depend on the way in which $\tilde{\mu}$ is specified. For instance, if $\tilde{\mu}(\tilde{v}) = \tilde{v}(N)$ we have to exclude games with $\tilde{v}(N) \leq 0$. In the remaining of this paper, let $\mathcal{FG}_{\tilde{\mu}}$ denote the class of $\tilde{\mu}$ -positive games, i.e.

$$\mathcal{FG}_{\tilde{\mu}} = \{\tilde{v} \in \mathcal{FG} | \tilde{\mu}(\tilde{v}) > 0\}.$$

For a positive constant $\alpha > 0$ and a function $\tilde{\mu}$ it holds that $\mathcal{FG}_{\tilde{\mu}} = \mathcal{FG}_{\alpha\tilde{\mu}}$. Moreover, for an additive function $\tilde{\mu}$ we have that the class $\mathcal{FG}_{\tilde{\mu}}$ is additive, i.e., the game $\tilde{v} + \tilde{w}$ is $\tilde{\mu}$ -additive if both \tilde{v}, \tilde{w} are $\tilde{\mu}$ -positive.

As shown in Theorem 4.1, $\tilde{\rho}^S$ is defined on the class $\mathcal{FG}_{\tilde{\mu}}$ with $\tilde{\mu}(\tilde{v}) = \tilde{v}(N) > 0$. Clearly on this class we have the Shapley value $\tilde{\phi}^S(\tilde{v})$ of player i is equal to his Shapley share $\tilde{\rho}_i^S(\tilde{v})$ times the value $\tilde{v}(N)$ of grand coalition.

Theorem 4.1. Let the function $\tilde{\mu}^S$ be defined by $\tilde{\mu}^S = \tilde{v}(N) = I(\tilde{v})$ and let $\mathcal{FC} \subset \mathcal{FG}_{\tilde{\mu}^S}$ be a subclass of games in \mathcal{FG} . Then the fuzzy Shapley share function $\tilde{\rho}^S$ is the unique fuzzy share function satisfying the axioms of efficient shares, null player property, symmetry and $\tilde{\mu}^S$ -linearity on the class \mathcal{FC} .

Proof. For $T \subset N$ with $T = t$, take $\omega_t = \frac{t!(n-t-1)!}{n!}$. Then, we have that $\tilde{\mu}^\omega$ as defined in Theorem 3.2 is given by

$$\tilde{\mu}^\omega = \sum_{i \in N} \sum_{\{T|i \in T\}} \omega_t m_T^i(\tilde{v}) = \tilde{v}(N) = \tilde{\mu}^S(\tilde{v}) = I(\tilde{v}).$$

Further, the share function $\tilde{\rho}^\omega$ is given by

$$\rho_i^\omega(\tilde{v}) = \frac{\sum_{\{T|i \in T\}} \omega_t m_T^i(\tilde{v})}{I(\tilde{v})} = \frac{\sum_{\{T|i \in T\}} \frac{t!(n-t-1)!}{n!} m_T^i(\tilde{v})}{I(\tilde{v})} = \frac{\phi_i^S(\tilde{v})}{I(\tilde{v})} = \rho_i^S(\tilde{v}), i \in N.$$

Since all positively scaled unanimity games belong to \mathcal{FC} and $\tilde{\mu}^S$ is linear, it follows from Theorem 3.1 that $\tilde{\rho}^\omega = \tilde{\rho}^S$ is the unique share function on \mathcal{FC} that satisfies the axioms. \square

Since $\mathcal{FG}_{\tilde{\mu}^S} \subset \mathcal{FG}$, Theorem 4.1 holds on the class $\mathcal{FG}_{\tilde{\mu}^S}$ and the restriction to the class of $\tilde{\mu}^S$ -positive games only requires that the value of the grand coalition is positive. Therefore the class of essential zero normalized games is a subset of $\tilde{\mu}^S$ -positive games, so that Theorem 4.1 also holds on this class of games.

Theorem 4.2. Let the function $\tilde{\mu}^B$ be defined by $\tilde{\mu}^B(\tilde{v}) = I(\tilde{v})$ and let $\mathcal{FC} \subset \mathcal{FG}_{\tilde{\mu}^B}$ be a subclass of games in \mathcal{FG} . Then the fuzzy Banzhaf share function $\tilde{\rho}^B$ is the unique share function satisfying the axioms of efficient shares, null player property, symmetry and $\tilde{\mu}^B$ -linearity on the class \mathcal{FC} .

Proof. The function $\tilde{\mu}^\omega$ as defined in Theorem 3.2 is given by

$$\tilde{\mu}^\omega(\tilde{v}) = \sum_{i \in N} \sum_{\{T|i \in T\}} \omega_t m_T^i(\tilde{v}) = \tilde{\mu}^B(\tilde{v}) = I(\tilde{v}).$$

Further, the share function $\tilde{\rho}^\omega$ as defined in Theorem 3.2 is given by

$$\tilde{\rho}_i^\omega(\tilde{v}) = \frac{\sum_{\{T|i \in T\}} \omega_t m_T^i(\tilde{v})}{I(\tilde{v})} = \frac{\sum_{\{T|i \in T\}} \frac{1}{2^{n-1}} m_T^i(\tilde{v})}{I(\tilde{v})} = \tilde{\rho}_i^B(\tilde{v}), i \in N.$$

Since all positively scaled unanimity games belong to \mathcal{FC} and $\tilde{\mu}^B$ is linear, it follows from Theorem 3.1 that $\tilde{\rho}^\omega = \tilde{\rho}^B$ is the unique share function on \mathcal{FC} that satisfies the axioms. \square

Example 4.1. Consider a joint production model in which three decision makers pool three resources to make seven finished products. Three decision makers, named 1, 2 and 3, possess three different initial resources. Decision maker i has 10 tons of resource R_i and can produce n_i tons of Product P_{ii} , $i = 1, 2, 3$. Now, decision makers decide to undertake a joint project: if decision makers i and j cooperate, they will produce n_{ij} tons of product P_{ij} , and if all three cooperate, n_{123} tons of product P_{123} can be produced. The effective output of each finished product is shown in Table 1.

It is natural for the three decision makers to try to evaluate the revenue of the joint project in the early period of the project in order to decide whether the project can be realized or not. However, the average profit per ton of each product is dependent on a number of factors such as product market price, product cost, consumer demand, the relation of commodity supply and demand, etc. Hence, the average profit of each product is an approximate evaluation, which is represented by triangular fuzzy numbers as shown in Table 1.

Table 1. The effective output and the average profit of each finished product

Product	Output of product(tons)	Average Profit(thousands of dollars)
P_{11}	8.0	(1.8,2.0,2.2)
P_{12}	18.0	(2.9,3.1,3.3)
P_{13}	17.5	(2.0,2.3,2.6)
P_{22}	9.0	(2.9,3.0,3.1)
P_{23}	18.0	(3.0,3.2,3.4)
P_{33}	10.0	(0.9,1.0,1.2)
P_{123}	28.0	(3.2,3.5,3.8)

Now, we can make an imprecise assessment of the worth of each crisp coalition (i.e., the fuzzy worth of each crisp coalition) as follows:

$$\tilde{v}(\{1\}) = 8.0 \cdot (1.8, 2.0, 2.2) = (14.4, 16.0, 17.6),$$

$$\tilde{v}(\{2\}) = 9.0 \cdot (2.9, 3.0, 3.1) = (26.1, 27.0, 27.9),$$

$$\tilde{v}(\{3\}) = 10.0 \cdot (0.9, 1.0, 1.2) = (9.0, 10.0, 12.0),$$

$$\tilde{v}(\{1, 2\}) = 18.0 \cdot (2.9, 3.1, 3.3) = (52.2, 55.8, 59.4),$$

$$\tilde{v}(\{1, 3\}) = 17.5 \cdot (2.0, 2.3, 2.6) = (35.0, 40.25, 45.5),$$

$$\tilde{v}(\{2, 3\}) = 18.0 \cdot (3.0, 3.2, 3.4) = (54.0, 57.6, 61.2),$$

$$\tilde{v}(\{1, 2, 3\}) = 28.0 \cdot (3.2, 3.5, 3.8) = (89.6, 98.0, 106.4),$$

Fuzzy share Shapley function:

We can employ the proposed Hukuhara - Shapley function in Eq.(4.1) to estimate each decision maker's share in crisp coalition $T \subseteq \{1, 2, 3\}$.

For example, decision maker 1 in the grand coalition $\{1, 2, 3\}$ has the profit share $\rho_1(\tilde{v})(\{1, 2, 3\})$,

$$\begin{aligned}\phi_1^S(\tilde{v})(\{1, 2, 3\}) &= \frac{1}{3}\tilde{v}(\{1\}) + \frac{1}{6}[\tilde{v}(\{1, 2\}) -_H \tilde{v}(\{2\})] + \frac{1}{6}[\tilde{v}(\{1, 3\}) -_H \tilde{v}(\{3\})] \\ &\quad + \frac{1}{3}[\tilde{v}(\{1, 2, 3\}) -_H \tilde{v}(\{2, 3\})] \\ &= \frac{1}{3}(14.4, 16.0, 17.6) + \frac{1}{6}(26.1, 28.8, 31.5) + \frac{1}{6}(26.0, 30.25, 33.5) \\ &\quad + \frac{1}{3}(35.6, 40.4, 45.2) \\ &= (25.35, 28.64, 31.77).\end{aligned}$$

$$\tilde{v}(N) = I(\tilde{v}) = \frac{89.6 + 98.0 + 106.4}{3} = 98$$

$$\tilde{\rho}_1^S(\tilde{v})(\{1, 2, 3\}) = \frac{\phi_1^S(\tilde{v})(\{1, 2, 3\})}{I(\tilde{v})} = \frac{(25.35, 28.64, 31.77)}{98} = (0.2587, 0.2922, 0.3242)$$

Using a similar method, the fuzzy share Shapley value for this game can be obtained as shown in Table 2.

Table 2. The fuzzy share Shapley values of game with fuzzy characteristic function

Coalition	Decision maker 1	Decision maker 2	Decision maker 3
$\{1\}$	(0.1469, 0.1633, 0.1796)	0	0
$\{2\}$	0	(0.2663, 0.2755, 0.2847)	0
$\{3\}$	0	0	(0.0918, 0.1020, 0.1224)
$\{1, 2\}$	(0.2066, 0.2286, 0.2505)	(0.3260, 0.3408, 0.3556)	0
$\{1, 3\}$	(0.2061, 0.2360, 0.2607)	0	(0.1510, 0.1747, 0.2036)
$\{2, 3\}$	0	(0.3628, 0.3806, 0.3934)	(0.1882, 0.2071, 0.2311)
$\{1, 2, 3\}$	(0.2587, 0.2922, 0.3242)	(0.4153, 0.4369, 0.4568)	(0.2403, 0.2708, 0.3047)

By judging the allocations in Table 2, decision makers can conclude whether the joint project can be realized or not. To do so, decision makers can investigate the problem by varying parameter λ , which is the degree of all the membership functions of the fuzzy numbers involved in the game, from 0.0 to 1.0. For example, consider the case of $\lambda = 0.7$. The expected worth of all the resources is the interval $\tilde{v}_{0.7}(\{1, 2, 3\}) = [0.9743, 1.0257]$, which is allocated among three decision makers. By Eq.(3.4), we estimate the interval Shapley function for each decision maker, i.e.,

$$\tilde{\rho}_i(\tilde{v}_{0.7})(\{1, 2, 3\}) = \tilde{\rho}_i(\tilde{v})(\{1, 2, 3\})_{0.7}, \quad i = 1, 2, 3.$$

Therefore, $\tilde{\rho}_1(\tilde{v}_{0.7})(\{1, 2, 3\}) = [0.2822, 0.3018]$,

$$\tilde{\rho}_2(\tilde{v}_{0.7})(\{1, 2, 3\}) = [0.4304, 0.4429],$$

$$\tilde{\rho}_3(\tilde{v}_{0.7})(\{1, 2, 3\}) = [0.2617, 0.2810].$$

In other words, the expected worth is interval $[0.9743, 1.0257]$, which is allocated among three decision makers, i.e., $[0.2822, 0.3018]$ for decision makers 1, $[0.4304, 0.4429]$ for decision makers 2, and $[0.2617, 0.2810]$ for decision makers 3.

Fuzzy share Banzhaf function:

From definition 4.2, we could know that

$$\begin{aligned}\phi_1^B(\tilde{v})(\{1, 2, 3\}) &= \frac{1}{4}\tilde{v}(\{1\}) + \frac{1}{4}[\tilde{v}(\{1, 2\}) -_H \tilde{v}(\{2\})] + \frac{1}{4}[\tilde{v}(\{1, 3\}) -_H \tilde{v}(\{3\})] \\ &\quad + \frac{1}{4}[\tilde{v}(\{1, 2, 3\}) -_H \tilde{v}(\{2, 3\})] \\ &= \frac{1}{4}(14.4, 16.0, 17.6) + \frac{1}{4}(26.1, 28.8, 31.5) + \frac{1}{4}(26.0, 30.25, 33.5) \\ &\quad + \frac{1}{4}(35.6, 40.4, 45.2) \\ &= (25.525, 28.8625, 31.95).\end{aligned}$$

Using the same way, we could get

$$\phi_2^B(\tilde{v})(\{1, 2, 3\}) = (40.875, 43.0375, 44.95), \phi_3^B(\tilde{v})(\{1, 2, 3\}) = (23.725, 26.7625, 30.05).$$

$$\text{so } \tilde{\mu}^B(\tilde{v}) = (90.125, 98.6625, 106.95),$$

$$\tilde{\mu}^B(\tilde{v}) = I(\tilde{v}) = \frac{90.125 + 98.6625 + 106.95}{3} = 98.5792,$$

$$\begin{aligned} \tilde{\rho}_1^B(\tilde{v})(\{1, 2, 3\}) &= \frac{\phi_1^B(\tilde{v})(\{1, 2, 3\})}{I(\tilde{v})} = \frac{(25.525, 28.8625, 31.95)}{98.5792} \\ &= (0.2589, 0.2928, 0.3241). \end{aligned}$$

Using a similar method, the fuzzy share Shapley value for this game can be obtained as shown in Table 3.

Table 3. The fuzzy share Banzhaf values of game with cooperative fuzzy game

Coalition	Decision maker 1	Decision maker 2	Decision maker 3
{1}	(0.1461, 0.1623, 0.1785)	0	0
{2}	0	(0.2648, 0.2739, 0.2830)	0
{3}	0	0	(0.0913, 0.1014, 0.1217)
{1, 2}	(0.2054, 0.2272, 0.2490)	(0.3241, 0.3388, 0.3535)	0
{1, 3}	(0.2049, 0.2346, 0.2592)	0	(0.1501, 0.1737, 0.2024)
{2, 3}	0	(0.3606, 0.3784, 0.3911)	(0.1872, 0.2059, 0.2298)
{1, 2, 3}	(0.2589, 0.2928, 0.3241)	(0.4146, 0.4366, 0.4560)	(0.2407, 0.2715, 0.3048)

We also consider the case of $\lambda = 0.7$. The expected worth of all the resources is the interval $\tilde{v}_{0.7}(\{1, 2, 3\}) = [0.9749, 1.0261]$, which is allocated among three decision makers. By Eq.(4.2), we estimate the interval Shapley function for each decision maker, i.e.,

$$\tilde{\rho}_i(\tilde{v}_{0.7})(\{1, 2, 3\}) = \tilde{\rho}_i(\tilde{v})(\{1, 2, 3\})_{0.7}, \quad i = 1, 2, 3.$$

Therefore, $\tilde{\rho}_1(\tilde{v}_{0.7})(\{1, 2, 3\}) = [0.2826, 0.3022]$,

$$\tilde{\rho}_2(\tilde{v}_{0.7})(\{1, 2, 3\}) = [0.4300, 0.4424],$$

$$\tilde{\rho}_3(\tilde{v}_{0.7})(\{1, 2, 3\}) = [0.2622, 0.2815].$$

In other words, the expected worth is interval $[0.9749, 1.0261]$, which is allocated among three decision makers, i.e., $[0.2826, 0.3022]$ for decision makers 1, $[0.4300, 0.4424]$ for decision makers 2, and $[0.2622, 0.2815]$ for decision makers 3, and it satisfies efficiency.

5. Conclusion

Game theoretic approaches to cooperative situations in fuzzy environments have given rise to several kinds of fuzzy games. We mention here only the games with fuzzy characteristic functions. In this paper, we have extended the share function introduced by Van der Laan et al. in [18] to a fuzzy environment, we generalize the axiom of additivity by introducing a positive fuzzy valued function $\tilde{\mu}$ on the class of cooperative fuzzy games in fuzzy characteristic function form. The so-called axiom of $\tilde{\mu}$ -additivity generalizes the classical axiom of additivity by putting the weight $\tilde{\mu}(\tilde{v})$ on the value of the game \tilde{v} . We show that any additive function $\tilde{\mu}$ determines a unique fuzzy share function satisfying the axioms of efficient shares, null player property, symmetry and $\tilde{\mu}$ -additivity on the subclass of games on which $\tilde{\mu}$ is positive and which contains all positively scaled unanimity games. Then we introduce fuzzy Shapley share functions and fuzzy Banzhaf share functions, and at last, we give an applicable example for the cooperative fuzzy games with fuzzy characteristic functions.

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On structures of IVF approximation spaces *

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March 2, 2015

Abstract: Rough set theory is a powerful mathematical tool for dealing with inexact, uncertain or vague information. An IVF rough set, which is the result of approximation of an IVF set with respect to an IVF approximation space, is an extension of fuzzy rough sets. In this paper, properties of IVF rough approximation operators and construction of IVF rough sets are investigated. Topological and lattice structures of IVF approximation spaces are given.

Keywords: IVF set; IVF relation; IVF approximate space; IVF rough set; Topology; Lattice.

1 Introduction

Rough set theory was proposed by Pawlak [14] as a mathematical tool to handle imprecision and uncertainty in data analysis. It has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert systems and many other fields [15, 16, 17, 18]. The foundation of its object classification is an equivalence relation. The upper and lower approximation operations are two core notions of this theory. They can also be seen as the closure operator and the interior operator of the topology induced by an equivalence relation on the universe, respectively. In the real world, the equivalence relation is, however, too restrictive for many practical applications. To address this issue, many interesting and meaningful extensions of Pawlak's rough sets have been presented in the literature. Equivalence relations can be replaced by tolerance relations [21], similarity relations [22], binary relations [10, 25].

By replacing crisp relations with fuzzy relations, various fuzzy generalizations of rough approximations have been proposed [1, 3, 9, 13, 19, 24, 28]. Dubois [3] first proposed the concept of rough fuzzy set and fuzzy rough set.

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Fuzzy rough set have been applied to solve a lot of practical problems. For example, medical time series, neural networks, case generation and descriptive dimensionality reduction.

As a generalization of Zadeh's fuzzy set, interval-valued fuzzy (IVF, for short) sets were introduced by Gorzalczany [5] and Turksen [23], and they were applied to the fields of approximate inference, signal transmission and controller, etc. Mondal et al. [12] defined IVF topologies and studied their properties.

By integrating Pawlak rough set theory with IVF set theory, Sun et al. [20] introduced IVF rough sets based on an IVF approximation space, defined IVF information systems and discussed their attribute reduction. Gong et al. [6] presented IVF rough sets based on approximation spaces, studied the knowledge discovery in IVF information systems. However, structures of IVF rough sets have not been deeply studied.

Topologies are widely used in the research field of machine learning and cybernetics. For example, Koretelainen [7, 8] used topologies to detect dependencies of attributes in information systems with respect to gradual rules. Choudhury et al. [2] applied topology to study the evolutionary impact of learning on social problems. Topological structures are the most powerful notions and are important bases in data and system analysis.

Lattices and ordered sets play an important role in many areas of computer science. These range from lattices as models for logics, which are fundamental to understanding computation, to the ordered sets as models for computation, to the role both lattices and ordered sets play in combinatorics, a fundamental aspect of computation. Some researchers investigated relationships between rough sets and lattices. For example, Yang et al. [26] studied lattice structures in generalized approximation spaces. Estaji et al. [4] considered rough set theory applied to lattice theory. Zheng al. [4] investigated topological structures in IVF approximation spaces where the universe may be infinite.

The purpose of this paper is to investigate construction of IVF rough sets and topological or lattice structures of IVF approximation spaces.

2 Preliminaries

Throughout this paper, "interval-valued fuzzy" denotes briefly by "IVF". U denotes a nonempty finite set called the universe of discourse. I denotes $[0, 1]$ and $[I]$ denotes $\{[a, b] : a, b \in I \text{ and } a \leq b\}$. $F^{(i)}(U)$ denotes the family of all IVF sets in U . \bar{a} denotes $[a, a]$ for each $a \in [0, 1]$.

2.1 IVF sets

For any $[a_j, b_j] \in [I]$ ($j = 1, 2$), we define

$$[a_1, b_1] = [a_2, b_2] \iff a_1 = a_2, b_1 = b_2;$$

$$[a_1, b_1] \leq [a_2, b_2] \iff a_1 \leq a_2, b_1 \leq b_2;$$

$$[a_1, b_1] < [a_2, b_2] \iff [a_1, b_1] \leq [a_2, b_2] \text{ and } [a_1, b_1] \neq [a_2, b_2];$$

$$\bar{1} - [a_1, b_1] \text{ or } [a_1, b_1]^c = [1 - b_1, 1 - a_1].$$

Obviously, $([a, b]^c)^c = [a, b]$ for each $[a, b] \in [I]$.

Definition 2.1 ([5, 23]). For each $\{[a_j, b_j] : j \in J\} \subseteq [I]$, we define

$$\bigvee_{j \in J} [a_j, b_j] = [\bigvee_{j \in J} a_j, \bigvee_{j \in J} b_j] \text{ and } \bigwedge_{j \in J} [a_j, b_j] = [\bigwedge_{j \in J} a_j, \bigwedge_{j \in J} b_j],$$

where $\bigvee_{j \in J} a_j = \sup \{a_j : j \in J\}$ and $\bigwedge_{j \in J} a_j = \inf \{a_j : j \in J\}$.

Definition 2.2 ([5, 23]). An IVF set A in U is defined by a mapping $A : U \rightarrow [I]$. Denote

$$A(x) = [A^-(x), A^+(x)] \quad (x \in U).$$

Then $A^-(x)$ (resp. $A^+(x)$) is called the lower (resp. upper) degree to which x belongs to A . A^- (resp. A^+) is called the lower (resp. upper) IVF set of A .

The set of all IVF sets in U is denoted by $F^{(i)}(U)$.

Let $a, b \in I$. $\widetilde{[a, b]}$ represents the IVF set which satisfies $\widetilde{[a, b]}(x) = [a, b]$ for each $x \in U$. We denoted $\widetilde{[a, a]}$ by \tilde{a} .

We recall some basic operations on $F^{(i)}(U)$ as follows ([5, 23]): for any $A, B \in F^{(i)}(U)$ and $[a, b] \in [I]$,

- (1) $A = B \iff A(x) = B(x)$ for each $x \in U$.
- (2) $A \subseteq B \iff A(x) \leq B(x)$ for each $x \in U$.
- (3) $A = B^c \iff A(x) = B(x)^c$ for each $x \in U$.
- (4) $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in U$.
- (5) $(A \cup B)(x) = A(x) \vee B(x)$ for each $x \in U$.
- (6) $([a, b]A)(x) = [a, b] \wedge [A^-(x), A^+(x)]$ for each $x \in U$.

Obviously,

$$A = B \iff A^- = B^- \text{ and } A^+ = B^+; (\widetilde{[a, b]})^c = \widetilde{[a, b]^c} \quad ([a, b] \in [I]).$$

Definition 2.3 ([12]). $A \in F^{(i)}(U)$ is called an IVF point in U , if there exist $[a, b] \in [I] - \{\bar{0}\}$ and $x \in U$ such that

$$A(y) = \begin{cases} [a, b], & y = x, \\ \bar{0}, & y \neq x. \end{cases}$$

We denote A by $x_{[a, b]}$.

Remark 2.4. $A = \bigcup_{x \in U} (A(x)x_{\bar{1}})$ ($A \in F^{(i)}(U)$).

2.2 IVF topologies

Definition 2.5 ([12]). $\tau \subseteq F^{(i)}(U)$ is called an IVF topology on U , if

- (i) $\tilde{0}, \tilde{1} \in \tau$,
- (ii) $A, B \in \tau \implies A \cap B \in \tau$,
- (iii) $\{A_j : j \in J\} \subseteq \tau \implies \bigcup_{j \in J} A_j \in \tau$.

The pair (U, τ) is called an IVF topological space. Every member of τ is called an IVF open set in U . Its complement is called an IVF closed set in U .

An IVF topology τ is called Alexandrov, if (ii) in Definition 2.5 is replaced by

- (ii)' $\{A_j : j \in J\} \subseteq \tau \implies \bigcap_{j \in J} A_j \in \tau$.

We denote $\tau^c = \{A : A^c \in \tau\}$.

The interior and closure of $A \in F^{(i)}(U)$ denoted respectively by $\text{int}(A)$ and $\text{cl}(A)$, are defined as follows:

$$\text{int}(A) \text{ or } \text{int}_\tau(A) = \bigcup \{B \in \tau : B \subseteq A\}, \quad \text{cl}(A) \text{ or } \text{cl}_\tau(A) = \bigcap \{B \in \tau^c : B \supseteq A\}.$$

Proposition 2.6 ([12]). Let τ be an IVF topology on U . Then for any $A, B \in F^{(i)}(U)$,

- (1) $\text{int}(\tilde{1}) = \tilde{1}, \text{cl}(\tilde{0}) = \tilde{0}$.
- (2) $\text{int}(A) \subseteq A \subseteq \text{cl}(A)$.
- (3) $A \subseteq B \implies \text{int}(A) \subseteq \text{int}(B), \text{cl}(A) \subseteq \text{cl}(B)$.
- (4) $\text{int}(A^c) = (\text{cl}(A))^c, \text{cl}(A^c) = (\text{int}(A))^c$.
- (5) $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B), \text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$.
- (6) $\text{int}(\text{int}(A)) = \text{int}(A), \text{cl}(\text{cl}(A)) = \text{cl}(A)$.

3 Construction of IVF rough sets

3.1 IVF rough sets and IVF rough approximation operators

Recall that R is called an IVF relation on U if $R \in F^{(i)}(U \times U)$.

Definition 3.1 ([20]). Let R be an IVF relation on U . Then R is called

- (1) *serial*, if $\bigvee_{y \in U} R(x, y) = \tilde{1}$ for each $x \in U$.
- (2) *reflexive*, if $R(x, x) = \tilde{1}$ for each $x \in U$.
- (3) *symmetric*, if $R(x, y) = R(y, x)$ for any $x, y \in U$.
- (4) *transitive*, if $R(x, z) \geq R(x, y) \wedge R(y, z)$ for any $x, y, z \in U$.
- (5) *Euclidian*, if $R(x, z) \geq R(y, x) \wedge R(y, z)$ for any $x, y, z \in U$.

Let R be an IVF relation on U . R^{-1} is called the inverse relation of R if $R^{-1}(x, y) = R(y, x)$ for each $(x, y) \in U \times U$. R is called preorder if R is reflexive and transitive (see [10]).

Definition 3.2 ([20]). Let R be an IVF relation on U . The pair (U, R) is called an IVF approximation space. For each $A \in F^{(i)}(U)$, the IVF lower and the IVF upper approximation of A with respect to (U, R) , denoted by $\underline{R}(A)$ and $\overline{R}(A)$, are two IVF sets and are respectively defined as follows:

$$\underline{R}(A)(x) = \bigwedge_{y \in U} (A(y) \vee (\bar{1} - R(x, y))), \quad \overline{R}(A)(x) = \bigvee_{y \in U} (A(y) \wedge R(x, y)) \quad (x \in U).$$

The pair $(\underline{R}(A), \overline{R}(A))$ is called the IVF rough set of A with respect to (U, R) .

$\underline{R} : F^{(i)}(U) \rightarrow F^{(i)}(U)$ and $\overline{R} : F^{(i)}(U) \rightarrow F^{(i)}(U)$ are called the IVF lower approximation operator and the IVF upper approximation operator, respectively.

Remark 3.3. Let (U, R) be an IVF approximation space. Then

(1) For each $x, y \in U$,

$$\overline{R}(x_{\bar{1}})(y) = R(y, x) \quad \text{and} \quad \underline{R}((x_{\bar{1}})^c)(y) = \bar{1} - R(y, x).$$

(2) For each $[a, b] \in [I]$, $\underline{R}(\widetilde{[a, b]}) \supseteq \widetilde{[a, b]} \supseteq \overline{R}(\widetilde{[a, b]})$.

Proposition 3.4 ([20]). Let (U, R) be an IVF approximation space. Then for each $A \in F^{(i)}(U)$,

$$(\underline{R}(A))^- = \underline{R}^+(A^-), \quad (\underline{R}(A))^+ = \underline{R}^-(A^+),$$

$$(\overline{R}(A))^- = \overline{R}^-(A^-) \quad \text{and} \quad (\overline{R}(A))^+ = \overline{R}^+(A^+).$$

3.2 Properties of IVF rough approximation operators

Theorem 3.5 ([27]). Let (U, R) be an IVF approximation space. Then for any $A, B \in F^{(i)}(U)$, $\{A_j : j \in J\} \subseteq F^{(i)}(U)$ and $[a, b] \in [I]$,

- (1) $\underline{R}(\bar{1}) = \bar{1}$, $\overline{R}(\bar{0}) = \bar{0}$.
- (2) $A \subseteq B \implies \underline{R}(A) \subseteq \underline{R}(B)$, $\overline{R}(A) \subseteq \overline{R}(B)$.
- (3) $\underline{R}(A^c) = (\overline{R}(A))^c$, $\overline{R}(A^c) = (\underline{R}(A))^c$.
- (4) $\underline{R}(\bigcap_{j \in J} A_j) = \bigcap_{j \in J} \underline{R}(A_j)$, $\overline{R}(\bigcup_{j \in J} A_j) = \bigcup_{j \in J} \overline{R}(A_j)$.
- (5) $\underline{R}(\widetilde{[a, b]} \cup A) = \widetilde{[a, b]} \cup \underline{R}(A)$, $\overline{R}([a, b]A) = [a, b]\overline{R}(A)$.

Theorem 3.6 ([27]). Let R be an IVF relation on U and let τ be an IVF topology on U . If one of the following conditions is satisfied, then R is preorder.

- (1) \underline{R} is the interior operator of τ .
- (2) \overline{R} is the closure operator of τ .

Theorem 3.7. *Let (U, R) be an IVF approximation space. Then*

- (1) R is serial $\iff (ILS^*) \forall [a, b] \in [I], \underline{R}(\widetilde{[a, b]}) = \widetilde{[a, b]}$.
 $\iff (IUS^*) \forall [a, b] \in [I], \overline{R}(\widetilde{[a, b]}) = \widetilde{[a, b]}$.
 $\iff (IUS^{**}) \overline{R}(\widetilde{1}) = \widetilde{1}$.
 $\iff (ILS^{**}) \underline{R}(\widetilde{1}) = \widetilde{1}$.
- (2) R is reflexive $\iff (ILR) \forall A \in F^{(i)}(U), \underline{R}(A) \subseteq A$.
 $\iff (IUR) \forall A \in F^{(i)}(U), A \subseteq \overline{R}(A)$.
- (3) R is symmetric $\iff (ILS) \forall (x, y) \in U \times U, \underline{R}((x_1)^c)(y) = \underline{R}((y_1)^c)(x)$.
 $\iff (IUS) \forall (x, y) \in U \times U, \overline{R}(x_1)(y) = \overline{R}(y_1)(x)$.
- (4) R is transitive $\iff (ILT) \forall A \in F^{(i)}(U), \underline{R}(A) \subseteq \underline{R}(\underline{R}(A))$.
 $\iff (IUT) \forall A \in F^{(i)}(U), \overline{R}(\overline{R}(A)) \subseteq \overline{R}(A)$.

Proof. (1) By Theorem 3.5(3), (ILS^*) and (IUS^*) are equivalent, (ILS^{**}) and (IUS^{**}) are equivalent. We only need to prove that the serialisation of R is equivalent to (IUS^*) or (IUS^{**}) .

For any $[a, b] \in [I]$ and $x \in U$, we have

$$\overline{R}(\widetilde{[a, b]})(x) = \bigvee_{y \in U} ([a, b] \wedge R(x, y)) = [a, b] \wedge (\bigvee_{y \in U} R(x, y)) \quad (\star).$$

Assume that R is serial. Then for each $x \in U$, $\bigvee_{y \in U} R(x, y) = \bar{1}$. By (\star) ,

$$\overline{R}(\widetilde{[a, b]})(x) = [a, b]. \text{ Thus } \overline{R}(\widetilde{[a, b]}) = \widetilde{[a, b]} \text{ and so } \overline{R}(\widetilde{1}) = \widetilde{1}.$$

Assume $\overline{R}(\widetilde{[a, b]}) = \widetilde{[a, b]}$ for each $[a, b] \in [I]$. For each $x \in U$, then $\underline{R}(\widetilde{[a, b]})(x) = [a, b]$. By (\star) , $\bigvee_{y \in U} R(x, y) \geq [a, b]$. Put $[a, b] = \bar{1}$, then $\bigvee_{y \in U} R(x, y) \geq \bar{1}$. Hence $\bigvee_{y \in U} R(x, y) = \bar{1}$. So R is serial.

Assume that $\overline{R}(\widetilde{1}) = \widetilde{1}$. For each $x \in U$, by (\star) , $\bigvee_{y \in U} R(x, y) = \bar{1}$. So R is serial.

(2), (3) and (4) hold by Theorem 13 in [27]. \square

Corollary 3.8 ([27]). *Let (U, R) be an IVF approximation space. If R is pre-order, then*

$$\underline{R}(\underline{R}(A)) = \underline{R}(A) \text{ and } \overline{R}(\overline{R}(A)) = \overline{R}(A) \quad (A \in F^{(i)}(U)).$$

3.3 Lower and upper sets in IVF approximation spaces

Definition 3.9. *Let (U, R) be an IVF approximation space.*

(1) $A \in F^{(i)}(U)$ is called an upper set if $A(x) \wedge R(x, y) \leq A(y)$ for any $x, y \in U$.

(2) $A \in F^{(i)}(U)$ is called a lower set if $A(y) \wedge R(x, y) \leq A(x)$ for any $x, y \in U$.

Proposition 3.10. *Let (U, R) be an IVF approximation space. Then the following are equivalent.*

- (1) $\overline{R}(A) \subseteq A$;
- (2) A is a lower set in (U, R) ;
- (3) A is an upper set in (U, R^{-1}) .

Proof. (1) \implies (2). Suppose that $\overline{R}(A) \subseteq A$. Since for each $x \in U$,

$$\bigvee_{y \in U} (A(y) \wedge R(x, y)) = \overline{R}(A)(x) \leq A(x),$$

$$A(y) \wedge R(x, y) \leq A(x) \quad (x, y \in U).$$

Then A is a lower set in (U, R) .

(2) \implies (3). This is obvious.

(3) \implies (1). Suppose that A is an upper set in (U, R^{-1}) . Then for any $x, y \in U$, $A(x) \wedge R^{-1}(x, y) \leq A(y)$. So $A(x) \wedge R(y, x) \leq A(y)$. Thus

$$\overline{R}(A)(y) = \bigvee_{x \in U} (A(x) \wedge R(y, x)) \leq A(y) \quad (y \in U).$$

Hence $\overline{R}(A) \subseteq A$. □

Corollary 3.11. *Let (U, R) be an IVF approximation space. If R is reflexive, then the following are equivalent.*

- (1) $\overline{R}(A) = A$;
- (2) A is a lower set in (U, R) ;
- (3) A is an upper set in (U, R^{-1}) .

Proof. This holds by Theorem 3.7(2) and Proposition 3.10. □

Let R be an IVF relation on U . For each $z \in U$, we define IVF sets $[z]^R : U \rightarrow [I]$, $[z]^R(x) = R(z, x)$ and $[z]_R : U \rightarrow [I]$, $[z]_R(x) = R(x, z)$.

Theorem 3.12. *Let (U, R) be an IVF approximation space. Then*

- (1) R is reflexive $\iff (ILS') \forall x \in U, [x]_R(x) = \bar{1}$.
 $\iff (IUR') \forall x \in U, [x]^R(x) = \bar{1}$.
- (2) R is symmetric $\iff (ILS') \forall x \in U, [x]_R = [x]^R$.
 $\implies \forall A \in F^{(i)}(U), A$ is a lower set if and only if A is an upper set.
- (3) R is transitive $\iff (ILT') \forall x \in U, [x]_R$ is a lower set.
 $\iff (IUT') \forall x \in U, [x]^R$ is an upper set.
 $\iff (IUT'') \forall A \in F^{(i)}(U), \overline{R}(A)$ is a lower set.
- (4) R is Euclidian $\iff (ILE) \forall x \in U, [x]_R$ is an upper set.

Proof. (1) and (2) are obvious.

(3) (IUT'')

$$\iff \forall A \in F^{(i)}(U), \overline{R}(\overline{R}(A)) \subseteq \overline{R}(A). \quad (\text{Proposition 3.10})$$

$$\iff R \text{ is transitive.} \quad (\text{Theorem 3.7(4)})$$

$$\iff \forall x, y, z \in U, R(x, y) \wedge R(y, z) \leq R(x, z).$$

$$\iff \forall x, y, z \in U, [z]_R(y) \wedge R(x, y) \leq [z]_R(x).$$

$$\iff (ILT') \forall x \in U, [x]_R \text{ is a lower set.}$$

$$\iff \forall x, y, z \in U, [x]^R(y) \wedge R(y, z) \leq [x]^R(z).$$

$$\iff (IUT') \forall x \in U, [x]^R \text{ is an upper set.}$$

(4) The proof is similar to (3). □

3.4 IVF rough equal relations

Definition 3.13. Let (U, R) be an IVF approximation space. Then for any $A, B \in F^{(i)}(U)$,

(1) If $\underline{R}(A) = \underline{R}(B)$, then A and B are called IVF lower rough equal. We denote it by $A \approx B$.

(2) If $\overline{R}(A) = \overline{R}(B)$, then A and B are called IVF upper rough equal. We denote it by $A \simeq B$.

(3) If $\underline{R}(A) = \underline{R}(B)$ and $\overline{R}(A) = \overline{R}(B)$, then A and B are called IVF rough equal. We denote it by $A \sim B$.

Proposition 3.14. Let (U, R) be an IVF approximation space. Then for any $A, B, C, D \in F^{(i)}(U)$,

$$(1) A \approx B \iff (A \cap B) \approx A, (A \cap B) \approx B.$$

$$(2) A \simeq B \iff (A \cup B) \simeq A, (A \cup B) \simeq B.$$

$$(3) A \approx B, C \approx D \implies (A \cap B) \approx (C \cap D), (A \cup B) \approx (C \cup D);$$

$$A \simeq B, C \simeq D \implies (A \cap B) \simeq (C \cap D), (A \cup B) \simeq (C \cup D).$$

$$(4) A \approx \tilde{0} \text{ or } B \approx \tilde{0} \implies (A \cap B) \approx \tilde{0};$$

$$A \simeq \tilde{1} \text{ or } B \simeq \tilde{1} \implies (A \cup B) \simeq \tilde{1}.$$

$$(5) A \subseteq B, B \approx \tilde{0} \implies A \approx \tilde{0};$$

$$A \subseteq B, A \simeq \tilde{1} \implies B \simeq \tilde{1}.$$

(6) If R is reflexive, then

$$a) A \approx \tilde{1} \iff A = \tilde{1}; \quad b) A \simeq \tilde{0} \iff A = \tilde{0}.$$

Proof. (1) Let $A \approx B$. Then $\underline{R}(A) = \underline{R}(B)$. By Theorem 3.5(4),

$$\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B) = \underline{R}(A) = \underline{R}(B).$$

Hence $(A \cap B) \approx A, (A \cap B) \approx B$.

Let $(A \cap B) \approx A, (A \cap B) \approx B$. Then $\underline{R}(A) = \underline{R}(A \cap B) = \underline{R}(B)$. So $A \approx B$.

(2) Let $A \simeq B$. Then $\overline{R}(A) = \overline{R}(B)$. By Theorem 3.5(4),

$$\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B) = \overline{R}(A) = \overline{R}(B).$$

Hence $(A \cup B) \simeq A$, $(A \cup B) \simeq B$.

Let $(A \cap B) \simeq A$, $(A \cap B) \simeq B$. Then $\underline{R}(A) = \underline{R}(A \cup B) = \underline{R}(B)$. So $A \simeq B$.

(3) This holds by Theorem 3.5(4).

(4) This holds by (1) and (2).

(5) This holds by Theorem 3.5(2).

(6) a) Obviously, $A = \bar{1}$ implies $A \approx \bar{1}$.

Let $A \approx \bar{1}$. Then $\underline{R}(A) = \underline{R}(\bar{1})$. By Theorem 3.5(1), $\underline{R}(\bar{1}) = \bar{1}$. Note that R is reflexive. Then for each $x \in U$,

$$A(x) = A(x) \vee (\bar{1} - R(x, x)) \geq \bigwedge_{y \in U} (A(y) \vee (\bar{1} - R(x, y))) = \underline{R}(A)(x) = \bar{1}(x) = \bar{1}.$$

Thus $A = \bar{1}$.

b) The proof is similar to a). \square

Theorem 3.15. Let (U, R) be an IVF approximation space. If R is preorder, then for each $A \in F^{(i)}(U)$,

(1) $\underline{R}(A) = \bigcap \{B \in F^{(i)}(U) : B \approx A\}$.

(2) $\bar{R}(A) = \bigcup \{B \in F^{(i)}(U) : B \simeq A\}$.

Proof. (1) By Theorem 3.7(2), $\underline{R}(A) \subseteq \bigcap \{B \in F^{(i)}(U) : B \approx A\}$. By Corollary 3.8, $\underline{R}(A) \supseteq \bigcap \{B \in F^{(i)}(U) : B \approx A\}$. Then $\underline{R}(A) = \bigcap \{B \in F^{(i)}(U) : B \approx A\}$.

(2) The proof is similar to (1). \square

4 Topological structures of IVF approximation spaces

Let (U, R) be an IVF approximation space. We denote

$$\tau_R = \{A \in F^{(i)}(U) : \underline{R}(A) = A\}, \quad \theta_R = \{\underline{R}(A) : A \in F^{(i)}(U)\}.$$

4.1 IVF topologies based on IVF relations

Theorem 4.1 ([27]). Let R be an IVF relation on U . If R is reflexive, then τ_R is an IVF topology on U .

Definition 4.2 ([27]). Let R be an IVF relation on U . If R is reflexive, then τ_R is called the IVF topology induced by R on U .

Theorem 4.3. Let R be a reflexive IVF relation on U and let τ_R be the IVF topology induced by R on U . Then the following properties hold.

(1) a) $\tau_R \subseteq \theta_R$.

b) For each $A \in F^{(i)}(U)$,

$$\text{int}_{\tau_R}(A) \subseteq \underline{R}(A) \subseteq A \subseteq \bar{R}(A) \subseteq \text{cl}_{\tau_R}(A).$$

c) For each $[a, b] \in [I]$, $[\widetilde{a}, \widetilde{b}] \in \tau_R \cap \tau_R^c$.

- (2) If R is transitive, then
- a) $\tau_R = \theta_R$.
 - b) \underline{R} is the interior operator of τ_R .
 - c) \overline{R} is the closure operator of τ_R .
 - d) $\text{int}_\tau(A) = \bigcap \{B \in F^{(i)}(U) : B \preceq A\}$.
 - e) $\text{cl}_\tau(A) = \bigcup \{B \in F^{(i)}(U) : B \succeq A\}$.

Proof. (1) holds by Theorem 17 in [27].

(2) a) b) and c) holds by Theorem 18 in [27].

d) This holds by (2) b) Proposition 3.18(1).

e) This holds by (2) c) Proposition 3.18(2). \square

Theorem 4.4. Let R be a preorder IVF relation on U and let τ_R be the IVF topology induced by R on U . Then for any $x, y \in U$

$$R(x, y) = \bigwedge_{A \in (y)_{\tau_R}} A(x) ,$$

where $(y)_{\tau_R} = \{A \in \tau_R^c : A(y) = \bar{1}\}$.

Proof. For any $x, y \in U$, by Remark 3.3(1) and Theorem 4.3(2),

$$\begin{aligned} R(x, y) &= \overline{R}(y_{\bar{1}})(x) = \text{cl}_{\tau_R}(y_{\bar{1}})(x) \\ &= \left(\bigcap \{A \in \tau_R^c : A \supseteq y_{\bar{1}}\} \right)(x) \\ &= \bigwedge \{A(x) : A \in \tau_R^c, A \supseteq y_{\bar{1}}\}. \end{aligned}$$

Note that $A \supseteq y_{\bar{1}}$ if and only if $A(y) = \bar{1}$. Thus

$$R(x, y) = \bigwedge \{A(x) : A^c \in \tau_R, A(y) = \bar{1}\} = \bigwedge_{A \in (y)_{\tau_R}} A(x).$$

\square

Theorem 4.5. Let R_1 and R_2 be two preorder IVF relations on U . Let τ_{R_1} and τ_{R_2} be the IVF topologies induced by R_1 and R_2 on U , respectively. Then the following properties hold.

- (1) If $R_1 \subseteq R_2$, then $\tau_{R_2} \subseteq \tau_{R_1}$.
- (2) $\tau_{R_1} = \tau_{R_2} \iff R_1 = R_2$.

Proof. (1) Let $R_1 \subseteq R_2$. For each $A \in \tau_{R_2}$, $\underline{R_2}(A) = A$. For each $x \in U$, by the

transitivity of R_2 ,

$$\begin{aligned}
\underline{R_1}(A)(x) &= \underline{R_1}(\underline{R_2}(A))(x) \\
&= \bigwedge_{y \in U} (\underline{R_2}(A)(y) \vee (\bar{1} - R_1(x, y))) \\
&= \bigwedge_{y \in U} ((\bigwedge_{z \in U} (A(z) \vee (\bar{1} - R_2(y, z)))) \vee (\bar{1} - R_1(x, y))) \\
&= \bigwedge_{y \in U} (\bigwedge_{z \in U} ((A(z) \vee (\bar{1} - R_2(y, z))) \vee (\bar{1} - R_1(x, y)))) \\
&= \bigwedge_{y \in U} (\bigwedge_{z \in U} (A(z) \vee ((\bar{1} - R_2(y, z)) \vee (\bar{1} - R_1(x, y))))) \\
&\geq \bigwedge_{y \in U} (\bigwedge_{z \in U} (A(z) \vee ((\bar{1} - R_2(y, z)) \vee (\bar{1} - R_2(x, y))))) \\
&= \bigwedge_{y \in U} (\bigwedge_{z \in U} (A(z) \vee (\bar{1} - R_2(x, y) \wedge R_2(y, z)))) \\
&\geq \bigwedge_{y \in U} (\bigwedge_{z \in U} (A(z) \vee (\bar{1} - R_2(x, z)))) \\
&= \bigwedge_{z \in U} (A(z) \vee (\bar{1} - R_2(x, z))) \\
&= \underline{R_2}(A)(x) = A(x).
\end{aligned}$$

Then $\underline{R_1}(A) \supseteq A$.

By Theorem 3.7(2), $\underline{R_1}(A) \subseteq A$.

Then $\underline{R_1}(A) = A$ and so $A \in \tau_{R_1}$. Thus $\tau_{R_2} \subseteq \tau_{R_1}$.

(2) Let $\tau_{R_1} = \tau_{R_2}$. By Remark 3.3(1) and Theorem 4.3(2),

$$R_1(x, y) = \overline{R_1}(y_{\bar{1}})(x) = cl_{R_{\tau_1}}(y_{\bar{1}})(x) = cl_{R_{\tau_2}}(y_{\bar{1}})(x) = R_2(x, y)$$

for any $x, y \in U$. Then $R_1 = R_2$.

Conversely, this is obvious. \square

4.2 IVF relations based on IVF topologies

4.2.1 IVF relations induced by IVF topologies

Definition 4.6. Let τ be an IVF topology. Define an IVF relation R_τ on U by

$$R_\tau(x, y) = cl_\tau(y_{\bar{1}})(x)$$

for each $(x, y) \in U \times U$. Then R_τ is called the IVF relation induced by τ on U .

Theorem 4.7. Let τ be an IVF topology on U and let R_τ be the IVF relation induced by τ on U . Then the following properties hold.

(1) R_τ is reflexive.

(2) If $\{\widetilde{[a, b]} : [a, b] \in [I]\} \subseteq \tau$, then

$$\underline{R}_\tau(A) \subseteq \text{int}_\tau(A) \subseteq A \subseteq \text{cl}_\tau(A) \subseteq \overline{R}_\tau(A) \quad (A \in F^{(i)}(U)).$$

Proof. (1) holds by Theorem 21 in [27].

(2) Since $\{\widetilde{[a, b]} : [a, b] \in [I]\} \subseteq \tau$, we have $\{\widetilde{[a, b]} : [a, b] \in [I]\} \subseteq \tau^c$. For each $A \in F^{(i)}(U)$, by Remark 2.4, Proposition 2.6 and Theorem 3.5,

$$\begin{aligned} \text{cl}_\tau(A) &= \text{cl}_\tau\left(\bigcup_{y \in U} (A(y)y_{\bar{1}})\right) = \bigcup_{y \in U} \text{cl}_\tau(A(y)y_{\bar{1}}) = \bigcup_{y \in U} \text{cl}_\tau(\widetilde{A(y)} \cap y_{\bar{1}}) \\ &\subseteq \bigcup_{y \in U} (\text{cl}_\tau(\widetilde{A(y)}) \cap \text{cl}_\tau(y_{\bar{1}})) = \bigcup_{y \in U} (\widetilde{A(y)} \cap \text{cl}_\tau(y_{\bar{1}})). \end{aligned}$$

Then for each $x \in U$,

$$\text{cl}_\tau(A)(x) \leq \bigvee_{y \in U} (\widetilde{A(y)}(x) \wedge \text{cl}_\tau(y_{\bar{1}})(x)) = \bigvee_{y \in U} (A(y) \wedge R_\tau(x, y)) = \overline{R}_\tau(A)(x).$$

Hence $\text{cl}_\tau(A) \subseteq \overline{R}_\tau(A)$.

By Proposition 2.6(4) and Theorem 3.5(3),

$$\text{int}_\tau(A) = (\text{cl}_\tau(A^c))^c \supseteq (\overline{R}_\tau(A^c))^c = \underline{R}_\tau(A).$$

So $\underline{R}_\tau(A) \subseteq \text{int}_\tau(A) \subseteq A \subseteq \text{cl}_\tau(A) \subseteq \overline{R}_\tau(A)$. □

Theorem 4.8. Let R be a reflexive IVF relation on U , let τ_R be the IVF topology induced by τ on U and let R_{τ_R} be the IVF relation induced by τ on U . If R is transitive, then $R_{\tau_R} = R$.

Proof. For each $(x, y) \in U \times U$, by Remark 3.3(1) and Theorem 4.3(2),

$$R(x, y) = \overline{R}(y_{\bar{1}})(x) = \text{cl}_{\theta_R}(y_{\bar{1}})(x) = \text{cl}_{\tau_R}(y_{\bar{1}})(x)$$

Note that $R_{\tau_R}(x, y) = \text{cl}_{\tau_R}(y_{\bar{1}})(x)$. Then $R_{\tau_R}(x, y) = R(x, y)$.

Thus $R_{\tau_R} = R$. □

4.2.2 The (CC) axiom

An IVF topology τ on U is said to satisfy the follows:

The (CC) axiom: for any $[a, b] \in [I]$ and $A \in F^{(i)}(U)$,

$$\text{cl}_\tau([a, b]A) = [a, b]\text{cl}_\tau(A).$$

Proposition 4.9. Let τ be an IVF topology on U . If τ satisfies the (CC) axiom, then

- (1) \overline{R}_τ is the closure operator of τ .
- (2) \underline{R}_τ is the interior operator of τ .
- (3) For each $[a, b] \in [I]$, $\widetilde{[a, b]} \in \tau$.
- (4) τ is Alexandrov.

Proof. (1) For each $A \in F^{(i)}(U)$, by Remark 2.4 and Proposition 2.6(5),

$$cl_\tau(A) = cl_\tau\left(\bigcup_{y \in U} (A(y)y_1)\right) = \bigcup_{y \in U} cl_\tau(A(y)y_1) = \bigcup_{y \in U} (A(y)cl_\tau(y_1)).$$

Then for each $x \in U$,

$$cl_\tau(A)(x) = \bigvee_{y \in U} (A(y)(x) \wedge cl_\tau(y_1)(x)) = \bigvee_{y \in U} (A(y) \wedge R_\tau(x, y)) = \overline{R_\tau}(A)(x).$$

Hence $\overline{R_\tau}(A) = cl_\tau(A)$. Thus $\overline{R_\tau}$ is the closure operator of τ .

(2) This holds by (1), Proposition 2.6(4) and Theorem 3.5(3).

(3) For each $[a, b] \in [I]$, by (2), Remark 3.3(2) and Proposition 2.6(2),

$$[a, b] \supseteq int_\tau([a, b]) = \underline{R}([a, b]) \supseteq [a, b].$$

Then $int_\tau([a, b]) = [a, b]$ and so $[a, b] \in \tau$.

(4) Let $\{A_j : j \in J\} \subseteq \tau$. By (2), then for each $j \in J$, $A_j = int_\tau(A_j) = \underline{R}(A_j)$. By Proposition 2.6 and Theorem 3.5,

$$\bigcap_{j \in J} A_j = \bigcap_{j \in J} \underline{R}(A_j) = \underline{R}\left(\bigcap_{j \in J} A_j\right) = int_\tau\left(\bigcap_{j \in J} A_j\right).$$

So $\bigcap_{j \in J} A_j \in \tau$. Hence τ is Alexandrov. \square

Proposition 4.10. *Let R be a preorder IVF relation on U . Then τ_R satisfies the (CC) axiom.*

Proof. For any $[a, b] \in [I]$ and $A \in F^{(i)}(U)$, by Theorems 4.3(2) and 3.5(5),

$$cl_{\tau_R}([a, b]A) = \overline{R}([a, b]A) = [a, b]\overline{R}(A) = [a, b]cl_{\tau_R}(A).$$

Thus τ_R satisfies the (CC) axiom. \square

Theorem 4.11. *Let τ be an IVF topology on U and $\{[a, b] : [a, b] \in [I]\} \subseteq \tau$. Let R_τ be the IVF relation induced by τ on U and let τ_{R_τ} be the IVF topology induced by R_τ on U . Then*

$$\tau_{R_\tau} = \tau \text{ if and only if } \tau \text{ satisfies the (CC) axiom.}$$

Proof. Necessity. Let $\tau_{R_\tau} = \tau$. By Theorems 4.7(1), R_τ is reflexive.

For each $A \in F^{(i)}(U)$, by Theorems 4.3(2) and 4.7(2),

$$int_\tau(A) = int_{\tau_{R_\tau}}(A) \subseteq \underline{R_\tau}(A) \subseteq int_\tau(A).$$

Then $int_\tau(A) = \underline{R_\tau}(A)$. So $\underline{R_\tau}$ is the interior operator of τ . By Theorem 3.6(1), R_τ is a preorder. By Proposition 4.10, τ satisfies the (CC) axiom.

Sufficiency. By Theorem 4.7(1), R_τ is reflexive. For any $x, y, z \in U$, put $cl(z_{\bar{1}})(y) = [a, b]$. By Remark 2.4 and Theorem 3.5(2),

$$\begin{aligned} [a, b]cl_\tau(y_{\bar{1}}) &= cl_\tau([a, b]y_{\bar{1}}) = cl_\tau(cl_\tau(z_{\bar{1}})(y)y_{\bar{1}}) \\ &\subseteq cl_\tau\left(\bigcup_{t \in U} (cl_\tau(z_{\bar{1}})(t)t_{\bar{1}})\right) = cl_\tau(cl_\tau(z_{\bar{1}})) = cl_\tau(z_{\bar{1}}). \end{aligned}$$

Then

$$\begin{aligned} R_\tau(x, y) \wedge R_\tau(y, z) &= cl_\tau(y_{\bar{1}})(x) \wedge cl_\tau(z_{\bar{1}})(y) = cl_\tau(y_{\bar{1}})(x) \wedge [a, b] \\ &= [a, b] \wedge cl_\tau(y_{\bar{1}})(x) = ([a, b]cl_\tau(y_{\bar{1}}))(x) \\ &\leq cl_\tau(z_{\bar{1}})(x) = R_\tau(x, z). \end{aligned}$$

So R is transitive.

So R_τ is preorder. For each $A \in F^{(i)}(U)$, by Theorem 4.3(2),

$$cl_{\tau_{R_\tau}}(A) = cl_{\theta_{R_\tau}}(A) = \overline{R_\tau}(A).$$

By Proposition 4.9(1), $\overline{R_\tau}(A) = cl_\tau(A)$. So $cl_{\tau_{R_\tau}}(A) = cl_\tau(A)$.

Thus $\tau_{R_\tau} = \tau$. □

Theorem 4.12. *Let*

$$\Sigma = \{R : R \text{ is a preorder IVF relation on } U\}$$

and

$$\Gamma = \{\tau : \tau \text{ is an IVF topology on } U \text{ satisfying the (CC) axiom}\}.$$

Then there exists a one-to-one correspondence between Σ and Γ .

Proof. Two mappings $f : \Sigma \rightarrow \Gamma$ and $g : \Gamma \rightarrow \Sigma$ are defined as follows:

$$f(R) = \tau_R \ (R \in \Sigma), \ g(\tau) = R_\tau \ (\tau \in \Gamma).$$

By Theorem 4.8, $g \circ f = i_\Sigma$, where $g \circ f$ is the composition of f and g , and i_Σ is the identity mapping on Σ .

By Proposition 4.9(3) and Theorem 4.11, $f \circ g = i_\Gamma$, where $f \circ g$ is the composition of g and f , and i_Γ is the identity mapping on Γ .

Hence f and g are two one-to-one correspondences. This prove that there exists a one-to-one correspondence between Σ and Γ . □

Theorem 4.13. *Let τ be an IVF topology on U . Then the following are equivalent.*

- (1) τ satisfies the (CC) axiom;
- (2) For any $[a, b] \in [I]$ and $A \in F^{(i)}(U)$,

$$int_\tau(\widetilde{[a, b]} \cup A) = \widetilde{[a, b]} \cup int_\tau(A);$$

- (3) *There exists a preorder IVF relation ρ on U such that $\bar{\rho}$ is the closure operator of τ ;*
 (4) *There exists a preorder IVF relation ρ on U such that $\underline{\rho}$ is the interior operator of τ ;*
 (5) \bar{R}_τ *is the closure operator of τ ;*
 (6) \underline{R}_τ *is the interior operator of τ .*

Proof. (1) \iff (2) is obvious.

(1) \implies (3). Suppose that τ satisfies the (CC) axiom. Pick $\rho = R_\tau$. By Proposition 4.9(1), $\bar{\rho}$ is the closure operator of τ . By Theorem 3.6(2), ρ is preorder.

(3) \implies (4). Let $\bar{\rho}$ be the closure operator of τ for some preorder IVF relation ρ on U . For each $A \in F^{(i)}(U)$, by Proposition 2.6(4) and Theorem 3.5(3),

$$\underline{\rho}(A) = (\bar{\rho}(A^c))^c = (cl_\tau(A^c))^c = int_\tau(A).$$

Thus, $\underline{\rho}$ is the interior operator of τ .

(4) \implies (6). Let $\underline{\rho}$ be the interior operator of τ for some preorder IVF relation ρ on U . For each $(x, y) \in U \times U$, by Remark 3.3(1),

$$\rho(x, y) = \bar{1} - \underline{\rho}((y_1)^c)(x) = \bar{1} - int_\tau((y_1)^c)(x) = cl_\tau(y_1)(x) = R_\tau(x, y).$$

Then $\rho = R_\tau$. Note that $\underline{\rho}$ is the interior operator of τ . Then \underline{R}_τ is the interior operator of τ .

(6) \iff (5) holds by Proposition 2.6(4) and Theorem 3.5(3).

(5) \implies (1). For any $[a, b] \in [I]$ and $A \in F^{(i)}(U)$, by Theorem 3.5(5),

$$cl_\tau([a, b]A) = \bar{R}_\tau([a, b]A) = [a, b]\bar{R}_\tau(A) = [a, b]cl_\tau(A).$$

Thus τ satisfies the (CC) axiom. \square

5 Lattice structures of IVF approximation spaces

Let (U, R) be an IVF approximation space. We denote

$$Fix(\underline{R}) = \{A \in F^{(i)}(U) : \underline{R}(A) = A\}, \quad Fix(\bar{R}) = \{A \in F^{(i)}(U) : \bar{R}(A) = A\};$$

$$Im(\underline{R}) = \{\underline{R}(A) : A \in F^{(i)}(U)\}, \quad Im(\bar{R}) = \{\bar{R}(A) : A \in F^{(i)}(U)\};$$

$$Def(R) = \{A \in F^{(i)}(U) : \underline{R}(A) = \bar{R}(A)\};$$

$$Fix(\underline{R} \circ \underline{R}) = \{A \in F^{(i)}(U) : \underline{R}(\underline{R}(A)) = A\}, \quad Fix(\bar{R} \circ \bar{R}) = \{A \in F^{(i)}(U) : \bar{R}(\bar{R}(A)) = A\};$$

$$\mathcal{O}(\underline{R}) = \{A \in F^{(i)}(U) : \underline{R}(\underline{R}(A)) = \underline{R}(A)\}, \quad \mathcal{O}(\bar{R}) = \{A \in F^{(i)}(U) : \bar{R}(\bar{R}(A)) = \bar{R}(A)\}.$$

For $\mathcal{A} \subseteq F^{(i)}(U)$, denote $\mathcal{A}^c = \{A : A^c \in \mathcal{A}\}$.

Proposition 5.1. *Let (U, R) be an IVF approximation space.*

(1)

$$\begin{aligned} \text{Fix}(\underline{R}) &\subseteq \text{Im}(\underline{R}), \quad \text{Fix}(\overline{R}) \subseteq \text{Im}(\overline{R}); \\ \text{Fix}(\underline{R}) &\subseteq \text{Fix}(\underline{R} \circ \underline{R}), \quad \text{Fix}(\overline{R}) \subseteq \text{Fix}(\overline{R} \circ \overline{R}). \end{aligned}$$

(2) *If R is reflexive, then*

a) *For each $[a, b] \in [I]$,*

$$\begin{aligned} [a, b] &\in \text{Fix}(\underline{R}) \cap \text{Fix}(\overline{R}) \cap \text{Fix}(\underline{R} \circ \underline{R}) \cap \text{Fix}(\overline{R} \circ \overline{R}) \\ &\cap \text{Im}(\underline{R}) \cap \text{Im}(\overline{R}) \cap \text{Def}(R) \cap \mathcal{O}(\underline{R}) \cap \mathcal{O}(\overline{R}). \end{aligned}$$

b)

$$\begin{aligned} (\text{Fix}(\underline{R}))^c &= \text{Fix}(\overline{R}); \\ (\text{Fix}(\underline{R} \circ \underline{R}))^c &= \text{Fix}(\overline{R} \circ \overline{R}); \\ (\text{Im}(\underline{R}))^c &= \text{Im}(\overline{R}); \\ (\mathcal{O}(\underline{R}))^c &= \mathcal{O}(\overline{R}). \end{aligned}$$

c)

$$\begin{aligned} \text{Fix}(\underline{R}) &= \text{Fix}(\underline{R} \circ \underline{R}) \subseteq \mathcal{O}(\underline{R}), \quad \text{Fix}(\overline{R}) = \text{Fix}(\overline{R} \circ \overline{R}) \subseteq \mathcal{O}(\overline{R}); \\ \text{Fix}(\underline{R} \circ \underline{R}) &\subseteq \text{Im}(\underline{R}), \quad \text{Fix}(\overline{R} \circ \overline{R}) \subseteq \text{Im}(\overline{R}). \end{aligned}$$

d)

$$\text{Def}(R) = \text{Fix}(\underline{R}) \cap \text{Fix}(\overline{R}).$$

(3) *If R is proorder, then*

$$\begin{aligned} \text{Fix}(\underline{R}) &= \text{Im}(\underline{R}), \quad \text{Fix}(\overline{R}) = \text{Im}(\overline{R}); \\ \mathcal{O}(\underline{R}) &= F^{(i)}(U) = \mathcal{O}(\overline{R}). \end{aligned}$$

Proof. These hold by Theorem 3.5, Theorem 3.7 and Corollary 3.8. \square

Theorem 5.2. *Let (U, R) be an IVF approximation space. If R is reflexive, then*

- (1) *$(\text{Fix}(\underline{R}), \cap, \cup)$ is a complete distributive lattice.*
- (2) *$(\text{Fix}(\overline{R}), \cap, \cup)$ is a complete distributive lattice.*

Proof. (1) By Proposition 5.1(1), $\text{Fix}(\underline{R}) \neq \emptyset$.

Let $\{A_j : j \in J\} \subseteq \text{Fix}(\underline{R})$. Then $\underline{R}(A_j) = A_j$ for each $j \in J$. By Theorem 3.5,

$$\underline{R}(\bigcap_{j \in J} A_j) = \bigcap_{j \in J} \underline{R}(A_j) = \bigcap_{j \in J} A_j, \quad \underline{R}(\bigcup_{j \in J} A_j) \supseteq \bigcup_{j \in J} \underline{R}(A_j) = \bigcup_{j \in J} A_j.$$

By Theorem 3.7(2), $\underline{R}(\bigcup_{j \in J} A_j) \subseteq \bigcup_{j \in J} A_j$. Then $\underline{R}(\bigcup_{j \in J} A_j) = \bigcup_{j \in J} A_j$. So $\bigcap_{j \in J} A_j, \bigcup_{j \in J} A_j \in \text{Fix}(\underline{R})$.

Thus $(\text{Fix}(\underline{R}), \cap, \cup)$ is a complete lattice. Note that $(\text{Fix}(\underline{R}), \cap, \cup)$ satisfies distributive law. Then $(\text{Fix}(\underline{R}), \cap, \cup)$ is a complete distributive lattice.

(2) The proof is similar to (1). \square

Theorem 5.3. *Let (U, R) be an IVF approximation space. If R is preorder, then $(\text{Im}(\underline{R}), \cap, \cup)$ and $(\text{Im}(\overline{R}), \cap, \cup)$ are both complete distributive lattice.*

Proof. This hold by Proposition 5.1(4) and Theorem 5.2. \square

Theorem 5.4. *Let (U, R) be an IVF approximation space. If R is reflexive, then $(\text{Def}(R), \cap, \cup)$ is a complete lattice.*

Proof. Let $\{A_j : j \in J\} \subseteq \text{Def}(R)$. Then $\underline{R}(A_j) = \overline{R}(A_j)$ for each $j \in J$. By Theorems 3.5 and 3.7(2),

$$\begin{aligned} \underline{R}\left(\bigcap_{j \in J} A_j\right) &= \bigcap_{j \in J} \underline{R}(A_j) = \bigcap_{j \in J} \overline{R}(A_j) \supseteq \overline{R}\left(\bigcap_{j \in J} A_j\right), \quad \underline{R}\left(\bigcap_{j \in J} A_j\right) \subseteq \overline{R}\left(\bigcap_{j \in J} A_j\right); \\ \underline{R}\left(\bigcup_{j \in J} A_j\right) &\supseteq \bigcup_{j \in J} \underline{R}(A_j) = \bigcup_{j \in J} \overline{R}(A_j) = \overline{R}\left(\bigcup_{j \in J} A_j\right), \quad \underline{R}\left(\bigcup_{j \in J} A_j\right) \subseteq \overline{R}\left(\bigcup_{j \in J} A_j\right). \end{aligned}$$

Then $\underline{R}\left(\bigcap_{j \in J} A_j\right) = \overline{R}\left(\bigcap_{j \in J} A_j\right)$, $\underline{R}\left(\bigcup_{j \in J} A_j\right) = \overline{R}\left(\bigcup_{j \in J} A_j\right)$. So $\bigcap_{j \in J} A_j, \bigcup_{j \in J} A_j \in \text{Def}(R)$.

Thus $(\text{Def}(R), \cap, \cup)$ is a complete lattice. \square

Theorem 5.5. *Let (U, R) be an IVF approximation space. If R is reflexive, then*

- (1) $(\text{Fix}(\underline{R} \circ \underline{R}), \cap, \cup)$ is a complete distributive lattice.
- (2) $(\text{Fix}(\overline{R} \circ \overline{R}), \cap, \cup)$ is a complete distributive lattice.

Proof. (1) By Proposition 5.1(1), $\text{Fix}(\underline{R} \circ \underline{R}) \neq \emptyset$.

Let $\{A_j : j \in J\} \subseteq \text{Fix}(\underline{R} \circ \underline{R})$. Then $\underline{R}(\underline{R}(A_j)) = A_j$ for each $j \in J$. By Theorem 3.5,

$$\begin{aligned} \underline{R}\left(\underline{R}\left(\bigcap_{j \in J} A_j\right)\right) &= \underline{R}\left(\bigcap_{j \in J} \underline{R}(A_j)\right) = \bigcap_{j \in J} \underline{R}(\underline{R}(A_j)) = \bigcap_{j \in J} A_j, \\ \underline{R}\left(\underline{R}\left(\bigcup_{j \in J} A_j\right)\right) &\supseteq \underline{R}\left(\bigcup_{j \in J} \underline{R}(A_j)\right) \supseteq \bigcup_{j \in J} \underline{R}(\underline{R}(A_j)) = \bigcup_{j \in J} A_j. \end{aligned}$$

By Theorem 3.7(2), $\underline{R}(\underline{R}(\bigcup_{j \in J} A_j)) \subseteq \underline{R}(\bigcup_{j \in J} A_j) \subseteq \bigcup_{j \in J} A_j$. Then $\underline{R}(\underline{R}(\bigcup_{j \in J} A_j)) = \bigcup_{j \in J} A_j$. So $\bigcap_{j \in J} A_j, \bigcup_{j \in J} A_j \in \text{Fix}(\underline{R} \circ \underline{R})$. Thus $(\text{Fix}(\underline{R} \circ \underline{R}), \cap, \cup)$ is a complete lattice.

Note that $(\text{Fix}(\underline{R} \circ \underline{R}), \cap, \cup)$ satisfies distributive law. Then $(\text{Fix}(\underline{R} \circ \underline{R}), \cap, \cup)$ is a complete distributive lattice.

(2) The proof is similar to (1). \square

Theorem 5.6. *Let (U, R) be an IVF approximation space. If R is reflexive, then*

- (1) $(\mathcal{O}(\underline{R}), \cap, \cup)$ is a complete distributive lattice.
- (2) $(\mathcal{O}(\overline{R}), \cap, \cup)$ is a complete distributive lattice.

Proof. (1) By Proposition 5.1(1), $\mathcal{O}(\underline{R}) \neq \emptyset$.

Let $\{A_j : j \in J\} \subseteq \mathcal{O}(\underline{R})$. Then $\underline{R}(\underline{R}(A_j)) = \underline{R}(A_j)$ for each $j \in J$. By Theorem 3.5,

$$\underline{R}(\underline{R}(\bigcap_{j \in J} A_j)) = \underline{R}(\bigcap_{j \in J} \underline{R}(A_j)) = \bigcap_{j \in J} \underline{R}(\underline{R}(A_j)) = \bigcap_{j \in J} \underline{R}(A_j) = \underline{R}(\bigcap_{j \in J} A_j),$$

$$\underline{R}(\underline{R}(\bigcup_{j \in J} A_j)) \supseteq \underline{R}(\bigcup_{j \in J} \underline{R}(A_j)) \supseteq \bigcup_{j \in J} \underline{R}(\underline{R}(A_j)) = \bigcup_{j \in J} \underline{R}(A_j).$$

By Theorem 3.7(2), $\underline{R}(\underline{R}(\bigcup_{j \in J} A_j)) \subseteq \underline{R}(\bigcup_{j \in J} A_j)$. Then $\underline{R}(\underline{R}(\bigcup_{j \in J} A_j)) = \bigcup_{j \in J} A_j$.

So $\bigcap_{j \in J} A_j, \bigcup_{j \in J} A_j \in \mathcal{O}(\underline{R})$.

Thus $(\mathcal{O}(\underline{R}), \cap, \cup)$ is a complete distributive lattice.

(3) The proof is similar to (2). □

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Stability of ternary m -derivations on ternary Banach algebras

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Abstract. In this paper, we use a fixed point method to prove the stability of ternary m -derivations on ternary Banach algebras.

1. Introduction and preliminaries

Consider the functional equation $\mathfrak{S}_1(f) = \mathfrak{S}_2(f)$ (\mathfrak{S}) in a certain general setting. A mapping g is an approximate solution of (\mathfrak{S}) if $\mathfrak{S}_1(g)$ and $\mathfrak{S}_2(g)$ are close in some sense. The Ulam stability problem asks whether or not there is a true solution of (\mathfrak{S}) near g . A functional equation is superstable if every approximate solution of the equation is an exact solution of it. For more details about various results concerning such problems the reader is referred to [3, 7, 9, 11, 14, 15, 18, 19, 20, 22, 28].

Ternary algebraic operations were considered in the 19th century by several mathematicians: Cayley [6] introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii in 1990 [13]. As an application in physics, the quark model inspired a particular brand of ternary algebraic systems. There are also some applications, although still hypothetical, in the fractional quantum Hall effect, the non-standard statistics (the anyons), supersymmetric theories, Yang-Baxter equation, etc, (cf. [1, 29]).

The comments on physical applications of ternary structures can be found in (see [10, 24, 26]).

The monomial $f(x) = ax^m$ ($x \in \mathbb{R}$, $m = 1, 2, 3, 4$) is a solution of the following functional equation

$$f(ax + y) + f(ax - y) = a^{m-2}[f(x + y) + f(x - y)] + 2(a^2 - 1)[a^{m-2}f(x) + \frac{(m-2)(1 - (m-2)^2)}{6}f(y)]. \quad (1.1)$$

For $m = 1, 2, 3, 4$, the functional equation (1.1) is equivalent to the additive, quadratic, cubic and quartic functional equation, respectively. The general solution of the functional equation (1.1) for any fixed integer a with $a \neq 0, \pm 1$, was obtained by Eshaghi Gordji et al. [8].

Let A be an algebra. An additive mapping $f : A \rightarrow A$ is called a derivation if $f(xy) = xf(y) + f(x)y$ holds for all $x, y \in A$. If, in addition, $f(\lambda x) = \lambda f(x)$ for all $x \in A$ and all $\lambda \in \mathbb{F}$, then f is called a linear derivation, where \mathbb{F} denotes the scalar field of A . The stability result concerning derivations between operator algebras was first obtained by Šemrl [23]. In [2], Badora proved the stability of functional equation $f(xy) = xf(y) + f(x)y$, where f is a mapping on normed algebra A with unit. Recently, Miura et al. [17] examined the stability of derivations on Banach algebras.

Suppose that A is a Banach algebra. Let θ, r be nonnegative real numbers. If $r \neq 1$ and $f : A \rightarrow A$ is a mapping such that

$$\|f(x + y) - f(x) - f(y)\| \leq \theta(\|x\|^r + \|y\|^r),$$

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$$\|f(xy) - xf(y) - f(x)y\| \leq \theta(\|x\|^r \cdot \|y\|^r)$$

for all $x, y \in A$. Then there exists a unique derivation $D : A \rightarrow A$ satisfying

$$\|f(x) - D(x)\| \leq \frac{2\theta}{|2 - 2^r|} \|x\|^r,$$

for all $x, y \in A$. In particular, if A is a Banach algebra, then f is a derivation.

The various problems of the stability of derivations have been studied during last few years (see, for instance, [10, 25, 27]).

Definition 1.1. Let A be a ternary algebra.

(i) A mapping $f : A \rightarrow A$ is called a ternary additive derivation (briefly, ternary 1-derivation) if f is an additive mapping satisfying $f([a, b, c]) = [f(a), b, c] + [a, f(b), c] + [a, b, f(c)]$ for all $a, b, c \in A$;

(ii) A mapping $f : A \rightarrow A$ is called a ternary quadratic derivation (briefly, ternary 2-derivation) if f is a quadratic mapping satisfying $f([a, b, c]) = [f(a), b, [b, c, c]] + [a, a, [f(b), c, c]] + [a, a, [b, b, f(c)]]$ for all $a, b, c \in A$;

(iii) A mapping $f : A \rightarrow A$ is called a ternary cubic derivation (briefly, ternary 3-derivation) if f is a cubic mapping satisfying $f([a, b, c]) = [f(a), b, [b, b, [c, c, c]]] + [a, a, [a, f(b), [c, c, c]]] + [a, a, [a, b, [b, b, f(c)]]]$ for all $a, b, c \in A$;

(iv) A mapping $f : A \rightarrow A$ is called a ternary quartic derivation (briefly, ternary 4-derivation) if f is a quartic mapping satisfying $f([a, b, c]) = [f(a), b, [b, b, [b, c, [c, c, c]]] + [a, a, [a, a, [f(b), c, [c, c, c]]] + [a, a, [a, a, [b, b, [b, b, f(c)]]]]$ for all $a, b, c \in A$.

The main theorem of [16], which is called the alternative of fixed point, plays an important role in proving the stability problem. Recently, Cădariu and Radu [4] applied the fixed point method to the investigation of the Cauchy additive functional equation (see also [5, 12, 21]).

In this paper, we adopt the idea of Cădariu and Radu to establish the stability of m -derivations on ternary Banach algebras related to the functional equation (1.1). In addition, we study the superstability of the functional equation (1.1) by suitable control functions.

2. Stability of ternary m -derivations on ternary Banach algebras via fixed point method

Throughout this section, we suppose that A is a ternary Banach algebra, and m is a fixed positive integer less than 5. For convenience, we use the following abbreviation for a given mapping $f : A \rightarrow A$

$$\begin{aligned} \Delta_m f(x, y) &= f(wx + y) + f(wx - y) \\ &\quad - w^{m-2} [f(x + y) + f(x - y)] - 2(w^2 - 1) [w^{m-2} f(x) + \frac{(m-2)(1-(m-2)^2)}{6} f(y)] \end{aligned}$$

for all $x, y \in A$ and any fixed integers $w \neq 0, \pm 1$.

Let

$$\begin{aligned} F_1 f(a, b, c) &:= [f(a), b, c] + [a, f(b), c] + [a, b, f(c)], \\ F_2 f(a, b, c) &:= [f(a), b, [b, c, c]] + [a, a, [f(b), c, c]] + [a, a, [b, b, f(c)]], \\ F_3 f(a, b, c) &:= [f(a), b, [b, b, [c, c, c]]] + [a, a, [a, f(b), [c, c, c]]] + [a, a, [a, b, [b, b, f(c)]]], \\ F_4 f(a, b, c) &:= [f(a), b, [b, b, [b, c, [c, c, c]]] + [a, a, [a, a, [f(b), c, [c, c, c]]] + [a, a, [a, a, [b, b, [b, b, f(c)]]]] \end{aligned}$$

for all $a, b, c \in A$.

Theorem 2.1. Let $f : A \rightarrow A$ be a mapping for which there exists function $\varphi_m : A^5 \rightarrow [0, \infty)$ such that

$$\|\Delta_m f(x, y) + f([a, b, c]) - F_m f(a, b, c)\| \leq \varphi_m(x, y, a, b, c) \quad (2.1)$$

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for all $x, y, a, b, c \in A$. If there exists a constant $0 < L < 1$ such that

$$\varphi_m \left(\frac{x}{w}, \frac{y}{w}, \frac{a}{w}, \frac{b}{w}, \frac{c}{w} \right) \leq \frac{L}{|w|^m} \varphi_m(x, y, a, b, c), \quad (2.2)$$

for all $x, y, a, b, c \in A$, then there exists a unique ternary m -derivation $D_m : A \rightarrow A$ such that

$$\|f(x) - D_m(x)\| \leq \frac{L}{2|w|^m(1-L)} \varphi_m(x, 0, 0, 0, 0), \quad (2.3)$$

for all $x \in A$.

Proof. First of all, if we take $x = y = a = b = c = 0$ in (2.2), then we obtain that $\varphi_m(0, 0, 0, 0, 0) = 0$, since $0 < L < 1$ and $w \neq 0, \pm 1$. Letting $x = y = a = b = c = 0$ in (2.1), we obtain $f(0) = 0$.

It follows from (2.2) that

$$\lim_{n \rightarrow \infty} |w|^{mn} \varphi_m \left(\frac{x}{w^n}, \frac{y}{w^n}, \frac{a}{w^n}, \frac{b}{w^n}, \frac{c}{w^n} \right) = 0 \quad (2.4)$$

for all $x, y, a, b, c \in A$.

Let us define Ω to be the set of all mappings $g : A \rightarrow A$ and introduce a generalized metric on Ω as follows:

$$d(g, h) = d_{\varphi_m}(g, h) = \inf \{ K \in (0, \infty) : \|g(x) - h(x)\| \leq K \varphi_m(x, 0, 0, 0, 0), x \in A \}$$

It is easy to show that (Ω, d) is a generalized complete metric space [4, 5].

Now we consider the mapping $T : \Omega \rightarrow \Omega$ defined by $Tg(x) = w^m g(\frac{x}{w})$ for all $x \in A$ and all $g \in \Omega$.

Note that for all $g, h \in \Omega$,

$$\begin{aligned} d(g, h) < K &\Rightarrow \|g(x) - h(x)\| \leq K \varphi_m(x, 0, 0, 0, 0) && \text{for all } x \in A, \\ &\Rightarrow \left\| w^m g \left(\frac{x}{w} \right) - w^m h \left(\frac{x}{w} \right) \right\| \leq |w|^m K \varphi_m \left(\frac{x}{w}, 0, 0, 0, 0 \right) && \text{for all } x \in A, \\ &\Rightarrow \left\| w^m g \left(\frac{x}{w} \right) - w^m h \left(\frac{x}{w} \right) \right\| \leq L K \varphi_m(x, 0, 0, 0, 0) && \text{for all } x \in A, \\ &\Rightarrow d(Tg, Th) \leq L K. \end{aligned}$$

Hence we see that

$$d(Tg, Th) \leq L d(g, h)$$

for all $g, h \in \Omega$, that is, T is a strictly contractive mapping of Ω with the Lipschitz constant L .

Putting $y = a = b = c = 0$ in (2.1), we have

$$\|2f(wx) - 2w^m f(x)\| \leq \varphi_m(x, 0, 0, 0, 0) \quad (2.5)$$

for all $x \in A$. So

$$\left\| f(x) - w^m f \left(\frac{x}{w} \right) \right\| \leq \frac{1}{2} \varphi_m \left(\frac{x}{w}, 0, 0, 0, 0 \right) \leq \frac{L}{2|w|^m} \varphi_m(x, 0, 0, 0, 0)$$

for all $x \in A$, that is, $d(f, Tf) \leq \frac{L}{2|w|^m} < \infty$.

Now, from the fixed point alternative, it follows that there exists a fixed point D_m of T in Ω such that

$$D_m(x) = \lim_{n \rightarrow \infty} w^{mn} f \left(\frac{x}{w^n} \right) \quad (2.6)$$

for all $x \in A$, since $\lim_{n \rightarrow \infty} d(T^n f, D_m) = 0$.

On the other hand, it follows from (2.1), (2.4) and (2.6) that

$$\|\Delta_m D_m(x, y)\| = \lim_{n \rightarrow \infty} |w|^{mn} \left\| \Delta_m f \left(\frac{x}{w^n}, \frac{y}{w^n} \right) \right\| \leq \lim_{n \rightarrow \infty} |w|^{mn} \varphi_m \left(\frac{x}{w^n}, \frac{y}{w^n}, 0, 0, 0 \right) = 0$$

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for all $x, y \in A$. So $\Delta_m D_m(x, y) = 0$. By [8], D_m is an m -mapping. So it follows from the definition of D_m , (2.2) and (2.4) that

$$\begin{aligned} \|D_m([a, b, c]) - F_m D_m(a, b, c)\| &= \lim_{n \rightarrow \infty} |w|^{3mn} \left\| f\left(\left[\frac{a}{w^n}, \frac{b}{w^n}, \frac{c}{w^n}\right]\right) - F_m f\left(\frac{a}{w^n}, \frac{b}{w^n}, \frac{c}{w^n}\right) \right\| \\ &\leq \lim_{n \rightarrow \infty} |w|^{3mn} \varphi_m\left(0, 0, \frac{a}{w^n}, \frac{b}{w^n}, \frac{c}{w^n}\right) = 0 \end{aligned}$$

for all $a, b, c \in A$. So $D_m([a, b, c]) = F_m D_m(a, b, c)$ for all $a, b, c \in A$.

According to the fixed point alternative, since D_m is the unique fixed point of T in the set $\Lambda = \{g \in \Omega : d(f, g) < \infty\}$, D_m is the unique mapping such that

$$\|f(x) - D_m(x)\| \leq K \varphi_m(x, 0, 0, 0, 0)$$

for all $x \in A$ and $K > 0$. Again using the fixed point alternative, we obtain

$$d(f, D_m) \leq \frac{1}{1-L} d(f, Tf) \leq \frac{L}{2|w|^m(1-L)}$$

and so we conclude that

$$\|f(x) - D_m(x)\| \leq \frac{L}{2|w|^m(1-L)} \varphi_m(x, 0, 0, 0, 0)$$

for all $x \in A$. This completes the proof. \square

Corollary 2.2. Let θ, r, s be nonnegative real numbers with $s > m$ and $r > m$. Suppose that $f : A \rightarrow A$ is a mapping such that

$$\|\Delta_m f(x, y) + f([a, b, c]) - F_m f(a, b, c)\| \leq \theta(\|x\|^r + \|y\|^r + \|a\|^s \cdot \|b\|^s \cdot \|c\|^s)$$

for all $x, y, a, b, c \in A$. Then there exists a unique ternary m -derivation $D_m : A \rightarrow A$ satisfying

$$\|f(x) - D_m(x)\| \leq \frac{\theta}{2(|w|^r - |w|^m)} \|x\|^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi_m(x, y, a, b, c) := \theta(\|x\|^r + \|y\|^r + \|a\|^s \cdot \|b\|^s \cdot \|c\|^s)$$

for all $x, y, a, b, c \in A$. Then we can choose $L = |w|^{m-r}$ and we get the desired results. \square

Remark 2.3. Let $f : A \rightarrow A$ be a mapping with $f(0) = 0$ for which there exist functions $\varphi_m : A^5 \rightarrow [0, \infty)$ satisfying (2.1) and (2.2). Let $0 < L < 1$ be a constant such that $\varphi_m(wx, wy, wa, wb, wc) \leq |w|^m L \varphi_m(x, y, a, b, c)$ for all $x, y, a, b, c \in A$. By a similar method to the proof of Theorem 2.1, we can show that there exists a unique ternary m -derivation $D_m : A \rightarrow X$ satisfying

$$\|f(x) - D_m(x)\| \leq \frac{1}{2|w|^m(1-L)} \varphi_m(x, 0, 0, 0, 0)$$

for all $x \in A$.

For the case $\varphi_m(x, y, a, b, c) := \delta + \theta(\|x\|^r + \|y\|^r + \|a\|^s \cdot \|b\|^s \cdot \|c\|^s)$ (where θ, δ are nonnegative real numbers and $0 < r, s < m$), there exists a unique ternary m -derivation $D_m : A \rightarrow A$ satisfying

$$\|f(x) - D_m(x)\| \leq \frac{\delta}{2(|w|^m - |w|^r)} + \frac{\theta}{2(|w|^m - |w|^r)} \|x\|^r$$

for all $x \in A$.

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On IF approximating spaces ^{*}

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Abstract: In this paper, a IF approximating space is introduced. It is a particular type of IF topological spaces which associate with IF relations. A characteristic condition for IF topological spaces to be IF approximating spaces is established.

Keywords: IF set; IF relation; IF approximate space; IF rough set; IF topology; IF approximating space.

1 Introduction

Rough set theory was proposed by Pawlak [16, 17] as a mathematical tool to handle imprecision and uncertainty in data analysis. Usefulness and versatility of this theory have amply been demonstrated by successful applications in a variety of problems [21, 22].

The basic structure of rough set theory is an approximation space. Based on it, rough approximations can be induced. Using them, knowledge hidden in information systems may be revealed and expressed in the form of decision rules [16].

Intuitionistic fuzzy (IF, for short) sets were originated by Atanassov [1, 2]. It is a straightforward extension of Zadeh's fuzzy sets [26]. IF sets have played an useful role in the research of uncertainty theories. Unlike a fuzzy set, which gives a degree of which element belongs to a set, an IF set gives both a membership degree and a nonmembership degree. Thus, an IF set is more objective than a fuzzy set to describe the vagueness of data or information.

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Recently, IF approximate spaces were introduced and then IF rough sets were presented [6, 7, 8, 20, 23, 27, 28, 29, 30]. For example, Zhou et al. [27, 28, 29, 30] studied structures of IF rough sets, Wu et al. [23] researched IF topologies based on preorder IF relations, Zhang et al. [31] investigated IF rough sets on two universes.

It is well known that topology is a branch of mathematics, whose concepts exist not only in almost all branches of mathematics but also in many real life applications. Topology and rough set theory have been widely used in research field of computer science. An interesting and natural research topic is to study the relationship between rough sets and topologies.

The purpose of this paper is to investigate IF approximating space where the given IF topology coincides with the IF topology induced by some reflexive IF relation.

2 Preliminaries

Throughout this paper, “Intuitionistic fuzzy” is briefly written “IF”, X denotes a infinite universe. I denotes $[0, 1]$, $J = \{\lambda \in I \times I : a + b \leq 1\}$, $F(X)$ denotes the family of all fuzzy sets in X and $IF(X)$ denotes the family of all IF sets in X .

In this section, we recall some basic notions and properties related to IF sets, IF topologies and fuzzy rough sets.

2.1 IF sets

Definition 2.1 ([11]). Let $(a, b), (c, d) \in I \times I$. Define

- (1) $(a, b) = (c, d) \iff a = c, b = d$.
- (2) $(a, b) \sqcup (c, d) = (a \vee c, b \wedge d)$, $(a, b) \sqcap (c, d) = (a \wedge c, b \vee d)$.
- (3) $(a, b)^c = (b, a)$.

Moreover, for $\{(a_\alpha, b_\alpha) : \alpha \in \Gamma\} \subseteq I \times I$,

$$\bigsqcup_{\alpha \in \Gamma} (a_\alpha, b_\alpha) = (\bigvee_{\alpha \in \Gamma} a_\alpha, \bigwedge_{\alpha \in \Gamma} b_\alpha) \text{ and } \bigsqcap_{\alpha \in \Gamma} (a_\alpha, b_\alpha) = (\bigwedge_{\alpha \in \Gamma} a_\alpha, \bigvee_{\alpha \in \Gamma} b_\alpha).$$

Definition 2.2 ([11]). Let $(a, b), (c, d) \in J$ and let $S \subseteq J \times J$. $(a, b)S(c, d)$, if $a \leq c$ and $b \geq d$. We denote S by \leq .

Obviously, $(a, b) = (c, d) \iff (a, b) \leq (c, d)$ and $(c, d) \leq (a, b)$.

Remark 2.3. (1) (J, \leq) be a poset with $0_J = (0, 1)$ and $1_J = (1, 0)$.

- (2) $(a, b)^{cc} = (a, b)$.
- (3) $((a, b) \sqcup (c, d)) \sqcap (e, f) = (a, b) \sqcup ((c, d) \sqcap (e, f))$,
 $((a, b) \sqcap (c, d)) \sqcap (e, f) = (a, b) \sqcap ((c, d) \sqcap (e, f))$.
- (4) $(a, b) \sqcup (c, d) = (c, d) \sqcup (a, b)$, $(a, b) \sqcap (c, d) = (c, d) \sqcap (a, b)$.
- (5) $((a, b) \sqcup (c, d)) \sqcap (e, f) = ((a, b) \sqcap (e, f)) \sqcup ((c, d) \sqcap (e, f))$.
 $((a, b) \sqcap (c, d)) \sqcup (e, f) = ((a, b) \sqcup (e, f)) \sqcap ((c, d) \sqcup (e, f))$.
- (6) $(\bigsqcup_{\alpha \in \Gamma} (a_\alpha, b_\alpha))^c = \bigsqcap_{\alpha \in \Gamma} (a_\alpha, b_\alpha)^c$, $(\bigsqcap_{\alpha \in \Gamma} (a_\alpha, b_\alpha))^c = \bigsqcup_{\alpha \in \Gamma} (a_\alpha, b_\alpha)^c$.

Proposition 2.4 ([11]). $(J, \leq, \sqcap, \sqcup)$ be a complete distributive lattice.

Definition 2.5 ([1]). An IF set A in X is an object having the form

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle : x \in X \},$$

where $\mu_A, \nu_A \in F(X)$ satisfying $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$, and $\mu_A(x), \nu_A(x)$ are used to define the degree of membership and the degree of non-membership of the element x to A , respectively.

For the sake of simplicity, we give the following definition.

Definition 2.6. A is called an IF set in X , if $A = (A^*, A_*) \in F(X) \times F(X)$ and for each $x \in X$, $A(x) = (A^*(x), A_*(x)) \in J$, where $A^*(x), A_*(x)$ are used to define the degree of membership and the degree of non-membership of the element x to A , respectively.

For each $\mathcal{A} \subseteq IF(X)$, we denote

$$\mathcal{A}^c = \{A^c : A \in \mathcal{A}\},$$

$$\mathcal{A}^* = \{A^* : A \in \mathcal{A}\} \text{ and } \mathcal{A}_* = \{A_* : A \in \mathcal{A}\}.$$

Let $\lambda \in J$. $\widehat{\lambda}$ represents a constant IF set which satisfies $\widehat{\lambda}(x) = \lambda$ for each $x \in X$. Denote $1_\sim = (\widehat{1}, 0)$ and $0_\sim = (0, \widehat{1})$.

Some IF relations and IF operations are defined as follows ([1, 2]): for any $A, B \in IF(X)$ and $\{A_\alpha : \alpha \in \Gamma\} \subseteq IF(X)$,

$$(1) A = B \iff A(x) = B(x) \text{ for each } x \in X.$$

$$(2) A \subseteq B \iff A(x) \leq B(x) \text{ for each } x \in X.$$

$$(3) (\bigcup_{\alpha \in \Gamma} A_\alpha)(x) = \bigcup_{\alpha \in \Gamma} A_\alpha(x) \text{ for each } x \in X.$$

$$(4) (\bigcap_{\alpha \in \Gamma} A_\alpha)(x) = \bigcap_{\alpha \in \Gamma} A_\alpha(x) \text{ for each } x \in X.$$

$$(5) A^c(x) = A(x)^c \text{ for each } x \in X.$$

$$(6) (\lambda A)(x) = \lambda \sqcap (A^*(x), A_*(x)) \text{ for any } x \in X \text{ and } \lambda \in J.$$

Obviously, $A = B \iff A^* = B^*$ and $A_* = B_* \iff A \subseteq B$ and $B \subseteq A$.

We define two special IF sets $1_y = ((1_y)^*, (1_y)_*)$ and $0_y = ((0_y)^*, (0_y)_*)$ for some $y \in X$ as follows:

$$(1_y)^*(x) = \begin{cases} 1, & x = y, \\ 0, & x \neq y. \end{cases} \quad (1_y)_*(x) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases}$$

$$(0_y)^*(x) = \begin{cases} 0, & x = y, \\ 1, & x \neq y. \end{cases} \quad (0_y)_*(x) = \begin{cases} 1, & x = y, \\ 0, & x \neq y. \end{cases}$$

Remark 2.7. For each $A \in IF(X)$,

$$A = \bigcup_{y \in X} (A(y)1_y).$$

2.2 IF topologies

Definition 2.8 ([5]). Let $\tau \subseteq IF(X)$. Then τ is called an IF topology on X , if

- (i) $0_\sim, 1_\sim \in \tau$,
- (ii) $A, B \in \tau$ implies $A \cap B \in \tau$,
- (iii) $\{A_\alpha : \alpha \in \Gamma\} \subseteq \tau$ implies $\bigcup \{A_\alpha : \alpha \in \Gamma\} \in \tau$.

The pair (X, τ) is called an IF topological space and every member of τ is called an IF open set in X . Its complement is called an IF closed set in X .

We denote $\tau^c = \{A : A^c \in \tau\}$.

The interior and closure of $A \in IF(X)$ denoted respectively by $int(A)$ and $cl(A)$, are defined as follows:

$$int(A) \text{ or } int_\tau(A) = \bigcup \{B \in \tau : B \subseteq A\},$$

$$cl(A) \text{ or } cl_\tau(A) = \bigcap \{B \in \tau^c : B \supseteq A\}.$$

An IF topology τ is called Alexandrov, if (i) and (ii) in Definition 2.8 are replaced by

- (i)' For each $\lambda \in J$, $\widehat{\lambda} \in \tau$.
- (ii)' $\{A_\alpha : \alpha \in \Gamma\} \subseteq \tau$ implies $\bigcap_{\alpha \in \Gamma} A_\alpha \in \tau$.

Proposition 2.9 ([5]). Let (X, τ) be an IF topological space. Then

- (1) τ^* is the fuzzy topology on X in Chang' sense.
- (2) $(\tau_*)^c = \{(A_*)^c : A \in \tau\}$ is the fuzzy topology on X in Chang' sense and τ_* is the family of all fuzzy closed sets in X .

Proposition 2.10 ([5]). Let (X, τ) be an IF topological space and $A \in IF(X)$. Then

- (1) If $A \in \tau$, then $A^* \in \tau^*$ and $A_* \in (\tau_*)^c$.
- (2) If $A \in \tau^c$, then $A^* \in (\tau_*)^c$ and $A_* \in \tau^*$.

2.3 Fuzzy rough sets

Recall that R is called a fuzzy relation on X if $R \in F(X \times X)$.

Definition 2.11 ([18]). Let R be a fuzzy relation on X . Then the pair (X, R) is called a fuzzy approximation space. Based on (X, R) , the fuzzy lower and the fuzzy upper approximation of $A \in F(X)$ with respect to (X, R) , denoted by $\underline{R}(A)$ and $\overline{R}(A)$ are respectively, defined as follows:

$$\underline{R}(A)(x) = \bigwedge_{y \in X} (A(y) \vee (1 - R(x, y))) \quad (x \in X)$$

and

$$\overline{R}(A)(x) = \bigvee_{y \in X} (A(y) \wedge R(x, y)) \quad (x \in X).$$

The pair $(\underline{R}(A), \overline{R}(A))$ is called the fuzzy rough set of A with respect to (X, R) .

$\underline{R} : F(X) \rightarrow F(X)$ and $\overline{R} : F(X) \rightarrow F(X)$ are called the fuzzy lower approximation operator and the fuzzy upper approximation operator, respectively. In general, we refer to \underline{R} and \overline{R} as the fuzzy rough approximation operators.

Proposition 2.12 ([18]). Let (X, R) be a fuzzy approximation space. Then for any $A, B \in F(X)$, $\{A_\alpha : \alpha \in \Gamma\} \subseteq F(X)$ and $\lambda \in I$,

- (1) $\underline{R}(\bar{1}) = \bar{1}$, $\overline{R}(\bar{0}) = \bar{0}$.
- (2) $A \subseteq B \implies \underline{R}(A) \subseteq \underline{R}(B)$, $\overline{R}(A) \subseteq \overline{R}(B)$.
- (3) $\underline{R}(A^c) = (\overline{R}(A))^c$, $\overline{R}(A^c) = (\underline{R}(A))^c$.
- (4) $\underline{R}(\bigcap_{\alpha \in \Gamma} A_\alpha) = \bigcap_{\alpha \in \Gamma} (\underline{R}(A_\alpha))$, $\overline{R}(\bigcup_{\alpha \in \Gamma} A_\alpha) = \bigcup_{\alpha \in \Gamma} (\overline{R}(A_\alpha))$.
- (5) $\underline{R}(\bar{\lambda} \cup A) = \bar{\lambda} \cup \underline{R}(A)$, $\overline{R}(\lambda A) = \lambda \overline{R}(A)$.

3 IF approximation spaces and IF rough sets

In this section, we investigate properties related to IF approximation spaces.

An IF relation R on X is an IF set in $X \times X$ ([3]), we write $R \in IF(X \times X)$, namely,

$$R = (R^*, R_*),$$

$$R(x, y) = (R^*(x, y), R_*(x, y)) \in J \text{ for any } x, y \in X$$

where

R^* and R_* are two fuzzy relations on X .

Let R be an IF relation on a finite set X . R may be represent by a matrix. That is, if $X = \{t_1, t_2, \dots, t_n\}$, then R may be represented by the following matrix

$$\begin{pmatrix} R(t_1, t_1) & R(t_1, t_2) & \dots & R(t_1, t_n) \\ R(t_2, t_1) & R(t_2, t_2) & \dots & R(t_2, t_n) \\ \vdots & \vdots & & \vdots \\ R(t_n, t_1) & R(t_n, t_2) & \dots & R(t_n, t_n) \end{pmatrix}.$$

Definition 3.1 ([3]). Let R be an IF relation on X . Then R is called

- (1) reflexive, if $R(x, x) = (1, 0)$ for each $x \in X$.
- (2) symmetric, if $R(x, y) = R(y, x)$ for any $x, y \in X$.
- (3) transitive, if $R(x, z) \geq R(x, y) \sqcap R(y, z)$ for any $x, y, z \in X$.

Remark 3.2. Let R be an IF relation on X . Then

- (1) If R is reflexive, then R^* and $(R_*)^c$ are reflexive.
- (2) If R is symmetric, then R^* and $(R_*)^c$ are symmetric.
- (3) If R is transitive, then R^* and $(R_*)^c$ are transitive.

Let R be an IF relation on X . R is called preorder if R is reflexive and transitive. R^c is called the dual of R if $R^c(x, y) = (R_*(x, y), R^*(x, y))$ for any $x, y \in X$.

Definition 3.3 ([29]). Let R be an IF relation on X . Then the pair (X, R) is called an IF approximation space. Based on (X, R) , the IF lower and the IF upper approximation of $A \in IF(X)$ with respect to (X, R) , denoted by $\underline{R}(A)$ and $\overline{R}(A)$, are two IF sets and are respectively defined as follows:

$$\begin{aligned}\underline{R}(A)(x) &= ((\underline{R}(A))^*(x), (\underline{R}(A))_*(x)) \quad (x \in X), \\ \overline{R}(A)(x) &= ((\overline{R}(A))^*(x), (\overline{R}(A))_*(x)) \quad (x \in X),\end{aligned}$$

where

$$\begin{aligned}(\underline{R}(A))^*(x) &= \bigwedge_{y \in X} (A^*(y) \vee R_*(x, y)), \quad (\underline{R}(A))_*(x) = \bigvee_{y \in X} (A_*(y) \wedge R^*(x, y)), \\ (\overline{R}(A))^*(x) &= \bigvee_{y \in X} (A^*(y) \wedge R^*(x, y)), \quad (\overline{R}(A))_*(x) = \bigwedge_{y \in X} (A_*(y) \vee R_*(x, y)).\end{aligned}$$

The pair $(\underline{R}(A), \overline{R}(A))$ is called the IF rough set of A with respect to (X, R) .

$\underline{R} : IF(X) \rightarrow IF(X)$ and $\overline{R} : IF(X) \rightarrow IF(X)$ are called the IF lower approximation operator and the IF upper approximation operator, respectively. In general, we refer to \underline{R} and \overline{R} as the IF rough approximation operators.

Remark 3.4 ([29]). Let (X, R) be an IF approximation space. Then

$$\overline{R}(1_x)(y) = R(y, x) \quad \text{and} \quad \underline{R}(0_x)(y) = R^c(y, x) \quad (x, y \in X).$$

Proposition 3.5. Let (X, R) be an IF approximation space. Then for any $A \in IF(X)$ and $x \in X$,

$$\begin{aligned}(1) \quad &(\underline{R}(A))^* = (\underline{R}_*)^c(A^*), \quad (\underline{R}(A))_* = \overline{R}^*(A_*), \\ &(\overline{R}(A))^* = \overline{R}^*(A^*), \quad (\overline{R}(A))_* = (\underline{R}_*)^c(A_*). \\ (2) \quad &\underline{R}(A)(x) = \bigcap_{y \in X} (A(y) \sqcup R^c(x, y)), \quad \overline{R}(A)(x) = \bigcup_{y \in X} (A(y) \sqcap R(x, y)).\end{aligned}$$

Proof. (1) This is obvious.

(2) For any $A \in IF(X)$ and $x \in X$, since

$$\begin{aligned}\bigcap_{y \in X} (A(y) \sqcup R^c(x, y)) &= \bigcap_{y \in X} ((A^*(y), A_*(y)) \sqcup (R_*(x, y), R^*(x, y))) \\ &= \bigcap_{y \in X} (A^*(y) \vee R_*(x, y), A_*(y) \wedge R^*(x, y)) \\ &= (\bigwedge_{y \in X} (A^*(y) \vee R_*(x, y)), \bigvee_{y \in X} (A_*(y) \wedge R^*(x, y))) \\ &= ((\underline{R}(A))^*(x), (\underline{R}(A))_*(x)) \\ &= \underline{R}(A)(x),\end{aligned}$$

we have $\underline{R}(A)(x) = \bigcap_{y \in X} (A(y) \sqcup R^c(x, y))$.

Similarly, we can prove $\overline{R}(A)(x) = \bigcup_{y \in X} (A(y) \sqcap R(x, y))$. \square

Proposition 3.6 ([29]). *Let (X, R) be an IF approximation space. Then for any $A, B \in IF(X)$, $\{A_\alpha : \alpha \in \Gamma\} \subseteq IF(X)$ and $\lambda \in J$,*

- (1) $\underline{R}(1_\sim) = 1_\sim, \overline{R}(0_\sim) = 0_\sim$;
- (2) $A \subseteq B \implies \underline{R}(A) \subseteq \underline{R}(B), \overline{R}(A) \subseteq \overline{R}(B)$;
- (3) $\underline{R}(A^c) = (\overline{R}(A))^c, \overline{R}(A^c) = (\underline{R}(A))^c$;
- (4) $\underline{R}(\bigcap_{\alpha \in \Gamma} A_\alpha) = \bigcap_{\alpha \in \Gamma} (\underline{R}(A_\alpha)), \overline{R}(\bigcup_{\alpha \in \Gamma} A_\alpha) = \bigcup_{\alpha \in \Gamma} (\overline{R}(A_\alpha))$;
- (5) $\underline{R}(\hat{\lambda} \cup A) = \hat{\lambda} \cup \underline{R}(A), \overline{R}(\lambda A) = \lambda \overline{R}(A)$.

Theorem 3.7 ([29]). *Let (X, R) be an IF approximation space. Then*

- (1) R is reflexive $\iff (ILR) \forall A \in IF(X), \underline{R}(A) \subseteq A$.
 $\iff (IUR) \forall A \in IF(X), A \subseteq \overline{R}(A)$.
- (2) R is symmetric $\iff (ILS) \forall x, y \in X, \underline{R}(0_x)(y) = \underline{R}(0_y)(x)$.
 $\iff (IUS) \forall x, y \in X, \overline{R}(1_x)(y) = \overline{R}(1_y)(x)$.
- (3) R is transitive $\iff (ILT) \forall A \in IF(X), \underline{R}(A) \subseteq \underline{R}(\underline{R}(A))$.
 $\iff (IUT) \forall A \in IF(X), \overline{R}(\overline{R}(A)) \subseteq \overline{R}(A)$.

Proposition 3.8. *Let (X, R) be an IF approximation space.*

- (1) *For each $\lambda \in J$,*

$$\underline{R}(\hat{\lambda}) \supseteq \hat{\lambda} \supseteq \overline{R}(\hat{\lambda}).$$

- (2) *If R is reflexive, then for each $\lambda \in J$,*

$$\underline{R}(\hat{\lambda}) = \hat{\lambda} = \overline{R}(\hat{\lambda}).$$

Proof. (1) For any $\lambda \in J$ and $x \in X$, by Proposition 3.5(2),

$$\overline{R}(\hat{\lambda})(x) = \bigsqcup_{y \in X} (\lambda \sqcap R(x, y)) = \lambda \sqcap (\bigsqcup_{y \in X} R(x, y)) \leq \lambda.$$

Hence $\hat{\lambda} \supseteq \overline{R}(\hat{\lambda})$. By Proposition 3.6(3),

$$\underline{R}(\hat{\lambda}) = (\overline{R}(\hat{\lambda}^c))^c = (\overline{R}(\hat{\lambda}^c))^c \supseteq (\hat{\lambda}^c)^c = \hat{\lambda}.$$

- (2) This holds by (1) and Theorem 3.7(1). □

Theorem 3.9. *Let R be an IF relation on X and let τ be an IF topology on X . If one of the following conditions is satisfied, then R is preorder.*

- (1) \overline{R} is the closure operator of τ .
- (2) \underline{R} is the interior operator of τ .

Proof. (1) By Remark 3.4, $\overline{R}(1_x)(y) = R(y, x)$ for any $x, y \in X$. Note that \underline{R} is the interior operator of τ . Then for each $x \in X$,

$$R(x, x) = \overline{R}(1_x)(x) = cl_\tau(1_x)(x) \geq 1_x(x) = 1,$$

Thus R is reflexive.

For any $x, y, z \in X$, denote $cl_\tau(1_z)(y) = \lambda$, by Remark 2.7, Remark 3.4 and Proposition 3.6(5),

$$\begin{aligned}
 R(x, y) \sqcap R(y, z) &= \overline{R}(1_y)(x) \sqcap \overline{R}(1_z)(y) = \overline{R}(1_y)(x) \sqcap cl_\tau(1_z)(y) \\
 &= \overline{R}(1_y)(x) \sqcap \lambda = \lambda \overline{R}(1_y)(x) = \overline{R}(\lambda 1_y)(x) \\
 &= cl_\tau(\lambda 1_y)(x) = cl_\tau(cl_\tau(1_z)(y) 1_y)(x) \\
 &\leq cl_\tau\left(\bigcup_{t \in X} (cl_\tau(1_z)(t) 1_t)\right)(x) \\
 &= cl_\tau(cl_\tau(1_z))(x) = cl_\tau(1_z)(x) = R(x, z).
 \end{aligned}$$

Then R is transitive. Hence R is preorder.

(2) The proof is similar to (1). □

4 Relationships between IF relations and IF topologies

In this section we establish relationships between IF relations and IF topologies.

4.1 IF topologies induced by IF relations

For $R \in IF(X \times X)$, we denote

$$\tau_R = \{A \in IF(X) : A = \underline{R}(A)\}, \quad \theta_R = \{R(A) : A \in IF(X)\}.$$

Proposition 4.1. *Let (X, R) be an IF approximation space. If R is preorder, then*

$$\tau_R = \theta_R.$$

Proof. Obviously, $\tau_R \subseteq \theta_R$. For each $\underline{R}(A) \in \theta_R$, by Theorem 3.7, $\underline{R}(\underline{R}(A)) = \underline{R}(A)$. So $\underline{R}(A) \in \tau_R$. Thus $\theta_R \subseteq \tau_R$. Hence $\tau_R = \theta_R$. □

Theorem 4.2 ([29]). *Let R be a preorder IF relation. Then*

- (1) θ_R is an IF topology on X .
- (2) \underline{R} is the interior operator of θ_R .
- (3) \overline{R} is the closure operator of θ_R .

Theorem 4.3. *Let R be a reflexive IF relation. Then*

- (1) τ_R is an Alexandrov IF topology on X .
- (2) For each $A \in IF(X)$,

$$int_{\tau_R}(A) \subseteq \underline{R}(A) \subseteq A \subseteq \overline{R}(A) \subseteq cl_{\tau_R}(A).$$

- (3) $A \in (\tau_R)^c \iff A = \overline{R}(A)$.
- (4) For each $\lambda \in J$, $\widehat{\lambda} \in (\tau_R)^c$.
- (5) $(\tau_R)^* = \tau_{(R_*)^c}$ where $\tau_{(R_*)^c} = \{V \in F(X) : (\underline{R}_*)^c(V) = V\}$.
 $(\tau_R)_* = (\tau_{R^*})^c$ where $\tau_{R^*} = \{V \in F(X) : \underline{R}^*(V) = V\}$.

Proof. (1) (i) For each $\lambda \in J$, by Proposition 3.8(2), $\underline{R}(\widehat{\lambda}) = \widehat{\lambda}$. Then $\widehat{\lambda} \in \tau_R$.

(ii) Let $\{A_\alpha : \alpha \in \Gamma\} \subseteq \tau_R$. Then $\underline{R}(A_\alpha) = A_\alpha$ for each $\alpha \in \Gamma$. By Proposition 3.6(4),

$$\underline{R}\left(\bigcap_{\alpha \in \Gamma} A_\alpha\right) = \bigcap_{\alpha \in \Gamma} \underline{R}(A_\alpha) = \bigcap_{i \in J} A_\alpha.$$

Hence $\bigcap_{\alpha \in \Gamma} A_\alpha \in \tau_R$.

(iii) Let $\{A_\alpha : \alpha \in \Gamma\} \subseteq \tau_R$. Then $\underline{R}(A_\alpha) = A_\alpha$ for each $\alpha \in \Gamma$. By the reflexivity of R and Theorem 3.7(1), $\underline{R}(\bigcup_{\alpha \in \Gamma} A_\alpha) \subseteq \bigcup_{\alpha \in \Gamma} A_\alpha$. Note that

$$\underline{R}\left(\bigcup_{\alpha \in \Gamma} A_\alpha\right) \supseteq \bigcup_{\alpha \in \Gamma} \underline{R}(A_\alpha) = \bigcup_{\alpha \in \Gamma} A_\alpha.$$

Then $\underline{R}(\bigcup_{\alpha \in \Gamma} A_\alpha) = \bigcup_{\alpha \in \Gamma} A_\alpha$. Hence $\bigcup_{\alpha \in \Gamma} A_\alpha \in \tau_R$.

So τ_R is an Alexandrov IF topology on X .

(2) For each $A \in IF(X)$, by Proposition 3.6(2),

$$\begin{aligned} \text{int}_{\tau_R}(A) &= \bigcup \{B \in \tau_R : B \subseteq A\} \subseteq \bigcup \{B \in \tau_R : \underline{R}(B) \subseteq \underline{R}(A)\} \\ &= \bigcup \{B \in IF(X) : B = \underline{R}(B) \subseteq \underline{R}(A)\} \subseteq \underline{R}(A). \end{aligned}$$

By Proposition 3.6(3),

$$\text{cl}_{\tau_R}(A) = (\text{int}_{\tau_R}(A^c))^c \supseteq (\underline{R}(A^c))^c = \overline{R}(A).$$

By the reflexivity of R and Proposition 3.6(1),

$$\text{int}_{\tau_R}(A) \subseteq \underline{R}(A) \subseteq A \subseteq \overline{R}(A) \subseteq \text{cl}_{\tau_R}(A).$$

(3) This holds by Proposition 3.6(3).

(4) This holds by (3) and Proposition 3.8(2).

(5) Let $V \in (\tau_R)^*$. Then $A^* = V$ for some $A \in \tau_R$ and so $\underline{R}(A) = A$. By Proposition 3.5(1),

$$(\underline{R}_*)^c(V) = (\underline{R}_*)^c(A^*) = (\underline{R}(A))^* = A^* = V.$$

So $V \in \tau_{(\underline{R}_*)^c}$. Thus $(\tau_R)^* \subseteq \tau_{(\underline{R}_*)^c}$.

Let $V \in \tau_{(\underline{R}_*)^c}$. Put $A = (V, \bar{0})$. By Remark 3.2, $(\underline{R}_*)^c$ is reflexive. Then $(\underline{R}_*)^c(\bar{0}) = \bar{0}$. Thus $A^* = V$, $A_* = \bar{0}$. By Proposition 3.5(1), we have

$$(\underline{R}(A))^* = (\underline{R}_*)^c(A^*) = (\underline{R}_*)^c(V) = V = A^*$$

and

$$(\underline{R}(A))_* = (\underline{R}^*)^c(A_*) = (\underline{R}^*)^c(\bar{0}) = \bar{0} = A_*.$$

Then $\underline{R}(A) = A$ and so $A \in \tau_R$. This implies that $V = A^* \in (\tau_R)^*$. Thus $(\tau_R)^* \supseteq \tau_{(\underline{R}_*)^c}$.

Hence $(\tau_R)^* = \tau_{(\underline{R}_*)^c}$.

Similarly, we can prove that $(\tau_R)_* = (\tau_{R^*})^c$. □

Definition 4.4. Let R be a reflexive IF relation. Then τ_R is called the IF topology induced by R on X .

Example 4.5. Let $U = \{x, y, z, w\}$ and let R be an IF relation on X where

$$R = \begin{pmatrix} (0, 1) & (1, 0) & (1, 0) & (1, 0) \\ (1, 0) & (0, 1) & (1, 0) & (1, 0) \\ (1, 0) & (1, 0) & (0, 1) & (1, 0) \\ (1, 0) & (1, 0) & (1, 0) & (0, 1) \end{pmatrix}.$$

Then R is not reflexive.

For any $A \in IF(X)$ and $t \in X$,

$$\underline{R}(A)(t) = \bigcap_{s \in X} (A(s) \sqcup R^c(t, s)) = \bigcap_{s \in X - \{t\}} A(s).$$

Suppose that $A(x) \leq A(y) \leq A(z) \leq A(w)$. Since $\underline{R}(A) = A$, we have

$$A(x) \wedge A(y) \wedge A(z) = A(w).$$

Then $A(t) \geq A(w)$ for each $t \in \{x, y, z\}$. So $A(x) = A(y) = A(z) = A(w)$.

Thus $\tau_R = \{\hat{\lambda} : \lambda \in J\}$.

Obviously, τ_R is an Alexandrov IF topology on X .

4.2 IF relations induced by IF topologies

Definition 4.6. Let τ be an IF topology on X . Define an IF relation R_τ on X by

$$R_\tau(x, y) = cl_\tau(1_y)(x)$$

for each $x, y \in X$. Then R_τ is called the IF relation induced by τ on X and (X, R_τ) is called the IF approximation space induced by τ on X .

Theorem 4.7. Let τ be an IF topology on X and let R_τ be the IF relation induced by τ on X . Then the following properties hold.

- (1) R_τ is reflexive.
- (2) If $\{\hat{\lambda} : \lambda \in J\} \subseteq \tau^c$, then for each $A \in IF(X)$,

$$\underline{R}_\tau(A) \subseteq int_\tau(A) \subseteq A \subseteq cl_\tau(A) \subseteq \overline{R}_\tau(A).$$

Proof. (1) For each $x \in X$,

$$R_\tau(x, x) = cl_\tau(1_x)(x) \geq (1_x)(x) = (1, 0).$$

Then R_τ is reflexive.

- (2) For each $A \in IF(X)$, by Remark 2.7 and Proposition 3.6(2),

$$\begin{aligned} cl_\tau(A) &= cl_\tau\left(\bigcup_{y \in X} (A(y)1_y)\right) = \bigcup_{y \in X} cl_\tau(A(y)1_y) = \bigcup_{y \in X} cl_\tau(\widehat{A(y)} \cap 1_y) \\ &\subseteq \bigcup_{y \in X} (cl_\tau(\widehat{A(y)}) \cap cl_\tau(1_y)) = \bigcup_{y \in X} (\widehat{A(y)} \cap cl_\tau(1_y)). \end{aligned}$$

Then for each $x \in X$,

$$cl_\tau(A)(x) \leq \bigcup_{y \in X} (\widehat{A(y)}(x) \sqcap cl_\tau(1_y)(x)) = \bigcup_{y \in X} (A(y) \sqcap R_\tau(x, y)) = \overline{R_\tau}(A)(x).$$

Hence $cl_\tau(A) \subseteq \overline{R_\tau}(A)$.

By Proposition 3.6(3),

$$int_\tau(A) = (cl_\tau(A^c))^c \supseteq (\overline{R_\tau}(A^c))^c = \underline{R_\tau}(A).$$

So

$$\underline{R_\tau}(A) \subseteq int_\tau(A) \subseteq A \subseteq cl_\tau(A) \subseteq \overline{R_\tau}(A).$$

□

Theorem 4.8 ([23]). *Let R be a reflexive IF relation on X , let τ_R be the IF topology by R on X and let R_{τ_R} be the IF relation induced by τ_R on X . If R is transitive, then $R_{\tau_R} = R$.*

4.3 (C_1) and (C_2) axioms

The following conditions for an IF topology τ on X are respectively called (C_1) axiom and (C_2) axiom: for any $\lambda \in J$, $A \in IF(X)$ and $\{A_\alpha : \alpha \in \Gamma\} \subseteq IF(X)$,

$$(C_1) \text{ axiom : } cl_\tau(\lambda A) = \lambda cl_\tau(A); \quad (C_2) \text{ axiom : } cl_\tau\left(\bigcup_{\alpha \in \Gamma} A_\alpha\right) = \bigcup_{\alpha \in \Gamma} cl_\tau(A_\alpha).$$

Proposition 4.9. *Let τ be an IF topology on U . If τ satisfies (C_1) and (C_2) axioms, then*

- (1) $\overline{R_\tau}$ is the closure operator of τ .
- (2) $\underline{R_\tau}$ is the interior operator of τ .
- (3) For each $\lambda \in J$, $\widehat{\lambda} \in \tau$.
- (4) τ is Alexandrov.

Proof. (1) For each $A \in IF(X)$, by Remark 2.7, (C_1) axiom and (C_2) axiom,

$$cl_\tau(A) = cl_\tau\left(\bigcup_{y \in X} (A(y)1_y)\right) = \bigcup_{y \in X} cl_\tau(A(y)1_y) = \bigcup_{y \in X} (A(y)cl_\tau(1_y)).$$

Then for each $x \in X$,

$$cl_\tau(A)(x) = \bigcup_{y \in X} (\widehat{A(y)}(x) \sqcap cl_\tau(1_y)(x)) = \bigcup_{y \in X} (A(y) \sqcap R_\tau(x, y)) = \overline{R_\tau}(A)(x).$$

Thus $\overline{R_\tau}(A) = cl_\tau(A)$. So $\overline{R_\tau}$ is the closure operator of τ .

- (2) This holds by (1) and Proposition 3.6(3).
- (3) For each $\lambda \in J$, by (2) and Proposition 3.8(1),

$$\widehat{\lambda} \supseteq int_\tau(\widehat{\lambda}) = \underline{R_\tau}(\widehat{\lambda}) \supseteq \widehat{\lambda}.$$

Then $int_\tau(\hat{\lambda}) = \hat{\lambda}$ and so $\hat{\lambda} \in \tau$.

(4) Suppose that τ satisfies (C_1) and (C_2) axioms. For each $\lambda \in J$, by (1) and Proposition 3.6(3),

$$int_\tau(\hat{\lambda}) = (cl_\tau(\hat{\lambda}^c))^c = (\overline{R}(\hat{\lambda}^c))^c = \underline{R}(\hat{\lambda}) \supseteq \hat{\lambda}.$$

Note that $int_\tau(\hat{\lambda}) \subseteq \hat{\lambda}$. Then $int_\tau(\hat{\lambda}) = \hat{\lambda}$. Thus $\hat{\lambda} \in \tau$.

For each $A \in IF(X)$, by (1), (C_1) axiom and (C_2) axiom, $\overline{R}(A^c) = cl_\tau(A^c)$. Then $\underline{R}(A) = int_\tau(A)$.

Let $\{A_\alpha : \alpha \in \Gamma\} \subseteq \tau$. Note that $\underline{R}(A_\alpha) = int_\tau(A_\alpha)$. Then $A_\alpha = \underline{R}(A_\alpha)$. By Proposition 3.6(4),

$$\bigcap_{\alpha \in \Gamma} A_\alpha = \bigcap_{\alpha \in \Gamma} \underline{R}(A_\alpha) = \underline{R}(\bigcap_{\alpha \in \Gamma} A_\alpha) = int_\tau(\bigcap_{\alpha \in \Gamma} A_\alpha).$$

So $\bigcap_{\alpha \in \Gamma} A_\alpha \in \tau$. Hence τ is Alexandrov. \square

Proposition 4.10 ([29]). *Let R be a preorder IF relation on X . Then τ_R satisfies (C_1) and (C_2) axioms.*

Theorem 4.11. *Let τ be an IF topology on X , let R_τ be the IF relation induced by τ on X and let τ_{R_τ} be the IF topology induced by R_τ on X . Then*

$$\tau_{R_\tau} = \tau \quad \text{if and only if } \tau \text{ satisfies } (C_1) \text{ and } (C_2) \text{ axioms.}$$

Proof. Necessity. For each $A \in IF(X)$, by Theorem 4.7(2), $\underline{R}_\tau(A) \subseteq int_\tau(A)$. By Theorem 4.3(2),

$$int_\tau(A) = int_{\tau_{R_\tau}}(A) \subseteq \underline{R}_\tau(A).$$

Then $int_\tau(A) = \underline{R}_\tau(A)$. So \underline{R}_τ is the interior operator of τ . By Theorem 3.9(2), R_τ is a preorder IF relation on X . By Theorem 4.10, τ_{R_τ} satisfies (C_1) and (C_2) axioms.

Sufficiency. By Theorem 4.7(1), R_τ is reflexive. For any $x, y, z \in X$, put $cl_\tau(1_z)(y) = \lambda$. By Remark 2.7, Proposition 3.6(2),

$$\begin{aligned} \lambda cl_\tau(1_y) &= cl_\tau(\lambda 1_y) = cl_\tau(cl_\tau(1_z)(y) 1_y) \\ &\subseteq cl_\tau\left(\bigcup_{t \in X} (cl_\tau(1_z)(t) 1_t)\right) = cl_\tau(cl_\tau(1_z)) = cl_\tau(1_z). \end{aligned}$$

Then

$$\begin{aligned} R_\tau(x, y) \sqcap R_\tau(y, z) &= cl_\tau(1_y)(x) \sqcap cl_\tau(1_z)(y) = cl_\tau(1_y)(x) \sqcap \lambda \\ &= \lambda \sqcap cl_\tau(1_y)(x) = (\lambda cl_\tau(1_y))(x) \\ &\leq cl_\tau(1_z)(x) = R_\tau(x, z). \end{aligned}$$

Then R_τ is transitive.

So R_τ is preorder. For each $A \in IF(X)$, by Proposition 4.1 and Theorem 4.2(3),

$$cl_{\tau_{R_\tau}}(A) = cl_{\theta_{R_\tau}}(A) = \overline{R_\tau}(A).$$

By (C_1) axiom, (C_2) axiom and Proposition 4.9(1), $\overline{R_\tau}(A) = cl_\tau(A)$. So $cl_{\tau_{R_\tau}}(A) = cl_\tau(A)$. Thus $\tau_{R_\tau} = \tau$. \square

Theorem 4.12. *Let*

$$\Sigma = \{R : R \text{ is a preorder IF relation on } X\}$$

and

$$\Gamma = \{\tau : \tau \text{ is an IF topology on } X \text{ satisfying } (C_1) \text{ and } (C_2) \text{ axioms}\}.$$

Then there exists a one-to-one correspondence between Σ and Γ .

Proof. Two mappings $f : \Sigma \rightarrow \Gamma$ and $g : \Gamma \rightarrow \Sigma$ are defined as follows:

$$f(R) = \tau_R \quad (R \in \Sigma),$$

$$g(\tau) = R_\tau \quad (\tau \in \Gamma).$$

By Theorem 4.8,

$$g \circ f = i_\Sigma,$$

where $g \circ f$ is the composition of f and g , and i_Σ is the identity mapping on Γ .

By Theorem 4.11,

$$f \circ g = i_\Gamma,$$

where $f \circ g$ is the composition of g and f , and i_Γ is the identity mapping on Σ .

Hence f and g are two one-to-one correspondences. This prove that there exists a one-to-one correspondence between Σ and Γ . \square

5 IF approximating spaces

As can be seen from Section 4, a reflexive IF relation yields an IF topology. In this section, we consider the reverse problem, that is, under which conditions can an IF topology be associated with an IF relation which produces the given IF topology?

Definition 5.1. *Let (X, τ) be an IF topological space. If there exists a reflexive IF relation R on X such that $\tau_R = \tau$, then (X, τ) is called an IF approximating space.*

Theorem 5.2. *Let τ be an IF topology on X . Then the following are equivalent.*

- (1) τ satisfies (C_1) and (C_2) axioms;
- (2) For any $\lambda \in J$, $A \in IF(X)$ and $\{A_\alpha : \alpha \in \Gamma\} \subseteq IF(X)$,

$$int_\tau(\widehat{\lambda} \cup A) = \widehat{\lambda} \cup int_\tau(A), \quad int_\tau\left(\bigcap_{\alpha \in \Gamma} A_\alpha\right) = \bigcap_{\alpha \in \Gamma} int_\tau(A_\alpha).$$

- (3) *There exists a preorder IF relation ρ on X such that $\bar{\rho}$ is the closure operator of τ ;*
 (4) *There exists a preorder IF relation ρ on X such that $\underline{\rho}$ is the interior operator of τ ;*
 (5) \bar{R}_τ *is the closure operator of τ ;*
 (6) \underline{R}_τ *is the interior operator of τ .*

Proof. (1) \iff (2). This is obvious.

(1) \implies (3). Suppose that τ satisfies (C_1) and (C_2) axioms. Pick $\rho = R_\tau$. By Theorem 4.9, $\bar{\rho}$ is the closure operator of τ . By Theorem 3.9(1), ρ is preorder.

(3) \implies (4). Let $\bar{\rho}$ be the closure operator of τ for some preorder IF relation ρ on X . For each $A \in IF(X)$, by Proposition 3.6(3),

$$\underline{\rho}(A) = (\bar{\rho}(A^c))^c = (cl_\tau(A^c))^c = int_\tau(A).$$

Thus, $\underline{\rho}$ is the interior operator of τ .

(4) \implies (6). Let $\underline{\rho}$ be the interior operator of τ for some preorder IF relation ρ on X .

For $x, y \in X$, by Remark 3.4,

$$\rho(x, y) = (\underline{\rho}((1_y)^c)(x))^c = (int_\tau((1_y)^c)(x))^c = cl_\tau(1_y)(x) = R_\tau(x, y).$$

Then $\rho = R_\tau$. Note that $\underline{\rho}$ is the interior operator of τ . Then \underline{R}_τ is the interior operator of τ .

(6) \implies (5) holds by Proposition 3.6(3).

(5) \implies (1). For any $\lambda \in J$ and $A \in IF(X)$, by Proposition 3.6,

$$cl_\tau(\lambda A) = \bar{R}_\tau(\lambda A) = \lambda \bar{R}_\tau(A) = \lambda cl_\tau(A),$$

and

$$cl_\tau\left(\bigcup_{\alpha \in \Gamma} A_\alpha\right) = \bar{R}_\tau\left(\bigcup_{\alpha \in \Gamma} A_\alpha\right) = \bigcup_{\alpha \in \Gamma} \bar{R}_\tau(A_\alpha) = \bigcup_{\alpha \in \Gamma} cl_\tau(A_\alpha)$$

Thus τ satisfies (C_1) and (C_2) axioms. \square

Theorem 5.3. *Let (X, τ) be an IF topological space. If one of the following conditions is satisfied, then (X, τ) is an IF approximating space.*

- (1) τ *satisfies (C_1) and (C_2) axioms.*
 (2) *For any $\lambda \in J$, $A \in IF(X)$ and $\{A_\alpha : \alpha \in \Gamma\} \subseteq IF(X)$,*

$$int_\tau(\hat{\lambda} \cup A) = \hat{\lambda} \cup int_\tau(A), \quad int_\tau\left(\bigcap_{\alpha \in \Gamma} A_\alpha\right) = \bigcap_{\alpha \in \Gamma} int_\tau(A_\alpha).$$

(3) *There exists a preorder IF relation R on X such that \bar{R} is the closure operator of τ .*

(4) *There exists a preorder IF relation R on X such that \underline{R} is the interior operator of τ .*

(5) \bar{R}_τ *is the closure operator of τ .*

(6) \underline{R}_τ *is the interior operator of τ .*

Proof. These hold by Theorems 4.11 and 5.2. \square

Example 5.4. $\{\hat{\lambda} : \lambda \in J\}$ *is an IF approximating space.*

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On Cauchy problems with Caputo Hadamard fractional derivatives

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Abstract

The current work is motivated by the so-called Caputo-type modification of the Hadamard or Caputo Hadamard fractional derivative discussed in [4]. The main aim of this paper is to study Cauchy problems for a differential equation with a left Caputo Hadamard fractional derivative in spaces of continuously differentiable functions. The equivalence of this problem to a nonlinear Volterra type integral equation of the second kind is shown. On the basis of the obtained results, the existence and uniqueness of the solution to the considered Cauchy problem is proved by using Banach's fixed point theorem. Finally, two examples are provided to explain the applications of the results.

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Keywords: Caputo Hadamard fractional derivatives, Cauchy problem, Volterra integral equation, continuously differentiable function, fixed point theorem.

1 Introduction

Fractional calculus, that is, the theory of derivatives and integrals of fractional non-integer order, are used in many fields like: mathematics, physics, chemistry, engineering, and other sciences.

Few years ago, many scholars started making deeper researches on fractional differential equations. Intensive development of this latter and its applications led to that. (e.g.; [1, 2, 3, 10, 11, 12]). Many definitions were supplied for the Fractional order differential operators and many reports on the existence and uniqueness of solutions to differential equations in the frame of these operators appeared. (see for example [14] and the references therein).

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J. Hadamard [6] in 1892, introduced a new definition of fractional derivatives and integrals in which he claims:

$$\left(\mathcal{J}_{a+}^{\alpha} g\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} g(\tau) \frac{d\tau}{\tau}, \quad (0 < a < t), \quad \operatorname{Re}(\alpha) > 0, \quad (1)$$

for suitable functions g , where Γ represents gamma function. This is the generalization of the n^{th} integral

$$\left(\mathcal{J}_{a+}^n g\right)(t) = \int_a^t \frac{d\tau_1}{\tau_1} \int_a^{\tau_1} \frac{d\tau_2}{\tau_2} \dots \int_a^{\tau_{n-1}} g(\tau_n) \frac{d\tau_n}{\tau_n} \equiv \frac{1}{\Gamma(n)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{n-1} g(\tau) \frac{d\tau}{\tau}, \quad (2)$$

where $n = [\operatorname{Re}(\alpha)] + 1$ and $[\operatorname{Re}(\alpha)]$ means the integer part of $\operatorname{Re}(\alpha)$.

The corresponding left-sided Hadamard fractional derivative of order α is defined by

$$\left(\mathcal{D}_{a+}^{\alpha} g\right)(t) = \delta^n \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{n-\alpha-1} g(\tau) \frac{d\tau}{\tau}, \quad \alpha \in [n-1, n), \quad (3)$$

where $\delta = t \frac{d}{dt}$. The main difference between Hadamard's definition and the previous ones is that the kernel integral contains logarithmic function of arbitrary exponent. The present paper follows the Caputo-type definition based on the modification of Hadamard fractional derivatives. This approach is given by the equality,

$$\left({}^c \mathcal{D}_{a+}^{\alpha} g\right)(t) = \left(\mathcal{D}_{a+}^{\alpha} g\right) \left[g(\tau) - \sum_{k=0}^{n-1} \frac{\delta^k g(a)}{k!} \left(\ln \frac{\tau}{a}\right)^k \right](t), \quad (0 < a < t). \quad (4)$$

We can use the following equivalent representation, which follows from (3) and (4)

$$\left({}^c \mathcal{D}_{a+}^{\alpha} g\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log \frac{t}{\tau}\right)^{n-\alpha-1} \delta^n g(\tau) \frac{d\tau}{\tau}. \quad (5)$$

The Caputo Hadamard derivative is obtained from the Hadamard derivative by changing the order of its differential and integral parts. Despite the different requirements on the function itself, the main difference between the Caputo Hadamard fractional derivative and the Hadamard fractional derivative is that the Caputo Hadamard derivative of a constant is zero [4]. The most important advantage of Caputo Hadamard is that it brought a new definition through which the integer order initial conditions can be defined for fractional order differential equations in the frame of the Hadamard fractional derivative.

In this article, we extend the approach of Kilbas et al. [10] to fractional Cauchy problems with a left Caputo Hadamard in spaces of continuously differentiable functions and prove the existence and uniqueness of solutions to these problems.

To get to our aim, the equivalence of the Cauchy type problems to a nonlinear Volterra type integral equation of the second kind is first proved. Once that is done, Banach's fixed point theorem is applied. By the end, some examples are given to illustrate the obtained results.

2 Preliminaries

Below, we recall some basic definitions, properties, theorems and lemmas needed in the rest of this paper.

Let $C^n([a, b], \mathbb{R})$ be the Banach space of all continuously differentiable functions from $[a, b]$ to \mathbb{R} . We will introduce the weighted space $C_{\gamma, \ln}[a, b]$, $C_{\delta, \gamma, \ln}^n[a, b]$ and $C_{\delta, \gamma, \ln}^{\alpha, r}[a, b]$ of the function g on the finite interval $[a, b]$.

Definition 2.1. If $\alpha \in (n-1, n]$ and $\gamma \in (0, 1]$, then

(1) The space $C_{\gamma, \ln} [a, b]$ is defined by

$$C_{\gamma, \ln} [a, b] = \left\{ g : \left(\ln \frac{t}{a} \right)^\gamma g(t) \in C[a, b] \right\}, C_{0, \ln} [a, b] = C[a, b],$$

and on this space we define the norm $\|\cdot\|_{C_{\gamma, \ln}}$ by

$$\|g\|_{C_{\gamma, \ln}} = \left\| \left(\ln \frac{t}{a} \right)^\gamma g(t) \right\|_C = \max_{t \in [a, b]} \left| \left(\ln \frac{t}{a} \right)^\gamma g(t) \right|.$$

(2) The space $C_{\delta, \gamma, \ln}^n [a, b]$ is defined by

$$C_{\delta, \gamma, \ln}^n [a, b] = \{g : \delta^k g \in C[a, b], k = 0, \dots, n-1 \text{ and } \delta^n g \in C_{\gamma, \ln} [a, b]\},$$

and on this space we define the norm $\|\cdot\|_{C_{\delta, \gamma, \ln}^n}$ by

$$\|g\|_{C_{\delta, \gamma, \ln}^n} = \sum_{k=0}^{n-1} \|\delta^k g\|_C + \|\delta^n g\|_{C_{\gamma, \ln}}, \|g\|_{C_\delta^n} = \sum_{k=0}^n \max_{t \in [a, b]} |\delta^k g(t)|.$$

(3) We denote by $C_{\delta, \gamma, \ln}^{\alpha, r} [a, b]$ the space of functions g given on $[a, b]$ and such that

$$C_{\delta, \gamma, \ln}^{\alpha, r} [a, b] = \left\{ g \in C_\delta^r [a, b] : \left({}^c \mathcal{D}_{a+}^\alpha g \right) \in C_{\gamma, \ln} [a, b], r \in \mathbb{N} \right\}, \\ C_{\delta, \gamma, \ln}^{r, r} [a, b] = C_{\delta, \gamma, \ln}^r [a, b].$$

Property 2.2 ([10]). The fractional integral operators $\left(\mathcal{J}_{a+}^\alpha \right)$ satisfy the semigroup property

$$\left(\mathcal{J}_{a+}^\alpha \mathcal{J}_{a+}^\beta g \right) (t) = \left(\mathcal{J}_{a+}^{\alpha+\beta} g \right) (t), \operatorname{Re}(\alpha) > 0, \operatorname{Re}(\beta) > 0.$$

The fractional derivative operators $\left(\mathcal{D}_{a+}^\alpha \right)$ fulfill the semigroup property

$$\left(\mathcal{D}_{a+}^\alpha \mathcal{J}_{a+}^\beta g \right) (t) = \left(\mathcal{J}_{a+}^{\beta-\alpha} g \right) (t).$$

Property 2.3 ([4]). Let $\operatorname{Re}(\alpha) \geq 0$, $n = [\operatorname{Re}(\alpha)] + 1$ and $\operatorname{Re}(\beta) > 0$, then

$$\left({}^c \mathcal{D}_{a+}^\alpha \left(\ln \frac{t}{a} \right)^{\beta-1} \right) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} \left(\ln \frac{t}{a} \right)^{\beta-\alpha-1}, \operatorname{Re}(\beta) > n.$$

On the other hand, for $k = 0, 1, \dots, n-1$,

$$\left({}^c \mathcal{D}_{a+}^\alpha \left(\ln \frac{t}{a} \right)^k \right) = 0.$$

Lemma 2.4 ([4]). Let $\alpha \in \mathbb{C}$, $n = [\operatorname{Re}(\alpha)] + 1$, let $g(t) \in AC_\delta^n [a, b]$ or $C_\delta^n [a, b]$, then

$$\left(\mathcal{J}_{a+}^\alpha \left({}^c \mathcal{D}_{a+}^\alpha g \right) \right) (t) = g(t) - \sum_{k=0}^{n-1} \frac{(\delta^k g)(a)}{k!} \left(\ln \frac{t}{a} \right)^k.$$

Lemma 2.5 ([10]). Let $n \in \mathbb{N}$ and $0 \leq \gamma < 1$. The space $C_{\delta, \gamma, \ln}^n [a, b]$ consists of those and only those functions g which are represented in the form

$$g(t) = \frac{1}{(n-1)!} \int_a^t \left(\ln \frac{t}{\tau} \right)^{n-1} \varphi(\tau) \frac{d\tau}{\tau} + \sum_{k=0}^{n-1} d_k \left(\ln \frac{t}{a} \right)^k,$$

where $\varphi \in C_{\gamma, \ln} [a, b]$ and d_k ($k = 0, 1, \dots, n-1$) are arbitrary constants, such that

$$\varphi(t) = \delta^n g(t), \quad d_k = \frac{\delta^k g(a)}{k!} \quad (k = 0, 1, \dots, n-1).$$

Lemma 2.6 ([10]). Let $0 < a < b < +\infty$, $\operatorname{Re}(\alpha) > 0$, and $0 \leq \gamma < 1$, then

a. If $\gamma > \alpha > 0$, then $(\mathcal{J}_{a+}^\alpha)$ is bounded from $C_{\gamma, \ln} [a, b]$ into $C_{\gamma-\alpha, \ln} [a, b]$:

$$\left\| \mathcal{J}_{a+}^\alpha g \right\|_{C_{\gamma-\alpha, \ln}} \leq k \|g\|_{C_{\gamma, \ln}}, \quad k = \left(\ln \frac{b}{a} \right)^{\operatorname{Re}(\alpha)} \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}.$$

In particular $(\mathcal{J}_{a+}^\alpha)$ is bounded in $C_{\gamma, \ln} [a, b]$.

b. If $\gamma \leq \alpha$, then $(\mathcal{J}_{a+}^\alpha)$ is bounded from $C_{\gamma, \ln} [a, b]$ into $C [a, b]$:

$$\left\| \mathcal{J}_{a+}^\alpha g \right\|_C \leq k \|g\|_{C_{\gamma, \ln}}, \quad k = \left(\ln \frac{b}{a} \right)^{\operatorname{Re}(\alpha)-\gamma} \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}.$$

In particular $(\mathcal{J}_{a+}^\alpha)$ is bounded in $C_{\gamma, \ln} [a, b]$.

Lemma 2.7 ([10]). The fractional operator $(\mathcal{J}_{a+}^\alpha)$ represents a mapping from $C [a, b]$ to $C [a, b]$ and

$$\left\| \mathcal{J}_{a+}^\alpha g \right\|_C \leq \frac{1}{\operatorname{Re}(\alpha) \Gamma(\alpha)} \left(\ln \frac{b}{a} \right)^{\operatorname{Re}(\alpha)} \|g\|_C.$$

Theorem 2.8 (Banach fixed point Theorem, [10]). Let (X, d) be a nonempty complete metric space, let $0 \leq w < 1$, and let $T : X \rightarrow X$ be a map such that for every $x, \tilde{x} \in X$, the relation

$$d(Tx, T\tilde{x}) \leq wd(x, \tilde{x}),$$

holds. Then the operator T has a uniquely defined fixed point $x^* \in X$.

Furthermore, if T^k ($k \in \mathbb{N}$) is the sequence defined by

$$T^1 = T, \quad T^k = TT^{k-1} \quad (k \in \mathbb{N} - \{1\}),$$

then, for any $x_0 \in X$ $\{T^k x_0\}_{k=1}^{k=\infty}$ converges to the above fixed point x^* .

Definition 2.9 ([10]). Let $l \in \mathbb{N}$, $G \subset \mathbb{R}^l$, $[a, b] \subset \mathbb{R}$, $g : [a, b] \times G \rightarrow \mathbb{R}$ be a function such that, for any $(x_1, \dots, x_l), (\tilde{x}_1, \dots, \tilde{x}_l) \in G$, g satisfies generalized Lipschitzian condition:

$$|g[t, x_1, \dots, x_l] - g[t, \tilde{x}_1, \dots, \tilde{x}_l]| \leq A_1 |x_1 - \tilde{x}_1| + \dots + A_l |x_l - \tilde{x}_l|, \quad A_j \geq 0, \quad j = 1, \dots, l. \quad (6)$$

In particular, g satisfies the Lipschitzian condition with respect to the second variable if for all $t \in (a, b]$ and for any $x, \tilde{x} \in G$ one has

$$|g[t, x] - g[t, \tilde{x}]| \leq A |x - \tilde{x}|, \quad A > 0. \quad (7)$$

3 Nonlinear Cauchy problem

In this section, we present the existence and uniqueness results in the space $C_{\delta, \gamma, \ln}^{\alpha, r}[a, b]$ of the Cauchy problem for the nonlinear fractional differential equation in the frame of Caputo Hadamard fractional derivative. That is we consider the equation

$$\left({}^c\mathcal{D}_{a+}^{\alpha}x\right)(t)=h[t,x(t)], \operatorname{Re}(\alpha)>0, t>a>0, \quad (8)$$

subject to the initial conditions

$$(\delta^k x)(a_+) = d_k, \quad d_k \in \mathbb{R}, \quad k=0, \dots, n-1, \quad n=[\operatorname{Re}(\alpha)]+1. \quad (9)$$

The Volterra type integral equation corresponding to problem (8)-(9) is :

$$x(t)=\sum_{j=0}^{n-1}\frac{d_j}{j!}\left(\ln\frac{t}{a}\right)^j+\frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{\tau}\right)^{\alpha-1}h[\tau,x(\tau)]\frac{d\tau}{\tau}, \quad a\leq t\leq b. \quad (10)$$

In particular, if $\alpha=n\in\mathbb{N}$ then the problem (8)-(9) is as follows:

$$(\delta^n x)(t)=h[t,x(t)], \quad a\leq t\leq b, \quad (\delta^k x)(a_+)=d_k\in\mathbb{R}, \quad k=0, 1, \dots, n-1. \quad (11)$$

The corresponding integral equation to the problem (11) has the form:

$$x(t)=\sum_{j=0}^{n-1}\frac{d_j}{j!}\left(\ln\frac{t}{a}\right)^j+\left(\mathcal{J}_{a+}^n h\right)(t), \quad a\leq t\leq b. \quad (12)$$

Firstly, we have to prove the equivalence of the Cauchy problem to the Volterra type integral equation in the sense that, if $x\in C_{\delta}^r[a, b]$ satisfies one of them, then it also satisfies the other one.

Theorem 3.1. *Let $\operatorname{Re}(\alpha)>0$, $n=[\operatorname{Re}(\alpha)]+1$, $(0<a<b<+\infty)$, and $0\leq\gamma<1$ be such that $\alpha\geq\gamma$. Let G be an open set in \mathbb{R} and let $h:[a, b]\times G\rightarrow\mathbb{R}$ be a function such that $h[t, x]\in C_{\gamma, \ln}[a, b]$ for any $x\in C_{\gamma, \ln}[a, b]$.*

- (i) *Let $r=n-1$ for $\alpha\notin\mathbb{N}$, if $x\in C_{\delta}^{n-1}[a, b]$ then x satisfies the relations (8) and (9) iff x satisfies equation (10).*
- (ii) *Let $r=n$ for $\alpha\in\mathbb{N}$, if $x\in C_{\delta}^n[a, b]$ then x satisfies the relation (11) if and only if, x satisfies equation (12).*

Proof. (i) Let $\alpha\notin\mathbb{N}$, $n-1<\alpha<n$ and $x\in C_{\delta}^{n-1}[a, b]$.

(i.a) Here we prove the necessity. From definition of ${}^c\mathcal{D}_{a+}^{\alpha}$ and (3) we obtain

$${}^c\mathcal{D}_{a+}^{\alpha}x(t)=(\delta^n)\left(\mathcal{J}_{a+}^{n-\alpha}\left[x(\tau)-\sum_{j=0}^{n-1}\frac{\delta^j x(a)}{j!}\left(\ln\frac{\tau}{a}\right)^j\right]\right)(t).$$

By hypothesis, $h[t, x]\in C_{\gamma, \ln}[a, b]$ and it follows from (8) that ${}^c\mathcal{D}_{a+}^{\alpha}x(t)\in C_{\gamma, \ln}[a, b]$, and hence, by applying Lemma 2.5, we have

$$\left(\mathcal{J}_{a+}^{n-\alpha}\left[x(\tau)-\sum_{j=0}^{n-1}\frac{\delta^j x(a)}{j!}\left(\ln\frac{t}{\tau}\right)^j\right]\right)(t)\in C_{\delta, \gamma, \ln}^n[a, b].$$

By using Lemma 2.4, we obtain

$$\mathcal{J}_{a+}^{\alpha} \left({}^C \mathcal{D}_{a+}^{\alpha} x \right) (t) = x(t) - \sum_{j=0}^{n-1} \frac{\delta^j x(a)}{j!} \left(\ln \frac{t}{a} \right)^j. \quad (13)$$

In view of Lemma 2.6-(b), $\mathcal{J}_{a+}^{\alpha} h[t, x]$ belongs to the $C[a, b]$ space, Applying $\left(\mathcal{J}_{a+}^{\alpha} \right)$ to the both sides of (8) and utilizing (13), with respect to the initial conditions (9), we deduce that there exists a unique solution $x \in C_{\delta}^{n-1}[a, b]$ to equation (10).

(i.b) Let $x \in C_{\delta}^{n-1}[a, b]$ satisfies the equation (10).

– We want to show that x satisfies equation (8). Applying $\left(\mathcal{D}_{a+}^{\alpha} \right)$ to both sides of (10), and taking into account (4), (9), Property 2.2 and Property 2.3, we get

$$\mathcal{D}_{a+}^{\alpha} \left(x(t) - \sum_{j=0}^{n-1} \frac{\delta^j x(a)}{j!} \left(\ln \frac{t}{a} \right)^j \right) = \mathcal{D}_{a+}^{\alpha} \left(\frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\tau} \right)^{\alpha-1} h[\tau, x(\tau)] \frac{d\tau}{\tau} \right),$$

then

$$\left({}^C \mathcal{D}_{a+}^{\alpha} x \right) (t) = \left(\mathcal{D}_{a+}^{\alpha} \right) \left(\mathcal{J}_{a+}^{\alpha} h \right) (t) \equiv h[t, x(t)].$$

– Now, we show that x satisfies the initial relations (9). We obtain by differentiation both sides of (10) that,

$$\delta^k x(t) = \sum_{j=k}^{n-1} \frac{d_j}{(j-k)!} \left(\ln \frac{t}{a} \right)^{j-k} + \frac{1}{\Gamma(\alpha-k)} \int_a^t \left(\ln \frac{t}{\tau} \right)^{\alpha-k-1} h[\tau, x(\tau)] d\tau.$$

Changing the variable $\tau = a \left(\frac{t}{a} \right)^s$, yieldys

$$\begin{aligned} \delta^k x(t) &= \sum_{j=k}^{n-1} \frac{d_j}{(j-k)!} \left(\ln \frac{t}{a} \right)^{j-k} + \frac{1}{\Gamma(\alpha-k)} \int_0^1 \left(\ln \frac{t}{a \left(\frac{t}{a} \right)^s} \right)^{\alpha-k-1} \\ &\quad \times h \left[a \left(\frac{t}{a} \right)^s, x \left(a \left(\frac{t}{a} \right)^s \right) \right] a \ln \left(\frac{t}{a} \right) \left(\frac{t}{a} \right)^s ds \\ &= \sum_{j=k}^{n-1} \frac{d_j}{(j-k)!} \left(\ln \frac{t}{a} \right)^{j-k} + \frac{\ln \left(\frac{t}{a} \right)^{\alpha-k}}{\Gamma(\alpha-k)} \int_0^1 (1-s)^{\alpha-k-1} h \left[a \left(\frac{t}{a} \right)^s, x \left(a \left(\frac{t}{a} \right)^s \right) \right] ds. \end{aligned}$$

for $k = 0, \dots, n-1$. Because $\alpha - k > n - 1 - k \geq 0$, using the continuity of h , Property 2.3 and Lemma 2.7 we get $\mathcal{J}_{a+}^{\alpha} h[t, x] \in C[a, b]$, and taking a limit as $t \rightarrow a_+$, we obtain $\delta^k x(a_+) = d_k$.

(ii) For $\alpha \in \mathbb{N}$ and $x(t) \in C_{\delta}^n[a, b]$ be the solution to the Cauchy problem (11).

(ii.a) Firstly, we prove the necessity. Applying $\left(\mathcal{J}_{a+}^n \right)$ to both sides of equation (11), using (4) and Lemma 2.4, we have

$$\mathcal{J}_{a+}^n \delta^n x(t) = x(t) - \sum_{k=0}^{n-1} \frac{\delta^k x(a)}{k!} \left(\ln \frac{t}{a} \right)^k = \mathcal{J}_{a+}^n h(t),$$

since $\delta^k x(a_+) = d_k$, we arrive at equation (12) and hence the necessity is proved.

(ii.b) If $x \in C_\delta^n[a, b]$ satisfies the equation (12), in addition, by term-by-term differentiation of (12) in the usual sense k times, we get

$$\delta^k x(t) = \sum_{j=k}^{n-1} \frac{d_j}{(j-k)!} \left(\ln \frac{t}{a} \right)^{j-k} + \frac{1}{(n-k-1)!} \int_a^t \left(\ln \frac{t}{\tau} \right)^{n-k-1} h[\tau, x(\tau)] \frac{d\tau}{\tau},$$

for $k = 0, \dots, n$. Using Property 2.3, taking the limit as $t \rightarrow a_+$, we obtain $\delta^k x(a_+) = d_k$, and $\delta^n x(t) = h[t, x(t)]$. Thus the Theorem 3.1 is proved for $\alpha \in \mathbb{N}$.

This completes the proof of the theorem. \square

Corollary 3.2. Under the hypotheses of Theorem 3.1, with $0 < \operatorname{Re}(\alpha) < 1$, if $x \in C_\delta[a, b]$ then $x(t)$ satisfies the relation

$$\left({}^c \mathcal{D}_{a+}^\alpha x \right)(t) = h[t, x(t)], \quad t > a > 0, \quad x(a) = d_0,$$

if and only if, x satisfies the equation

$$x(t) = d_0 + \left(\mathcal{J}_{a+}^\alpha h \right)(t), \quad a \leq t \leq b.$$

The next step is to prove the existence of a unique solution to the Cauchy problem (8)-(9) in the space of functions $C_{\delta, \gamma, \ln}^{\alpha, r}[a, b]$ by using the Banach's fixed point theorem.

Theorem 3.3. Let $\alpha > 0$, and $n = [\Re(\alpha)] + 1$, $0 \leq \gamma < 1$ be such that $\alpha \geq \gamma$. Let G be an open set in \mathbb{R} and $h :]a, b] \times G \rightarrow \mathbb{C}$ be a function such that, for any $x \in G$, $h[t, x] \in C_{\gamma, \ln}[a, b]$, $x \in C_{\gamma, \ln}[a, b]$, and the Lipschitz condition (7) holds with respect to the second variable.

- (i) If $n-1 < \alpha < n$, then there exists a unique solution x to (8)-(9) in the space $C_{\delta, \gamma, \ln}^{\alpha, n-1}[a, b]$.
- (ii) If $\alpha = n$, then there exists a unique solution $x \in C_{\delta, \gamma, \ln}^n[a, b]$.

Since the problem (8)-(9) and the equation (10) are equivalent, it is enough to prove that there exists only one solution to (10).

Proof. Here we prove (i) only as (ii) can be proved similarly.

Step 1. First we show that there exists a unique solution $x \in C_\delta^{n-1}[a, b]$.

Divide the interval $[a, b]$ into M subdivisions $[a, t_1], [t_1, t_2], \dots, [t_{M-1}, b]$ such that $a < t_1 < t_2 < \dots < t_{M-1} < b$.

(a) Choose $t_1 \in]a, b]$ such that the inequality

$$w_1 = A \sum_{k=0}^{n-1} \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)} \left(\ln \frac{t_1}{a} \right)^{\operatorname{Re}(\alpha)-k} < 1, \quad A > 0, \quad (14)$$

holds. Now we prove that there exists a unique solution $x(t) \in C_\delta^{n-1}[a, t_1]$ to equation (10) in the interval $[a, t_1]$.

It is easy to see that $C_\delta^{n-1}[a, t_1]$ is a complete metric space equipped with the distance

$$d(x_1, x_2) = \|x_1 - x_2\|_{C_\delta^{n-1}[a, t_1]} = \sum_{k=0}^{n-1} \|(\delta^k x_1 - \delta^k x_2)\|_{C[a, t_1]}.$$

Now, for any $x \in C_\delta^{n-1}[a, t_1]$, define operator T as follows

$$(Tx)(t) \equiv Tx(t) = x_0(t) + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} h[\tau, x(\tau)] \frac{d\tau}{\tau}, \quad (15)$$

with

$$x_0(t) = \sum_{j=0}^{n-1} \frac{d_j}{j!} \left(\ln \frac{t}{a}\right)^j. \quad (16)$$

Transforming the problem (10) into a fixed point problem, $x(t) = Tx(t)$, where T is defined by (15). One can see that the fixed points of T are nothing but solutions to problem (8)-(9). Applying the Banach contraction mapping, we shall prove that T has a unique fixed point.

Firstly, we have to show that:

- (a.i) if $x(t) \in C_\delta^{n-1}[a, t_1]$, then $(Tx)(t) \in C_\delta^{n-1}[a, t_1]$.
- (a.ii) $\forall x_1, x_2 \in C_\delta^{n-1}[a, t_1]$ the following inequality holds:

$$\|Tx_1 - Tx_2\|_{C_\delta^{n-1}[a, t_1]} \leq w_1 \|x_1 - x_2\|_{C_\delta^{n-1}[a, t_1]}, \quad 0 < w_1 < 1.$$

- (a.i) Let us prove that $Tx : C_\delta^{n-1}[a, t_1] \rightarrow C_\delta^{n-1}[a, t_1]$ is a continuous operator. Differentiating (15) k ($k = 0, \dots, n-1$) times, we arrive at the equality

$$(\delta^k Tx)(t) = \delta^k x_0(t) + \frac{1}{\Gamma(\alpha - k)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1-k} h[\tau, x(\tau)] \frac{d\tau}{\tau},$$

with

$$\delta^k x_0(t) = \sum_{j=k}^{n-1} \frac{d_j}{(j-k)!} \left(\ln \frac{t}{a}\right)^{j-k}.$$

It follows that $\delta^k x_0(t) \in C_\delta[a, t_1]$ because $x_0(t)$ might be further decomposed as a finite sum of functions in $C_\delta^{n-1}[a, t_1]$. When $x_0(t) \in C_\delta^{n-1}[a, t_1]$ then

$$\|x_0(t)\|_{C[a, t_1]} \leq \|x_0(t)\|_{C_\delta^{n-1}[a, t_1]} = \sum_{k=1}^{n-1} \|(\delta^k x_0(t))\|_{C[a, t_1]} + \|x_0(t)\|_{C[a, t_1]}.$$

On the other hand, we can apply Lemma 2.6-(b) with $\alpha \geq \gamma$, and α being replaced by $(\alpha - k)$, we have

$$\mathcal{J}_{a+}^{\alpha-k} h[\tau, x(\tau)](t) \in C_\delta[a, t_1].$$

In view of Lemma 2.6 and (7), for all $k = 0, \dots, n-1$, we have

$$\begin{aligned} \left\| \mathcal{J}_{a+}^{\alpha-k} h[\tau, x(\tau)] \right\|_{C[a, t_1]} &\leq \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-k-\gamma)} \left(\ln \frac{t_1}{a}\right)^{Re(\alpha)-k-\gamma} \|h[t, x(t)]\|_{C_{\gamma, \ln}[a, t_1]} \\ &\leq A \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-k-\gamma)} \left(\ln \frac{t_1}{a}\right)^{Re(\alpha)-k-\gamma} \|x(t)\|_{C_{\gamma, \ln}[a, t_1]} \\ &\leq A \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-k-\gamma)} \left(\ln \frac{t_1}{a}\right)^{Re(\alpha)-k} \|x(t)\|_{C[a, t_1]}. \end{aligned}$$

As fractional integrals are bounded in the space of functions continuous in interval $[a, t_1]$. The above implies that $Tx(t)$ belongs to the $C_\delta^{n-1}[a, t_1]$ space.

(a.ii) Next, we let $x_1, x_2 \in C_\delta^{n-1}[a, t_1]$ the following estimate holds:

$$\begin{aligned} \|Tx_1 - Tx_2\|_{C_\delta^{n-1}[a, t_1]} &= \left\| \mathcal{J}_{a+}^\alpha (h[\tau, x_1(\tau)] - h[\tau, x_2(\tau)])(t) \right\|_{C_\delta^{n-1}[a, t_1]} \\ &= \sum_{k=0}^{n-1} \left\| \mathcal{J}_{a+}^{\alpha-k} (h[\tau, x_1(\tau)] - h[\tau, x_2(\tau)])(t) \right\|_{C[a, t_1]} \\ &\leq \sum_{k=0}^{n-1} \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)} \left(\ln \frac{t_1}{a} \right)^{Re(\alpha)-k-\gamma} \|h[\tau, x_1(\tau)] - h[\tau, x_2(\tau)]\|_{C_{\gamma, \ln}[a, t_1]} \\ &\leq A \sum_{k=0}^{n-1} \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)} \left(\ln \frac{t_1}{a} \right)^{Re(\alpha)-k-\gamma} \|x_1(t) - x_2(t)\|_{C_{\gamma, \ln}[a, t_1]} \\ &\leq A \sum_{k=0}^{n-1} \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)} \left(\ln \frac{t_1}{a} \right)^{Re(\alpha)-k} \|x_1(t) - x_2(t)\|_{C[a, t_1]} \\ &\leq A \sum_{k=0}^{n-1} \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)} \left(\ln \frac{t_1}{a} \right)^{Re(\alpha)-k} \|x_1(t) - x_2(t)\|_{C_\delta^{n-1}[a, t_1]}. \end{aligned}$$

Thus

$$\|Tx_1 - Tx_2\|_{C_\delta^{n-1}[a, t_1]} \leq A \sum_{k=0}^{n-1} \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)} \left(\ln \frac{t_1}{a} \right)^{Re(\alpha)-k} \|x_1(t) - x_2(t)\|_{C_\delta^{n-1}[a, t_1]}.$$

The last estimate shows that the operator T is a contraction mapping from $C_\delta^{n-1}[a, t_1]$. Thus, the Banach fixed point theorem implies that there exists a unique function (solution) $x_0^* \in C_\delta^{n-1}[a, t_1]$ and this given as:

$$x_0^* = \lim_{m \rightarrow +\infty} T^m x_{00}^*, \quad (m \in \mathbb{N}^*),$$

where

$$(T^m x_{00}^*)(t) = x_0(t) + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\tau} \right)^{\alpha-1} h[\tau, (T^{m-1} x_{00}^*)(\tau)] \frac{d\tau}{\tau},$$

with $x_{00}^* \in C_\delta^{n-1}[a, t_1]$ is an arbitrary starting function.

Let us take $x_{00}^*(t) = x_0(t)$ when $d_k \neq 0$ with $x_0(t)$ defined by (16), if we denote by

$$x_m(t) = (T^m x_{00}^*)(t), \quad (m \in \mathbb{N}^*),$$

then

$$\lim_{m \rightarrow +\infty} \|x_m(t) - x_0^*(t)\|_{C_\delta^{n-1}[a, t_1]} = 0.$$

Now we show that this solution $x_0^*(t)$ is unique. Suppose that there exist two solutions $x_0^*(t)$, $\tilde{x}_0^*(t)$ of equation (10) on $[a, t_1]$. Using Lemma 2.6 and substituting them into (10), we get

$$\|x_0^*(t) - \tilde{x}_0^*(t)\|_{C_\delta^{n-1}[a, t_1]} \leq A \sum_{k=0}^{n-1} \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)} \left(\ln \frac{t_1}{a} \right)^{Re(\alpha)-k} \|x_0^*(t) - \tilde{x}_0^*(t)\|_{C_\delta^{n-1}[a, t_1]}.$$

This relation yields

$$A \sum_{k=0}^{n-1} \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)} \left(\ln \frac{t_1}{a} \right)^{Re(\alpha)-k} \geq 1,$$

which contradicts the assumption (14). Thus there is a unique solution $x_0^*(t) \in C_\delta^{n-1}[a, t_1]$.

(b) We prove the existence of an unique solution $x(t) \in C_\delta^{n-1}[t_1, b]$. analogously

Further, if we consider the closed interval $[t_1, b]$, we can rewrite equation (10) in the form $x(t) = (Tx)(t)$ where

$$(Tx)(t) = x_{01}(t) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} \frac{h[\tau, x(\tau)]}{\tau} d\tau, \quad (17)$$

where $x_{01}(t)$ defined by

$$x_{01}(t) = x_0(t) + \frac{1}{\Gamma(\alpha)} \int_a^{t_1} \left(\ln \frac{t}{\tau}\right)^{\alpha-1} \frac{h[\tau, x(\tau)]}{\tau} d\tau,$$

is a known function.

We note that $x_{01}(t) \in C_\delta^{n-1}[t_1, b]$. Differentiating (17) k ($k = 0, \dots, n-1$) times, we arrive at the equality

$$(\delta^k Tx)(t) = \delta^k x_{01}(t) + \frac{1}{\Gamma(\alpha-k)} \int_a^t \left(\ln \frac{t}{\tau}\right)^{\alpha-k-1} h[\tau, x(\tau)] \frac{d\tau}{\tau}.$$

It follows that $\delta^k x_{01}(t) \in C_\delta[t_1, b]$ and $\mathcal{J}_{a+}^{\alpha-k} h[\tau, x(\tau)] \in C_\delta[t_1, b]$ thus $(Tx)(t) \in C_\delta^{n-1}[t_1, b]$.

(b.i) Choose $t_2 \in]t_1, b]$ such that the inequality

$$w_2 = A \sum_{k=1}^{n-1} \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)} \left(\ln \frac{t_2}{t_1}\right)^{Re(\alpha)-k} < 1,$$

hold. Let $x_1, x_2 \in C_\delta^{n-1}[t_1, t_2]$ the following estimate holds:

$$\begin{aligned} \|Tx_1 - Tx_2\|_{C_\delta^{n-1}[t_1, t_2]} &\leq \sum_{k=0}^{n-1} \left\| \mathcal{J}_{a+}^{\alpha-k} (h[\tau, x_1(\tau)] - h[\tau, x_2(\tau)])(t) \right\|_{C[t_1, t_2]} \\ &\leq A \sum_{k=0}^n \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-k+1)} \left(\ln \frac{t_2}{t_1}\right)^{Re(\alpha)-k} \|x_1(t) - x_2(t)\|_{C_\delta^{n-1}[t_1, t_2]}. \end{aligned}$$

Hence Tx is a contraction in $C_\delta^{n-1}[t_1, t_2]$.

By Lemma 2.6-(b) and α being replaced by $\alpha-k$, we obtain that $\mathcal{J}_{t_1+}^{\alpha-k} (h[\tau, x_1(\tau)] - h[\tau, x_2(\tau)])$ is continuous in $[t_1, t_2]$. Then, the Banach fixed point theorem implies that there exists a unique solution $x_1^* \in C_\delta^{n-1}[t_1, t_2]$ to the equation (10) on the interval $[t_1, t_2]$.

Notice that $x_1^*(t_1) = x_0^*(t_1) = x_{01}(t_1)$. Further, Theorem 2.8 guarantees that this solution $x_1^*(t)$ is the limit of the convergent sequence $T^m x_{01}^*$. Thus, we have

$$\lim_{m \rightarrow +\infty} \|T^m x_{01}^* - x_1^*\|_{C_\delta^{n-1}[t_1, t_2]} = 0,$$

with

$$(T^m x_{01}^*)(t) = x_{01}(t) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t \left(\ln \frac{t}{\tau}\right)^{\alpha-1} h[\tau, (T^{m-1} x_{01}^*)(\tau)] \frac{d\tau}{\tau}, (m \in \mathbb{N}^*).$$

If $x_0(t) \neq 0$ then we can take $x_{01}^*(t) = x_0(t)$, therefore,

$$\lim_{m \rightarrow +\infty} \|x_m(t) - x_1^*(t)\|_{C_\delta^{n-1}[t_1, t_2]} = 0, \quad x_m(t) = (T^m x_{01}^*)(t).$$

Now let

$$x^*(t) = \begin{cases} x_0^*(t) & t \in [t_1, t_2], \\ x_1^*(t) & t \in [a, t_1]. \end{cases}$$

Moreover, since $x^* \in C_\delta^{n-1}[a, t_1]$ and $x^* \in C_\delta^{n-1}[t_1, t_2]$, we have $x^* \in C_\delta^{n-1}[a, t_2]$, and hence there is a unique solution $x^* \in C_\delta^{n-1}[a, t_2]$ to the equation (10) on the interval $[a, t_2]$.

(b.ii) Finally, we prove that a unique solution $x(t) \in C_\delta^{n-1}[t_2, b]$ exists.

If $t_2 \neq b$, we choose $t_{i+1} \in]t_i, b]$ such that the relation

$$w_{i+1} = A \sum_{k=0}^{n-1} \frac{\Gamma(1-\gamma)}{\Gamma(\alpha-k-\gamma+1)} \left(\ln \frac{t_{i+1}}{t_i} \right)^{Re(\alpha)-k} < 1, \quad i = 2, 3, \dots, M, \quad b = t_M.$$

Repeating the above process i times, we also deduce that there exists a unique solution $x_i^* \in C_\delta^{n-1}[t_i, t_{i+1}]$ given as a limit of a convergent sequence $T^m x_{0i}^*$ i.e.,

$$\lim_{m \rightarrow +\infty} \|T^m x_{0i}^* - x_i^*\|_{C_\delta^{n-1}[t_i, t_{i+1}]} = 0, \quad i = 2, 3, \dots, M.$$

Consequently, the previous relation can be rewritten as

$$\lim_{m \rightarrow +\infty} \|x_m(t) - x^*(t)\|_{C_\delta^{n-1}[a, b]} = 0, \quad (18)$$

with

$$x_m(t) = T^m x_{0i}^*, \quad x_{0i}^*(t) = x_0(t), \quad x^*(t) = x_i^*(t), \quad i = 0, 1, \dots, M,$$

and

$$x_i^*(t_{i+1}) = x_{i+1}^*(t_{i+1}), \quad [a, b] = \cup [t_i, t_{i+1}], \quad a = t_0 < \dots < t_M = b.$$

Step 2. Now we show that $({}^c\mathcal{D}_{a+}^\alpha x^*)(t) \in C_{\gamma, \ln}[a, b]$.

By (8), (18) and the Lipschitzian condition (7), we have that

$$\begin{aligned} \lim_{m \rightarrow +\infty} \left\| \left({}^c\mathcal{D}_{a+}^\alpha x_m \right)(t) - \left({}^c\mathcal{D}_{a+}^\alpha x^* \right)(t) \right\|_{C_{\gamma, \ln}[a, b]} &= \lim_{m \rightarrow +\infty} \|h[t, x_m(t)] - h[t, x^*(t)]\|_{C_{\gamma, \ln}[a, b]} \\ &\leq A \lim_{m \rightarrow +\infty} \|x_m(t) - x^*(t)\|_{C_{\gamma, \ln}[a, b]} \\ &\leq A \left(\ln \frac{b}{a} \right)^\gamma \lim_{m \rightarrow +\infty} \|x_m(t) - x^*(t)\|_{C[a, b]} \\ &\leq A \left(\ln \frac{b}{a} \right)^\gamma \lim_{m \rightarrow +\infty} \|x_m(t) - x^*(t)\|_{C_\delta^{n-1}[a, b]}. \end{aligned}$$

It is obvious that the right hand side of the above inequality approaches to zero independently, thus

$$\lim_{m \rightarrow +\infty} \left\| \left({}^c\mathcal{D}_{a+}^\alpha x_m \right)(t) - \left({}^c\mathcal{D}_{a+}^\alpha x^* \right)(t) \right\|_{C_{\gamma, \ln}[a, b]} = 0.$$

By hypothesis, $({}^c\mathcal{D}_{a+}^\alpha x_m)(t) = h[t, x_m(t)]$ and $h[t, x(t)] \in C_{\gamma, \ln}[a, b]$ for $x \in C_\delta^{n-1}[a, b]$, we have $({}^c\mathcal{D}_{a+}^\alpha x^*)(t) \in C_{\gamma, \ln}[a, b]$.

Consequently, $x^* \in C_{\delta, \gamma, \ln}^{\alpha, n-1}[a, b]$ is the unique solution to the problem (8)-(9). \square

Corollary 3.4. *Under the hypotheses of Theorem 3.3, with $\gamma = 0$, there exists a unique solution x to the problem (8)-(9) in the space $C_\delta^{\alpha, n-1}[a, b]$ and to the problem (11) in the space $C_\delta^n[a, b]$.*

Proof. The above Corollary can be demonstrated in a similar way to that of Theorem 3.3, using the following inequality

$$w_{i+1} = A \sum_{k=0}^{n-1} \frac{1}{\operatorname{Re}(\alpha - k) \Gamma(\alpha - k + 1)} \left(\ln \frac{t_{i+1}}{t_i} \right)^{\operatorname{Re}(\alpha) - k} < 1, \quad i = 0, \dots, M, \quad a = t_0, \quad b = t_M,$$

where $t_i \in [a, b]$ and we observe that T is a contractive mapping when the following inequality holds, indeed, for any $x_1, x_2 \in C_\delta^{n-1}[t_i, t_{i+1}]$

$$\begin{aligned} \|Tx_1 - Tx_2\|_{C_\delta^{n-1}[t_i, t_{i+1}]} &= \sum_{k=0}^{n-1} \left\| \mathcal{J}_{t_i+}^{\alpha-k} (h[\tau, x_1(\tau)] - h[\tau, x_2(\tau)])(t) \right\|_{C[t_i, t_{i+1}]} \\ &\leq \sum_{k=0}^{n-1} \frac{\left(\ln \frac{t_{i+1}}{t_i} \right)^{\operatorname{Re}(\alpha) - k}}{\operatorname{Re}(\alpha - k) \Gamma(\alpha - k + 1)} \|h[t, x_1(t)] - h[t, x_2(t)]\|_{C[t_i, t_{i+1}]} \\ &\leq A \sum_{k=0}^{n-1} \frac{\left(\ln \frac{t_{i+1}}{t_i} \right)^{\operatorname{Re}(\alpha) - k}}{\operatorname{Re}(\alpha - k) \Gamma(\alpha - k + 1)} \|x_1(t) - x_2(t)\|_{C[t_i, t_{i+1}]} \\ &\leq A \sum_{k=0}^{n-1} \frac{\left(\ln \frac{t_{i+1}}{t_i} \right)^{\operatorname{Re}(\alpha) - k}}{\operatorname{Re}(\alpha - k) \Gamma(\alpha - k + 1)} \|x_1(t) - x_2(t)\|_{C_\delta^{n-1}[t_i, t_{i+1}]} . \end{aligned}$$

□

4 The Generalized Cauchy type problem

The results in the previous section can be extended to the following equation, which is more general than (8) :

$$\left({}^c \mathcal{D}_{a+}^\alpha x \right)(t) = h \left[t, x(t), \left({}^c \mathcal{D}_{a+}^{\alpha_1} x \right)(t), \dots, \left({}^c \mathcal{D}_{a+}^{\alpha_l} x \right)(t) \right], \quad (19)$$

with $\alpha_j \in (j-1, j]$, $j = 1, 2, \dots, l$, $\alpha_0 = 0$, and $({}^c \mathcal{D}_{a+}^{\alpha_j})$ denotes the Caputo Hadamard operator of order α_j .

The initial conditions for (19) are

$$(\delta^k x)(a_+) = d_k, \quad d_k \in \mathbb{R} \quad (k = 0, \dots, n-1). \quad (20)$$

For simplicity, we denote by $h[t, \varphi(t, x)]$ instead of $h \left[t, x(t), \left({}^c \mathcal{D}_{a+}^{\alpha_1} x \right)(t), \dots, \left({}^c \mathcal{D}_{a+}^{\alpha_l} x \right)(t) \right]$.

Similar to the things discussed in the previous, our investigations are based on reducing the problem (19)-(20) to the Volterra equation

$$x(t) = \sum_{j=0}^{n-1} \frac{d_j}{j!} \left(\ln \frac{t}{a} \right)^j + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\tau} \right)^{\alpha-1} h[\tau, \varphi(\tau, x)] \frac{d\tau}{\tau}, \quad (t > a). \quad (21)$$

Theorem 4.1. *Let $\alpha > 0$, $n = [\operatorname{Re}(\alpha)] + 1$ and $\alpha_j \in \mathbb{C}$ ($j = 0, \dots, l$) be such that*

$$0 = \operatorname{Re}(\alpha_0) < \operatorname{Re}(\alpha_1) < \dots < \operatorname{Re}(\alpha_l) < n-1. \quad (22)$$

Let $G \in \mathbb{R}^{l+1}$ be open subsets and let $h : (a, b] \times G \rightarrow \mathbb{R}$ be a function such that $h[t, x, x_1, \dots, x_l] \in C_{\gamma, \ln}[a, b]$ for arbitrary $x, x_1, \dots, x_l \in C_{\gamma, \ln}[a, b]$ and the Lipschitz condition (6) is fulfilled.

(i) If $x \in C_{\delta, \gamma, \ln}^{\alpha, n-1}[a, b]$, then x holds the relations (19)-(20) if and only if x holds the equation (21).

(ii) If $0 < \alpha < 1$, then $x \in C_{\delta, \gamma, \ln}^{\alpha}[a, b]$ satisfies the relations

$$\left({}^c\mathcal{D}_{a+}^{\alpha} x\right)(t) = h[t, \varphi(t, x)], \quad x(a_+) = d_0, \quad d_0 \in \mathbb{R}, \quad (23)$$

iff x satisfies the equation

$$x(t) = d_0 + \left(\mathcal{J}_{a+}^{\alpha}\right) h[\tau, \varphi(\tau, x)](t), \quad (t > a). \quad (24)$$

Proof. Let $\alpha \in (n-1, n]$ and $x \in C_{\delta}^{n-1}[a, b]$ satisfies the relations (19)-(20).

(i.a) According to (4) and (19),

$$\left({}^c\mathcal{D}_{a+}^{\alpha} x\right)(t) = \left(\mathcal{D}_{a+}^{\alpha}\right) \left[x(\tau) - \sum_{k=0}^{n-1} \frac{\delta^k x(a)}{k!} \left(\ln \frac{\tau}{a}\right)^k \right](t).$$

We have $\left({}^c\mathcal{D}_{a+}^{\alpha} x\right)(t) \in C_{\gamma, \ln}[a, b]$ and hence

$$\delta^n \mathcal{J}_{a+}^{n-\alpha} \left(x(\tau) - \sum_{j=0}^{n-1} \frac{\delta^j x(a)}{j!} \left(\ln \frac{\tau}{a}\right)^j \right) \in C_{\gamma, \ln}[a, b].$$

Thus,

$$\mathcal{J}_{a+}^{n-\alpha} \left(x(\tau) - \sum_{j=0}^{n-1} \frac{\delta^j x(a)}{j!} \left(\ln \frac{\tau}{a}\right)^j \right) \in C_{\delta, \gamma, \ln}^n[a, b],$$

and by Lemma 2.4

$$\left(\mathcal{J}_{a+}^{\alpha}\right) \left({}^c\mathcal{D}_{a+}^{\alpha} x\right)(t) = x(t) - \sum_{j=1}^{n-1} \frac{\delta^j x(a)}{(j-1)!} \left(\ln \frac{t}{a}\right)^{j-1},$$

Then, from (19), (20) and the last relation, we obtain

$$x(t) = \sum_{j=0}^{n-1} \frac{d_j}{j!} \left(\ln \frac{t}{a}\right)^j + \left(\mathcal{J}_{a+}^{\alpha}\right) h[\tau, \varphi(\tau, x)](t), \quad (t > a).$$

That is $x \in C_{\delta}^{n-1}[a, b]$ satisfy the equation (21).

(i.b) Now we prove the sufficiency. Let $x \in C_{\delta}^{n-1}[a, b]$ satisfies equation (21).

– From (21) we have

$$x(t) - \sum_{j=0}^{n-1} \frac{d_j}{j!} \left(\ln \frac{t}{a}\right)^j = \left(\mathcal{J}_{a+}^{\alpha}\right) h\left[\tau, x(\tau), \left({}^c\mathcal{D}_{a+}^{\alpha_1} x\right)(\tau), \dots, \left({}^c\mathcal{D}_{a+}^{\alpha_l} x\right)(\tau)\right](t).$$

Applying $(\mathcal{D}_{a+}^\alpha)$ on both sides of this relation, taking into account the conditions for h and using Property 2.2, we get

$$\begin{aligned} (\mathcal{D}_{a+}^\alpha) \left(x(t) - \sum_{j=0}^{n-1} \frac{d_j}{j!} \left(\ln \frac{t}{a} \right)^j \right) &= (\mathcal{D}_{a+}^\alpha) (\mathcal{J}_{a+}^\alpha) h[\tau, \varphi(\tau, x)](t) \\ &= h[t, \varphi(t, x)]. \end{aligned}$$

By (4), the left hand of the above expression is $({}^c\mathcal{D}_{a+}^\alpha)$ and thus

$$({}^c\mathcal{D}_{a+}^\alpha) x(t) = h \left[t, x(t), ({}^c\mathcal{D}_{a+}^{\alpha_1} x)(t), \dots, ({}^c\mathcal{D}_{a+}^{\alpha_l} x)(t) \right].$$

Hence $x \in C_\delta^{n-1}[a, b]$ satisfies (19).

– Applying δ^k ($k = 0, \dots, n-1$) to both sides of (21), we have

$$\delta^k x(t) = \sum_{j=k}^{n-1} \frac{d_j}{(j-k)!} \left(\ln \frac{t}{a} \right)^{j-k} + (\delta^k) (\mathcal{J}_{a+}^\alpha) h[\tau, \varphi(\tau, x)](t), \quad (t > a), \quad (25)$$

Since $x \in C_\delta^{n-1}[a, b]$ for any $(({}^c\mathcal{D}_{a+}^{\alpha_1} x), \dots, ({}^c\mathcal{D}_{a+}^{\alpha_l} x)) \in \mathbb{R}^{n-1}$ and $\alpha - k > \gamma - (n-1) > 0$, we have

$$(\mathcal{J}_{a+}^{\alpha-k}) h[\tau, x(\tau), ({}^c\mathcal{D}_{a+}^{\alpha_1} x)(\tau), \dots, ({}^c\mathcal{D}_{a+}^{\alpha_l} x)(\tau)] \in C[a, b]. \quad (26)$$

On the other hand, by Lemma 2.3, we let $\tau \longrightarrow a_+$ on the both sides of (25), then we obtain

$$\delta^k x(\tau)|_{\tau=a_+} = d_k, \quad k = 0, \dots, n-1.$$

Hence, x satisfying (21) satisfies the initial condition (20). That is $x \in C_\delta^{n-1}[a, b]$ satisfies the Cauchy problem (19)-(20).

Similarly, we prove the second part of the Theorem. \square

Theorem 4.2. Let $\alpha \in \mathbb{C}$, $n = [\operatorname{Re}(\alpha)] + 1$, $0 \leq \gamma < 1$ be such that $\gamma \leq \alpha$. Let $\alpha_j > 0$ ($j = 1, \dots, l$) be such that conditions in (22) are satisfied. Let G be an open set in \mathbb{R}^{l+1} and let $h : (a, b] \times G \longrightarrow \mathbb{R}$ be a function such that $h[t, x, x_1, \dots, x_l] \in C_{\gamma, \ln}[a, b]$ for any $x, x_1, \dots, x_l \in C_{\gamma, \ln}[a, b]$ and the Lipschitz condition (6) is fulfilled.

(i) If $n-1 < \alpha < n$, then there is a unique solution x to the problem (19)-(20) in the space $C_{\delta, \gamma, \ln}^{\alpha, n-1}[a, b]$.

(ii) If $0 < \alpha < 1$, then there is a unique solution $x \in C_{\delta, \gamma, \ln}^\alpha[a, b]$ to (19) with the condition

$$x(a_+) = d_0 \in \mathbb{R}.$$

Proof. By Theorem 4.1 it is sufficient to establish the existence of a unique solution $x \in C_{\delta, \gamma, \ln}^{\alpha, n-1}[a, b]$ to the integral equation (21).

Step 1. First we show that there exists a unique solution $x \in C_\delta^{n-1}[a, b]$.

- (a) We choose $t_1 \in]a, b]$, we prove the existence of a unique solution $x \in C_\delta^{n-1}[a, t_1]$, so that the conditions

$$w_1 = \sum_{k=0}^{n-1} \sum_{j=0}^l A_j \left(\ln \frac{t_1}{a} \right)^{Re(\alpha - \alpha_j) - k} \frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma + \alpha - \alpha_j - k)} < 1 \quad \text{if } \gamma \leq \alpha,$$

holds, and apply the Banach fixed point theorem to prove the existence of a unique solution $x \in C_\delta^{n-1}[a, t_1]$ of the integral equation (21).

We rewrite the equation (21) in the form $x(t) = (Tx)(t)$, where

$$(Tx)(t) = x_0(t) + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{\tau} \right)^{\alpha-1} h[\tau, \varphi(\tau, x)] \frac{d\tau}{\tau},$$

with

$$x_0(t) = \sum_{j=0}^{n-1} \frac{d_j}{j!} \left(\ln \frac{t}{a} \right)^j.$$

It follows that $x_0(t) \in C_\delta^{n-1}[a, t_1]$ because $x_0(t)$ may be further decomposed as a finite sum of functions in $C_\delta^{n-1}[a, t_1]$,

$$h[\tau, \varphi(\tau, x)] \in C_{\gamma, \ln}[a, b] \implies h[\tau, \varphi(\tau, x)] \in C_{\gamma, \ln}[a, t_1],$$

and, from Lemma 2.6-(b), we have, using the fact that $\alpha > 0$ and $0 \leq \gamma < 1$,

$$\mathcal{J}_{a+}^\alpha h[\tau, \varphi(\tau, x)] \in C[a, t_1] \quad \text{if } \gamma \leq \alpha.$$

Let $x \in C_\delta^{n-1}[a, t_1]$, by Lemma 2.7, the integral in the right-hand side of (21) also belongs to $C_\delta^{n-1}[a, t_1]$ i.e., $\mathcal{J}_{a+}^\alpha h[\tau, \varphi(\tau, x)] \in C_\delta^{n-1}[a, t_1]$, and hence $Tx \in C_\delta^{n-1}[a, t_1]$, this proves T is continuous on $C_\delta^{n-1}[a, t_1]$.

To show that T is a contraction we have to prove that, for any $x_1, x_2 \in C_\delta^{n-1}[a, t_1]$ there exists $w_1 > 0$ such that

$$\|Tx_1 - Tx_2\|_{C_\delta^{n-1}[a, t_1]} \leq w_1 \|x_1 - x_2\|_{C_\delta^{n-1}[a, t_1]}.$$

By Lipschitzian condition (6), Property 2.2 and Lemma 2.4, thus

$$\begin{aligned} & \left\| \left(\mathcal{J}_{a+}^\alpha \left(h \left[\tau, x_1, {}^c\mathcal{D}_{a+}^{\alpha_1} x_1, \dots, {}^c\mathcal{D}_{a+}^{\alpha_l} x_1 \right] - h \left[\tau, x_2, {}^c\mathcal{D}_{a+}^{\alpha_1} x_2, \dots, {}^c\mathcal{D}_{a+}^{\alpha_l} x_2 \right] \right) \right) (t) \right\| \\ & \leq \mathcal{J}_{a+}^\alpha \left(\left\| h \left[\tau, x_1, {}^c\mathcal{D}_{a+}^{\alpha_1} x_1, \dots, {}^c\mathcal{D}_{a+}^{\alpha_l} x_1 \right] - h \left[\tau, x_2, {}^c\mathcal{D}_{a+}^{\alpha_1} x_2, \dots, {}^c\mathcal{D}_{a+}^{\alpha_l} x_2 \right] \right\| \right) (t) \\ & \leq \sum_{j=0}^l A_j \left\| \left(\mathcal{J}_{a+}^{\alpha - \alpha_j} \right) \mathcal{J}_{a+}^{\alpha_j} ({}^c\mathcal{D}_{a+}^{\alpha_j}) (x_1 - x_2) \right\| (t) \\ & = \left(\sum_{j=0}^l A_j \mathcal{J}_{a+}^{\alpha - \alpha_j} \left\| \mathcal{J}_{a+}^{\alpha_j} ({}^c\mathcal{D}_{a+}^{\alpha_j}) (x_1 - x_2) \right\| \right) (t) \\ & = \left[\left(\sum_{j=0}^l A_j \mathcal{J}_{a+}^{\alpha - \alpha_j} \|x_1 - x_2\| \right) (\tau) - \sum_{k_j=0}^{n_j-1} \frac{\delta^{k_j} (x_1 - x_2)(a_+)}{k_j!} \left(\ln \frac{t}{a} \right)^{k_j} \right]. \end{aligned}$$

By the hypothesis and Lemma 2.4, $\delta^{k_j} x_1(a_+) = \delta^{k_j} (x_2)(a_+)$, $k_j = 0, \dots, n_j - 1$, $n_j = Re(\alpha_j) + 1$, thus

$$\begin{aligned} \left\| \mathcal{J}_{a+}^{\alpha_j} ({}^c\mathcal{D}_{a+}^{\alpha_j}) (x_1 - x_2) (t) \right\| &= \left\| (x_1 - x_2) (t) - \sum_{k_j=0}^{n_j-1} \frac{\delta^{k_j} (x_1 - x_2)(a_+)}{k_j!} \left(\ln \frac{t}{a} \right)^{k_j} \right\| \\ &= \|(x_1 - x_2)(t)\| \end{aligned}$$

for arbitrary $t \in [a, t_1]$. Thus we may continue our estimation above according to

$$\left\| \left(\mathcal{J}_{a+}^{\alpha} \{h[\tau, \varphi(\tau, x_1)] - h[\tau, \varphi(\tau, x_2)]\} \right) (t) \right\| \leq \sum_{j=0}^l A_j \left(\mathcal{J}_{a+}^{\alpha-\alpha_j} (\|x_1 - x_2\|) \right) (t). \quad (27)$$

Moreover by Lemma 2.6-(b), (27) and by (a.ii) in Theorem 3.3 the following holds, indeed, for $x_1, x_2 \in C_{\delta}^{n-1}[a, t_1]$

$$\begin{aligned} \left\| \mathcal{J}_{a+}^{\alpha} (h[\tau, \varphi(\tau, x_1)] - h[\tau, \varphi(\tau, x_2)]) (t) \right\|_{C_{\delta}^{n-1}[a, t_1]} &\leq \left\| \sum_{k=0}^{n-1} \mathcal{J}_{a+}^{\alpha-k} (h[\tau, \varphi(\tau, x_1)] - h[\tau, \varphi(\tau, x_2)]) (t) \right\|_{C_{\delta}[a, t_1]} \\ &\leq \sum_{k=0}^{n-1} \sum_{j=0}^l A_j \left(\ln \frac{t_1}{a} \right)^{Re(\alpha-\alpha_j)-k} \frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma+\alpha-\alpha_j-k)} \|x_1(t) - x_2(t)\|_{C_{\delta}^{n-1}[a, t_1]}. \end{aligned}$$

We conclude that mapping T satisfies

$$\|Tx_1 - Tx_2\|_{C_{\delta}^{n-1}[a, t_1]} \leq w_1 \|x_1 - x_2\|'_{C_{\delta}^{n-1}[a, t_1]}$$

for any functions $x_1, x_2 \in C_{\delta}^{n-1}[a, t_1]$. Hence, a unique fixed point in space $C_{\delta}^{n-1}[a, t_1]$ exists and it is explicitly given as a limit of iterations of the mapping T i.e., $\exists x_0^* \in C_{\delta}^{n-1}[a, t_1]$ such that

$$\lim_{m \rightarrow +\infty} \|x_m(t) - x_0^*(t)\|_{C_{\delta}^{n-1}[a, t_1]} = 0,$$

Thus we deduce that a unique solution $x^*(t) \in C_{\delta}^{n-1}[a, b]$ exists such that

$$\lim_{m \rightarrow +\infty} \|x_m(t) - x^*(t)\|_{C_{\delta}^{n-1}[a, b]} = 0,$$

where

$$x_m(t) = T^m x_{0i}^*, \quad x_{0i}^*(t) = x_0(t), \quad x^*(t) = x_i^*(t), \quad i = 0, 1, \dots, M,$$

and

$$x_i^*(t_{i+1}) = x_{i+1}^*(t_{i+1}), \quad [a, b] = \cup [t_i, t_{i+1}], \quad a = t_0 < \dots < t_M = b.$$

Step 2. To complete the proof of Theorem 4.2, we show that this unique solution $x(t) = x^*(t) \in C_{\delta}^{n-1}[a, b]$ belongs to the space $C_{\delta, \gamma, \ln}^{\alpha, n-1}[a, b]$. It is sufficient to prove that $({}^c\mathcal{D}_{a+}^{\alpha} x)(t) \in C_{\delta, \gamma, \ln}^{\alpha}[a, b]$. Using the estimate (27), we have

$$\begin{aligned} \left\| ({}^c\mathcal{D}_{a+}^{\alpha} x_m)(t) - ({}^c\mathcal{D}_{a+}^{\alpha} x^*)(t) \right\|_{C_{\gamma, \ln}[a, b]} &= \|h[t, \varphi(t, x_m)] - h[t, \varphi(t, x^*)]\|_{C_{\gamma, \ln}[a, b]} \\ &\leq \sum_{j=0}^l A_j \|{}^c\mathcal{D}_{a+}^{\alpha_j} (x_m(t) - x^*(t))\|_{C_{\gamma, \ln}[a, b]} \\ &\leq \sum_{j=0}^l A_j \left\| \mathcal{J}_{a+}^{n-1-\alpha_j} \delta^{n-1} (x_m(t) - x^*(t)) \right\|_{C_{\gamma, \ln}[a, b]} \\ &\leq \sum_{j=0}^l A_j \left(\ln \frac{b}{a} \right)^{\gamma} \left\| \mathcal{J}_{a+}^{n-1-\alpha_j} \delta^{n-1} (x_m(t) - x^*(t)) \right\|_{C[a, b]} \\ &\leq \sum_{j=0}^l A_j \frac{\left(\ln \frac{b}{a} \right)^{\gamma}}{Re(n-1-\alpha_j)\Gamma(n-1-\alpha_j)} \|\delta^{n-1} (x_m(t) - x^*(t))\|_{C[a, b]} \\ &\leq \sum_{j=0}^l A_j \frac{\left(\ln \frac{b}{a} \right)^{\gamma}}{Re(n-1-\alpha_j)\Gamma(n-1-\alpha_j)} \|x_m(t) - x^*(t)\|_{C^{n-1}[a, b]}, \end{aligned}$$

It is clear that the right hand side of the above inequality approaches to zero independently. Hence,

$$\lim_{m \rightarrow +\infty} \left\| \left({}^c\mathcal{D}_{a+}^{\alpha} x_m \right) (t) - \left({}^c\mathcal{D}_{a+}^{\alpha} x^* \right) (t) \right\|_{C_{\gamma, \ln}[a, b]} = 0.$$

Consequently, a unique solution $x^* \in C_{\delta, \gamma, \ln}^{\alpha, n-1}[a, b]$ of equation (21) exists. The second part of the theorem can be proved analogously. \square

Corollary 4.3. *Under the hypotheses of Theorem 4.2 with $\gamma = 0$. Then there exists a unique solution $x^*(t) \in C_{\delta}^{m-1}[a, b]$ to the Cauchy problem (19)-(20).*

Proof. The above Corollary can be demonstrated in a similar way to that of Theorem 4.2, using the following inequality

$$\begin{aligned} & \left\| \mathcal{J}_{a+}^{\alpha} (h[\tau, \varphi(\tau, x_1)] - h[\tau, \varphi(\tau, x_2)])(t) \right\|_{C[t_i, t_{i+1}]} \\ & \leq \sum_{k=0}^{n-1} \sum_{j=0}^l A_j \frac{\left(\ln \frac{t_i}{t_{i+1}} \right)^{Re(\alpha - \alpha_j) - k}}{\Re(\alpha - \alpha_j - k) \Gamma(\alpha - \alpha_j - k)} \|x_1(t) - x_2(t)\|_{C[t_i, t_{i+1}]}, \end{aligned}$$

for $i = 0, 1, \dots, M$, $a = t_0$, $b = t_M$, and

$$\begin{aligned} & \left\| \left({}^c\mathcal{D}_{a+}^{\alpha} x_m \right) (t) - \left({}^c\mathcal{D}_{a+}^{\alpha} x^* \right) (t) \right\|_{C_{\gamma, \ln}[a, b]} \leq \\ & \sum_{j=0}^l A_j \frac{\left(\ln \frac{b}{a} \right)^{\gamma}}{Re(n-1-\alpha_j) \Gamma(n-1-\alpha_j)} \|x_m(t) - x^*(t)\|_{C^{n-1}[a, b]}. \end{aligned}$$

\square

We can derive the corresponding results for the Cauchy problems for linear fractional equations.

Corollary 4.4. *Let $\alpha > 0$, $n = [Re(\alpha)] + 1$ and $0 \leq \gamma < 1$ be such that $\alpha \geq \gamma$. Let $l \in \mathbb{N}$, $\alpha_j > 0$ ($j = 1, \dots, l$) be such that conditions in (22) are satisfied and let $d_j(t) \in C[a, b]$ ($j = 1, \dots, l$) and $f(t) \in C_{\gamma, \ln}[a, b]$.*

Then the Cauchy problem for the following linear differential equation of order α

$$\left({}^c\mathcal{D}_{a+}^{\alpha} x \right) (t) + \sum_{j=1}^l d_j(t) \left({}^c\mathcal{D}_{a+}^{\alpha_j} x \right) (t) + d_0(t) x(t) = f(t) \quad (t > a),$$

with the initial conditions (9) has a unique solution $x(t)$ in the space $C_{\delta, \gamma, \ln}^{\alpha, n-1}[a, b]$.

In particular, there exists a unique solution $x(t)$ in the space $C_{\delta, \gamma, \ln}^{\alpha, n-1}[a, b]$ to the Cauchy problem for the equation with $\lambda_j \in \mathbb{R}$ and $\beta_j \geq 0$ ($j = 1, \dots, l$):

$$\left({}^c\mathcal{D}_{a+}^{\alpha} x \right) (t) + \sum_{j=1}^l \lambda_j \left(\ln \frac{t}{a} \right)^{\beta_j} \left({}^c\mathcal{D}_{a+}^{\alpha_j} x \right) (t) + \lambda_0 \left(\ln \frac{t}{a} \right)^{\beta_0} x(t) = f(t) \quad (t > a).$$

Proof. The proof is a direct consequence of Theorem 4.2. \square

5 Illustrative Examples

We give here some applications of the above results to Cauchy problems with the Caputo Hadamard derivative.

Example 5.1. We consider the fractional differential equation of the form

$$\left({}^c\mathcal{D}_{a+}^{\alpha}x\right)(t)=\lambda\left(\ln\frac{t}{a}\right)^{\beta}[x(t)]^m; \quad t>a>0; \quad \operatorname{Re}(\alpha)>0, \quad m>0; \quad m\neq 1, \quad (28)$$

with $\lambda, \beta \in \mathbb{R}$ ($\lambda \neq 0$), with the initial conditions

$$(\delta^k x)(a_+) = 0, \quad k = 0, \dots, n-1. \quad (29)$$

(a) Suppose that the solution has the following form:

$$x(t) = c \left(\ln \frac{t}{a} \right)^{\nu},$$

then, this equation has the explicit solution

$$x(t) = \left[\frac{\Gamma(\gamma - \alpha + 1)}{\lambda \Gamma(\gamma + 1)} \right]^{\frac{1}{(m-1)}} \left(\ln \frac{t}{a} \right)^{\alpha - \gamma}, \quad \gamma = \frac{(\beta + m\alpha)}{(m-1)}. \quad (30)$$

Moreover, the condition (29) is satisfied.

Hence $x(t)$ is an eigenfunction if both of $\gamma + 1$ and $\gamma - \alpha + 1$ are not equal to 0 or negative integer. also using Property 2.3 it is easily verified that if the condition

$$\frac{(\beta + \alpha)}{(m-1)} \geq -1, \quad (31)$$

holds, this solution $x(t)$ belongs to $C_{\gamma}[a, b]$ and to $C[a, b]$ in the respective cases $0 \leq \alpha$ and $\gamma - \alpha \leq 0$.

$$x(t) \in C_{\gamma}[a, b] \quad \text{if} \quad 0 \leq \gamma < 1 \quad \text{and} \quad 0 \leq \alpha, \quad (32)$$

$$x(t) \in C[a, b] \quad \text{if} \quad \gamma - \alpha \leq 0.$$

The right-hand side of the equation (28) takes the form

$$h[t, x(t)] = \left[\frac{\Gamma(\gamma - \alpha + 1)}{\lambda \Gamma(\gamma + 1)} \right]^{\frac{m}{(m-1)}} \left(\ln \frac{t}{a} \right)^{-\gamma}. \quad (33)$$

The function $h[t, x(t)] \in C_{\gamma}[a, b]$ when $0 \leq \gamma < 1$ and $h[t, x(t)] \in C[a, b]$ when $\gamma \leq 0$

$$h[t, x(t)] \in C_{\gamma}[a, b] \quad \text{if} \quad 0 \leq \gamma < 1, \quad (34)$$

$$h[t, x(t)] \in C[a, b] \quad \text{if} \quad \gamma \leq 0.$$

In accordance with (31), the following case is possible for the space of the right-hand side (33) and of the solution (30) :

1. When $\alpha > 0$ and

$$\begin{aligned} m > 1, \quad -m\alpha \leq \beta < m - 1 - m\alpha, \quad \beta \leq -\alpha, \\ \text{or} \\ 0 < m < 1, \quad m - 1 - m\alpha < \beta \leq -m\alpha, \quad \beta \geq -\alpha. \end{aligned}$$

2. If $0 < \alpha < 1$ these conditions take the following forms

$$m > 1, \quad -m\alpha \leq \beta \leq -\alpha \text{ or } 0 < m < 1, \quad -\alpha \leq \beta \leq -m\alpha. \quad (35)$$

3. If $\alpha \geq 1$ then

$$m > 1, \quad -m\alpha \leq \beta < m - 1 - m\alpha \text{ or } 0 < m < 1, \quad m - 1 - m\alpha < \beta \leq -m\alpha. \quad (36)$$

(b) Now we establish the conditions for the uniqueness of the solution (30) to the above problem (28)-(29). For this we have to choose the domain G and check when the Lipschitz condition (7) with the right-hand side of (28) is valid.

We choose the following domain:

$$G = \left\{ (t, x) \in \mathbb{R}^2 : 0 < a < t \leq b, \quad 0 < x < p \left(\ln \frac{t}{a} \right)^q, \quad q \in \mathbb{R}, \quad p > 0 \right\}. \quad (37)$$

To prove the Lipschitz condition (7) with

$$h[t, x(t)] = \lambda \left(\ln \frac{t}{a} \right)^\beta (x(t))^m, \quad (38)$$

we have, for any $(t, x_1), (t, x_2) \in G$:

$$|h[t, x_1] - h[t, x_2]| \leq |\lambda| \left(\ln \frac{t}{a} \right)^\beta |x_1^m - x_2^m|. \quad (39)$$

By definition (37), we have

$$|x_1^m - x_2^m| < mK \left(\ln \frac{t}{a} \right)^q |x_1 - x_2|, \quad m > 0.$$

Substituting this estimate into (39), we obtain

$$|h[t, x_1] - h[t, x_2]| \leq |\lambda| mK \left(\ln \frac{t}{a} \right)^{\beta + (m-1)q} |x_1 - x_2|.$$

Then the functions $h[t, x(t)]$ fulfil the Lipschitzian condition provided that $\beta + (m-1)q \geq 0$.

Proposition 5.2. Let $\lambda, \beta \in \mathbb{R}$ ($\lambda \neq 0$) and $m > 0$ ($m \neq 1$), $\gamma = (\beta + m\alpha) \setminus (m-1)$. Let G be the domain (37), where $q \in \mathbb{R}$ is such that $\beta + (m-1)q \geq 0$.

(i) Let $0 < \alpha < 1$, if either of the conditions (35) holds, then the Cauchy problem

$$\left({}^c \mathcal{D}_{a+}^\alpha x \right) (t) = \lambda \left(\ln \frac{t}{a} \right)^\beta [x(t)]^m \text{ and } x(a_+) = 0, \quad (40)$$

has a unique solution $x(t) \in C_{\delta, \gamma, \ln}^\alpha[a, b]$ and this solution is given by (30).

(ii) Let $n-1 < \alpha < n$ ($n \in \mathbb{N} \setminus \{1\}$), if either of the conditions (36) holds, then the problem

$$\left({}^c\mathcal{D}_{a+}^{\alpha}x\right)(t) = \lambda \left(\ln \frac{t}{a}\right)^{\beta} [x(t)]^m \text{ and } (\delta^k x)(a_+) = 0, \quad k = 0, \dots, n-1, \quad (41)$$

has a unique solution $x(t) \in C_{\delta, \gamma, \ln}^{\alpha, n-1}[a, b]$ and this solution is given by (30).

Remark 5.3. If $\beta = 0$, $0 < \operatorname{Re}(\alpha) < 1$ then the Lipschitz condition is violated in the domain (37). The Cauchy problem (41) admits of two continuous solutions $x = 0$ and

$$x(t) = \left[\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} \right]^{\frac{1}{(m-1)}} \left(\ln \frac{t}{a} \right)^{\gamma}, \quad \gamma = \frac{\alpha}{(1-m)}.$$

Example 5.4. Let us consider the following problem of order α ($\operatorname{Re}(\alpha) > 0$)

$$\left({}^c\mathcal{D}_{a+}^{\alpha}x\right)(t) = \lambda \left(\ln \frac{t}{a}\right)^{\beta} [x(t)]^m + c \left(\ln \frac{t}{a}\right)^{\nu}, \quad \lambda, c \in \mathbb{R} \quad (\lambda \neq 0) \text{ and } \nu, \beta \in \mathbb{R}. \quad (42)$$

Then it is verified that the equation (42) has the solution of the form

$$x(t) = \mu \left(\ln \frac{t}{a} \right)^{\gamma}, \quad \gamma = (\beta + \alpha) \setminus (1-m). \quad (43)$$

In this case the right-hand side of 42 takes the form

$$h[t, x(t)] = (\lambda + c) \left(\ln \frac{t}{a} \right)^{(\beta + \alpha m) \setminus (1-m)}. \quad (44)$$

Using the same arguments as in the proof of Proposition 5.2 we derive the uniqueness result for the Cauchy problem 42.

Proposition 5.5. Let $\lambda, \beta \in \mathbb{R}$ ($\lambda \neq 0$) and $m > 0$ ($m \neq 1$), $\gamma = (\beta + m\alpha) \setminus (m-1)$. Let G be the domain (37), where $q \in \mathbb{R}$ is such that $\beta + (m-1)q \geq 0$. Let $\nu = -\gamma$ and let the transcendental equation

$$\Gamma \left(\frac{\alpha + \beta}{1-m} + 1 - \alpha \right) [\lambda y^m + c] - \Gamma \left(\frac{\alpha + \beta}{1-m} + 1 \right) y = 0,$$

have a unique solution $y = \mu$.

(i) Let $0 < \alpha < 1$, if either of the conditions (35) holds, then the Cauchy problem

$$\left({}^c\mathcal{D}_{a+}^{\alpha}x\right)(t) = \lambda \left(\ln \frac{t}{a}\right)^{\beta} [x(t)]^m + c \left(\ln \frac{t}{a}\right)^{\nu}, \quad x(a_+) = 0,$$

has a unique solution $x(t) \in C_{\delta, \gamma, \ln}^{\alpha}[a, b]$ and this solution is given by (43).

(ii) Let $n-1 < \alpha < n$, if either of the conditions (36) holds, then the problem (42)-(29) has a unique solution $x(t) \in C_{\delta, \gamma, \ln}^{\alpha, n-1}[a, b]$ and this solution is given by (43).

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Multivalued Generalized Contractive Maps and Fixed Point Results

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Abstract

In this paper, we prove some fixed point results for generalized contractive multimaps with respect to generalized distance. Consequently, several known fixed point results either generalized or improved including the corresponding recent fixed point results of Ćirić, BinDehaish-Latif, Latif-Albar, Klim-Wardowski, Feng-Liu.

1 Introduction and Preliminaries

Let (X, d) be a metric space, 2^X a collection of nonempty subsets of X , and $CB(X)$ a collection of nonempty closed bounded subsets of X , $Cl(X)$ a collection of nonempty closed subsets of X , $K(X)$ a collection of nonempty compact subsets of X and H the Hausdorff metric induced by d . Then for any $A, B \in CB(X)$,

$$H(A, B) = \max\left\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\right\},$$

where $d(x, B) = \inf_{y \in B} d(x, y)$.

An element $x \in X$ is called a *fixed point* of a multivalued map $T : X \rightarrow 2^X$ if $x \in T(x)$. We denote $Fix(T) = \{x \in X : x \in T(x)\}$. A sequence $\{x_n\}$ in X is called an *orbit* of T at $x_0 \in X$ if $x_n \in T(x_{n-1})$ for all $n \geq 1$. A map $f : X \rightarrow \mathbb{R}$

0

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is called *T-orbitally lower semicontinuous* if for any orbit $\{x_n\}$ of T and $x \in X$, $x_n \rightarrow x$ imply that $f(x) \leq \liminf_{n \rightarrow \infty} f(x_n)$.

Using the concept of Hausdorff metric, Nadler [13] introduced a notion of multivalued contraction maps and proved a multivalued version of the well-known Banach contraction principle, which states that each closed bounded valued contraction map on a complete metric space has a fixed point. Since then various fixed point results concerning multivalued contractions have appeared. Feng and Liu [4] extended Nadler's fixed point theorem without using the concept of Hausdorff metric. While in [7] Klim and Wardowski generalized their result. Ćirić [3] obtained some interesting fixed point results which extend and generalize these cited results.

In [6], Kada et al. introduced the concept of w -distance on a metric space and studied the properties, examples and some classical results with respect to w -distance. Using this generalized distance, Suzuki and Takahashi [14] have introduced notions of single-valued and multivalued weakly contractive maps and proved fixed point results for such maps. Consequently, they generalized the Banach contraction principle and Nadler's fixed point result. Some other fixed point results concerning w -distance can be found in [8, 9, 10, 16, 18].

In [15], Suzuki generalized the concept of w -distance by introducing the notion of τ -distance on metric space (X, d) . In [15], Suzuki improved several classical results including the Caristi's fixed point theorem for single-valued maps with respect to τ -distance.

In the literature, several other kinds of distances and various versions of known results are appeared. Most recently, Ume [17] generalized the notion of τ -distance by introducing a concept of u -distance as follows:

A function $p : X \times X \rightarrow \mathbb{R}_+$ is called u -distance on X if there exists a function $\theta : X \times X \times \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that the following hold for each $x, y, z \in X$:

- (u₁) $p(x, z) \leq p(x, y) + p(y, z)$.
- (u₂) $\theta(x, y, 0, 0) = 0$ and $\theta(x, y, s, t) \geq \min\{s, t\}$ for each $s, t \in \mathbb{R}_+$,
and for every $\epsilon > 0$, there exists $\delta > 0$ such that $|s - s_0| < \delta$,
 $|t - t_0| < \delta$, $s, s_0, t, t_0 \in \mathbb{R}_+$ and $y \in X$ imply

$$|\theta(x, y, s, t) - \theta(x, y, s_0, t_0)| < \epsilon.$$

$$(u_3) \quad \lim_{n \rightarrow \infty} x_n = x$$

$$\lim_{n \rightarrow \infty} \sup \{ \theta(w_n, z_n, p(w_n, x_m), p(z_n, x_m)) : m \geq n \} = 0$$

imply

$$p(y, x) \leq \liminf_{n \rightarrow \infty} p(y, x_n)$$

$$(u_4)$$

$$\lim_{n \rightarrow \infty} \sup \{ p(x_n, w_m) : m \geq n \} = 0,$$

$$\lim_{n \rightarrow \infty} \sup \{ p(y_n, z_m) : m \geq n \} = 0,$$

$$\lim_{n \rightarrow \infty} \theta(x_n, w_n, s_n, t_n) = 0,$$

$$\lim_{n \rightarrow \infty} \theta(y_n, z_n, s_n, t_n) = 0$$

imply

$$\lim_{n \rightarrow \infty} \theta(w_n, z_n, s_n, t_n) = 0$$

or

$$\lim_{n \rightarrow \infty} \sup \{ p(w_n, x_m) : m \geq n \} = 0,$$

$$\lim_{n \rightarrow \infty} \sup \{ p(z_m, y_n) : m \geq n \} = 0,$$

$$\lim_{n \rightarrow \infty} \theta(x_n, w_n, s_n, t_n) = 0,$$

$$\lim_{n \rightarrow \infty} \theta(y_n, z_n, s_n, t_n) = 0$$

imply

$$\lim_{n \rightarrow \infty} \theta(w_n, z_n, s_n, t_n) = 0;$$

$$(u_5)$$

$$\lim_{n \rightarrow \infty} \theta(w_n, z_n, p(w_n, x_n), p(z_n, x_n)) = 0,$$

$$\lim_{n \rightarrow \infty} \theta(w_n, z_n, p(w_n, y_n), p(z_n, y_n)) = 0$$

imply

$$\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$$

or

$$\lim_{n \rightarrow \infty} \theta(a_n, b_n, p(x_n, a_n), p(x_n, b_n)) = 0,$$

$$\lim_{n \rightarrow \infty} \theta(a_n, b_n, p(y_n, a_n), p(y_n, b_n)) = 0$$

imply

$$\lim_{n \rightarrow \infty} d(x_n, y_n) = 0.$$

Remark 1.1 [17] (a) Suppose that θ from $X \times X \times \mathbb{R}_+ \times \mathbb{R}_+$ into \mathbb{R}_+ is a mapping satisfying (u2) to (u5). Then there exists a mapping η from $X \times X \times \mathbb{R}_+ \times \mathbb{R}_+$ into \mathbb{R}_+ such that η is nondecreasing in its third and fourth variable, respectively, satisfying (u2) η to (u5) η , where (u2) η to (u5) η stand for substituting η for θ in (u2) to (u5), respectively.

(b) In the light of (a), we may assume that θ is nondecreasing in its third and fourth variables, respectively, for a function θ from $X \times X \times \mathbb{R}_+ \times \mathbb{R}_+$ into \mathbb{R}_+ satisfying (u2) to (u5).

(c) Each τ -distance p on a metric space (X, d) is also a u -distance on X .

Here we present some examples of u -distance which are not τ -distance. (For the detail, see [17]).

Example 1.2. Let $X = \mathbb{R}_+$ with the usual metric. Define $p : X \times X \rightarrow \mathbb{R}_+$ by $p(x, y) = (\frac{1}{4})x^2$. Then p is a u -distance on X but not a τ distance on X .

Example 1.3. Let X be a normed space with norm $\|\cdot\|$. Then a function $p : X \times X \rightarrow \mathbb{R}_+$ defined by $p(x, y) = \|x\|$ for every $x, y \in X$ is a u -distance on X but not a τ -distance.

It follows from the above examples and Remark 1.1(c) that u -distance is a proper extension of τ -distance. Other useful examples on u -distance are also given in [17].

Let (X, d) be a metric space and let p be a u -distance on X . A sequence $\{x_n\}$ in X is called p -Cauchy [17] if there exists a function θ from $X \times X \times \mathbb{R}_+ \times \mathbb{R}_+$ into \mathbb{R}_+ satisfying (u2)~(u5) and a sequence $\{z_n\}$ of X such that

$$\lim_{n \rightarrow \infty} \sup\{\theta(z_n, z_n, p(z_n, x_m), p(z_n, x_m)) : m \geq n\} = 0,$$

or

$$\lim_{n \rightarrow \infty} \sup \{ \theta(z_n, z_n, p(x_m, z_n), p(x_m, z_n)) : m \geq n \} = 0.$$

The following lemmas concerning u -distance are crucial for the proofs of our results.

Lemma 1.4 [17] *Let (X, d) be a metric space and let p be a u -distance on X . If $\{x_n\}$ is a p -Cauchy sequence in X , then $\{x_n\}$ is a Cauchy sequence.*

Lemma 1.5 [5] *Let (X, d) be a metric space and let p be a u -distance on X . If $\{x_n\}$ is a p -Cauchy sequence and $\{y_n\}$ is a sequence satisfying*

$$\lim_{n \rightarrow \infty} \sup \{ p(x_n, y_m) : m \geq n \} = 0,$$

then $\{y_n\}$ is also a p -Cauchy sequence and $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$.

Lemma 1.6 [17] *Let (X, d) be a metric space and let p be a u -distance on X . Suppose that a sequence $\{x_n\}$ of X satisfies*

$$\lim_{n \rightarrow \infty} \sup \{ p(x_n, x_m) : m > n \} = 0,$$

or

$$\lim_{n \rightarrow \infty} \sup \{ p(x_m, x_n) : m > n \} = 0.$$

Then $\{x_n\}$ is a p -Cauchy sequence.

The aim of this paper is to present some more general fixed point results with respect to u -distance for multivalued maps satisfying certain conditions. Our results unify and generalize the corresponding results of Mizoguchi and Takahashi [12], Klim and Wardowski [7], Latif and Abdou [10], BinDehaish and Latif [2], Ćirić [3], Feng and Liu [4], and several others.

2 The Results

Using the u -distance, we prove a general result on the existence of fixed points for multivalued maps.

Theorem 2.1 *Let (X, d) be a complete metric space. Let $T : X \rightarrow Cl(X)$ be a multivalued map and let $\varphi : [0, \infty) \rightarrow [0, 1)$ be such that $\limsup_{r \rightarrow t^+} \varphi(r) < 1$*

for each $t \in [0, \infty)$. Let p be a u -distance on X and assume that the following conditions hold:

(I) for any $x \in X$, there exists $y \in T(x)$ satisfying

$$p(x, y) \leq (2 - \varphi(p(x, y)))p(x, T(x)),$$

and

$$p(y, T(y)) \leq \varphi(p(x, y))p(x, y)$$

(II) the map $f : X \rightarrow \mathbb{R}$, defined by $f(x) = p(x, T(x))$ is T -orbitally lower semicontinuous.

Then there exists $v_0 \in X$ such that $f(v_0) = 0$. Further if $p(v_0, v_0) = 0$, then $v_0 \in T(v_0)$.

Proof. let $x_0 \in X$ be an arbitrary but fixed element in X . Then there exists $x_1 \in T(x_0)$ such that

$$p(x_0, x_1) \leq (2 - \varphi(p(x_0, x_1)))p(x_0, T(x_0)), \quad (1)$$

and

$$p(x_1, T(x_1)) \leq \varphi(p(x_0, x_1))p(x_0, x_1). \quad (2)$$

From (1) and (2), we get

$$p(x_1, T(x_1)) \leq \varphi(p(x_0, x_1))(2 - \varphi(p(x_0, x_1)))p(x_0, T(x_0)). \quad (3)$$

Define a function $\psi : [0, \infty) \rightarrow [0, \infty)$ by

$$\psi(t) = \varphi(t)(2 - \varphi(t)) = 1 - (1 - \varphi(t))^2. \quad (4)$$

Using the facts that for each $t \in [0, \infty)$, $\varphi(t) < 1$ and $\lim_{r \rightarrow t^+} \sup \varphi(r) < 1$, we have

$$\psi(t) < 1 \quad (5)$$

and

$$\limsup_{r \rightarrow t^+} \psi(r) < 1, \quad \text{for all } t \in [0, \infty) \quad (6)$$

From (3) and (4), we have

$$p(x_1, T(x_1)) \leq \psi(p(x_0, x_1))p(x_0, T(x_0)). \quad (7)$$

Similarly, for $x_1 \in X$, there exists $x_2 \in T(x_1)$ such that

$$p(x_1, x_2) \leq (2 - \varphi(p(x_1, x_2)))p(x_1, T(x_1)),$$

and

$$p(x_2, T(x_2)) \leq \varphi(p(x_1, x_2))p(x_1, x_2).$$

Thus

$$p(x_2, T(x_2)) \leq \psi(p(x_1, x_2))p(x_1, T(x_1)).$$

Continuing this process we can get an orbit $\{x_n\}$ of T in X satisfying the following

$$p(x_n, x_{n+1}) \leq (2 - \varphi(p(x_n, x_{n+1})))p(x_n, T(x_n)) \quad (8)$$

and

$$p(x_{n+1}, T(x_{n+1})) \leq \psi(p(x_n, x_{n+1}))p(x_n, T(x_n)), \quad (9)$$

for each integer $n \geq 0$. Since $\psi(t) < 1$ for each $t \in [0, \infty)$ and from (9), we have for all $n \geq 0$

$$p(x_{n+1}, T(x_{n+1})) < p(x_n, T(x_n)). \quad (10)$$

Thus the sequence of non-negative real numbers $\{p(x_n, T(x_n))\}$ is decreasing and bounded below, thus convergent. Therefore, there is some $\delta \geq 0$ such that

$$\lim_{n \rightarrow \infty} p(x_n, T(x_n)) = \delta. \quad (11)$$

From (8), as $\varphi(t) < 1$ for all $t \geq 0$, we get

$$p(x_n, T(x_n)) \leq p(x_n, x_{n+1}) < 2p(x_n, T(x_n)), \quad (12)$$

Thus, we conclude that the sequence of non-negative reals $\{p(x_n, x_{n+1})\}$ is bounded. Therefore, there is some $\theta \geq 0$ such that

$$\liminf_{n \rightarrow \infty} p(x_n, x_{n+1}) = \theta. \quad (13)$$

Note that $p(x_n, x_{n+1}) \geq p(x_n, T(x_n))$ for each $n \geq 0$, so we have $\theta \geq \delta$. Now we shall show that $\theta = \delta$. If $\delta = 0$. Then we get

$$\lim_{n \rightarrow \infty} p(x_n, x_{n+1}) = 0.$$

Now consider $\delta > 0$. Suppose to the contrary, that $\theta > \delta$. Then $\theta - \delta > 0$ and so from (11) and (13) there is a positive integer n_0 such that

$$p(x_n, T(x_n)) < \delta + \frac{\theta - \delta}{4} \quad \text{for all } n \geq n_0 \quad (14)$$

and

$$\theta - \frac{\theta - \delta}{4} < p(x_n, x_{n+1}) \quad \text{for all } n \geq n_0 \quad (15)$$

Then from (15), (8) and (14), we get

$$\begin{aligned} \theta - \frac{\theta - \delta}{4} &< p(x_n, x_{n+1}) \\ &\leq (2 - \varphi(p(x_n, x_{n+1})))p(x_n, T(x_n)) \\ &< (2 - \varphi(p(x_n, x_{n+1}))) \left[\delta + \frac{\theta - \delta}{4} \right]. \end{aligned}$$

Thus for all $n \geq n_0$,

$$(2 - \varphi(p(x_n, x_{n+1}))) > \frac{3\theta + \delta}{3\delta + \theta},$$

that is;

$$1 + (1 - \varphi(p(x_n, x_{n+1}))) > 1 + \frac{2(\theta - \delta)}{3\delta + \theta},$$

and we get

$$-(1 - \varphi(p(x_n, x_{n+1})))^2 < -\left[\frac{2(\theta - \delta)}{3\delta + \theta} \right]^2.$$

Thus for all $n \geq n_0$,

$$\begin{aligned} \psi(p(x_n, x_{n+1})) &= 1 - (1 - \varphi(p(x_n, x_{n+1})))^2 \\ &< 1 - \left[\frac{2(\theta - \delta)}{3\delta + \theta} \right]^2. \end{aligned} \quad (16)$$

Thus, from (9) and (16), we get

$$p(x_{n+1}, T(x_{n+1})) \leq h p(x_n, T(x_n)) \quad \text{for all } n \geq n_0, \quad (17)$$

where $h = 1 - \left[\frac{2(\theta - \delta)}{3\delta + \theta} \right]^2$. Clearly $h < 1$ as $\theta > \delta$. From (14) and (17), we have for any $k \geq 1$

$$p(x_{n_0+k}, T(x_{n_0+k})) \leq h^k p(x_{n_0}, T(x_{n_0})). \quad (18)$$

Since $\delta > 0$ and $h < 1$, there is a positive integer k_0 such that $h^{k_0}p(x_{n_0}, T(x_{n_0})) < \delta$. Now, since $\delta \leq p(x_n, T(x_n))$ for each $n \geq 0$, by (18) we have

$$\delta \leq p(x_{n_0+k_0}, T(x_{n_0+k_0})) \leq h^{k_0}p(x_{n_0}, T(x_{n_0})) < \delta.$$

a contradiction. Hence, our assumption $\theta > \delta$ is wrong. Thus $\delta = \theta$. Now we show that $\theta = 0$. Since $\theta = \delta \leq p(x_n, T(x_n)) \leq p(x_n, x_{n+1})$, then from (13) we can read as

$$\liminf_{n \rightarrow \infty} p(x_n, x_{n+1}) = \theta +,$$

so, there exists a subsequence $\{p(x_{n_k}, x_{n_k+1})\}$ of $\{p(x_n, x_{n+1})\}$ such that

$$\lim_{k \rightarrow \infty} p(x_{n_k}, x_{n_k+1}) = \theta +.$$

Now from (6) we have

$$\limsup_{p(x_{n_k}, x_{n_k+1}) \rightarrow \theta+} \psi(p(x_{n_k}, x_{n_k+1})) < 1, \quad (19)$$

and from (9), we have

$$p(x_{n_k}, T(x_{n_k+1})) \leq \psi(p(x_{n_k}, x_{n_k+1}))p(x_{n_k}, T(x_{n_k})).$$

Taking the limit as $k \rightarrow \infty$ and using (11), we get

$$\begin{aligned} \delta &= \limsup_{k \rightarrow \infty} p(x_{n_k+1}, T(x_{n_k+1})) \\ &\leq (\limsup_{k \rightarrow \infty} \psi(p(x_{n_k}, x_{n_k+1}))) (\limsup_{k \rightarrow \infty} p(x_{n_k}, T(x_{n_k}))) \\ &= (\limsup_{p(x_{n_k}, x_{n_k+1}) \rightarrow \theta+} \psi(p(x_{n_k}, x_{n_k+1}))) \delta. \end{aligned}$$

If we suppose that $\delta > 0$, then from last inequality, we have

$$\limsup_{p(x_{n_k}, x_{n_k+1}) \rightarrow \theta+} \psi(p(x_{n_k}, x_{n_k+1})) \geq 1,$$

which contradicts with (19). Thus $\delta = 0$. Then from (11) and (12), we have

$$\lim_{n \rightarrow \infty} p(x_n, T(x_n)) = 0+, \quad (20)$$

and thus

$$\lim_{n \rightarrow \infty} p(x_n, x_{n+1}) = 0+. \quad (21)$$

Now, let

$$\alpha = \lim_{p(x_{n_k}, x_{n_k+1}) \rightarrow 0^+} \sup \psi(p(x_{n_k}, x_{n_k+1})).$$

Then by (6), $\alpha < 1$. Let q be such that $\alpha < q < 1$. Then there is some $n_0 \in \mathbb{N}$ such that

$$\psi(p(x_n, x_{n+1})) < q, \quad \text{for all } n \geq n_0.$$

Thus it follows from (9)

$$p(x_{n+1}, T(x_{n+1})) \leq qp(x_n, T(x_n)) \quad \text{for all } n \geq n_0.$$

By induction we get

$$p(x_{n+1}, T(x_{n+1})) \leq q^{n+1-n_0} p(x_{n_0}, T(x_{n_0})) \quad \text{for all } n \geq n_0. \quad (22)$$

Now, using (12) and (22), we have

$$p(x_n, x_{n+1}) \leq 2q^{n-n_0} p(x_{n_0}, T(x_{n_0})) \quad \text{for all } n \geq n_0. \quad (23)$$

Now, we show that $\{x_n\}$ is a Cauchy sequence, for all $m \geq n \geq n_0$, we get

$$\begin{aligned} p(x_n, x_m) &\leq \sum_{k=n}^{m-1} p(x_k, x_{k+1}) \\ &\leq 2 \sum_{k=n}^{m-1} q^{k-n_0} p(x_{n_0}, T(x_{n_0})) \\ &\leq 2 \left(\frac{q^{n-n_0}}{1-q} \right) p(x_{n_0}, T(x_{n_0})). \end{aligned} \quad (24)$$

and hence

$$\lim_{n \rightarrow \infty} \sup \{p(x_n, x_m) : m \geq n\} = 0.$$

Thus, by Lemma 1.6, $\{x_n\}$ is a p -Cauchy sequence and hence by Lemma 1.4, $\{x_n\}$ is a Cauchy sequence. Due to the completeness of X , there exists some $v_0 \in X$ such that $\lim_{n \rightarrow \infty} x_n = v_0$. Since f is T -orbitally lower semicontinuous and from (20), we have

$$0 \leq f(v_0) \leq \liminf_{n \rightarrow \infty} f(x_n) = \liminf_{n \rightarrow \infty} p(x_n, T(x_n)) = 0,$$

and thus, $f(v_0) = p(v_0, T(v_0)) = 0$. Thus there exists a sequence $\{v_n\} \subset T(v_0)$ such that $\lim_{n \rightarrow \infty} p(v_0, v_n) = 0$. It follows that

$$0 \leq \lim_{n \rightarrow \infty} \sup \{p(x_n, v_m) : m \geq n\} \leq \lim_{n \rightarrow \infty} \sup \{p(x_n, v_0) + p(v_0, v_m) : m \geq n\} = 0. \quad (25)$$

Since $\{x_n\}$ is a p -Cauchy sequence, thus it follows from (25) and Lemma 1.5 that $\{v_n\}$ is also a p -Cauchy sequence and $\lim_{n \rightarrow \infty} d(x_n, v_n) = 0$. Thus, by Lemma 1.4, $\{v_n\}$ is a Cauchy sequence in the complete space. Due to closedness of $T(v_0)$, there exists $z_0 \in X$ such that $\lim_{n \rightarrow \infty} v_n = z_0 \in T(v_0)$. Consequently, using (u_3) we get

$$p(v_0, z_0) \leq \liminf_{n \rightarrow \infty} p(v_0, v_n) = 0,$$

and thus $p(v_0, z_0) = 0$. But, since $\lim_{n \rightarrow \infty} x_n = v_0$, $\lim_{n \rightarrow \infty} v_n = z_0$ and $\lim_{n \rightarrow \infty} d(x_n, v_n) = 0$, we have $v_0 = z_0$. Hence $v_0 \in \text{Fix}(T)$ and $p(v_0, v_0) = 0$.

Remarks 2.2 Theorem 2.1 generalizes fixed point theorems of Latif and Abdou [10, Theorem 2.1], Ćirić [3, Theorem 5], Bin Dehaish and Latif [2, Theorem 2.2], Latif and Abdou [8, Theorem 2.2], Suzuki [15, Theorem 2], Bin Dehaish and Latif [1, Theorem 2.2], Suzuki and Takahashi [14, Theorem 1], Klim and Wardowski [7, Theorem 2.1] and Feng and Liu [4, Theorem 3.1] which contains Nadler's fixed point theorem.

We also have the following interesting result by replacing the hypothesis (II) of Theorem 2.1 with another suitable condition.

Theorem 2.3 Suppose that all the hypotheses of Theorem 2.1 except (II) hold. Assume that

$$\inf\{p(x, v) + p(x, T(x)) : x \in X\} > 0,$$

for every $v \in X$ with $v \notin T(v)$. Then $\text{Fix}(T) \neq \emptyset$.

Proof. Following the proof of Theorem 2.1, there exists there exists an orbit $\{x_n\}$ of T , which is Cauchy sequence in a complete metric space X . Thus, there exists $v_0 \in X$ such that $\lim_{n \rightarrow \infty} x_n = v_0$. Thus, using (u_3) and (24) we have for all $n \geq n_0$

$$p(x_n, v_0) \leq \liminf_{m \rightarrow \infty} p(x_n, x_m) \leq \left(\frac{2q^{n-n_0}}{1-q}\right)p(x_{n_0}, T(x_{n_0})),$$

and

$$p(x_n, T(x_n)) \leq p(x_n, x_{n+1}) \leq 2q^{n-n_0}p(x_{n_0}, T(x_{n_0})).$$

Assume that $v_0 \notin T(v_0)$. Then, we have

$$0 < \inf\{p(x, v_0) + p(x, T(x)) : x \in X\}$$

$$\begin{aligned}
&\leq \inf\{p(x_n, v_0) + p(x_n, T(x_n)) : n \geq n_0\} \\
&\leq \inf\left\{\left(\frac{2q^{n-n_0}}{1-q}\right)p(x_{n_0}, T(x_{n_0})) + 2q^{n-n_0}p(x_{n_0}, T(x_{n_0})) : n \geq n_0\right\} \\
&= \frac{2(2-q)}{(1-q)q^{n_0}} p(x_{n_0}, T(x_{n_0})) \inf\{q^n : n \geq n_0\} = 0,
\end{aligned}$$

which is impossible and hence $v_0 \in Fix(T)$.

Remarks 2.4 Theorem 2.3 generalizes [8, Theorem 2.4], [10, Theorem 3.3] and [2, Theorem 2.5].

Competing interests

The author declares that he has no competing interests.

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ESSENTIAL COMMUTATIVITY AND ISOMETRY OF COMPOSITION OPERATOR AND DIFFERENTIATION OPERATOR

GENG-LEI LI

ABSTRACT. In this paper, we characterize the essential commutativity and isometry of composition operator and differentiation operator from the Bloch type space to space of all weighted bounded analytic functions in the disk.

1. INTRODUCTION

Let \mathbb{D} be the unit disk of the complex plane, and $S(\mathbb{D})$ be the set of analytic self-maps of \mathbb{D} . The algebra of all holomorphic functions with domain \mathbb{D} will be denoted by $H(\mathbb{D})$.

A positive continuous function v on $[0, 1)$ is called normal (see, e.g., [17]), if there exist three constants $0 \leq \delta < 1$, and $0 < a < b < \infty$, such that for $r \in [\delta, 1)$

$$\frac{v(r)}{(1-r)^a} \downarrow 0, \quad \frac{v(r)}{(1-r)^b} \uparrow \infty$$

as $r \rightarrow 1$.

Assume v is normal, the weighted-type space H_v^∞ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H_v^\infty} = \sup_{z \in \mathbb{D}} v(z)|f(z)| < \infty.$$

When $v(z) = 1$, we know that $H_v^\infty = H^\infty$, that is

$$H^\infty = \{f \in H(\mathbb{D}), \sup_{z \in \mathbb{D}} |f(z)| < \infty\}.$$

We recall that the Bloch type space \mathcal{B}^α ($\alpha > 0$) consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}^\alpha} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty,$$

then $\|\cdot\|_{\mathcal{B}^\alpha}$ is a complete semi-norm on \mathcal{B}^α , which is Möbius invariant.

It is well known that \mathcal{B}^α is a Banach space under the norm

$$\|f\| = |f(0)| + \|f\|_{\mathcal{B}^\alpha}.$$

Let φ be an analytic self-map of \mathbb{D} , the composition operator C_φ induced by φ is defined by

$$(C_\varphi f)(z) = f(\varphi(z))$$

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for $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$.

Let D be the differentiation operator on $H(\mathbb{D})$, that is $Df(z) = f'(z)$. For $f \in H(\mathbb{D})$, the products of composition and differentiation operators DC_φ and $C_\varphi D$ are defined by

$$\begin{aligned} C_\varphi D(f) &= f'(\varphi) \\ DC_\varphi(f) &= (f \circ \varphi)' = f'(\varphi)\varphi' \end{aligned}$$

The boundedness and compactness of DC_φ on the Hardy space were discussed by Hirschweiler and Portnoy in [7] and by Ohno in [14].

We write T_φ for the operators $DC_\varphi - C_\varphi D$, which is from the Bloch type space \mathcal{B}^α to H_v^∞ . Generally speaking, it is clear that $DC_\varphi \neq C_\varphi D$, but it is interesting to study when

$$DC_\varphi(\mathcal{B}^\alpha \rightarrow H_v^\infty) \equiv C_\varphi D(\mathcal{B}^\alpha \rightarrow H_v^\infty), \text{mod } K$$

where K denotes the collection of all compact operators from Bloch type space \mathcal{B}^α to H_v^∞ . If the upper properties is satisfied, we say they are essential commutative.

In the past few decades, boundedness, compactness, isometries and essential norms of composition and closely related operators between various spaces of holomorphic functions have been studied by many authors, see, e.g., [1, 3, 5, 9, 12, 15, 16, 21, 22]; the results about difference and other properties can be seen [?, 4, 6, 10, 11, 13, 18, 20] and the related references therein. Recently, many interests focused on studying the essential commutativity of various different composition operators.

In [23], Zhou and Zhang studied the essential commutativity of the integral operators and composition operators from a mixed-norm space to Bloch type space. In [19, ?], Tong and Zhou characterized the intertwining relations for Volterra operators on the Bergman space, and compact intertwining relations for composition operators between the weighted Bergman spaces and the weighted Bloch spaces, respectively.

The paper continues this line of research, and discusses the essential commutativity of composition operator and differentiation operator from the Bloch type space to the space of all weighted bounded analytic functions in the disk.

2. NOTATION AND LEMMAS

To begin the discussion, let us introduce some notations and state a couple of lemmas.

For $a \in \mathbb{D}$, the involution φ_a which interchanges the origin and point a , is defined by

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z}.$$

For z, w in \mathbb{D} , the pseudo-hyperbolic distance between z and w is given by

$$\rho(z, w) = |\varphi_z(w)| = \left| \frac{z - w}{1 - \bar{z}w} \right|,$$

and the hyperbolic metric is given by

$$\beta(z, w) = \inf_{\gamma} \int_{\gamma} \frac{|d\xi|}{1 - |\xi|^2} = \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)},$$

where γ is any piecewise smooth curve in \mathbb{D} from z to w .

The following lemma is well known [24].

Lemma 1. *For all $z, w \in \mathbb{D}$, we have*

$$1 - \rho^2(z, w) = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \bar{z}w|^2}.$$

For $\varphi \in S(D)$, the Schwarz-Pick lemma shows that $\rho(\varphi(z), \varphi(w)) \leq \rho(z, w)$, and if equality holds for some $z \neq w$, then φ is an automorphism of the disk. It is also well known that for $\varphi \in S(\mathbb{D})$, C_φ is always bounded on \mathcal{B} .

Lemma 2. [8, Lemma 3] *Assume that $f \in H^\infty(\mathbb{D})$, then for each $n \in \mathbb{N}$, there is a positive constant C independent of f such that*

$$\sup_{z \in D} (1 - |z|)^n |f^{(n)}(z)| < C \|f\|_\infty.$$

A little modification of Lemma 1 in [?] shows the following lemma.

Lemma 3. *There exists a constant $C > 0$ such that*

$$\left| \left(1 - |z|^2\right)^\alpha f'(z) - \left(1 - |w|^2\right)^\alpha f'(w) \right| \leq C \|f\|_{\mathcal{B}^\alpha} \cdot \rho(z, w)$$

for all $z, w \in \mathbb{D}$ and $f \in \mathcal{B}^\alpha$.

The following lemma is an easy modification of Proposition 3.11 in [2].

Lemma 4. *Let $0 < \alpha < \infty$, $g \in H(\mathbb{B})$ and φ be a holomorphic self-map of \mathbb{B} . Then $P_\varphi^g : H^\infty \rightarrow \mathcal{B}^\alpha$ is compact if and only if $P_\varphi^g : H^\infty \rightarrow \mathcal{B}^\alpha$ is bounded and for any bounded sequence $(f_k)_{k \in \mathbb{N}}$ in H^∞ which converges to zero uniformly on \mathbb{B} as $k \rightarrow \infty$, we have $\|(P_{\varphi_1}^{g_1} - P_{\varphi_2}^{g_2})f_k\|_{\mathcal{B}^\alpha} \rightarrow 0$ as $k \rightarrow \infty$.*

Throughout the remainder of this paper, C will denote a positive constant, the exact value of which will vary from one appearance to the next.

3. MAIN THEOREMS

Theorem 1. *Let $0 < \alpha < \infty$ and φ be a analytic self map of the unit disk. Then $T_\varphi = DC_\varphi - C_\varphi D$ is a bounded operator from \mathcal{B}^α to H_v^∞ if and only if*

$$\sup_{z \in D} \frac{v(z) |\varphi'(z) - 1|}{(1 - |\varphi(z)|^2)^\alpha} < \infty. \quad (1)$$

Proof. We prove the sufficiency first.

Assume that (1) is true, for every $f \in \mathcal{B}^\alpha$, we have

$$\begin{aligned} \|T_\varphi f\|_{H_v^\infty} &= \sup_{z \in D} v(z) |f'(\varphi(z))\varphi'(z) - f'(\varphi(z))| \\ &= \sup_{z \in \mathbb{D}} \frac{v(z) |\varphi'(z) - 1|}{(1 - |\varphi(z)|^2)^\alpha} (1 - |\varphi(z)|^2)^\alpha |f'(\varphi(z))| \\ &\leq C \|f\|_{\mathcal{B}^\alpha}. \end{aligned}$$

This means that $T_\varphi = DC_\varphi - C_\varphi D$ is a bounded operator from \mathcal{B}^α to H_v^∞ .

Now we turn to the necessity.

Suppose that $T_\varphi : \mathcal{B}^\alpha \rightarrow H_v^\infty$ is a bounded operator, that is, there exists a constant C such that $\|T_\varphi f\|_{H_v^\infty} \leq C \|f\|_{\mathcal{B}^\alpha}$, for any $f \in \mathcal{B}^\alpha$.

For any $a \in \mathbb{D}$, we begin by taking test function

$$f_a(z) = \int_0^z \frac{(1-|a|^2)^\alpha}{(1-\bar{a}t)^{2\alpha}} dt.$$

It is clear that $f'_a(z) = \frac{(1-|a|^2)^\alpha}{(1-\bar{a}z)^{2\alpha}}$. Using Lemma 1, we have

$$(1-|z|^2)^\alpha |f'_a(z)| = \frac{(1-|z|^2)^\alpha (1-|a|^2)^\alpha}{|1-\bar{a}z|^{2\alpha}} = (1-\rho^2(a, z))^\alpha.$$

So

$$\|f_a\|_{\mathcal{B}^\alpha} = \sup_{z \in \mathbb{D}} (1-|z|^2)^\alpha |f'_a(z)| \leq 1.$$

that is $f_a(z) \in \mathcal{B}^\alpha$.

Therefore

$$\begin{aligned} \infty &> C \|T_\varphi\|_{\mathcal{B}^\alpha \rightarrow H_v^\infty} > \|T_\varphi f_{\varphi(a)}\|_{H_v^\infty} \\ &= \sup_{z \in \mathbb{D}} \frac{v(z) |\varphi'(z) - 1|}{(1-|\varphi(z)|^2)^\alpha} (1-|\varphi(z)|^2)^\alpha \left| f'_{\varphi(a)}(\varphi(z)) \right| \\ &\geq \frac{v(z) |\varphi'(a) - 1|}{(1-|\varphi(a)|^2)^\alpha}. \end{aligned}$$

So (1) follows by noticing a is arbitrary.

This completes the proof of this theorem. \square

Theorem 2. Let $0 < \alpha < \infty$ and φ be a analytic self map of the unit disk. Then $T_\varphi = DC_\varphi - C_\varphi D$ is operator from \mathcal{B}^α to H_v^∞ . Then C_φ and D are essential commutative if and only if T_φ is bounded and

$$\lim_{|\varphi(z)| \rightarrow 1} \frac{v(z) |\varphi'(z) - 1|}{(1-|\varphi(z)|^2)^\alpha} = 0. \quad (2)$$

Proof. We prove the sufficiency first.

Assume that T_φ is bounded and condition (2) holds. By the Theorem 1, we have

$$\sup_{z \in \mathbb{D}} v(z) |\varphi'(z) - 1| < \infty \quad (3)$$

for any $z \in \mathbb{D}$.

Let $\{f_k\}_{k \in \mathbb{N}}$ be a arbitrary sequence in \mathcal{B}^α which converges to zero uniformly on compact subset of \mathbb{D} as $k \rightarrow \infty$, and its norm $\|f_k\|_{\mathcal{B}^\alpha} \leq C$.

Then, it follows from (2) that for any $\varepsilon > 0$, there is a $\delta > 0$, with $\delta < |\varphi(z)| < 1$, such that

$$\sup_{z \in \mathbb{D}} \frac{v(z) |\varphi'(z) - 1|}{(1-|\varphi(z)|^2)^\alpha} < \frac{\varepsilon}{C}. \quad (4)$$

Let $A = \{z \in \mathbb{D} : |\varphi(z)| \leq \delta\}$ and $B = \{w : |w| \leq \delta\}$, then B is a compact subset of \mathbb{D} .

The boundedness of T_φ implies (1) is true by the Theorem 1. Combining (3) and (4), it follows from Lemma 2 that

$$\begin{aligned} \|T_\varphi f_k\|_{H_v^\infty} &= \sup_{z \in \mathbb{D}} v(z) |f'_k(\varphi(z))\varphi'(z) - f'_k(\varphi(z))| \\ &= \sup_{z \in \mathbb{D}} \frac{v(z) |\varphi'(z) - 1|}{(1 - |\varphi(z)|^2)^\alpha} (1 - |\varphi(z)|^2)^\alpha |f'_k(\varphi(z))| \\ &\leq \sup_{z \in A} v(z) |\varphi'(z) - 1| |f'_k(\varphi(z))| \\ &\quad + \sup_{z \in \mathbb{D} \setminus A} \frac{v(z) |\varphi'(z) - 1|}{(1 - |\varphi(z)|^2)^\alpha} (1 - |\varphi(z)|^2)^\alpha |f'_k(\varphi(z))| \\ &\leq C \sup_{w \in B} |f_k(w)| + \varepsilon. \end{aligned}$$

As we assume that $f_k \rightarrow 0$ on compact subset of \mathbb{D} as $k \rightarrow \infty$, and ε is an arbitrary positive number. Letting $k \rightarrow \infty$, we have $\|T_\varphi f_k\|_{H_v^\infty} \rightarrow 0$. Therefore, the operator T_φ is a compact operator by Lemma 3, so the operators C_φ and D are essentially commutative.

Now we turn to the necessity.

Assume that C_φ and D are essentially commutative. Then $T_\varphi = DC_\varphi - C_\varphi D$ is obvious bounded since it is a compact operator.

Nest, let $\{z_k\}_{k \in \mathbb{N}}$ is a arbitrary sequence in \mathbb{D} such that $|\varphi(z_k)| \rightarrow 1$ as $k \rightarrow \infty$. we will show (2) holds.

For any z_k , we begin by taking test function

$$f_k(z) = \int_0^z \frac{(1 - |\varphi(z_k)|^2)^\alpha}{(1 - \varphi(z_k)t)^{2\alpha}} dt.$$

It is clear that $f'_k(z) = \frac{(1 - |\varphi(z_k)|^2)^\alpha}{(1 - \varphi(z_k)z)^{2\alpha}}$. Using Lemma 1, we have

$$(1 - |z|^2)^\alpha |f'_k(z)| = \frac{(1 - |z|^2)^\alpha (1 - |\varphi(z_k)|^2)^\alpha}{|1 - \varphi(z_k)z|^{2\alpha}} = (1 - \rho^2(\varphi(z_k), z))^\alpha.$$

So

$$\|f_k\|_{\mathcal{B}^\alpha} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'_k(z)| \leq 1.$$

that is $f_k(z) \in \mathcal{B}^\alpha$, and the sequence $\{f_k\}$ converges to 0 uniformly on compact subset of \mathbb{D} as $k \rightarrow \infty$. As the operator $T_\varphi = DC_\varphi - C_\varphi D$ is a compact operator, it follows from Lemma 3 that

$$\lim_{k \rightarrow \infty} \|T_\varphi f_k\|_{H_v^\infty} = 0. \quad (5)$$

So, we have

$$\begin{aligned} \|T_\varphi f_k\|_{H_v^\infty} &= \sup_{z \in \mathbb{D}} (v(z) |f'_k(\varphi(z))\varphi'(z) - f'_k(\varphi(z))|) \\ &\geq v(z_k) |f'_k(\varphi(z_k))\varphi'(z_k) - f'_k(\varphi(z_k))| \\ &= v(z_k) |\varphi'(z_k) - 1| |f'_k(\varphi(z_k))| \\ &= v(z_k) |\varphi'(z_k) - 1| \frac{1}{(1 - |\varphi(z_k)|^2)^\alpha} \end{aligned}$$

So, the condition (2) is followed by combining (5) and the above result.

This completes the proof of this theorem. \square

Remark If $\alpha = 1$, $v(z) = 1$ then the space \mathcal{B}^α and H_v^∞ will be Bloch space \mathcal{B} and H^∞ . The similar results from Bloch space \mathcal{B} to the H^∞ corresponding to Theorems 1 and 2 also hold.

In the next, we study the isometry of the operator $T_\varphi = DC_\varphi - C_\varphi$, which is from \mathcal{B}^α to space H_{β}^∞ , and give the following theorem.

Theorem 3. Let $0 < \alpha < \infty$ and φ be a analytic self maps of the unit disk. Then the operator $T_\varphi = DC_\varphi - C_\varphi D : \mathcal{B}^\alpha \rightarrow H_v^\infty$ is an isometry in the semi-norm if and only if the following conditions hold:

- (a) $\sup_{z \in D} \frac{v(z)|\varphi'(z)-1|}{(1-|\varphi(z)|^2)^\alpha} \leq 1$;
 (b) For every $a \in \mathbb{D}$, there at least exists a sequence $\{z_n\}$ in \mathbb{D} , such that $\lim_{n \rightarrow \infty} \rho(\varphi(z_n), a) = 0$ and $\lim_{n \rightarrow \infty} \frac{(1-|z_n|^2)^\beta |\varphi'(z_n)-1|}{(1-|\varphi(z_n)|^2)^\alpha} = 1$.

Proof. We prove the sufficiency first.

As condition (a), for every $f \in \mathcal{B}^\alpha$, we have

$$\begin{aligned} \|T_\varphi f\|_{H_v^\infty} &= \sup_{z \in D} v(z) |f'(\varphi(z))\varphi'(z) - f'(\varphi(z))| \\ &= \sup_{z \in D} \frac{v(z)|\varphi'(z)-1|}{(1-|\varphi(z)|^2)^\alpha} (1-|\varphi(z)|^2)^\alpha |f'(\varphi(z))| \\ &\leq \|f\|_{\mathcal{B}^\alpha}. \end{aligned}$$

Next we show that the property (b) implies $\|T_\varphi f\|_{H_v^\infty} \geq \|f\|_{\mathcal{B}^\alpha}$.

Given any $f \in \mathcal{B}^\alpha$, then $\|f\|_{\mathcal{B}^\alpha} = \lim_{m \rightarrow \infty} (1-|a_m|^2)^\alpha |f'(a_m)|$ for some sequence $\{a_m\} \subset D$. For any fixed m , by property (b), there is a sequence $\{z_k^m\} \subset \mathbb{D}$ such that

$$\rho(\varphi(z_k^m), a_m) \rightarrow 0 \text{ and } \frac{v(z_k^m)|\varphi'(z_k^m)-1|}{(1-|\varphi(z_k^m)|^2)^\alpha} \rightarrow 1$$

as $k \rightarrow \infty$. By Lemma 3, for all m and k ,

$$|(1-|\varphi(z_k^m)|^2)^\alpha f'(\varphi(z_k^m)) - (1-|a_m|^2)^\alpha f'(a_m)| \leq C\|f\|_{\mathcal{B}^\alpha} \cdot \rho(\varphi(z_k^m), a_m).$$

Hence

$$(1-|\varphi(z_k^m)|^2)^\alpha |f'(\varphi(z_k^m))| \geq (1-|a_m|^2)^\alpha |f'(a_m)| - C\|f\|_{\mathcal{B}^\alpha} \cdot \rho(\varphi(z_k^m), a_m).$$

Therefore,

$$\begin{aligned} \|T_\varphi f\|_{H_v^\infty} &= \sup_{z \in D} v(z) |f'(\varphi(z))\varphi'(z) - f'(\varphi(z))| \\ &= \sup_{z \in \mathbb{D}} \frac{v(z)|\varphi'(z)-1|}{(1-|\varphi(z)|^2)^\alpha} (1-|\varphi(z)|^2)^\alpha |f'(\varphi(z))| \\ &\geq \limsup_{k \rightarrow \infty} \frac{v(z_k^m)|\varphi'(z_k^m)-1|}{(1-|\varphi(z_k^m)|^2)^\alpha} (1-|\varphi(z_k^m)|^2)^\alpha |f'(\varphi(z_k^m))| \\ &= (1-|a_m|^2)^\alpha |f'(a_m)|. \end{aligned}$$

The inequality $\|T_\varphi f\|_{H_v^\infty} \geq \|f\|_{\mathcal{B}^\alpha}$ follows by letting $m \rightarrow \infty$.

From the above discussions, we have $\|T_\varphi f\|_{H_v^\infty} = \|f\|_{\mathcal{B}^\alpha}$, which means that T_φ is an isometry operator in the semi-norm from \mathcal{B}^α to H_v^∞ .

Now we turn to the necessity.

For any $a \in \mathbb{D}$, we begin by taking test function

$$f_a(z) = \int_0^z \frac{(1 - |a|^2)^\alpha}{(1 - \bar{a}t)^{2\alpha}} dt. \quad (6)$$

It is clear that $f'_a(z) = \frac{(1 - |a|^2)^\alpha}{(1 - \bar{a}z)^{2\alpha}}$. Using Lemma 1, we have

$$(1 - |z|^2)^\alpha |f'_a(z)| = \frac{(1 - |z|^2)^\alpha (1 - |a|^2)^\alpha}{|1 - \bar{a}z|^{2\alpha}} = (1 - \rho^2(a, z))^\alpha. \quad (7)$$

So

$$\|f_a\|_{\mathcal{B}^\alpha} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'_a(z)| \leq 1.$$

On the other hand, since $(1 - |a|^2)^\alpha |f'_a(a)| = \frac{(1 - |a|^2)^{2\alpha}}{(1 - |a|^2)^{2\alpha}} = 1$, we have $\|f_a\|_{\mathcal{B}^\alpha} = 1$. By isometry assumption, for any $a \in \mathbb{D}$, we have

$$\begin{aligned} 1 &= \|f_{\varphi(a)}\|_{\mathcal{B}^\alpha} = \|T_\varphi f_{\varphi(a)}\|_{H_v^\infty} \\ &= \sup_{z \in D} v(z) \left| f'_{\varphi(a)}(\varphi(z)) \varphi'(z) - f'_{\varphi(a)}(\varphi(z)) \right| \\ &= \sup_{z \in D} \frac{v(z) |\varphi'(z) - 1|}{(1 - |\varphi(z)|^2)^\alpha} (1 - |\varphi(z)|^2)^\alpha |f'_{\varphi(a)}(\varphi(z))| \\ &\geq \frac{v(a) |\varphi'(a) - 1|}{(1 - |\varphi(a)|^2)^\alpha}. \end{aligned}$$

So (a) follows by noticing a is arbitrary.

Since $\|T_\varphi f_a\|_{H_v^\infty} = \|f_a\|_{\mathcal{B}^\alpha} = 1$, there exists a sequence $\{z_m\} \subset \mathbb{D}$ such that

$$v(z_m) |(T_\varphi f_a)(z_m)| = v(z_m) |f'_a(\varphi(z_m))| |\varphi'(z_m) - 1| \rightarrow 1 \quad (8)$$

as $m \rightarrow \infty$.

It follows from (a) that

$$\begin{aligned} &v(z_m) |f'_a(\varphi(z_m))| |\varphi'(z_m) - 1| \\ &= \frac{v(z_m) |\varphi'(z_m) - 1|}{(1 - |\varphi(z_m)|^2)^\alpha} (1 - |\varphi(z_m)|^2)^\alpha |f'_a(\varphi(z_m))| \end{aligned} \quad (9)$$

$$\leq (1 - |\varphi(z_m)|^2)^\alpha |f'_a(\varphi(z_m))|. \quad (10)$$

Combining (8) and (10), it follows that

$$\begin{aligned} 1 &\leq \liminf_{m \rightarrow \infty} (1 - |\varphi(z_m)|^2)^\alpha |f'_a(\varphi(z_m))| \\ &\leq \limsup_{m \rightarrow \infty} (1 - |\varphi(z_m)|^2)^\alpha |f'_a(\varphi(z_m))| \leq 1. \end{aligned}$$

The last inequality follows by (7) since $\varphi(z_m) \in \mathbb{D}$.

Consequently,

$$\lim_{m \rightarrow \infty} (1 - |\varphi(z_m)|^2)^\alpha |f'_a(\varphi(z_m))| = \lim_{m \rightarrow \infty} (1 - \rho^2(\varphi(z_m), a))^\alpha = 1. \quad (11)$$

That is, $\lim_{m \rightarrow \infty} \rho(\varphi(z_m), a) = 0$.

Combining (8), (9) and (11), we know

$$\lim_{m \rightarrow \infty} \frac{(1 - |z_m|^2)^\beta |\varphi'(z_m) - 1|}{(1 - |\varphi(z_m)|^2)^\alpha} = 1.$$

This completes the proof of this theorem. \square

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APPROXIMATION OF JENSEN TYPE QUADRATIC-ADDITIVE MAPPINGS VIA THE FIXED POINT THEORY *

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ABSTRACT. In this article, we investigate the stability results of a Jensen type quadratic-additive functional equation

$$f(x+y) + f(x-y) + 2f(z) = 2f(x) + f(z+y) + f(z-y)$$

via the fixed point theory. And then, we present two counter-examples which do not satisfy the stability results.

1. INTRODUCTION

A classical question in the theory of functional equations is “when is it true that a mapping, which satisfies approximately a functional equation, must be somehow close to an exact solution of the equation?” Such a problem, called a stability problem of functional equations, was formulated by S. M. Ulam [31] in 1940 as follows: Let G_1 be a group and G_2 a metric group with metric $\rho(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G_1 \rightarrow G_2$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $h : G_1 \rightarrow G_2$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G_1$? When this problem has a solution, we say that the homomorphisms from G_1 to G_2 are *stable*. In 1941, D. H. Hyers [16] considered the case of approximately additive mappings between Banach spaces and proved the following result. Suppose that E_1 and E_2 are Banach spaces and $f : E_1 \rightarrow E_2$ satisfies the following condition: there is a constant $\epsilon \geq 0$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in E_1$. Then the limit $h(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E_1$ and it is a unique additive mapping $h : E_1 \rightarrow E_2$ such that $\|f(x) - h(x)\| \leq \epsilon$.

The method which was provided by Hyers, and which produces the additive mapping h , was called a direct method. This method is the most important and most powerful tool for studying the stability of various functional equations. Hyers' Theorem was generalized by T. Aoki [1] and D.G. Bourgin [3] for additive mappings by considering an unbounded Cauchy difference. In 1978, Th.M. Rassias [26] also provided a generalization of Hyers' Theorem for linear mappings which allows the Cauchy difference to be unbounded like this $\|x\|^p + \|y\|^p$. A generalized result of

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Th.M. Rassias' theorem was obtained by P. Găvruta in [10] and S. Jung in [18]. In 1990, Th.M. Rassias [27] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Z. Gajda [9] following the same approach as in [26], gave an affirmative solution to this question for $p > 1$. It was shown by Z. Gajda [9], as well as by Th.M. Rassias and P. Šemrl [28], that one cannot prove a Th.M. Rassias' type theorem when $p = 1$. In 2003-2007 J.M. Rassias and M.J. Rassias [23, 24] and J.M. Rassias [25] solved the above Ulam problem for Jensen and Jensen type mappings.

During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians, see [6, 14, 17, 29, 22, 15]. Almost all subsequent proofs, in this very active area, have used Hyers' direct method, namely, the mapping F , which is a solution of the functional equation, is explicitly constructed by the limit function of a Cauchy sequence starting from the given approximate mapping f .

The first result of the generalized Hyers-Ulam stability for Jensen equation was given in the paper [19] by the direct method. In 2003, L. Cădariu and V. Radu [4] observed that the existence of the solution F for a Cauchy functional equation and the estimation of the mapping F with the approximate mapping f of the equation can be obtained from the alternative fixed point theorem. This method is called a *fixed point method*. In addition, they applied this method to prove stability theorems of the *Jensen's functional equation*:

$$(1.1) \quad 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) = 0 \Leftrightarrow 2f(x) - f(x+y) - f(x-y) = 0.$$

On the other hand, some properties of generalized Hyers-Ulam stability for a functional equation of Jensen type were obtained in [7] by the fixed point method. Further, the authors [5] obtained the stability of the *quadratic functional equation*:

$$(1.2) \quad f(x+y) + f(x-y) - 2f(x) - 2f(y) = 0$$

by using the fixed point method. Notice that if we consider the functions $f_1, f_2 : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f_1(x) = ax + b$ and $f_2(x) = cx^2$, respectively, where a, b and c are real constants, then f_1 satisfies the equation (1.1) and f_2 is a solution of the equation (1.2), respectively.

Associating the equation (1.1) with the equation (1.2), we see the following well known Drygas functional equation:

$$(1.3) \quad f(x+y) + f(x-y) = 2f(x) + f(y) + f(-y),$$

which has quadratic solutions Q of equation (1.2) in the class of even functions, and has additive solutions A of equation (1.1) in the class of odd functions. Hence the general solution f of (1.3) is given by $f(x) = Q(x) + A(x)$ [30].

Now, adding the equation (1.3) and the following Drygas functional equation

$$(1.4) \quad 2f(z) + f(y) + f(-y) = f(z+y) + f(z-y),$$

we get the Jensen type quadratic-additive functional equation:

$$(1.5) \quad f(x+y) + f(x-y) + 2f(z) = f(z+y) + f(z-y) + 2f(x),$$

of which the general solution function $f(x) - f(0)$ has of the form $f(x) - f(0) = Q(x) + A(x)$, where $Q(x) := \frac{f(x)+f(-x)}{2} - f(0)$ is a quadratic mapping satisfying the equation (1.2) and $A(x) := \frac{f(x)-f(-x)}{2}$ is a Jensen mapping satisfying the equation (1.1). In the paper, without splitting the given approximate mapping $f : X \rightarrow Y$ of the equation (1.5) into two approximate even and odd parts, we are going to derive the desired approximate solution F near f at once. Precisely, we introduce a strictly contractive mapping with Lipschitz constant $0 < L < 1$, and then, we show that the contractive mapping has the fixed point F in a generalized metric function space by using the fixed point method in the sense of L. Cădariu and V. Radu, where, the fixed point F yields the precise solution of the equation (1.5) near f . In Section 2, we prove several stability results of the functional equation (1.5) using the fixed point method under suitable conditions. In Section 3, we use the results in the previous section to get stability results of the Jensen's functional equation (1.1) and to get that of the quadratic functional equation (1.2), respectively.

2. GENERALIZED HYERS–ULAM STABILITY OF (1.5)

In this section, we prove the generalized Hyers–Ulam stability of the Jensen type quadratic-additive functional equation (1.5). We recall the following fundamental result of the fixed point theory by Margolis and Diaz [20].

Theorem 2.1. *Suppose that a complete generalized metric space (X, d) , which means that the metric d may assume infinite values, and a strictly contractive mapping $\Lambda : X \rightarrow X$ with the Lipschitz constant $0 < L < 1$ are given. Then, for each given element $x \in X$, either*

$$d(\Lambda^n x, \Lambda^{n+1} x) = +\infty, \quad \forall n \in \mathbf{N} \cup \{0\},$$

or there exists a nonnegative integer k such that

- $d(\Lambda^n x, \Lambda^{n+1} x) < +\infty$ for all $n \geq k$;
- the sequence $\{\Lambda^n x\}$ is convergent to a fixed point y^* of Λ ;
- y^* is the unique fixed point of Λ in $X_1 := \{y \in X, d(\Lambda^k x, y) < +\infty\}$;
- $d(y, y^*) \leq \frac{1}{1-L} d(y, \Lambda y)$ for all $y \in X_1$.

Throughout this paper, let V be a (real or complex) linear space and Y a Banach space. For a given mapping $f : V \rightarrow Y$, we use the following abbreviation

$$Df(x, y, z) := f(x + y) + f(x - y) + 2f(z) - f(z + y) - f(z - y) - 2f(x)$$

for all $x, y, z \in V$.

In the following theorem, we prove the stability of the Jensen type quadratic-additive functional equation (1.5) using the fixed point method.

Theorem 2.2. *Let $f : V \rightarrow Y$ be a mapping for which there exists a mapping $\varphi : V^3 \rightarrow \mathbb{R}^+ := [0, \infty)$ such that*

$$(2.1) \quad \|Df(x, y, z)\| \leq \varphi(x, y, z)$$

for all $x, y, z \in V$. If $\varphi(x, y, z) = \varphi(-x, -y, -z)$ for all $x, y, z \in V$ and there exists a constant $0 < L < 1$ such that

$$(2.2) \quad \varphi(2x, 2y, 2z) \leq 2L\varphi(x, y, z),$$

for all $x, y, z \in V$, then there exists a unique Jensen type quadratic-additive mapping $F : V \rightarrow Y$ such that $DF(x, y, z) = 0$ for all $x, y, z \in V$ and

$$(2.3) \quad \|f(x) - F(x)\| \leq \frac{\varphi(x, x, 0)}{2(1 - L)}$$

for all $x \in V$. In particular, F is represented by

$$(2.4) \quad F(x) = f(0) + \lim_{n \rightarrow \infty} \left[\frac{f(2^n x) + f(-2^n x)}{2 \cdot 4^n} + \frac{f(2^n x) - f(-2^n x)}{2^{n+1}} \right]$$

for all $x \in V$.

Proof. If we consider the mapping $\tilde{f} := f - f(0)$, then $\tilde{f} : V \rightarrow Y$ satisfies $\tilde{f}(0) = 0$ and

$$(2.5) \quad \|D\tilde{f}(x, y, z)\| = \|Df(x, y, z)\| \leq \varphi(x, y, z)$$

for all $x, y, z \in V$. Let S be the set of all mappings $g : V \rightarrow Y$ with $g(0) = 0$, and then we introduce a generalized metric d on S by

$$(2.6) \quad d(g, h) := \inf \{ K \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq K\varphi(x, x, 0) \quad \forall x \in V \}.$$

It is easy to show that (S, d) is a generalized complete metric space. Now, we consider an operator $\Lambda : S \rightarrow S$ defined by

$$(2.7) \quad \Lambda g(x) := \frac{g(2x) - g(-2x)}{4} + \frac{g(2x) + g(-2x)}{8}$$

for all $g \in S$ and all $x \in V$. Then we notice that

$$(2.8) \quad \Lambda^n g(x) = \frac{g(2^n x) - g(-2^n x)}{2^{n+1}} + \frac{g(2^n x) + g(-2^n x)}{2 \cdot 4^n}$$

for all $n \in \mathbb{N}$ and $x \in V$.

First, we show that Λ is a strictly contractive self-mapping of S with the Lipschitz constant L . Let $g, h \in S$ and let $K \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq K$. From the definition of d , we have

$$(2.9) \quad \begin{aligned} \|\Lambda g(x) - \Lambda h(x)\| &= \frac{3}{8} \|g(2x) - h(2x)\| + \frac{1}{8} \|g(-2x) - h(-2x)\| \\ &\leq \frac{1}{2} K\varphi(2x, 2x, 0) \leq LK\varphi(x, x, 0) \end{aligned}$$

for all $x \in V$, which implies that

$$(2.10) \quad d(\Lambda g, \Lambda h) \leq Ld(g, h)$$

for any $g, h \in S$. That is, Λ is a strictly contractive self-mapping of S with the Lipschitz constant L . Moreover, by (2.5) we see that

$$(2.11) \quad \begin{aligned} \|\tilde{f}(x) - \Lambda\tilde{f}(x)\| &= \frac{1}{8} \| -3D\tilde{f}(x, x, 0) + D\tilde{f}(-x, -x, 0) \| \\ &\leq \frac{1}{2} \varphi(x, x, 0) \end{aligned}$$

for all $x \in V$. It means that $d(\tilde{f}, \Lambda\tilde{f}) \leq \frac{1}{2} < \infty$ by the definition of d . Therefore, according to Theorem 2.1, the sequence $\{\Lambda^n \tilde{f}\}$ converges to the unique fixed point $\tilde{F} : V \rightarrow Y$ of Λ in the set $S_1 = \{g \in S | d(\tilde{f}, g) < \infty\}$, which is represented by

$$(2.12) \quad \tilde{F}(x) = \lim_{n \rightarrow \infty} \left[\frac{\tilde{f}(2^n x) - \tilde{f}(-2^n x)}{2^{n+1}} + \frac{\tilde{f}(2^n x) + \tilde{f}(-2^n x)}{2 \cdot 4^n} \right]$$

for all $x \in V$. Putting $F := \tilde{F} + f(0)$, then we have the equality $\|f(x) - F(x)\| = \|\tilde{f}(x) - \tilde{F}(x)\|$ for all $x \in V$. Thus one notes that

$$(2.13) \quad d(\tilde{f}, \tilde{F}) \leq \frac{1}{1-L} d(\tilde{f}, \Lambda\tilde{f}) \leq \frac{1}{2(1-L)},$$

which implies (2.3) and (2.4).

By the definitions of F and \tilde{F} , together with (2.5) and (2.2), we have that

$$\begin{aligned} \|DF(x, y, z)\| &= \|D\tilde{F}(x, y, z)\| \\ &= \lim_{n \rightarrow \infty} \left\| \frac{Df(2^n x, 2^n y, 2^n z) - f(-2^n x, -2^n y, -2^n z)}{2^{n+1}} \right. \\ &\quad \left. + \frac{Df(2^n x, 2^n y, 2^n z) + Df(-2^n x, -2^n y, -2^n z)}{2 \cdot 4^n} \right\| \\ &\leq \lim_{n \rightarrow \infty} \frac{2^n + 1}{2 \cdot 4^n} \left(\varphi(2^n x, 2^n y, 2^n z) + \varphi(-2^n x, -2^n y, -2^n z) \right) \\ &= 0 \end{aligned}$$

for all $x, y, z \in V$. Thus, the mapping F satisfies the Jensen type quadratic-additive functional equation (2.3). This completes the proof of this theorem. \square

Theorem 2.3. *Let $f : V \rightarrow Y$ be a mapping for which there exists a mapping $\varphi : V^3 \rightarrow [0, \infty)$ such that*

$$(2.14) \quad \|Df(x, y, z)\| \leq \varphi(x, y, z)$$

for all $x, y, z \in V$. If $\varphi(x, y, z) = \varphi(-x, -y, -z)$ for all $x, y, z \in V$ and there exists a constant $0 < L < 1$ such that the mapping φ has the property

$$(2.15) \quad \varphi(x, y, z) \leq \frac{L}{4} \varphi(2x, 2y, 2z)$$

for all $x, y, z \in V$, then there exists a unique Jensen type quadratic-additive mapping $F : V \rightarrow Y$ such that $DF(x, y, z) = 0$ for all $x, y, z \in V$ and

$$(2.16) \quad \|f(x) - F(x)\| \leq \frac{L\varphi(x, x, 0)}{4(1-L)}$$

for all $x \in V$. In particular, F is represented by

$$(2.17) \quad F(x) := f(0) + \lim_{n \rightarrow \infty} \left[\frac{4^n}{2} \left(f\left(\frac{x}{2^n}\right) + f\left(\frac{-x}{2^n}\right) - 2f(0) \right) + 2^{n-1} \left(f\left(\frac{x}{2^n}\right) - f\left(\frac{-x}{2^n}\right) \right) \right]$$

for all $x \in V$.

Proof. The proof is similarly verified by the same argument as that of Theorem 2.2. \square

3. APPLICATIONS

For a given mapping $f : V \rightarrow Y$, we use the following abbreviations

$$\begin{aligned} Jf(x, y) &:= 2f\left(\frac{x+y}{2}\right) - f(x) - f(y), \\ Qf(x, y) &:= f(x+y) + f(x-y) - 2f(x) - 2f(y) \end{aligned}$$

for all $x, y \in V$. Using Theorem 2.2 and Theorem 2.3,

we will obtain the stability results of the quadratic functional equation $Qf \equiv 0$ and the Jensen's functional equation $Jf \equiv 0$ in the following corollaries.

Corollary 3.1. *Suppose that each $f_i : V \rightarrow Y, i = 1, 2$, and $\phi_i : V^2 \rightarrow [0, \infty), i = 1, 2$, are given functions satisfying*

$$\|Qf_i(x, y)\| \leq \phi_i(x, y)$$

and $\phi_i(x, y) = \phi_i(-x, -y)$ for all $x, y \in V$, respectively. If there exists $0 < L < 1$ such that

$$(3.1) \quad \phi_1(2x, 2y) \leq 2L\phi_1(x, y),$$

$$(3.2) \quad \phi_2(x, y) \leq \frac{L}{4}\phi_2(2x, 2y)$$

for all $x, y \in V$, then we have unique quadratic mappings $F_1, F_2 : V \rightarrow Y$ such that

$$(3.3) \quad \|f_1(x) - f_1(0) - F_1(x)\| \leq \frac{\phi_1(0, x) + \phi_1(x, x)}{2(1-L)},$$

$$(3.4) \quad \|f_2(x) - F_2(x)\| \leq \frac{L[\phi_2(0, x) + \phi_2(x, x)]}{4(1-L)}$$

for all $x \in V$. In particular, F_1 and F_2 are represented by

$$(3.5) \quad F_1(x) = \lim_{n \rightarrow \infty} \frac{f_1(2^n x)}{4^n},$$

$$(3.6) \quad F_2(x) = \lim_{n \rightarrow \infty} 4^n f_2\left(\frac{x}{2^n}\right)$$

for all $x \in V$. Moreover, if $0 < L < \frac{1}{2}$ and ϕ_1 is continuous, then $f_1 - f_1(0)$ is itself a Jensen type quadratic-additive mapping.

Proof. Notice that

$$\|Df_i(x, y, z)\| = \|Qf_i(x, y) - Qf_i(z, y)\| \leq \phi_i(x, y) + \phi_i(z, y)$$

for all $x, y, z \in V$ and $i = 1, 2$. Put

$$\varphi_i(x, y, z) := \phi_i(x, y) + \phi_i(z, y)$$

for all $x, y, z \in V$. Then φ_1 satisfies (2.2) and φ_2 satisfies (2.15). According to Theorem 2.2, there exists a unique mapping $F_1 : V \rightarrow Y$ satisfying (3.3) which is represented by

$$F_1(x) = \lim_{n \rightarrow \infty} \left(\frac{f_1(2^n x) + f_1(-2^n x)}{2 \cdot 4^n} + \frac{f_1(2^n x) - f_1(-2^n x)}{2^{n+1}} \right).$$

Observe that

$$\begin{aligned} \lim_{n \rightarrow \infty} \left\| \frac{f_1(2^n x) - f_1(-2^n x)}{2^{n+1}} \right\| &= \lim_{n \rightarrow \infty} \frac{1}{2^{n+1}} \|Qf_1(0, 2^n x)\| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{2^{n+1}} \phi_1(0, 2^n x) \\ &\leq \lim_{n \rightarrow \infty} \frac{L^n}{2} \phi_1(0, x) = 0 \end{aligned}$$

as well as

$$\lim_{n \rightarrow \infty} \left\| \frac{f_1(2^n x) + f_1(-2^n x)}{2 \cdot 4^n} \right\| \leq \lim_{n \rightarrow \infty} \frac{L^n}{2^{n+1}} \phi_1(0, x) = 0$$

for all $x \in V$. From these two properties, we lead to the mapping (3.5) for all $x \in V$. Moreover, we have

$$\left\| \frac{Qf_1(2^n x, 2^n y)}{4^n} \right\| \leq \frac{\phi_1(2^n x, 2^n y)}{4^n} \leq \frac{L^n}{2^n} \phi_1(x, y)$$

for all $x, y \in V$. Taking the limit as $n \rightarrow \infty$ in the above inequality, we get

$$QF_1(x, y) = 0$$

for all $x, y \in V$ and so $F_1 : V \rightarrow Y$ is a quadratic mapping.

On the other hand, since $L\phi_2(0, 0) \geq 4\phi_2(0, 0)$ and

$$\|2f_2(0)\| = \|Qf_2(0, 0)\| \leq \phi_2(0, 0)$$

we can show that $\phi_2(0, 0) = 0$ and $f_2(0) = 0$. According to Theorem 2.3, there exists a unique mapping $F_2 : V \rightarrow Y$ satisfying (3.4), which is represented by (2.17). We have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{4^n}{2} \left\| -f_2\left(\frac{x}{2^n}\right) + f_2\left(-\frac{x}{2^n}\right) \right\| &= \lim_{n \rightarrow \infty} \frac{4^n}{2} \left\| Qf_2\left(0, \frac{x}{2^n}\right) \right\| \\ &\leq \lim_{n \rightarrow \infty} \frac{4^n}{2} \phi_2\left(0, \frac{x}{2^n}\right) \\ &\leq \lim_{n \rightarrow \infty} \frac{L^n}{2} \phi_2(0, x) = 0 \end{aligned}$$

as well as

$$\lim_{n \rightarrow \infty} 2^{n-1} \left\| f_2\left(\frac{x}{2^n}\right) - f_2\left(-\frac{x}{2^n}\right) \right\| = 0$$

for all $x \in V$. From these and (2.8), we get (3.6). Notice that

$$\left\| 4^n Qf_2 \left(\frac{x}{2^n}, \frac{y}{2^n} \right) \right\| \leq 4^n \phi_2 \left(\frac{x}{2^n}, \frac{y}{2^n} \right) \leq L^n \phi_2(x, y)$$

for all $x, y \in V$. Taking the limit as $n \rightarrow \infty$, then we have shown that

$$QF_2(x, y) = 0$$

for all $x, y \in V$ and so $F_2 : V \rightarrow Y$ is a quadratic mapping. This completes the corollary. \square

Now, we obtain a stability result of Jensen functional equations.

Corollary 3.2. *Let $f_i : V \rightarrow Y$, $i = 1, 2$, be mappings for which there exist functions $\phi_i : V^2 \rightarrow [0, \infty)$, $i = 1, 2$, such that*

$$(3.7) \quad \|Jf_i(x, y)\| \leq \phi_i(x, y)$$

and $\phi_i(x, y) = \phi_i(-x, -y)$ for all $x, y \in V$, respectively. If there exists $0 < L < 1$ such that the mapping ϕ_1 has the property (3.1) and ϕ_2 holds (3.2) for all $x, y \in V$, then there exist unique Jensen mappings $F_i : V \rightarrow Y$, $i = 1, 2$, such that

$$(3.8) \quad \|f_1(x) - F_1(x)\| \leq \frac{\phi_1(0, 2x) + \phi_1(x, -x)}{2(1 - L)},$$

$$(3.9) \quad \|f_2(x) - F_2(x)\| \leq \frac{L(\phi_2(0, 2x) + \phi_2(x, -x))}{4(1 - L)}$$

for all $x \in V$. In particular, the mappings F_1, F_2 are represented by

$$(3.10) \quad F_1(x) = \lim_{n \rightarrow \infty} \frac{f_1(2^n x)}{2^n} + f_1(0),$$

$$(3.11) \quad F_2(x) = \lim_{n \rightarrow \infty} 2^n \left(f_2 \left(\frac{x}{2^n} \right) - f_2(0) \right) + f_2(0)$$

for all $x \in V$.

Proof. The proof is similar to that of Theorem 3.1. \square

Now, we obtain generalized Hyers-Ulam stability results in the framework of normed spaces using Theorem 2.2 and Theorem 2.3.

Corollary 3.3. *Let X be a normed space, $\theta \geq 0$, and $p \in (0, 1) \cup (2, \infty)$. Suppose that a mapping $f : X \rightarrow Y$ satisfies the inequality*

$$\|Df(x, y, z)\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in X$. Then there exists a unique Jensen type quadratic-additive mapping $F : X \rightarrow Y$ such that

$$\|f(x) - F(x)\| \leq \begin{cases} \frac{2\theta}{2-2^p} \|x\|^p, & \text{if } 0 < p < 1; \\ \frac{2\theta}{2^p-4} \|x\|^p, & \text{if } p > 2, \end{cases}$$

for all $x \in X$.

Proof. It follows from Theorem 2.2 and Theorem 2.3, by putting $L := 2^{p-1} < 1$ if $0 < p < 1$, and $L := 2^{2-p} < 1$ if $p > 2$. \square

In the following, we present counter-examples for the singular cases $p = 1$ and $p = 2$ in Corollary 3.3.

Example 3.1. We remark that if we consider an odd function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \sum_{i=1}^{\infty} \frac{\phi(2^i x)}{2^i}, \quad \phi(x) = \begin{cases} \mu x, & \text{if } -1 < x < 1; \\ \mu, & \text{if } x \geq 1; \\ -\mu, & \text{if } x \leq -1, \end{cases} \quad (\mu > 0),$$

which is the same type as that in the paper [9], then it follows from [28] that

$$\begin{aligned} |f(x+y) + f(x-y) - 2f(x)| &\leq \theta(|x| + |y|), \\ |f(z+y) + f(z-y) - 2f(z)| &\leq \theta(|z| + |y|), \end{aligned}$$

and so

$$|Df(x, y, z)| \leq 2\theta(|x| + |y| + |z|),$$

for all x, y, z and for some constant $\theta > 0$. However, there doesn't exist Jensen type quadratic-additive function $F : \mathbb{R} \rightarrow \mathbb{R}$ such that $|f(x) - F(x)| \leq K(\theta)|x|$ for all x and for some constant $K(\theta)$. Hence, there exists a counter-example for the case $p = 1$ in Corollary 3.3.

Also, we remark that if we consider an even function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \sum_{i=1}^{\infty} \frac{\phi(2^i x)}{4^i}, \quad \phi(x) = \begin{cases} \mu x^2, & \text{if } |x| < 1; \\ \mu, & \text{if } |x| \geq 1, \end{cases} \quad (\mu > 0),$$

which is the same type as that in the paper [8], then it is well known that

$$\begin{aligned} |f(x+y) + f(x-y) - 2f(x) - 2f(y)| &\leq \theta(|x|^2 + |y|^2), \\ |f(z+y) + f(z-y) - 2f(z) - 2f(y)| &\leq \theta(|z|^2 + |y|^2), \end{aligned}$$

and so

$$|Df(x, y, z)| \leq 2\theta(|x|^2 + |y|^2 + |z|^2),$$

for all x, y, z and for some constant $\theta > 0$. However, there doesn't exist Jensen type quadratic-additive function $F : \mathbb{R} \rightarrow \mathbb{R}$ such that $|f(x) - F(x)| \leq K(\theta)|x|^2$ for all x and for some constant $K(\theta)$. Hence, there exists a counter-example for the case $p = 2$ in Corollary 3.3.

Corollary 3.4. Let X be a normed space, $\theta \geq 0$ and $p, q, r > 0$ be reals with $p+q+r \in (-\infty, 1) \cup (2, \infty)$. If a mapping $f : X \rightarrow Y$ satisfies

$$\|Df(x, y, z)\| \leq \theta \|x\|^p \|y\|^q \|z\|^r$$

for all $x, y, z \in X$, then f is itself a Jensen type quadratic-additive mapping.

Proof. It follows from Theorem 2.2 and Theorem 2.3, by putting $L := 2^{p+q+r-1} < 1$ if $p+q+r < 1$, and $L := 2^{2-p-q-r} < 1$ if $p+q+r > 2$. \square

Corollary 3.5. *Let X be a normed space, $\theta_i \geq 0$, ($i = 1, 2, 3$) and $p, q, r > 0$ be reals such that either $\max\{p+q, q+r, p+r\} < 1$ or $\min\{p+q, q+r, p+r\} > 2$. If a mapping $f : X \rightarrow Y$ satisfies*

$$\|Df(x, y, z)\| \leq \theta_1 \|x\|^p \|y\|^q + \theta_2 \|y\|^q \|z\|^r + \theta_3 \|x\|^p \|z\|^r$$

for all $x, y, z \in X$, Then there exists a unique Jensen type quadratic-additive mapping $F : X \rightarrow Y$ such that

$$\|f(x) - F(x)\| \leq \begin{cases} \frac{\theta_1}{2-2\max\{p+q, q+r, p+r\}} \|x\|^{p+q}, & \text{if } \max\{p+q, q+r, p+r\} < 1; \\ \frac{\theta_1}{2\min\{p+q, q+r, p+r\}-4} \|x\|^{p+q}, & \text{if } \min\{p+q, q+r, p+r\} > 2, \end{cases}$$

for all $x \in X$.

Proof. It follows from Theorem 2.2 and Theorem 2.3, by putting

$$L := 2^{\max\{p+q, q+r, p+r\}-1} < 1, \text{ if } \max\{p+q, q+r, p+r\} < 1,$$

and

$$L := 2^{2-\min\{p+q, q+r, p+r\}} < 1, \text{ if } \min\{p+q, q+r, p+r\} > 2.$$

□

In the following, we present counter-examples for the singular cases $\max\{p+q, q+r, p+r\} = 1$ and $\min\{p+q, q+r, p+r\} = 2$ in Corollary 3.5.

Example 3.2. We remark that if we consider an odd function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} x \ln|x|, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0, \end{cases}$$

then for any p with $0 < p < 1$ it follows from [11, 12] that there exists a constant $c > 0$ such that

$$|f(x+y) - f(x) - f(y)| \leq c|x|^p|y|^{1-p},$$

and so

$$\begin{aligned} |f(x-y) - f(x) + f(y)| &\leq c|x|^p|y|^{1-p}, \\ |f(x+y) + f(x-y) - 2f(x)| &\leq 2c|x|^p|y|^{1-p}, \\ |f(z+y) + f(z-y) - 2f(z)| &\leq 2c|z|^p|y|^{1-p}, \end{aligned}$$

which yield

$$|Df(x, y, z)| \leq 2c(|x|^p|y|^{1-p} + |y|^{1-p}|z|^p),$$

for all x, y, z . However, there doesn't exist Jensen type quadratic-additive function $F : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$|f(x) - F(x)| \leq K(c, p)|x|$$

for all x and for some constant $K(c, p)$. Hence, there exists a counter-example for the case $\max\{p+q, q+r, p+r\} = 1$ in Corollary 3.5.

Also, we remark that if we consider an even function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \ln|x|, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0, \end{cases}$$

then for any p with $0 < p < 2$ it follows from [13] that there exists a constant $k > 0$ such that

$$\begin{aligned} |f(x+y) + f(x-y) - 2f(x) - 2f(y)| &\leq k|x|^p|y|^{2-p}, \\ |f(z+y) + f(z-y) - 2f(z) - 2f(y)| &\leq k|z|^p|y|^{2-p}, \end{aligned}$$

which yield

$$|Df(x, y, z)| \leq k(|x|^p|y|^{2-p} + |y|^{2-p}|z|^p),$$

for all x, y, z . However, there doesn't exist Jensen type quadratic-additive function $F : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$|f(x) - F(x)| \leq K(k, p)|x|^2$$

for all x and for some constant $K(k, p)$. Hence, there exists a counter-example for the case $\min\{p+q, q+r, p+r\} = 2$ in Corollary 3.5.

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On certain subclasses of p -valent analytic functions involving Saitoh operator

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Abstract: The object of the present investigation is to solve Fekete-Szegő problem for a new class $\mathcal{V}_p^\lambda(a, c, A, B)$ of p -valent analytic functions involving the Saitoh operator in the unit disk. We also obtain subordination results and some interesting corollaries for functions in \mathcal{A}_p involving this operator. Relevant connections of the results obtained here with those given by earlier workers on the subject are also mentioned.

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1. INTRODUCTION AND PRELIMINARIES

Let \mathcal{A}_p be the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (1.1)$$

analytic in the unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ with $p \in \mathbb{N} = \{1, 2, 3, \dots\}$. Let \mathcal{S} be the subclass of $\mathcal{A}_1 = \mathcal{A}$ consisting of univalent functions.

A function $f \in \mathcal{A}_p$ is said to be p -valent starlike of order α , denoted by $\mathcal{S}_p^*(\alpha)$, if and only if

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (0 \leq \alpha < p; z \in \mathcal{U}). \quad (1.2)$$

Similarly, a function $f \in \mathcal{A}_p$ is said to be p -valent convex of order α , denoted by $\mathcal{C}_p(\alpha)$, if and only if

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (0 \leq \alpha < p; z \in \mathcal{U}). \quad (1.3)$$

From (1.2) and (1.3), it follows that

$$f(z) \in \mathcal{C}_p(\alpha) \iff \frac{zf'(z)}{p} \in \mathcal{S}_p^*(\alpha).$$

Furthermore, we say that a function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{R}_p(\alpha)$, if and only if

$$\operatorname{Re} \left\{ \frac{f'(z)}{z^{p-1}} \right\} > \alpha \quad (0 \leq \alpha < p; z \in \mathcal{U}). \quad (1.4)$$

For functions f and g , analytic in the unit disk \mathcal{U} , we say the f is said to be subordinate to g , written as $f \prec g$ or $f(z) \prec g(z)$ ($z \in \mathcal{U}$), if there exists an analytic function ω in \mathcal{U} with $\omega(0) = 0, |\omega(z)| \leq |z|$ ($z \in \mathcal{U}$) and $f(z) = g(\omega(z))$ for all $z \in \mathcal{U}$. In particular, if g is univalent in \mathcal{U} , then we have the following equivalence (cf., e.g., [14]):

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(\mathcal{U}) \subset g(\mathcal{U}).$$

For the functions f and g given by the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n, \quad (z \in \mathcal{U})$$

their Hadamard product (or convolution), denoted by $f \star g$ is defined as

$$(f \star g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n = (g \star f)(z) \quad (z \in \mathcal{U}).$$

Note that $f \star g$ is analytic in \mathcal{U} .

By making use of the Hadamard product, Saitoh [18] defined a linear operator $\mathcal{L}_p(a, c) : \mathcal{A}_p \longrightarrow \mathcal{A}_p$ in terms of the function φ_p as

$$\mathcal{L}_p(a, c)f(z) = \varphi_p(a, c; z) \star f(z) \quad (f \in \mathcal{A}_p; z \in \mathcal{U}), \quad (1.5)$$

where

$$\varphi_p(a, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{p+k} \quad (a \in \mathbb{C}, c \in \mathbb{C} \setminus \{\dots, -2, -1, 0\}; z \in \mathcal{U}). \quad (1.6)$$

and $(x)_k$ is the Pochhammer symbol (or shifted factorial) given by

$$(x)_k = \begin{cases} 1, & n = 0, \\ x(x+1) \cdots (x+k-1), & k \in \mathbb{N}. \end{cases}$$

For $p = 1$, the operator $\mathcal{L}_p(a, c)$ reduces to the Carlson-Shaffer operator $\mathcal{L}(a, c)$ [1]. If $f \in \mathcal{A}_p$ is given by (1.1), then it follows from (1.5) and (1.6) that

$$\mathcal{L}_p(a, c)f(z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} a_{p+k} z^{p+k} \quad (z \in \mathcal{U}) \quad (1.7)$$

and

$$z(\mathcal{L}_p(a, c)f)'(z) = a\mathcal{L}_p(a+1, c)f(z) - (a-p)\mathcal{L}_p(a, c)f(z) \quad (z \in \mathcal{U}). \quad (1.8)$$

It is easily seen that for $f \in \mathcal{A}_p$

$$(i) \quad \mathcal{L}_p(a, a)f(z) = f(z),$$

$$(ii) \quad \mathcal{L}_p(p+1, p)f(z) = \frac{zf'(z)}{p},$$

(iii) $\mathcal{L}_p(n+p, p)f(z) = \mathcal{D}^{n+p-1}f(z)$ ($n \in \mathbb{Z}; n > -p$), the operator studied by Goel and Sohi [5]. For the case $p = 1$, \mathcal{D}^n is the Ruscheweyh derivative operator [17].

(iv) $\mathcal{L}_p(p+1, n+p)f(z) = \mathcal{J}_{n,p}f(z)$ ($n \in \mathbb{Z}; n > -p$), the extended Noor integral operator considered by Liu and Noor [10].

(v) $\mathcal{L}_p(p+1, p+1-\mu)f(z) = \Omega_z^{(\mu,p)}f(z)$ ($-\infty < \mu < p+1$), the extended fractional differintegral operator studied by Patel and Mishra [16]. Note that

$$\Omega_z^{(0,p)}f(z) = f(z), \quad \Omega_z^{(1,p)}f(z) = \frac{zf'(z)}{p} \text{ and } \Omega_z^{(2,p)}f(z) = \frac{z^2f''(z)}{p(p-1)} \quad (p \geq 2).$$

As a special case, we get the operator $\Omega_z^\mu f(z)$ ($0 \leq \mu < 1$) for $p = 1$ introduced and studied by Owa-Srivastava [15].

With the aid of the operator $\mathcal{L}_p(a, c)$, we introduce a subclass of \mathcal{A}_p as follows.

Definition 1.1. For the fixed parameters A, B ($-1 \leq B < A \leq 1$), $a > 0$ and $c > 0$, we say that a function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{V}_p^\lambda(a, c, A, B)$, if it satisfies the following subordination relation

$$(1 - \lambda) \frac{\mathcal{L}_p(a, c)f(z)}{z^p} + \frac{\lambda}{p} \frac{z(\mathcal{L}_p(a, c)f(z))'(z)}{\mathcal{L}_p(a, c)f(z)} \prec \frac{1 + Az}{1 + Bz} \quad (0 \leq \lambda \leq 1; z \in \mathcal{U}). \quad (1.9)$$

We note that the class $\mathcal{V}_p^\lambda(a, c, A, B)$ includes many known subclasses of \mathcal{A}_p as mentioned below.

$$\begin{aligned} \text{(i)} \quad \mathcal{V}_p^1\left(a, c, 1 - \frac{2\alpha}{p}, -1\right) &= \mathcal{S}_p(a, c; \alpha) \quad (0 \leq \alpha < p) \\ &= \left\{ f \in \mathcal{A}_p : \operatorname{Re} \left(\frac{z(\mathcal{L}_p(a, c)f(z))'(z)}{\mathcal{L}_p(a, c)f(z)} \right) > \alpha, z \in \mathcal{U} \right\}. \end{aligned}$$

Note that $\mathcal{S}_p(a, a; \alpha) = \mathcal{S}_p^*(\alpha)$, the class of p -valent starlike functions of order α and $\mathcal{S}_p(p+1, p; \alpha) = \mathcal{C}_p(\alpha)$, the class of p -valent convex functions of order α .

$$\begin{aligned} \text{(ii)} \quad \mathcal{V}_p^0\left(a, c, 1 - \frac{2\alpha}{p}, -1\right) &= \mathcal{R}_p(a, c; \alpha) \quad (0 \leq \alpha < p) \\ &= \left\{ f \in \mathcal{A}_p : \operatorname{Re} \left(\frac{\mathcal{L}_p(a, c)f(z)}{z^p} \right) > \frac{\alpha}{p}, z \in \mathcal{U} \right\} \end{aligned}$$

which, in turn yields the class $\mathcal{R}_p(\alpha)$ for $a = p+1$ and $c = p$.

For $0 \leq \alpha < 1$, the functions in the class $\mathcal{R}_1(\alpha) = \mathcal{R}(\alpha)$ are called functions of bounded turning. By the Nashiro-Warschowski Theorem [3], the functions in $\mathcal{R}(\alpha)$ are univalent and also close-to-convex in \mathcal{U} . It is well-known that $\mathcal{R}(\alpha) \subsetneq \mathcal{S}_1^*(0) = \mathcal{S}^*$ and $\mathcal{S}^* \subsetneq \mathcal{R}(\alpha)$. For more information on the class $\mathcal{R}(0) = \mathcal{R}$ (cf., e.g., [12]).

Fekete and Szegő [4] proved a remarkable result that the estimate

$$|a_3 - \gamma a_2^2| \leq 1 + 2 \exp\left(-\frac{2\gamma}{1-\gamma}\right)$$

is sharp and holds for each $\gamma \in [0, 1]$ over the class \mathcal{S} consisting of functions $f \in \mathcal{A}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathcal{U}). \quad (1.10)$$

The coefficient functional

$$\Phi_\gamma(f) = a_3 - \gamma a_2^2 = \frac{1}{6} \left\{ f'''(0) - \frac{3\gamma}{2} (f''(0))^2 \right\}$$

on the functions in \mathcal{A} represents various geometrical properties, for example, when $\gamma = 1$, $\Phi_\gamma(f) = a_3 - a_2^2$ becomes $S_f(0)/6$, where S_f denote the Schwarzian derivative

$$S_f(z) = \left\{ \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2 \right\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2} \left(\frac{f''(z)}{f'(z)} \right)^2$$

of locally univalent functions in \mathcal{U} . For a family \mathcal{F} of functions in \mathcal{A} of the form (1.10), the more general problem of maximizing the absolute value for the functional $\Phi_\gamma(f)$ for some γ (real as well as complex) is popularly known as Fekete-Szegő problem for the class \mathcal{F} . In literature, there exists a large number of results about the inequalities for $|\Phi_\gamma(f)|$ corresponding to various subclasses of \mathcal{S} (see, e.g., [4, 7, 8, 9]).

The object of the present study is to solve Fekete-Szegő problem for a new class $\mathcal{V}_p^\lambda(a, c, A, B)$ of p -valent analytic functions in \mathcal{U} involving the Saitoh operator. We also obtain some subordination results along with some interesting corollaries for functions in \mathcal{A}_p involving this operator. Relevant connections of the results obtained here with those given by earlier workers on the subject are pointed out.

2. PRELIMINARIES

Let \mathcal{P} denote the family of all functions of the form

$$\varphi(z) = 1 + q_1 z + q_2 z^2 + \cdots \quad (2.1)$$

analytic in \mathcal{U} and satisfying the condition $\operatorname{Re}\{\varphi(z)\} > 0$ in \mathcal{U} .

To establish our main results, we need the following lemmas.

Lemma 2.1. *If the function φ , given by (2.1) belongs to the class \mathcal{P} , then for any complex number γ ,*

$$|q_k| \leq 2 \quad (2.2)$$

and

$$|q_2 - \gamma q_1^2| \leq 2 \max\{1, |2\gamma - 1|\}. \quad (2.3)$$

The result in (2.2) is sharp for the function $\varphi_1(z) = (1+z)/(1-z)$ ($z \in \mathcal{U}$), where as, the result in (2.3) is sharp for the functions $\varphi_2(z) = (1+z^2)/(1-z^2)$ ($z \in \mathcal{U}$) and $\varphi_1(z)$.

We note that the estimate (2.2) is contained in [3], the estimate (2.3) is due to Ma and Minda [11].

The following lemma is due to Miller and Mocanu [14].

Lemma 2.2. *Let q be univalent in \mathcal{U} and let θ and ϕ be analytic in a domain Ω containing $q(\mathcal{U})$ with $\phi(w) \neq 0$, when $w \in q(\mathcal{U})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that either*

- (i) h is convex, or
- (ii) Q is starlike.

In addition, assume that

$$(iii) \operatorname{Re} \left\{ \frac{zh'(z)}{Q(z)} \right\} > 0 \quad (z \in \mathcal{U}).$$

If ψ is analytic in \mathcal{U} with $\psi(0) = q(0)$, $\psi(\mathcal{U}) \subset \Omega$ and

$$\theta(\psi(z)) + z\psi'(z)\phi(\psi(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) \quad (z \in \mathcal{U}),$$

then $\psi(z) \prec q(z)$ ($z \in \mathcal{U}$) and the function q is the best dominant.

3. MAIN RESULTS

Unless otherwise mentioned, we assume throughout the sequel that $a > 0, c > 0, 0 \leq \lambda \leq 1$ and $-1 \leq B < A \leq 1$.

We first solve the Fekete-Szegő problem for the class $\mathcal{V}_p^\lambda(a, c, A, B)$.

Theorem 3.1. If $\gamma \in \mathbb{R}$ and the function f , given by (1.1) belongs to the class $\mathcal{V}_p^\lambda(a, c, A, B)$, then

$$\left| a_{p+2} - \gamma a_{p+1}^2 \right| \leq \begin{cases} \frac{-p(A-B)cQ}{(p+\lambda(1-p))^2(p+\lambda(2-p))a(a)_2}, & \gamma < \rho_1 \\ \frac{p(A-B)(c)_2}{\{p+(2-p)\lambda\}(a)_2}, & \rho_1 \leq \gamma \leq \rho_2 \\ \frac{p(A-B)cQ}{(p+\lambda(1-p))^2(p+\lambda(2-p))a(a)_2}, & \gamma > \rho_2, \end{cases} \quad (3.1)$$

where

$$Q = \left[\gamma p(p+\lambda(2-p))(A-B)(a+1)c + \{B(p+\lambda(1-p))^2 - \lambda p(A-B)\}a(c+1) \right],$$

$$\rho_1 = \frac{[\lambda p(A-B) - (1+B)\{p+\lambda(1-p)\}^2] a(c+1)}{p\{p+\lambda(2-p)\}(A-B)(a+1)c},$$

and

$$\rho_2 = \frac{[\lambda p(A-B) + (1-B)\{p+\lambda(1-p)\}^2] a(c+1)}{p\{p+\lambda(2-p)\}(A-B)(a+1)c}.$$

All these results are sharp.

Proof. From (1.9), it follows that

$$(1-\lambda) \frac{\mathcal{L}_p(a, c)f(z)}{z^p} + \frac{\lambda}{p} \frac{z(\mathcal{L}_p(a, c)f(z))'(z)}{\mathcal{L}_p(a, c)f(z)} = \frac{1-A+(1+A)\varphi(z)}{1-B+(1+B)\varphi(z)} \quad (z \in \mathcal{U}),$$

where the function φ defined by (2.1) belongs to the class \mathcal{P} . Substituting the power series expansion of $\mathcal{L}_p(a, c)f$ and φ in the above expression, we deduce that

$$a_{p+1} = \frac{p(A-B)c}{2(p+\lambda(1-p))a} q_1, \quad (3.2)$$

and

$$\begin{aligned} a_{p+2} &= \frac{p(A-B)(c)_2}{2(p+\lambda(2-p))(a)_2} \left\{ q_2 - \left(\frac{1+B}{2} \right) q_1^2 + \frac{\lambda p(A-B)}{2(p+\lambda(1-p))^2} q_1^2 \right\} \\ &= q_2 + \frac{\lambda p(A-B) - (1+B)(p+\lambda(1-p))^2}{2(p+\lambda(1-p))^2} q_1^2. \end{aligned} \quad (3.3)$$

With the aid of (3.2) and (3.3), we get

$$\left| a_{p+2} - \gamma a_{p+1}^2 \right| = \frac{p(A-B)(c)_2}{2(p+\lambda(2-p))(a)_2} \\ \times \left| q_2 - \frac{\gamma p(p+\lambda(2-p))(A-B)(a+1)c + \{(1+B)(p+\lambda(1-p))^2 - \lambda p(A-B)\}a(c+1)}{2(p+\lambda(1-p))^2 a(c+1)} q_1^2 \right|,$$

which in view of Lemma 2.1 yields

$$\left| a_{p+2} - \gamma a_{p+1}^2 \right| = \frac{p(A-B)(c)_2}{(p+\lambda(2-p))(a)_2} \\ \times \max \left\{ 1, \frac{\left| \gamma p(p+\lambda(2-p))(A-B)(a+1)c + \{B(p+\lambda(1-p))^2 - \lambda p(A-B)\}a(c+1) \right|}{(p+\lambda(1-p))^2 a(c+1)} \right\}. \quad (3.4)$$

Now, we consider the following cases.

(i) If

$$\left| \frac{\gamma p(p+\lambda(2-p))(A-B)(a+1)c + \{B(p+\lambda(1-p))^2 - \lambda p(A-B)\}a(c+1)}{(p+\lambda(1-p))^2 a(c+1)} \right| \leq 1,$$

then it is easily seen that $\rho_1 \leq \gamma \leq \rho_2$ and (3.4) gives the second estimate in (3.1).

(ii) For

$$\left| \frac{\gamma p(p+\lambda(2-p))(A-B)(a+1)c + \{B(p+\lambda(1-p))^2 - \lambda p(A-B)\}a(c+1)}{(p+\lambda(1-p))^2 a(c+1)} \right| > 1,$$

we have either

$$\frac{\gamma p(p+\lambda(2-p))(A-B)(a+1)c + \{B(p+\lambda(1-p))^2 - \lambda p(A-B)\}a(c+1)}{(p+\lambda(1-p))^2 a(c+1)} < -1$$

or

$$\frac{\gamma p(p+\lambda(2-p))(A-B)(a+1)c + \{B(p+\lambda(1-p))^2 - \lambda p(A-B)\}a(c+1)}{(p+\lambda(1-p))^2 a(c+1)} > 1.$$

The above inequalities implies that either $\gamma < \rho_1$ or $\gamma > \rho_2$. Thus, again by use of (3.4), we get the first and the third estimate in (3.1).

We note that the results are sharp for the function f defined in \mathcal{U} by

$$(1-\lambda) \frac{\mathcal{L}_p(a, c)f(z)}{z^p} + \frac{\lambda z (\mathcal{L}_p(a, c)f)'(z)}{p \mathcal{L}_p(a, c)f(z)} = \begin{cases} \frac{1+Az}{1+Bz}, & \text{if } \gamma < \rho_1 \text{ or } \gamma > \rho_2 \\ \frac{1+Az^2}{1+Bz^2}, & \text{if } \rho_1 \leq \gamma \leq \rho_2, \end{cases}$$

where $0 \leq \lambda \leq 1, a > 0, c > 0$ and $-1 \leq B < A \leq 1$. This completes the proof of Theorem 3.1. \square

Taking $\lambda = 1, A = 1 - (2\alpha/p)$ ($0 \leq \alpha < p$) and $B = -1$ in Theorem 3.1, we obtain

Corollary 3.1. If $\gamma \in \mathbb{R}$ and the function f , given by (1.1) belongs to the class $\mathcal{S}_p(a, c; \alpha)$, then

$$\left| a_{p+2} - \gamma a_{p+1}^2 \right| \leq \begin{cases} \frac{(p-\alpha)\{(2(p-\alpha)+1)a(c)_2 - 4\gamma(p-\alpha)(a+1)c^2\}}{a(a)_2}, & \gamma < \frac{a(c+1)}{2(a+1)c} \\ \frac{(p-\alpha)(c)_2}{(a)_2}, & \frac{a(c+1)}{2(a+1)c} \leq \gamma \leq \frac{(p+1-\alpha)a(c+1)}{2(p-\alpha)(a+1)c} \\ \frac{(p-\alpha)\{4\gamma(p-\alpha)(a+1)c^2 - (2(p-\alpha)+1)a(c)_2\}}{a(a)_2}, & \gamma > \frac{(p+1-\alpha)a(c+1)}{2(p-\alpha)(a+1)c}. \end{cases}$$

These results are sharp for the function $f \in \mathcal{A}_p$ defined in \mathcal{U} by

$$f(z) = \begin{cases} \varphi_p(c, a; z) * \frac{z^p}{(1-z)^{2(p-\alpha)}}, & \gamma < \frac{a(c+1)}{2(a+1)c} \quad \text{or} \quad \gamma > \frac{(p+1-\alpha)a(c+1)}{2(p-\alpha)(a+1)c}, \\ \varphi_p(c, a; z) * \frac{z^p}{(1-z^2)^{p-\alpha}}, & \frac{a(c+1)}{2(a+1)c} \leq \gamma \leq \frac{(p+1-\alpha)a(c+1)}{2(p-\alpha)(a+1)c}. \end{cases}$$

Remark 3.1. (i) Setting $c = a$ ($a = p+1$ and $c = p$, respectively) in Corollary 3.1, we get the corresponding results obtained by Hayami and Owa [6, Theorem 3 and Theorem 4].

(ii) Using the fact that $|q_1| \leq 2$ in (3.2) and Lemma 2.1 in (3.3), we get the following coefficient estimates for a function f , given by (1.1) in the class $\mathcal{V}_p^\lambda(a, c, A, B)$,

$$|a_{p+1}| \leq \frac{p(A-B)c}{\{p+\lambda(1-p)\}a}$$

and

$$|a_{p+2}| \leq \frac{p(A-B)(c)_2}{\{p+\lambda(2-p)\}(a)_2} \max \left\{ 1, \frac{|B\{p+\lambda(1-p)\}^2 - \lambda p(A-B)|}{\{p+\lambda(1-p)\}^2} \right\}.$$

Both the estimates are sharp.

For the case $\lambda = 0$, $A = 1 - (2\alpha/p)$ ($0 \leq \alpha < p$) and $B = -1$, Theorem 3.1 yields the following result.

Corollary 3.2. If $\gamma \in \mathbb{R}$ and the function f , given by (1.1) belongs to the class $\mathcal{R}_{a,c}(\alpha)$, then

$$\left| a_{p+2} - \gamma a_{p+1}^2 \right| \leq \begin{cases} -\frac{2\left(1-\frac{\alpha}{p}\right)\left\{2\gamma\left(1-\frac{\alpha}{p}\right)(a+1)c - a(c+1)\right\}}{a(a)_2}, & \gamma < 0 \\ \frac{2\left(1-\frac{\alpha}{p}\right)(c)_2}{(a)_2}, & 0 \leq \gamma \leq \frac{\left(1-\frac{\alpha}{p}\right)^{-1}a(c+1)}{(a+1)c} \\ \frac{2\left(1-\frac{\alpha}{p}\right)\left\{2\gamma\left(1-\frac{\alpha}{p}\right)(a+1)c - a(c+1)\right\}}{a(a)_2}, & \gamma > \frac{\left(1-\frac{\alpha}{p}\right)^{-1}a(c+1)}{(a+1)c}. \end{cases}$$

The results are sharp for the function $f \in \mathcal{A}_p$ defined in \mathcal{U} by

$$f(z) = \begin{cases} \varphi_p(c, a; z) * \frac{z^p \left\{ 1 + \left(1 - \frac{2\alpha}{p} \right) z \right\}}{1 - z}, & \gamma < 0 \text{ or } \gamma > \frac{\left(1 - \frac{\alpha}{p} \right)^{-1} a(c+1)}{(a+1)c} \\ \varphi_p(c, a; z) * \frac{z^p \left\{ 1 + \left(1 - \frac{2\alpha}{p} \right) z^2 \right\}}{1 - z^2}, & 0 \leq \gamma \leq \frac{\left(1 - \frac{\alpha}{p} \right)^{-1} a(c+1)}{(a+1)c}. \end{cases}$$

Letting $c = a$ in Corollary 3.2, we get

Corollary 3.3. If $\gamma \in \mathbb{R}$ and the function $f \in \mathcal{A}$, given by (1.1) satisfies the condition

$$\operatorname{Re} \left\{ \frac{f(z)}{z^p} \right\} > \frac{\alpha}{p} \quad (0 \leq \alpha < p; z \in \mathcal{U}),$$

then

$$\left| a_{p+2} - \gamma a_{p+1}^2 \right| \leq \begin{cases} -2 \left(1 - \frac{\alpha}{p} \right) \left\{ 2\gamma \left(1 - \frac{\alpha}{p} \right) - 1 \right\}, & \gamma < 0 \\ 2 \left(1 - \frac{\alpha}{p} \right), & 0 \leq \gamma \leq \left(1 - \frac{\alpha}{p} \right)^{-1} \\ 2 \left(1 - \frac{\alpha}{p} \right) \left\{ 2\gamma \left(1 - \frac{\alpha}{p} \right) - 1 \right\}, & \gamma > \left(1 - \frac{\alpha}{p} \right)^{-1}. \end{cases}$$

These results are sharp for the function $f \in \mathcal{A}$ defined in \mathcal{U} by

$$f(z) = \begin{cases} \frac{z^p \left\{ 1 + \left(1 - \frac{2\alpha}{p} \right) z \right\}}{1 - z}, & \gamma < 0 \text{ or } \gamma > \left(1 - \frac{\alpha}{p} \right)^{-1} \\ \frac{z^p \left\{ 1 + \left(1 - \frac{2\alpha}{p} \right) z^2 \right\}}{1 - z^2}, & 0 \leq \gamma \leq \left(1 - \frac{\alpha}{p} \right)^{-1}. \end{cases}$$

For the choice $a = p + 1$ and $c = p$ in Corollary 3.2, we obtain

Corollary 3.4. If $\gamma \in \mathbb{R}$ and the function $f \in \mathcal{A}$, given by (1.1) satisfies the condition

$$\operatorname{Re} \left\{ \frac{f'(z)}{z^{p-1}} \right\} > \alpha \quad (0 \leq \alpha < p; z \in \mathcal{U}),$$

then

$$\left| a_{p+2} - \gamma a_{p+1}^2 \right| \leq \begin{cases} -\frac{2(p-\alpha)\{2\gamma(p+2)(p-\alpha) - (p+1)^2\}}{(p+1)^2(p+2)}, & \gamma < 0 \\ \frac{2(p-\alpha)}{p+2}, & 0 \leq \gamma \leq \frac{(p+1)^2}{(p+2)(p-\alpha)} \\ \frac{2(p-\alpha)\{2\gamma(p+2)(p-\alpha) - (p+1)^2\}}{(p+1)^2(p+2)}, & \gamma > \frac{(p+1)^2}{(p+2)(p-\alpha)}. \end{cases}$$

These results are sharp for the function $f \in \mathcal{A}$ defined in \mathcal{U} by

$$f(z) = \begin{cases} \varphi_p(p, p+1; z) * \frac{z^p \left\{ 1 + \left(1 - \frac{2\alpha}{p} \right) z \right\}}{1 - z}, & \gamma < 0 \text{ or } \gamma > \left(1 - \frac{\alpha}{p} \right)^{-1} \\ \varphi_p(p, p+1; z) * \frac{z^p \left\{ 1 + \left(1 - \frac{2\alpha}{p} \right) z^2 \right\}}{1 - z^2}, & 0 \leq \gamma \leq \left(1 - \frac{\alpha}{p} \right)^{-1}. \end{cases}$$

Next, we prove the following subordination result.

Theorem 3.2. *If a function $f \in \mathcal{A}_p$ satisfies the subordination relation*

$$(1-\lambda)\frac{\mathcal{L}_p(a,c)f(z)}{z^p} + \frac{\lambda}{p}\frac{z(\mathcal{L}_p(a,c)f)'(z)}{\mathcal{L}_p(a,c)f(z)} \prec 1 + (1-\lambda)\frac{(A-B)z}{1+Bz} + \frac{\lambda(A-B)z}{p(1+Az)(1+Bz)} \quad (0 < \lambda \leq 1, -1 \leq B < A \leq 1; z \in \mathcal{U}), \quad (3.5)$$

then

$$\frac{\mathcal{L}_p(a,c)f(z)}{z^p} \prec \frac{1+Az}{1+Bz} = \tilde{q}(z) \text{ (say)} \quad (z \in \mathcal{U}) \quad (3.6)$$

and the function \tilde{q} is the best dominant of (3.6).

Proof. Setting

$$q(z) = \frac{1+Az}{1+Bz} \quad (z \in \mathcal{U}), \theta(w) = \lambda + (1-\lambda)w \quad (w \in \mathbb{C}) \quad \text{and} \quad \phi(w) = \frac{\lambda}{pw} \quad (0 \neq w \in \mathbb{C}),$$

we see that

$$Q(z) = \frac{\lambda z q'(z)}{p q(z)} = \frac{\lambda(A-B)z}{p(1+Az)(1+Bz)}$$

and

$$\operatorname{Re} \left\{ \frac{z Q'(z)}{Q(z)} \right\} = \operatorname{Re} \left\{ \frac{1}{1+Az} - \frac{Bz}{1+Bz} \right\} > 0,$$

so that Q is starlike in \mathcal{U} . Further, letting $h(z) = \theta(q(z)) + Q(z)$, we get

$$\operatorname{Re} \left\{ \frac{z h'(z)}{Q(z)} \right\} = \frac{(1-\lambda)p}{\lambda} \operatorname{Re}\{q(z)\} + \operatorname{Re} \left\{ \frac{z Q'(z)}{Q(z)} \right\} > 0 \quad (z \in \mathcal{U}).$$

Suppose that

$$\psi(z) = \frac{\mathcal{L}_p(a,c)f(z)}{z^p} \quad (z \in \mathcal{U}).$$

Then the hypothesis (3.5) implies that

$$\theta(\psi(z)) + z\psi'(z)\phi(\psi(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) \quad (z \in \mathcal{U}),$$

which in view of Lemma 2.2 gives the required assertion (3.6) and the function \tilde{q} is the best dominant. The proof of Theorem 3.2 is thus completed. \square

Taking $\lambda = 1$, $A = -\alpha/p$ and $B = -1$ in Theorem 3.2, we get

Corollary 3.5. *If a function $f \in \mathcal{A}_p$ satisfies the subordination relation*

$$\frac{z(\mathcal{L}_p(a,c)f)'(z)}{\mathcal{L}_p(a,c)f(z)} \prec p + \frac{(p-\alpha)z}{(p-\alpha z)(1-z)} \quad (0 \leq \alpha < p; z \in \mathcal{U}),$$

then

$$\operatorname{Re} \left\{ \frac{\mathcal{L}_p(a,c)f(z)}{z^p} \right\} > \frac{p+\alpha}{2p} \quad (z \in \mathcal{U})$$

and the result is the best possible.

Putting $A = 1 - (2\alpha/p)$ ($0 \leq \alpha < p$) and $B = -1$ in Theorem 3.2, we obtain

Corollary 3.6. *If a function $f \in \mathcal{A}_p$ satisfies the subordination relation*

$$(1 - \lambda) \frac{\mathcal{L}_p(a, c)f(z)}{z^p} + \frac{\lambda z (\mathcal{L}_p(a, c)f)'(z)}{p \mathcal{L}_p(a, c)f(z)} \\ \prec 1 + \frac{2(1 - \lambda)(p - \alpha)}{p} \frac{z}{1 - z} + \frac{2\lambda(p - \alpha)z}{p\{p + (p - 2\alpha)z\}(1 - z)} \quad (0 < \lambda \leq 1; z \in \mathcal{U}),$$

then

$$\operatorname{Re} \left\{ \frac{\mathcal{L}_p(a, c)f(z)}{z^p} \right\} > \frac{\alpha}{p} \quad (z \in \mathcal{U})$$

and the result is the best possible.

For the choice $c = a$ ($a = p + 1$ and $c = p$, respectively), Corollary 3.5 yields the following result.

Corollary 3.7. *For $0 < \lambda \leq 1$ and $0 \leq \alpha < p$, let*

$$\Phi_p(\lambda, \alpha; z) = 1 + \frac{2(1 - \lambda)(p - \alpha)}{p} \frac{z}{1 - z} + \frac{2\lambda(p - \alpha)z}{p\{p + (p - 2\alpha)z\}(1 - z)} \quad (z \in \mathcal{U}).$$

(i) *If a function $f \in \mathcal{A}_p$ satisfies*

$$(1 - \lambda) \frac{f(z)}{z^p} + \frac{\lambda}{p} \frac{zf'(z)}{f(z)} \prec \Phi_p(\lambda, \alpha; z) \quad (z \in \mathcal{U}),$$

then

$$\operatorname{Re} \left\{ \frac{f(z)}{z^p} \right\} > \frac{\alpha}{p} \quad (z \in \mathcal{U}).$$

(ii) *If a function $f \in \mathcal{A}_p$ satisfies*

$$(1 - \lambda) \frac{f'(z)}{z^{p-1}} + \lambda \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \prec p \Phi_p(\lambda, \alpha; z) \quad (z \in \mathcal{U}),$$

then

$$\operatorname{Re} \left\{ \frac{f'(z)}{z^{p-1}} \right\} > \alpha \quad (z \in \mathcal{U}).$$

The results in (i) and (ii) are the best possible.

Remark 3.2. 1. Letting $a = p + 1, c = p$ in Corollary 3.5 and noting that

$$p + \operatorname{Re} \left\{ \frac{(p - \alpha)z}{(p - \alpha z)(1 - z)} \right\} > \frac{(2p - 1)(p + \alpha) + 2\alpha}{2(p + \alpha)} \quad (0 \leq \alpha < p; z \in \mathcal{U}),$$

we get the corresponding result obtained by Deniz [2, Theorem 2.1].

2. Setting $p = 1$ and $\alpha = 0$ ($p = 1$ and $\alpha = 1/2$, respectively) in Corollary 3.6, we get the following the following results due to Singh et al. [19, Theorem 1 and Theorem 2].

(i) *If $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re} \left\{ (1 - \lambda)f'(z) + \lambda \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \lambda \quad (0 < \lambda < 1; z \in \mathcal{U}),$$

then

$$\operatorname{Re}\{f'(z)\} > 0 \quad (z \in \mathcal{U})$$

and the result is sharp for the function $f(z) = -z - 2 \log(1 - z)$, $z \in \mathcal{U}$.

(ii) If $f \in \mathcal{A}$ satisfies

$$\operatorname{Re} \left\{ (1-\lambda)f'(z) + \lambda \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \frac{1}{2} \quad (0 < \lambda \leq 1; z \in \mathcal{U}),$$

then

$$\operatorname{Re}\{f'(z)\} > \frac{1}{2} \quad (z \in \mathcal{U})$$

and the result is sharp for the function $f(z) = -\log(1-z)$, $z \in \mathcal{U}$.

Theorem 3.3. If $0 < \lambda < 1$ and a function $f \in \mathcal{A}_p$ satisfies

$$\operatorname{Re} \left\{ \frac{\mathcal{L}_p(a, c)f(z)}{z^p} \right\} > \frac{\alpha}{p} \quad (0 \leq \alpha < p; z \in \mathcal{U}), \quad (3.7)$$

then

$$(1-\lambda) \frac{\mathcal{L}_p(a, c)f(z)}{z^p} + \frac{\lambda z (\mathcal{L}_p(a, c)f)'(z)}{p \mathcal{L}_p(a, c)f(z)} > (1-\lambda) \frac{\alpha}{p} + \lambda \quad (|z| < R_p(\lambda, \alpha)), \quad (3.8)$$

where

$$R_p(\lambda, \alpha) = \begin{cases} \frac{\{\lambda + (1-\lambda)(p-\alpha)\} - \sqrt{\lambda^2 + 2\lambda(1-\lambda)(p-\alpha)}}{(1-\lambda)(p-2\alpha)}, & \alpha \neq \frac{p}{2} \\ \frac{(1-\lambda)p}{2\lambda + (1-\lambda)p}, & \alpha = \frac{p}{2}. \end{cases}$$

The result is the best possible.

Proof. From (3.7), it follows that

$$\frac{\mathcal{L}_p(a, c)f(z)}{z^p} = \frac{\alpha}{p} + \left(1 - \frac{\alpha}{p}\right) \varphi(z) \quad (z \in \mathcal{U}),$$

where $\varphi \in \mathcal{P}$. Differentiating the above expression logarithmically followed by a simple calculations, we deduce that

$$\begin{aligned} & (1-\lambda) \frac{\mathcal{L}_p(a, c)f(z)}{z^p} + \frac{\lambda z (\mathcal{L}_p(a, c)f)'(z)}{p \mathcal{L}_p(a, c)f(z)} - (1-\lambda) \frac{\alpha}{p} - \lambda \\ &= (1-\lambda) \left(1 - \frac{\alpha}{p}\right) \left\{ \varphi(z) + \frac{\lambda z \varphi'(z)}{(1-\lambda)\{\alpha + (p-\alpha)\varphi(z)\}} \right\} \end{aligned}$$

so that

$$\begin{aligned} & \operatorname{Re} \left\{ (1-\lambda) \frac{\mathcal{L}_p(a, c)f(z)}{z^p} + \frac{\lambda z (\mathcal{L}_p(a, c)f)'(z)}{p \mathcal{L}_p(a, c)f(z)} \right\} - (1-\lambda) \frac{\alpha}{p} - \lambda \\ & \geq (1-\lambda) \left(1 - \frac{\alpha}{p}\right) \left[\operatorname{Re}\{\varphi(z)\} - \frac{\lambda |z\varphi'(z)|}{(1-\lambda)\{|\alpha + (p-\alpha)\varphi(z)|\}} \right]. \end{aligned} \quad (3.9)$$

Using the estimates [13]

$$\frac{|z\varphi'(z)|}{\operatorname{Re}\{\varphi(z)\}} \leq \frac{2r}{1-r^2} \quad \text{and} \quad \operatorname{Re}\{\varphi(z)\} \geq \frac{1-r}{1+r} \quad (|z| = r)$$

in (3.9), we get

$$\begin{aligned} & \operatorname{Re} \left\{ (1-\lambda) \frac{\mathcal{L}_p(a, c)f(z)}{z^p} + \frac{\lambda}{p} \frac{z (\mathcal{L}_p(a, c)f(z))'(z)}{\mathcal{L}_p(a, c)f(z)} \right\} - (1-\lambda) \frac{\alpha}{p} - \lambda \\ & \geq (1-\lambda) \left(1 - \frac{\alpha}{p} \right) \operatorname{Re}\{\varphi(z)\} \left[1 - \frac{2\lambda r}{(1-\lambda) \{ \alpha(1-r^2) + (p-\alpha)(1-r)^2 \}} \right]. \end{aligned} \quad (3.10)$$

We note that the right hand side of (3.10) is positive, provided $r < R_p(\lambda, \alpha)$, where $R_p(\lambda, \alpha)$ is defined as in the theorem.

To show that the bound $R_p(\lambda, \alpha)$ is the best possible, we consider the function $f \in \mathcal{A}_p$ defined by

$$f(z) = \varphi_p(c, a; z) \star \frac{z^p \left\{ 1 + \left(1 - \frac{2\alpha}{p} \right) z \right\}}{1-z} \quad (0 \leq \alpha < p; z \in \mathcal{U}).$$

It follows that

$$\frac{\mathcal{L}_p(a, c)f(z)}{z^p} = \frac{\left\{ 1 + \left(1 - \frac{2\alpha}{p} \right) z \right\}}{1-z} \quad (0 \leq \alpha < p; z \in \mathcal{U}),$$

which on differentiating logarithmically followed by a routine calculation yields

$$\begin{aligned} & (1-\lambda) \frac{\mathcal{L}_p(a, c)f(z)}{z^p} + \frac{\lambda}{p} \frac{z (\mathcal{L}_p(a, c)f(z))'(z)}{\mathcal{L}_p(a, c)f(z)} - (1-\lambda) \frac{\alpha}{p} - \lambda \\ & = (1-\lambda) \left(1 - \frac{\alpha}{p} \right) \frac{1+z}{1-z} \left[1 + \frac{2\lambda z}{\alpha(1-z^2) + (p-\alpha)(1-z)^2} \right] \\ & = 0 \quad \text{as } z \rightarrow -R_p(\lambda, \alpha). \end{aligned}$$

This completes the proof of Theorem 3.3. □

For the choice $c = a, p = 1$ and $\alpha = 0$ ($a = 2, c = p = 1$ and $\alpha = 0$, respectively), Theorem 3.3 yields the following result.

Corollary 3.8. *Let $0 < \lambda < 1$. If a function $f \in \mathcal{A}$ satisfies*

$$\operatorname{Re} \left\{ \frac{f(z)}{z} \right\} > 0 \quad (z \in \mathcal{U}),$$

then

$$\operatorname{Re} \left\{ (1-\lambda) \frac{f(z)}{z} + \lambda \frac{zf'(z)}{f(z)} \right\} > \lambda \quad (|z| < \tilde{R}(\lambda)),$$

and if it satisfies

$$\operatorname{Re}\{f'(z)\} > 0 \quad (z \in \mathcal{U}),$$

then

$$\operatorname{Re} \left\{ (1-\lambda)f'(z) + \lambda \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \lambda \quad (|z| < \tilde{R}(\lambda)),$$

where

$$\tilde{R}(\lambda) = \frac{1 - \sqrt{\lambda(2-\lambda)}}{1-\lambda}.$$

The results are the best possible.

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GENERALIZED φ -WEAK CONTRACTIVE FUZZY MAPPINGS AND RELATED FIXED POINT RESULTS ON COMPLETE METRIC SPACE

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ABSTRACT. In this paper, we discuss the existence and uniqueness of a (common) fixed point of generalized φ -weak contractive fuzzy mappings on complete metric spaces. We present some examples to illustrate the obtained results.

1. INTRODUCTION AND PRELIMINARIES

1.1. Fuzzy fixed points of fuzzy mappings. In fixed point theory, the importance of various contractive inequalities cannot be overemphasized. Existence theorems of fixed points have been established for mappings defined on various types of spaces and satisfying different types of contractive inequalities. The notion of fuzzy sets was introduced by Zadeh [27] in 1965. Following this initial result, Weiss [24] and Butnariu [9] studied on the characterization of several notion in the sense of fuzzy numbers. Heilpern [14] introduced the fuzzy mapping and further he established fuzzy Banach contraction principle on a complete metric space. Subsequently several other researchers studied the existence of fixed points and common fixed points of fuzzy mappings satisfying a contractive type condition on a metric space (see [1, 3, 4, 7, 8, 10, 16, 19, 20, 22, 25]).

The following are some definitions and concepts required for our discussion in the paper. In fact most of these are discussed in [13, 14, 17] in metric linear spaces. We discuss them in metric spaces.

Suppose that (X, d) is a metric space. A fuzzy set A over X is defined by a function μ_A ,

$$\mu_A : X \rightarrow [0, 1],$$

where μ_A is called a membership function of A , and the value $\mu_A(x)$ is called the grade of membership of x in X . The value represents the degree of x belonging to the fuzzy set X . The α -level set of A is denoted by $[A]_\alpha$, and is defined as follows:

$$\begin{aligned} [A]_\alpha &= \{x : A(x) \geq \alpha\} \text{ if } \alpha \in (0, 1], \\ [A]_0 &= \overline{\{x : A(x) > 0\}}, \end{aligned}$$

where \overline{B} denotes the closure of the set B .

Let $\mathfrak{F}(X)$ be the collection of all fuzzy sets in a metric space X . For $A, B \in \mathfrak{F}(X)$, $A \subset B$ means $A(x) \leq B(x)$ for each $x \in X$. A fuzzy set A in a metric linear space V is said to be an approximate quantity if and only if $[A]_\alpha$ is compact and convex in V for each $\alpha \in [0, 1]$ and $\sup_{x \in V} A(x) = 1$. We

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denote the collection of all approximate quantities in a metric linear space V by $W(V)$. Clearly when X is a metric linear space $W(X) \subset \mathfrak{F}(X)$.

Let X be an arbitrary set and (Y, d) be a metric space. A mapping G is called fuzzy mapping if G is a mapping from X into $\mathfrak{F}(Y)$. A fuzzy mapping G is a fuzzy subset on $X \times Y$ with membership function $G(x)(y)$. The function $G(x)(y)$ is the grade of membership of y in $G(x)$. For convenience, we denote α -level set of $G(x)$ by $[Gx]_\alpha$ instead of $[G(x)]_\alpha$.

Definition 1. Let G, H be fuzzy mappings from X into $\mathfrak{F}(X)$. A point z in X is called an α -fuzzy fixed point of H if $z \in [Hz]_\alpha$. The point z is called a common α -fuzzy fixed point of G and H if $z \in [Gz]_\alpha \cap [Hz]_\alpha$.

1.2. Fixed point theory on metric spaces. Let (X, d) be a metric space, $B(X)$ and $CB(X)$ be the sets of all nonempty bounded and closed subsets of X , respectively. For $P, Q \in B(X)$ we define

$$\delta(P, Q) = \sup\{d(p, q) : p \in P, q \in Q\}$$

and

$$D(P, Q) = \inf\{d(p, q) : p \in P, q \in Q\}.$$

If $P = \{p\}$, we write $\delta(P, Q) = \delta(p, Q)$, and if $Q = \{q\}$, then $\delta(p, Q) = d(p, q)$. For P, Q, R in $B(X)$ one can easily prove the following properties.

$$\begin{aligned} \delta(P, Q) &= \delta(P, Q) \geq 0, \\ \delta(P, Q) &\leq \delta(P, R) + \delta(R, Q), \\ \delta(P, P) &= \sup\{d(p, r) : p, r \in P\} = \text{diam } P \\ \delta(P, Q) &= 0 \text{ implies that } P = Q = \{p\}. \end{aligned}$$

Let $\{A_n\}$ be a sequence in $B(X)$. Then the sequence $\{A_n\}$ converges to A if and only if

- (i) $a \in A$ implies that $a_n \rightarrow a$ for some sequence $\{a_n\}$ with $a_n \in A_n$ for $n \in N$,
- and
- (ii) for any $\varepsilon > 0$, there exist $n, m \in N$ with $n > m$ such that

$$A_n \subseteq A_\varepsilon = \{x \in X : d(x, a) < \varepsilon \text{ for some } a \in A\}.$$

See [10, 11].

The following results will be useful in the proof of our main result.

Lemma 1. [11] Let $\{A_n\}$ and $\{B_n\}$ be sequences in $B(X)$ and (X, d) be a complete metric space. If $A_n \rightarrow A \in B(X)$ and $B_n \rightarrow B \in B(X)$, then $\delta(A_n, B_n) \rightarrow \delta(A, B)$.

Lemma 2. [15] Let (X, d) be a complete metric space. If $\{A_n\}$ is a sequence of nonempty bounded subsets in (X, d) and if $\delta(A_n, y) \rightarrow 0$ for some $y \in X$, then $A_n \rightarrow \{y\}$.

Theorem 1. [21] Let (X, d) be a complete metric space and T be a φ -weak contraction on X ; that is, for each $x, y \in X$, there exists a function $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that φ is positive on $(0, \infty)$ and $\varphi(0) = 0$, and

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) \quad (1)$$

Also if φ is a continuous and nondecreasing function, then T has a unique fixed point.

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A weakly contractive mapping is a map satisfying the inequality (1) which was first defined by Alber and Guerre-Delabriere [2]. For more results on these mappings, see [5, 6, 12, 18, 23] and the related references therein. Zhang and Song [26] gave the following theorem.

Theorem 2. [26] *Let (X, d) be a complete metric space and $T, S : X \rightarrow X$ be two mappings such that for each $x, y \in X$,*

$$d(Tx, Sy) \leq m(x, y) - \varphi(m(x, y)),$$

where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a lower semi-continuous function with $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$, and

$$m(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2} [d(y, Tx) + d(x, Sy)] \right\}$$

Then there exists a unique point $u \in X$ such that $u = Tu = Su$.

2. MAIN RESULTS

This section includes the main theorem of the paper. More precisely, we find out a common fixed point of fuzzy mappings which is also unique. Let (X, d) be a complete metric space. Then we define and use the following notations:

$$\begin{aligned} \xi^X &= \{A : A \text{ is the subset of } X\}, \\ B(\xi^X) &= \{A \in \xi^X : A \text{ is nonempty bounded}\}, \\ CB(\xi^X) &= \{A \in \xi^X : A \text{ is nonempty closed and bounded}\}. \end{aligned}$$

Theorem 3. *Let (X, d) be a complete metric space and $S, T : X \rightarrow \mathfrak{F}(X)$ and for $x \in X$, there exist $\alpha_S(x), \alpha_T(x) \in (0, 1]$ such that $[Sx]_{\alpha_S(x)}, [Tx]_{\alpha_T(x)} \in B(\xi^x)$, such that for all $x, y \in X$.*

$$\delta([Sx]_{\alpha_S(x)}, [Ty]_{\alpha_T(y)}) \leq M(x, y) - \varphi(M(x, y)) \quad (2)$$

where, $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a lower semicontinuous function with $\varphi(t) > 0$ for $t \in (0, \infty)$ and $\varphi(0) = 0$ and

$$M(x, y) = \max \left\{ d(x, y), D(x, [Sx]_{\alpha_S(x)}), D(y, [Ty]_{\alpha_T(y)}), \frac{1}{2} [D(y, [Sx]_{\alpha_S(x)}) + D(x, [Ty]_{\alpha_T(y)})] \right\} \quad (3)$$

Then there exists a unique $z \in [Sx]_{\alpha_S(x)}$ and $z \in [Ty]_{\alpha_T(y)}$.

Proof. Take $a_0 \in X$. According to the given condition, there exists $\alpha(a_0) \in (0, 1]$ such that $[Sa_0]_{\alpha(a_0)} \in CB(\xi^X)$. Let us denote $\alpha(a_0)$ by α_1 . We set $a_1 \in [Sa_0]_{\alpha_1}$, for this a_1 there exists $\alpha_2 \in (0, 1]$ such that, $[Ta_1]_{\alpha_2} \in CB(\xi^x)$. Iteratively, we shall construct a sequence $\{a_n\}$ in X in a way that

$$\begin{aligned} a_{2k+1} &\in [Sa_{2k}]_{\alpha_{2k+1}}, \\ a_{2k+2} &\in [Ta_{2k+1}]_{\alpha_{2k+2}} \end{aligned}$$

It is clear that if $M(a_n, a_{n+1}) = 0$, then the proof is completed. Consequently, throughout the proof, we suppose that

$$M(a_n, a_{n+1}) > 0 \text{ for all } n \geq 0. \quad (4)$$

We shall prove that

$$d(a_{2n+1}, a_{2n+2}) \leq d(a_{2n}, a_{2n+1}) \text{ for all } n \geq 0. \quad (5)$$

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Suppose, on the contrary, that there exists $\tilde{n} \geq 0$ such that

$$d(a_{2\tilde{n}+1}, a_{2\tilde{n}+2}) > d(a_{2\tilde{n}}, a_{2\tilde{n}+1}),$$

which yields that

$$M(a_{2\tilde{n}}, a_{2\tilde{n}+1}) \leq d(a_{2\tilde{n}+1}, a_{2\tilde{n}+2}).$$

Regarding (2), we derive that

$$\begin{aligned} d(a_{2\tilde{n}+1}, a_{2\tilde{n}+2}) &\leq \delta([Sa_{2\tilde{n}}]_{\alpha(a_{2\tilde{n}})}, [Ta_{2\tilde{n}+1}]_{\alpha(a_{2\tilde{n}+1})}) \\ &\leq M(a_{2\tilde{n}}, a_{2\tilde{n}+1}) - \varphi(M(a_{2\tilde{n}}, a_{2\tilde{n}+1})) \\ &\leq d(a_{2\tilde{n}}, a_{2\tilde{n}+1}) - \varphi(M(a_{2\tilde{n}}, a_{2\tilde{n}+1})). \end{aligned}$$

Consequently, we obtain that $\varphi(M(a_{2\tilde{n}}, a_{2\tilde{n}+1})) = 0$ and so we have $M(a_{2\tilde{n}}, a_{2\tilde{n}+1}) = 0$. This contradicts the observation (4). Hence we have the inequality (5). In an analogous way, one can conclude that

$$d(a_{2n+2}, a_{2n+3}) \leq d(a_{2n+1}, a_{2n+2}) \text{ for all } n \geq 0. \quad (6)$$

By combining (5) and (6), we get that

$$d(a_{n+1}, a_{n+2}) \leq d(a_n, a_{n+1}) \text{ for all } n \geq 0.$$

Hence we derive that the sequence $\{d(a_n, a_{n+1})\}$ is non-increasing and bounded below. Since (X, d) is complete, there exists $l \geq 0$ such that

$$\lim_{n \rightarrow \infty} d(a_n, a_{n+1}) = l. \quad (7)$$

Due to hypothesis, we observe that

$$\begin{aligned} d(a_{2n}, a_{2n+1}) &\leq M(a_{2n}, a_{2n+1}) \\ &= \max \left\{ d(a_{2n}, a_{2n+1}), D(a_{2n}, [Sa_{2n}]_{\alpha(a_{2n})}), D(a_{2n+1}, [Ta_{2n+1}]_{\alpha(a_{2n+1})}), \right. \\ &\quad \left. \frac{1}{2} [D(a_{2n+1}, [Sa_{2n}]_{\alpha(a_{2n})}) + D(a_{2n}, [Ta_{2n+1}]_{\alpha(a_{2n+1})})] \right\} \\ &\leq \max \left\{ d(a_{2n}, a_{2n+1}), d(a_{2n+1}, a_{2n+2}), \frac{1}{2} [d(a_{2n}, a_{2n+1}) + d(a_{2n+1}, a_{2n+2})] \right\}. \end{aligned}$$

Thus we have

$$l \leq \lim_{n \rightarrow \infty} M(a_{2n}, a_{2n+1}) \leq l.$$

Hence we get

$$\lim_{n \rightarrow \infty} M(a_{2n}, a_{2n+1}) = l. \quad (8)$$

Analogously, we have

$$\lim_{n \rightarrow \infty} M(a_{2n+1}, a_{2n+2}) = l. \quad (9)$$

By combining (7), (8) and (9), we derive that

$$\lim_{n \rightarrow \infty} d(a_n, a_{n+1}) = \lim_{n \rightarrow \infty} M(a_n, a_{n+1}) = l.$$

By the lower semi-continuity of φ , we find

$$\varphi(l) \leq \liminf_{n \rightarrow \infty} \varphi(M(a_n, a_{n+1})).$$

Now we claim that $l = 0$. From (2), we have

$$\begin{aligned} d(a_{2n+1}, a_{2n+2}) &\leq \delta([Sa_{2n}]_{\alpha(a_{2n})}, [Ta_{2n+1}]_{\alpha(a_{2n+1})}) \\ &\leq M(a_{2n}, a_{2n+1}) - \varphi(M(a_{2n}, a_{2n+1})) \end{aligned}$$

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By letting the upper limit as $n \rightarrow \infty$ in the inequality above, we obtain

$$\begin{aligned} l &\leq l - \liminf_{n \rightarrow \infty} \varphi(M(a_{2n}, a_{2n+1})) \\ &\leq l - \varphi(l), \end{aligned}$$

that is, $\varphi(l) = 0$. Regarding the property of φ , we conclude that $l = 0$.

As a next step, we shall show that $\{a_n\}$ is Cauchy. For this purpose, it is sufficient to get that $\{a_{2n}\}$ is Cauchy. Suppose, on the contrary, that $\{a_{2n}\}$ is not Cauchy. Then there is an $\epsilon > 0$ such that for an even integer $2k$ there exist even integers $2m(k) > 2n(k) > 2k$ such that

$$d(a_{2n(k)}, a_{2m(k)}) > \epsilon. \quad (10)$$

For every even integer $2k$, let $2m(k)$ be the least positive integer exceeding $2n(k)$, satisfying (10), and such that

$$d(a_{2n(k)}, a_{2m(k)-2}) < \epsilon. \quad (11)$$

Now

$$\begin{aligned} \epsilon &\leq d(a_{2n(k)}, a_{2m(k)}) \\ &\leq d(a_{2n(k)}, a_{2m(k)-2}) + d(a_{2m(k)-2}, a_{2m(k)-1}) \\ &\quad + d(a_{2m(k)-1}, a_{2m(k)}). \end{aligned}$$

By (10) and (11), we get

$$\lim_{k \rightarrow \infty} d(a_{2n(k)}, a_{2m(k)}) = \epsilon. \quad (12)$$

Due to the triangle inequality, we have

$$|d(a_{2n(k)}, a_{2m(k)-1}) - d(a_{2n(k)}, a_{2m(k)})| < d(a_{2m(k)-1}, a_{2m(k)}).$$

By (12), we get

$$d(a_{2n(k)}, a_{2m(k)-1}) = \epsilon. \quad (13)$$

Now by (3) we observe that

$$\begin{aligned} d(a_{2n(k)}, a_{2m(k)-1}) &\leq M(a_{2n(k)}, a_{2m(k)-1}) \\ &= \max \left\{ \begin{array}{l} d(a_{2n(k)}, a_{2m(k)-1}), D(a_{2n(k)}, [Sa_{2n(k)}]_{\alpha(a_{2n(k)})}), \\ D(a_{2m(k)-1}, [Ta_{2m(k)-1}]_{\alpha(a_{2m(k)-1})}), \\ \frac{1}{2} D(a_{2m(k)-1}, [Sa_{2n(k)}]_{\alpha(a_{2n(k)})}) + D(a_{2n(k)}, [Ta_{2m(k)-1}]_{\alpha(a_{2m(k)-1})}) \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{l} d(a_{2n(k)}, a_{2m(k)-1}), d(a_{2n(k)}, a_{2n(k)+1}), d(a_{2m(k)-1}, a_{2m(k)}) \\ \frac{1}{2} [d(a_{2m(k)-1}, a_{2n(k)+1}) + d(a_{2n(k)}, a_{2m(k)})] \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{l} d(a_{2n(k)}, a_{2m(k)-1}), d(a_{2n(k)}, a_{2n(k)+1}), d(a_{2m(k)-1}, a_{2m(k)}) \\ \frac{1}{2} [d(a_{2m(k)-1}, a_{2n(k)}) + d(a_{2n(k)}, a_{2n(k)+1}) + d(a_{2n(k)}, a_{2m(k)})] \end{array} \right\}. \end{aligned}$$

By letting $k \rightarrow \infty$ in the inequality above and taking (12) and (13) into account, we conclude that

$$\epsilon \leq \lim_{k \rightarrow \infty} M(x_{2n(k)}, x_{2m(k)-1}) \leq \epsilon.$$

Consequently, we have

$$\lim_{k \rightarrow \infty} M(x_{2n(k)}, x_{2m(k)-1}) = \epsilon.$$

By the lower semi-continuity of φ , we derive that

$$\varphi(\epsilon) \leq \liminf_{k \rightarrow \infty} \varphi(M(x_{2n(k)}, x_{2m(k)-1})).$$

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Now by (2), we get

$$\begin{aligned} & d(x_{2n(k)}, x_{2m(k)}) \\ & \leq d(x_{2n(k)}, x_{2n(k)+1}) + \delta([Sx_{2n(k)}]_{\alpha(x_{2n(k)})}, [Tx_{2m(k)-1}]_{\alpha(x_{2m(k)-1})}) \\ & \leq d(x_{2n(k)}, x_{2n(k)+1}) + M(x_{2n(k)}, x_{2m(k)-1}) - \varphi(M(x_{2n(k)}, x_{2m(k)-1})). \end{aligned}$$

Letting the upper limit $k \rightarrow \infty$ in the inequality above, we have

$$\begin{aligned} \epsilon & \leq \epsilon - \liminf_{k \rightarrow \infty} \varphi(M(a_{2n(k)}, a_{2m(k)-1})) \\ & \leq \epsilon - \varphi(\epsilon), \end{aligned}$$

which is a contradiction. Hence we conclude that $\{a_{2n}\}$ is a Cauchy sequence. It follows from the completeness of X that there exists $c \in X$ such that $a_n \rightarrow c$ as $n \rightarrow \infty$. Furthermore, $a_{2n} \rightarrow c$ and $a_{2n+1} \rightarrow c$.

We shall prove that $c \in [Sc]_{\alpha_S(c)}$.

$$\begin{aligned} & D(c, [Sc]_{\alpha_S(c)}) \leq M(c, a_{2n-1}) \\ & = \max \left\{ d(c, a_{2n-1}), D(c, [Sc]_{\alpha_S(c)}), D(a_{2n-1}, [Ta_{2n-1}]_{\alpha_T(a_{2n-1})}), \right. \\ & \quad \left. \frac{1}{2}[D(a_{2n-1}, [Sc]_{\alpha_S(c)}) + D(c, [Ta_{2n-1}]_{\alpha_T(a_{2n-1})})] \right\} \\ & \leq \max \left\{ d(c, a_{2n-1}), D(c, [Sc]_{\alpha_S(c)}), d(a_{2n-1}, a_{2n}) \right\} \\ & \quad \frac{1}{2}[D(a_{2n-1}, [Sc]_{\alpha_S(c)}) + d(c, a_{2n})] \end{aligned}$$

Letting $n \rightarrow \infty$, we have $\lim_{n \rightarrow \infty} M(c, a_{2n-1}) = D(c, [Sc]_{\alpha_S(c)})$. Due to the lower semi-continuity of φ , we have

$$\varphi(D(c, [Sc]_{\alpha_S(c)})) \leq \lim_{n \rightarrow \infty} \varphi(M(c, a_{2n-1})). \quad (14)$$

On the other hand, from (2)

$$\begin{aligned} \delta([Sc]_{\alpha_S(c)}, a_{2n}) & \leq \delta([Sc]_{\alpha_S(c)}, [Ta_{2n-1}]_{\alpha_T(a_{2n-1})}) \\ & \leq M(c, a_{2n-1}) - \varphi(M(c, a_{2n-1})) \end{aligned}$$

and letting $n \rightarrow \infty$, we have

$$\delta([Sc]_{\alpha_S(c)}, c) \leq D(c, [Sc]_{\alpha_S(c)}) - \lim_{n \rightarrow \infty} \varphi(M(c, a_{2n-1})). \quad (15)$$

This shows that $\lim_{n \rightarrow \infty} \varphi(M(c, a_{2n-1})) = 0$ and so from (14), we have $\varphi(D(c, [Sc]_{\alpha_S(c)})) = 0$, that is, $D(c, [Sc]_{\alpha_S(c)}) = 0$. This implies, from (15), that $\{c\} = [Sc]_{\alpha_S(c)}$. Now, from (3) it is easy to see that $M(c, c) = D(c, [Tc]_{\alpha_T(c)})$, and so from (2) we have

$$\begin{aligned} \delta(c, [Tc]_{\alpha_T(c)}) & \leq \delta([Sc]_{\alpha_S(c)}, [Tc]_{\alpha_T(c)}) \\ & \leq M(c, c) - \varphi(M(c, c)) \\ & = D(c, [Tc]_{\alpha_T(c)}) - \varphi(D(c, [Tc]_{\alpha_T(c)})). \end{aligned}$$

Therefore, we have $c \in [Tc]_{\alpha_T(c)}$ and so $\{c\} = [Tc]_{\alpha_T(c)}$. As a consequence, we have $\{c\} = [Sc]_{\alpha_S(c)} = [Tc]_{\alpha_T(c)}$, that is, c is a common fixed point of S and T . Now we will show that this common fixed point is unique. Assume that a and b are two common fixed points of S and T . Then $a \in [Sa]_{\alpha_S(a)}$, a

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$\in [Ta]_{\alpha_T(a)}$ and $b \in [Sb]_{\alpha_S(b)}$, $b \in [Tb]_{\alpha_T(b)}$. Therefore, from (3) we have $M(a, b) \leq d(a, b)$ and so from (2) we have

$$\begin{aligned} d(a, b) &\leq \delta([Sa]_{\alpha_S(a)}, [Tb]_{\alpha_T(b)}) \\ &\leq M(a, b) - \varphi(M(a, b)) \\ &\leq d(a, b) - \varphi(M(a, b)). \end{aligned}$$

This shows that $M(a, b) = 0$ and so $a = b$. □

Example 1. Let $X = [0, 1]$, $d(a, b) = |a - b|$, when $a, b \in X$ and let $G, H : X \rightarrow \mathfrak{F}(X)$ be fuzzy mappings defined as:

$$G(a)(t) = \begin{cases} 1 & \text{if } 0 \leq t < \frac{a}{6} \\ \frac{1}{2} & \text{if } \frac{a}{6} \leq t \leq \frac{a}{4} \\ \frac{1}{3} & \text{if } \frac{a}{4} \leq t < \frac{a}{3} \\ 0 & \text{if } \frac{a}{3} \leq t < \infty \end{cases}$$

$$H(a)(t) = \begin{cases} 1 & \text{if } 0 \leq t < \frac{a}{6} \\ \frac{1}{4} & \text{if } \frac{a}{6} \leq t \leq \frac{a}{3} \\ \frac{1}{6} & \text{if } \frac{a}{3} \leq t \leq \frac{a}{2} \\ 0 & \text{if } \frac{a}{2} < t < \infty \end{cases}$$

$$[Ga]_{\frac{1}{3}} = \left\{ t \in X : G(a)(t) \geq \frac{1}{3} \right\} = \left[0, \frac{a}{3} \right),$$

$$[Ha]_{\frac{1}{4}} = \left\{ t \in X : H(a)(t) \geq \frac{1}{4} \right\} = \left[0, \frac{a}{3} \right]$$

It is clear that $[Ga]_{\frac{1}{3}}$ and $[Ha]_{\frac{1}{4}}$ are nonempty bounded for all $a \in X$. We will show that the condition (2) of Theorem 3 is satisfied with $\varphi(t) = \frac{t}{2}$. Indeed, for all $a, b \in X$, we have

$$\begin{aligned} \delta([Ga]_{\frac{1}{3}}, [Hb]_{\frac{1}{4}}) &= \delta\left(\left[0, \frac{a}{3}\right), \left[0, \frac{b}{3}\right]\right) \\ &= \frac{b}{3} = \frac{1}{2} \frac{2b}{3} = \frac{1}{2} D\left(b, \left[0, \frac{b}{3}\right]\right) \\ &= \frac{1}{2} D\left(b, [Hb]_{\frac{1}{4}}\right) \leq \frac{1}{2} M(a, b) = M(a, b) - \frac{1}{2} M(a, b) \\ &= M(a, b) - \varphi(M(a, b)). \end{aligned}$$

All conditions of Theorem 3 are satisfied and so these mappings have a unique common fixed point in X .

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Example 2. Let $X = [0, 1]$, $d(a, b) = |a - b|$, where $a, b \in X$, $\lambda, \mu \in (0, 1]$ and let $G, H : X \rightarrow \mathfrak{F}(X)$ be fuzzy mappings defined as:

$$\begin{aligned} & \text{if } a = 0, \\ & G(a)(t) = \begin{cases} 1 & \text{if } t = 0 \\ \frac{1}{2} & \text{if } 0 < t \leq \frac{1}{100} \\ 0 & t > \frac{1}{100} \end{cases} \quad T(a)(t) = \begin{cases} 1 & \text{if } t = 0 \\ \frac{1}{3} & \text{if } 0 < t \leq \frac{1}{150} \\ 0 & t > \frac{1}{150} \end{cases} \\ & \text{if } a \neq 0, \\ & G(a)(t) = \begin{cases} \lambda & \text{if } 0 \leq t < \frac{a}{16} \\ \frac{\lambda}{2} & \text{if } \frac{a}{16} \leq t \leq \frac{a}{10} \\ \frac{\lambda}{3} & \text{if } \frac{a}{10} \leq t < a \\ 0 & \text{if } a \leq t < \infty \end{cases} \quad T(x)(t) = \begin{cases} \mu & \text{if } 0 \leq t < \frac{a}{16} \\ \frac{\mu}{4} & \text{if } \frac{a}{16} \leq t \leq \frac{a}{10} \\ \frac{\mu}{10} & \text{if } \frac{a}{10} \leq t < a \\ 0 & \text{if } a \leq t < \infty \end{cases} \end{aligned}$$

Note that

$$[G0]_{\lambda_S(0)} = [H0]_{\lambda_T(0)} = \{0\}, \text{ if } \lambda_G(0) = \lambda_H(0) = 1,$$

and for $a \neq 0$,

$$[Ga]_{\lambda} = \left[0, \frac{a}{16}\right) \text{ and } [Ha]_{\mu} = \left[0, \frac{a}{16}\right),$$

$$[Ga]_{\frac{\lambda}{2}} = \left[0, \frac{a}{10}\right] \text{ and } [Ha]_{\frac{\mu}{4}} = \left[0, \frac{a}{10}\right].$$

Since X is not linear and also $[Ga]_{\lambda}$ and $[Ha]_{\lambda}$ are not compact for each λ , all the previous fixed point results [4, 9, 15, 16] for fuzzy mappings on complete linear metric spaces are not applicable. However, G and H satisfy the conditions of Theorem 3.

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ON CARLITZ'S DEGENERATE EULER NUMBERS AND POLYNOMIALS

DAE SAN KIM, TAEKYUN KIM, AND DMITRY V. DOLGY

ABSTRACT. In this paper, a p -adic measure is constructed by using the generalized distribution relation of degenerate Euler numbers and polynomials generalizing those satisfied by E_k and $E_k(x)$. Furthermore, a family of p -adic measures are obtained by regularizing that one.

1. INTRODUCTION

Let p be a fixed odd prime number. Throughout this paper, \mathbb{Z}_p , \mathbb{Q}_p and \mathbb{C}_p will denote the ring of p -adic integers, the field of p -adic rational numbers and the completion of the algebraic closure of \mathbb{Q}_p . Let $|\cdot|_p$ be the p -adic norm with $|p|_p = \frac{1}{p}$. In [2], Carlitz defined degenerate Euler numbers and polynomials and proved some properties generalizing those satisfied by E_k and $E_k(x)$. Recently, D. S. Kim and T. Kim gave some formulae and identities of degenerate Euler polynomials which are derived from the fermionic p -adic integrals on \mathbb{Z}_p (see [2, 4]). In this note, we use those properties of them, especially the distribution relation for the degenerate Euler polynomials, to construct p -adic measures.

For $\lambda, t \in \mathbb{C}_p$ with $|\lambda t|_p < p^{-\frac{1}{p-1}}$, the degenerate Euler polynomials are given by the generating function to be

$$(1.1) \quad \frac{2}{(1 + \lambda t)^{\frac{1}{\lambda}} + 1} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_n(x | \lambda) \frac{t^n}{n!}, \quad (\text{see [1, 2]}).$$

Note that $\lim_{\lambda \rightarrow 0} \mathcal{E}_n(x | \lambda) = E_n(x)$, where $E_n(x)$ are the Euler polynomials defined by the generating function

$$(1.2) \quad \left(\frac{2}{e^t + 1} \right) e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (\text{see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]})$$

When $x = 0$, $\mathcal{E}_n(\lambda) = \mathcal{E}_n(0 | \lambda)$ are called degenerate Euler numbers. From (1.1), we can derive the following equation:

$$(1.3) \quad \mathcal{E}_n(x | \lambda) = \sum_{l=0}^n \binom{n}{l} E_l(\lambda) (x | \lambda)_{n-l},$$

where $(x | \lambda)_n = x(x - \lambda)(x - 2\lambda) \cdots (x - (n-1)\lambda)$.

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The degenerate Euler polynomials satisfy the following generalized distribution relation [4] :

$$(1.4) \quad \mathcal{E}_n(x | \lambda) = d^n \sum_{a=0}^{d-1} (-1)^a \mathcal{E}_n\left(\frac{a+x}{d} \middle| \frac{\lambda}{d}\right),$$

where $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$ and $n \in \mathbb{N} \cup \{0\}$.

2. DEGENERATE EULER MEASURES

Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, and let p be a fixed odd prime number.

Proposition 2.1.

$$\begin{aligned} X_d &= \varprojlim_N \mathbb{Z}/dp^N \mathbb{Z}; \\ a + dp^N \mathbb{Z}_p &= \{x \in X_d \mid x \equiv a \pmod{dp^N}\}; \\ X_d^* &= \bigcup_{\substack{0 < a < dp \\ (a,p)=1}} a + dp \mathbb{Z}_p. \end{aligned}$$

We shall always take $0 \leq a < dp^N$ when we write $a + dp^N \mathbb{Z}_p$.

Theorem 2.2. For $k \geq 0$, let $\mu_{k,\varepsilon}$ be given by

$$(2.1) \quad \mu_{k,\varepsilon}(a + dp^N \mathbb{Z}_p) = (dp^N)^k (-1)^a \mathcal{E}_k\left(\frac{a}{dp^N} \middle| \frac{\lambda}{dp^N}\right).$$

Then $\mu_{k,\varepsilon}$ extends to a \mathbb{C}_p -valued measure on compact open sets $U \subset X_d$.

Proof. It is enough to show that

$$\sum_{i=0}^{p-1} \mu_{k,\varepsilon}(a + idp^N + dp^{N+1} \mathbb{Z}_p) = \mu_{k,\varepsilon}(a + dp^N \mathbb{Z}_p).$$

From (2.1), we note that

$$\begin{aligned} & \sum_{i=0}^{p-1} \mu_{k,\varepsilon}(a + idp^N + dp^{N+1} \mathbb{Z}_p) \\ &= (dp^{N+1})^k \sum_{i=0}^{p-1} (-1)^{a+idp^N} \mathcal{E}_k\left(\frac{a+idp^N}{dp^{N+1}} \middle| \frac{\lambda}{dp^{N+1}}\right) \\ &= (-1)^a (dp^N)^k p^k \sum_{i=0}^{p-1} (-1)^i \mathcal{E}_k\left(\frac{\frac{a}{dp^N} + i}{p} \middle| \frac{\frac{\lambda}{dp^N}}{p}\right) \\ &= (-1)^a (dp^N)^k \mathcal{E}_k\left(\frac{a}{dp^N} \middle| \frac{\lambda}{dp^N}\right) \\ &= \mu_{k,\varepsilon}(a + dp^N \mathbb{Z}_p). \end{aligned}$$

We easily see that $|\mu_{k,\varepsilon}| \leq M$ for some constant M . \square

Definition 2.3. Let $\alpha \in X_d^*$, $\alpha \neq 1$, $k \geq 1$. For compact-open $U \subset X_d$, we define

$$\mu_{k,\alpha}(U) = \mu_{k,\varepsilon}(U) - \alpha^{-k} \mu_{k,\varepsilon}(\alpha U).$$

Remark. We note that $\mu_{k,\alpha}$ are (bounded) \mathbb{C}_p -valued measures on X_d for all $k \geq 0$.

For $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, let χ be a Dirichlet character with conductor d . Then, we define the generalized degenerate Euler number attached to χ as follows:

$$(2.2) \quad \frac{2}{(1+\lambda t)^{d/\lambda} + 1} \sum_{a=0}^{d-1} (-1)^a \chi(a) (1+\lambda t)^{\frac{a}{\lambda}} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\chi}(\lambda) \frac{t^n}{n!}.$$

Note that

$$(2.3) \quad \begin{aligned} & \sum_{n=0}^{\infty} \lim_{\lambda \rightarrow 0} \mathcal{E}_{n,\chi}(\lambda) \frac{t^n}{n!} \\ &= \lim_{\lambda \rightarrow 0} \frac{2}{(1+\lambda t)^{d/\lambda} + 1} \sum_{a=0}^{d-1} (-1)^a \chi(a) (1+\lambda t)^{\frac{a}{\lambda}} \\ &= \frac{2}{e^{dt} + 1} \sum_{a=0}^{d-1} (-1)^a \chi(a) e^{at} \\ &= \sum_{n=0}^{\infty} E_{n,\chi} \frac{t^n}{n!}, \end{aligned}$$

where $E_{n,\chi}$ are called the generalized Euler numbers attached to χ .

From (2.3), we have $\lim_{\lambda \rightarrow 0} \mathcal{E}_{n,\chi} = E_{n,\chi}$.

By (1.1) and (2.2), we get

$$(2.4) \quad \begin{aligned} & \sum_{n=0}^{\infty} \mathcal{E}_{n,\chi}(\lambda) \frac{t^n}{n!} \\ &= \sum_{a=0}^{d-1} (-1)^a \chi(a) \frac{2}{(1+\lambda t)^{d/\lambda} + 1} (1+\lambda t)^{\frac{a}{\lambda}} \\ &= \sum_{n=0}^{\infty} d^n \left(\sum_{a=0}^{d-1} (-1)^a \chi(a) \mathcal{E}_n \left(\frac{a}{d} \middle| \frac{\lambda}{d} \right) \right) \frac{t^n}{n!}. \end{aligned}$$

By comparing the coefficients of both sides of (2.4), we have

$$(2.5) \quad \mathcal{E}_{n,\chi}(\lambda) = d^n \sum_{a=0}^{d-1} (-1)^a \chi(a) \mathcal{E}_n \left(\frac{a}{d} \middle| \frac{\lambda}{d} \right).$$

The locally constant function χ on X_d can be integrated against the measure $\mu_{k,\mathcal{E}}$ defined by (2.1), and the result is given by

$$(2.6) \quad \begin{aligned} & \int_{X_d} \chi(x) d\mu_{k,\mathcal{E}}(x) \\ &= \lim_{N \rightarrow \infty} \sum_{a=0}^{dp^N-1} \chi(a) \mu_{k,\mathcal{E}}(a + dp^N \mathbb{Z}_p) \\ &= \lim_{N \rightarrow \infty} (dp^N)^k \sum_{a=0}^{dp^N-1} \chi(a) (-1)^a \mathcal{E}_k \left(\frac{a}{dp^N} \middle| \frac{\lambda}{dp^N} \right) \\ &= d^k \sum_{a=0}^{d-1} \chi(a) (-1)^a \lim_{N \rightarrow \infty} (p^N)^k \sum_{x=0}^{p^N-1} (-1)^x \mathcal{E}_k \left(\frac{\frac{a}{d} + x}{p^N} \middle| \frac{\lambda}{p^N} \right) \end{aligned}$$

$$\begin{aligned}
&= d^k \sum_{a=0}^{d-1} \chi(a) (-1)^a \mathcal{E}_k \left(\frac{a}{d} \middle| \frac{\lambda}{d} \right) \\
&= \mathcal{E}_{k,\chi}(\lambda).
\end{aligned}$$

Note that

$$\begin{aligned}
(2.7) \quad & \int_{pX_d} \chi(x) d\mu_{k,\mathcal{E}}(x) \\
&= (pd)^k \sum_{a=0}^{d-1} \chi(pa) (-1)^{pa} \mathcal{E}_k \left(\frac{pa}{pd} \middle| \frac{\lambda}{pd} \right) \\
&= p^k \chi(p) d^k \sum_{a=0}^{d-1} \chi(a) (-1)^a \mathcal{E}_k \left(\frac{a}{d} \middle| \frac{\lambda}{d} \right) \\
&= p^k \chi(p) \mathcal{E}_{k,\chi} \left(\frac{\lambda}{p} \right),
\end{aligned}$$

$$(2.8) \quad \int_{X_d} \chi(x) d\mu_{k,\mathcal{E}}(\alpha x) = \chi \left(\frac{1}{\alpha} \right) \mathcal{E}_{k,\chi}(\lambda),$$

and

$$(2.9) \quad \int_{pX_d} \chi(x) d\mu_{k,\mathcal{E}}(\alpha x) = p^k \chi \left(\frac{p}{\alpha} \right) \mathcal{E}_{k,\chi} \left(\frac{\lambda}{p} \right).$$

Hence, by definition of $\mu_{k,\alpha}$, we get

$$\begin{aligned}
(2.10) \quad & \int_{X_d^*} \chi(x) d\mu_{k,\alpha}(x) \\
&= \mathcal{E}_{k,\chi}(\lambda) - p^k \chi(p) \mathcal{E}_{k,\chi} \left(\frac{\lambda}{p} \right) - \frac{1}{\alpha^k} \chi \left(\frac{1}{\alpha} \right) \mathcal{E}_{k,\chi}(\lambda) \\
&\quad + \frac{p^k}{\alpha^k} \chi \left(\frac{p}{\alpha} \right) \mathcal{E}_{k,\chi} \left(\frac{\lambda}{p} \right) \\
&= \left(1 - \alpha^{-k} \chi \left(\frac{1}{\alpha} \right) \right) \left(\mathcal{E}_{k,\chi}(\lambda) - p^k \chi(p) \mathcal{E}_{k,\chi} \left(\frac{\lambda}{p} \right) \right).
\end{aligned}$$

Therefore, by (2.6), (2.7), (2.8), (2.9) and (2.10), we obtain the following theorem.

Theorem 2.4. *For $k \geq 0$, we have*

$$\begin{aligned}
& \int_{X_d} \chi(x) d\mu_{k,\mathcal{E}}(x) = \mathcal{E}_{k,\chi}(\lambda), \\
& \int_{pX_d} \chi(x) d\mu_{k,\mathcal{E}}(x) = p^k \chi(p) \mathcal{E}_{k,\chi} \left(\frac{\lambda}{p} \right), \\
& \int_{X_d} \chi(x) d\mu_{k,\mathcal{E}}(\alpha x) = \chi \left(\frac{1}{\alpha} \right) \mathcal{E}_{k,\chi}(\lambda), \\
& \int_{pX_d} \chi(x) d\mu_{k,\mathcal{E}}(\alpha x) = p^k \chi \left(\frac{p}{\alpha} \right) \mathcal{E}_{k,\chi} \left(\frac{\lambda}{p} \right),
\end{aligned}$$

and

$$\int_{X_d^*} \chi(x) d\mu_{k,\alpha}(x) = \left(1 - \alpha^{-k} \chi\left(\frac{1}{\alpha}\right)\right) \left(\mathcal{E}_{k,\chi}(\lambda) - p^k \chi(p) \mathcal{E}_{k,\chi}\left(\frac{\lambda}{p}\right)\right).$$

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Dynamics and Behavior of the Higher Order Rational Difference equation

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Abstract

The main objective of this paper is to study the periodic character and the global stability of the positive solutions of the difference equation

$$x_{n+1} = ax_n + bx_{n-k} + cx_{n-l} - \frac{dx_{n-s}}{ex_{n-s} - \alpha x_{n-t}}, \quad n = 0, 1, \dots,$$

where the parameters a, b, c, d, e and α are positive real numbers and the initial conditions $x_{-\sigma}, x_{-\sigma+1}, \dots, x_{-1}, x_0$ are positive real numbers where $\sigma = \max\{s, t, l, k\}$. Some numerical examples were given to illustrate our results.

Keywords: difference equations, stability, global stability, periodic solutions.

Mathematics Subject Classification: 39A10

1 Introduction

In this paper, we study the global stability character, the boundedness and the periodicity of the positive solutions of the nonlinear difference equation

$$x_{n+1} = ax_n + bx_{n-k} + cx_{n-l} - \frac{dx_{n-s}}{ex_{n-s} - \alpha x_{n-t}}, \quad n = 0, 1, \dots, \quad (1)$$

where the parameters a, b, c, d, e and α are positive real numbers and the initial conditions $x_{-\sigma}, x_{-\sigma+1}, \dots, x_{-1}, x_0$ are positive real numbers where $\sigma = \max\{s, t, l, k\}$. Here, we recall some notations and results, which will be useful in our investigation.

Let I be some interval of real numbers and let

$$F : I^{k+1} \rightarrow I, \quad k \in N$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, \dots, x_0 \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots, \quad (2)$$

has a unique solution $\{x_n\}_{n=-k}^\infty$.

Definition 1 (*Equilibrium Point*)

A point $\bar{x} \in I$ is called an equilibrium point of Eq.(2) if

$$\bar{x} = F(\bar{x}, \bar{x}, \dots, \bar{x}).$$

That is, $x_n = \bar{x}$ for $n \geq 0$ is a solution of Eq.(2), or equivalently, \bar{x} is a fixed point of f .

Definition 2 (*Stability*)

(i) The equilibrium point \bar{x} of Eq.(2) is called locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, \dots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \bar{x}| + \dots + |x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \delta,$$

we have

$$|x_n - \bar{x}| < \epsilon \quad \text{for all } n \geq -k.$$

(ii) The equilibrium point \bar{x} of Eq.(2) is called locally asymptotically stable if \bar{x} is locally stable solution of Eq.(2) and there exists $\gamma > 0$, such that for all $x_{-k}, \dots, x_{-1}, x_0 \in I$ with

$$|x_{-k} - \bar{x}| + \dots + |x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \gamma,$$

we have

$$\lim_{n \rightarrow \infty} x_n = \bar{x}.$$

(iii) The equilibrium point \bar{x} of Eq.(2) is called global attractor if for all $x_{-k}, \dots, x_{-1}, x_0 \in I$, we have

$$\lim_{n \rightarrow \infty} x_n = \bar{x}.$$

(iv) The equilibrium point \bar{x} of Eq.(2) is called globally asymptotically stable if \bar{x} is locally stable, and \bar{x} is also a global attractor of Eq.(2).

(v) The equilibrium point \bar{x} of Eq.(2) is called unstable if \bar{x} is not locally stable.

Definition 3 (*Boundedness*)

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be bounded and persisting if there exist positive constants m and M such that

$$m \leq x_n \leq M \quad \text{for all } n \geq -k.$$

Definition 4 (*Periodicity*)

A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with period p if $x_{n+p} = x_n$ for all $n \geq -k$. A sequence $\{x_n\}_{n=-k}^{\infty}$ is said to be periodic with prime period p if p is the smallest positive integer having this property.

Definition 5 The linearized equation of Eq.(2) about the equilibrium \bar{x} is the linear difference equation

$$y_{n+1} = \sum_{i=0}^k \frac{\partial F(\bar{x}, \bar{x}, \dots, \bar{x})}{\partial x_{n-i}} y_{n-i}. \quad (3)$$

Now, assume that the characteristic equation associated with (3) is

$$p(\lambda) = p_0 \lambda^k + p_1 \lambda^{k-1} + \dots + p_{k-1} \lambda + p_k = 0, \quad (4)$$

where

$$p_i = \frac{\partial F(\bar{x}, \bar{x}, \dots, \bar{x})}{\partial x_{n-i}}.$$

Theorem A [1]: Assume that $p_i \in R$, $i = 1, 2, \dots, k$ and k is non-negative integer. Then

$$\sum_{i=1}^k |p_i| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+k} + p_1 x_{n+k-1} + \dots + p_k x_n = 0, \quad n = 0, 1, \dots$$

Theorem B [2]: Let $g : [a, b]^{k+1} \rightarrow [a, b]$ be a continuous function, where k is a positive integer, and $[a, b]$ is an interval of real numbers and consider the difference equation

$$x_{n+1} = g(x_n, x_{n-1}, \dots, x_{n-k}), \quad n = 0, 1, \dots \quad (5)$$

Suppose that g satisfies the following conditions:

- (i) For every integer i with $1 \leq i \leq k+1$, the function $g(z_1, z_2, \dots, z_{k+1})$ is weakly monotonic in z_i , for fixed $z_1, z_2, \dots, z_{i-1}, z_{i+1}, \dots, z_{k+1}$.
- (ii) If (m, M) is a solution of the system

$$m = g(m_1, m_2, \dots, m_{k+1}) \text{ and } M = g(M_1, M_2, \dots, M_{k+1}),$$

then $m = M$, where for each $i = 1, 2, \dots, k + 1$, we set

$$m_i = \begin{cases} m & \text{if } g \text{ is non-decreasing in } z_i \\ M & \text{if } g \text{ is non-increasing in } z_i \end{cases},$$

and

$$M_i = \begin{cases} M & \text{if } g \text{ is non-decreasing in } z_i \\ m & \text{if } g \text{ is non-increasing in } z_i \end{cases}.$$

Then, there exists exactly one equilibrium point \bar{x} of the difference equation (5), and every solution of (5) converges to \bar{x} .

Many research have been done to study the global attractivity, boundedness character, periodicity and the solution form of nonlinear difference equations. For example, Agarwal et al. [3] investigated the global stability, periodicity character and gave the solution form of some special cases of the recursive sequence

$$x_{n+1} = ax_n + \frac{bx_n x_{n-3}}{cx_{n-2} + dx_{n-3}}, \quad n = 0, 1, \dots,$$

where a, b, c, d and the initial conditions $x_{-3}, x_{-2}, x_{-1}, x_0$ are positive real numbers.

Sun et al [4] studied the behavior of the solutions of the difference equation

$$x_{n+1} = p + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, \dots$$

where initial values $x_{-1}, x_0 \in (0, \infty)$ and $0 < p < 1$, and obtain the set of all initial values $(x_{-1}, x_0) \in (0, +\infty) \times (0, +\infty)$ such that the positive solutions $\{x_n\}_{n=-1}^{\infty}$ are bounded.

Elsayed and El-Dessoky [5] studied the global convergence, boundedness, and periodicity of solutions of the difference equation

$$x_{n+1} = ax_{n-s} + \frac{bx_{n-l} + cx_{n-k}}{dx_{n-l} + ex_{n-k}}, \quad n = 0, 1, \dots,$$

where the parameters a, b, c, d and e are positive real numbers and the initial conditions $x_{-t}, x_{-t+1}, \dots, x_{-1}, x_0$ are positive real numbers where $t = \max\{s, l, k\}$.

Zayed [6] studied the global asymptotic properties of the solutions of the following difference equations

$$x_{n+1} = Ax_n + Bx_{n-k} + \frac{px_n + x_{n-k}}{q + x_{n-k}}.$$

Elsayed [7] studied the global stability character and the periodicity of solutions of the difference equation

$$x_{n+1} = ax_n + \frac{bx_{n-1} + cx_{n-2}}{dx_{n-1} + ex_{n-2}}, \quad n = 0, 1, \dots,$$

where the parameters a, b, c, d and e are positive real numbers and the initial conditions x_{-2}, x_{-1} and x_0 are positive real numbers.

El-Moneam [8] investigated the periodicity, the boundedness and the global stability of the positive solutions of the following nonlinear difference equation

$$x_{n+1} = Ax_n + Bx_{n-k} + Cx_{n-l} + Dx_{n-s} + \frac{bx_{n-k}}{dx_{n-k} - ex_{n-l}}, \quad n = 0, 1, \dots,$$

where the coefficients $A, B, C, D, b, d, e \in (0, \infty)$, while k, l and s are positive integers. The initial conditions $x_{-s}, \dots, x_{-l}, \dots, x_{-k}, \dots, x_{-1}, x_0$ are arbitrary positive real numbers such that $k < l < s$.

Yalçinkaya [9] investigated the global behaviour of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-m}}{x_n^k}, \quad n = 0, 1, \dots,$$

where the parameter $\alpha, k \in (0, \infty)$ and the initial values are arbitrary positive real numbers.

Elabbasy et al. [10] studied the dynamics, the global stability, periodicity character and the solution of special case of the recursive sequence

$$x_{n+1} = ax_n - \frac{bx_n}{cx_n - dx_{n-1}}, \quad n = 0, 1, \dots,$$

where the initial conditions x_{-1}, x_0 are arbitrary real numbers and a, b, c, d are positive constants.

In [11] Berenhaut et al. studied the existence of positive prime periodic solutions of higher order for rational recursive equations of the form

$$y_{n+1} = A + \frac{y_{n-1}}{y_{n-m}}, \quad n = 0, 1, \dots,$$

with $y_{-m}, y_{-m+1}, \dots, y_{-1} \in (0, 1)$ and $m = \{2, 3, 4, \dots\}$.

Papaschinopoulos et al. [12] investigated the asymptotic behavior and the periodicity of the positive solutions of the nonautonomous difference equation:

$$x_{n+1} = A_n + \frac{x_{n-1}^p}{x_n^q}, \quad n = 0, 1, \dots,$$

where A_n is a positive bounded sequence, $p, q \in (0, \infty)$ and x_1, x_0 are positive numbers.

For some related results see [13-28].

2 Local Stability of the Equilibrium Point of Eq.(1)

In this section, we study the local stability character of the equilibrium point of Eq.(1).

Eq.(1) has an equilibrium point given by

$$\bar{x} = a\bar{x} + b\bar{x} + c\bar{x} - \frac{d\bar{x}}{e\bar{x} - \alpha\bar{x}},$$

and hence

$$(e - \alpha)(1 - a - b - c)\bar{x}^2 + d\bar{x} = 0.$$

Then if $a + b + c < 1$ and $\alpha > e$, the only positive equilibrium point of Eq.(1) is given by

$$\bar{x} = \frac{d}{(\alpha - e)(1 - a - b - c)}.$$

Theorem 1 *The equilibrium \bar{x} of Eq. (1) is locally asymptotically stable if*

$$\alpha - e > 2d.$$

Proof: Let $f : (0, \infty)^5 \longrightarrow (0, \infty)$ be a continuous function defined by

$$f(u_1, u_2, u_3, u_4, u_5) = au_1 + bu_2 + cu_3 + \frac{du_4}{eu_4 - \alpha u_5}. \quad (6)$$

Therefore, it follows that

$$\begin{aligned} \frac{\partial f(u_1, u_2, u_3, u_4, u_5)}{\partial u_1} &= a, \\ \frac{\partial f(u_1, u_2, u_3, u_4, u_5)}{\partial u_2} &= b, \\ \frac{\partial f(u_1, u_2, u_3, u_4, u_5)}{\partial u_3} &= c, \\ \frac{\partial f(u_1, u_2, u_3, u_4, u_5)}{\partial u_4} &= \frac{\alpha du_5}{(eu_4 - \alpha u_5)^2}, \\ \frac{\partial f(u_1, u_2, u_3, u_4, u_5)}{\partial u_5} &= -\frac{\alpha du_4}{(eu_4 - \alpha u_5)^2}. \end{aligned}$$

So, we can write

$$\begin{aligned} \frac{\partial f(\bar{x}, \bar{x}, \bar{x}, \bar{x}, \bar{x})}{\partial u_1} &= a = p_1, \\ \frac{\partial f(\bar{x}, \bar{x}, \bar{x}, \bar{x}, \bar{x})}{\partial u_2} &= b = p_2, \\ \frac{\partial f(\bar{x}, \bar{x}, \bar{x}, \bar{x}, \bar{x})}{\partial u_3} &= c = p_3, \end{aligned}$$

$$\begin{aligned}\frac{\partial f(\bar{x}, \bar{x}, \bar{x}, \bar{x}, \bar{x})}{\partial u_4} &= \frac{d(1-a-b-c)}{(e-\alpha)} = p_4, \\ \frac{\partial f(\bar{x}, \bar{x}, \bar{x}, \bar{x}, \bar{x})}{\partial u_5} &= -\frac{d(1-a-b-c)}{(e-\alpha)} = p_5.\end{aligned}$$

Then the linearized equation of Eq.(1) about \bar{x} is

$$y_{n+1} - p_1 y_n - p_2 y_{n-k} - p_3 y_{n-l} - p_4 y_{n-s} - p_5 y_{n-t} = 0.$$

It follows by Theorem A that, Eq.(1) is asymptotically stable if and only if

$$|p_1| + |p_2| + |p_3| + |p_4| + |p_5| < 1.$$

Thus,

$$|a| + |b| + |c| + \left| \frac{d(1-a-b-c)}{(\alpha-e)} \right| + \left| \frac{d(1-a-b-c)}{(\alpha-e)} \right| < 1,$$

and so

$$2 \left| \frac{d(1-a-b-c)}{(\alpha-e)} \right| < 1 - b - a - c,$$

or

$$2d < \alpha - e.$$

The proof is complete.

Example 1. The solution of the difference equation (1) is local stability if $k = 2$, $l = 1$, $s = 3$, $t = 2$, $a = 0.23$, $b = 0.12$, $c = 0.3$, $d = 0.1$, $e = 0.6$ and $\alpha = 0.9$ and the initial conditions $x_{-3} = 11.1$, $x_{-2} = 1.1$, $x_{-1} = 1.4$ and $x_0 = 1.9$ (See Fig. 1).

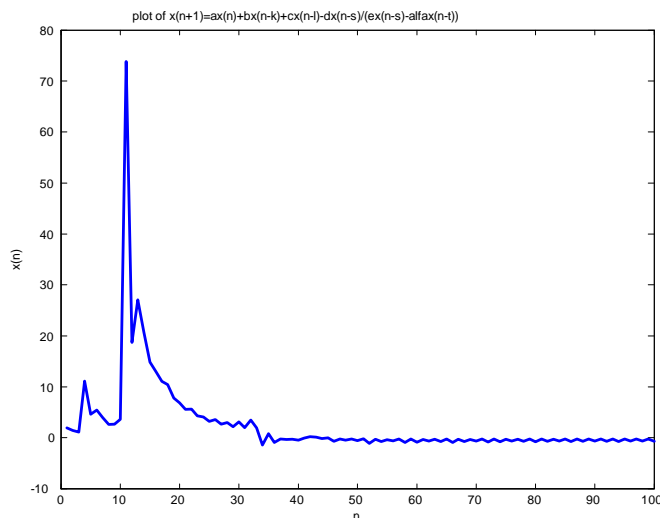


Figure 1. Plot the behavior of the solution of equation (1).

Example 2. The solution of the difference equation (1) if $k = 2$, $l = 1$, $s = 3$, $t = 2$, $a = 0.4$, $b = 0.2$, $c = 0.5$, $d = 0.1$, $e = 0.6$ and $\alpha = 0.9$ and the initial conditions $x_{-3} = 11.1$, $x_{-2} = 1.1$, $x_{-1} = 1.4$ and $x_0 = 1.9$ (See Fig. 2).

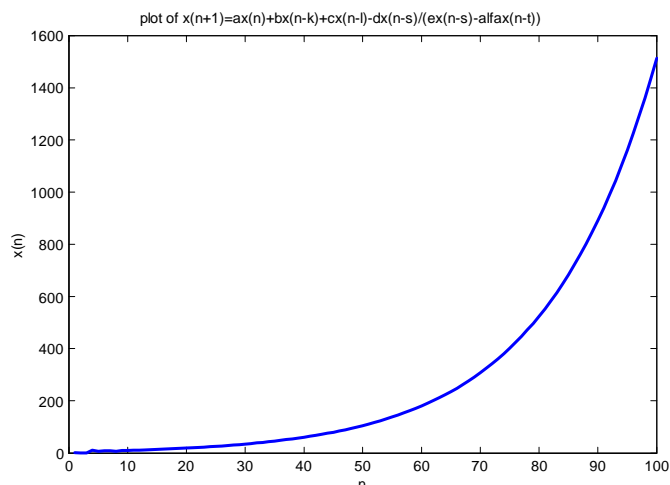


Figure 2. Plot the behavior of the solution of equation (1).

3 Global Attractivity of the Equilibrium Point of Eq.(1)

In this section, the global asymptotic stability of Eq.(1) will be studied.

Theorem 2 *The equilibrium point \bar{x} is a global attractor of Eq.(1) if $a + b + c < 1$.*

Proof: Suppose that ζ and η are real numbers and assume that $g : [\zeta, \eta]^5 \longrightarrow [\zeta, \eta]$ is a function defined by

$$g(u_1, u_2, u_3, u_4, u_5) = au_1 + bu_2 + cu_3 - \frac{du_4}{eu_4 - \alpha u_5}.$$

Then

$$\begin{aligned} \frac{\partial g(u_1, u_2, u_3, u_4, u_5)}{\partial u_1} &= a, \quad \frac{\partial g(u_1, u_2, u_3, u_4, u_5)}{\partial u_2} = b, \\ \frac{\partial g(u_1, u_2, u_3, u_4, u_5)}{\partial u_3} &= c, \\ \frac{\partial g(u_1, u_2, u_3, u_4, u_5)}{\partial u_4} &= \frac{\alpha du_5}{(eu_4 - \alpha u_5)^2}, \\ \frac{\partial g(u_1, u_2, u_3, u_4, u_5)}{\partial u_5} &= -\frac{\alpha du_4}{(eu_4 - \alpha u_5)^2}. \end{aligned}$$

Now, we can see that the function $g(u_1, u_2, u_3, u_4, u_5)$ increasing in u_1, u_2, u_3, u_4 and decreasing in u_5 .

Let (m, M) be a solution of the system $M = g(M, M, M, M, m)$ and $m = g(m, m, m, m, M)$. Then from Eq.(1), we see that

$$M = aM + bM + cM - \frac{dM}{eM - \alpha m} \text{ and } m = am + bm + cm - \frac{dm}{em - \alpha M},$$

and then

$$M(1 - a - b - c) = -\frac{dM}{eM - \alpha m} \text{ and } m(1 - a - b - c) = -\frac{dm}{em - \alpha M},$$

thus

$$e(1 - a - b - c)M^2 - \alpha(1 - a - b - c)Mm = -dM$$

and

$$e(1 - a - b - c)m^2 - \alpha(1 - a - b - c)Mm = -dm.$$

Subtracting we obtain

$$e(1 - a - b - c)(M^2 - m^2) + d(M - m) = 0,$$

then

$$(M - m)\{e(1 - a - b - c)(M + m) + d\} = 0$$

under the condition $a + b + c < 1$, we see that

$$M = m.$$

It follows by Theorem B that \bar{x} is a global attractor of Eq.(1). This completes the proof.

Example 3. The solution of the difference equation (1) is global stability if $k = 2$, $l = 1$, $s = 3$, $t = 2$, $a = 0.2$, $b = 0.2$, $c = 0.5$, $d = 0.12$, $e = 0.6$ and $\alpha = 0.9$ and the initial conditions $x_{-3} = 11.1$, $x_{-2} = 1.1$, $x_{-1} = 1.4$ and $x_0 = 1.9$ (See Fig. 3).

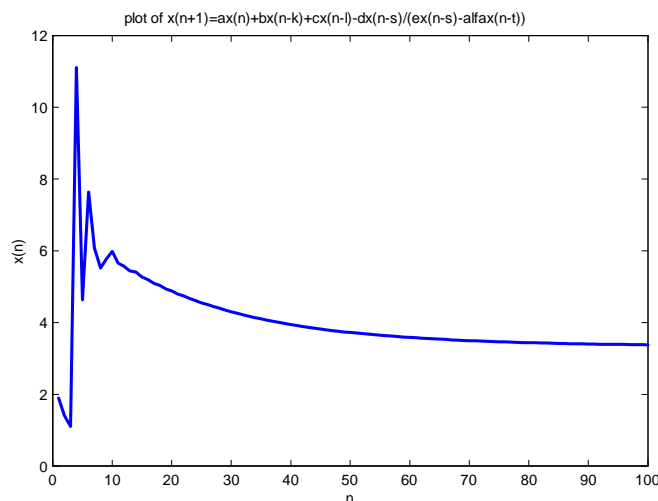


Figure 3. Plot the behavior of the solution of equation (1).

4 Existence of Periodic Solutions

In this section, we investigate the existence of periodic solutions of Eq.(1). The following theorem states the necessary and sufficient conditions for the Eq.(1) to be periodic solutions of prime period two.

Theorem 3 *Equation (1) has a prime period two solutions if and only if one of the following conditions satisfies*

- (i) $(e - 3\alpha)(a + b + c) + e + \alpha > 0$, l, k, t - even and s - odd.
- (ii) $(e + \alpha)(a + b + c + 1) - 4\alpha > 0$, l, k, s - even and t - odd.
- (iii) $(\alpha + e)(a + c - b + 1) - 4\alpha(a + c) > 0$, l, t - even and k, s - odd.
- (iv) $(\alpha + e)(b - a - c - 1) - 4\alpha(b - 1) > 0$, l, s - even and k, t - odd.
- (v) $(\alpha + e)(b - a - c - 1) - 4\alpha\alpha > 0$, l, k, s - odd, and t - even.
- (vi) $(\alpha + e)(b + c - a - 1) - 4\alpha(b + c - 1) > 0$, l, k, t - odd and s - even.
- (vii) $(\alpha + e)(b + a + c - 1) - 4\alpha(c - 1) > 0$, l, t - odd and k, s - even.
- (viii) $(\alpha + e)(b + a - c + 1) - 4\alpha(a + b) > 0$, l, s - odd and k, t - even.

Proof: We prove first case when l, k and t are even and s odd (the other cases are similar and will be left to readers).

First suppose that there exists a prime period two solution

$$\dots p, q, p, q, \dots,$$

of Equation (1). We will prove that Inequality (i) holds.

We see from Equation (1) when l, k and t are even and s odd that

$$p = aq + bq + cq - \frac{dp}{ep - \alpha q},$$

and

$$q = ap + bp + cp - \frac{dq}{eq - \alpha p}.$$

Therefore,

$$ep^2 - \alpha pq = e(a + b + c)pq - \alpha(a + b + c)q^2 - dp, \quad (7)$$

and

$$eq^2 - \alpha pq = e(a + b + c)pq - \alpha(a + b + c)p^2 - dq. \quad (8)$$

Subtracting (8) from (7) gives

$$e(p^2 - q^2) - \alpha(a + b + c)(p^2 - q^2) + d(p - q) = 0,$$

then

$$(p - q)[e - \alpha(a + b + c)](p + q) + d = 0$$

Since $p \neq q$, then

$$(p + q) = \frac{d}{\alpha(a + b + c) - e}. \quad (9)$$

Again, adding (7) and (8) yields

$$e(q^2 + p^2) - 2\alpha pq = 2e(a + b + c)pq - \alpha(a + b + c)(q^2 + p^2) - d(q + p),$$

then

$$2(e(a + b + c) + \alpha)pq = (e + \alpha(a + b + c))(q^2 + p^2) + d(q + p). \quad (10)$$

By using (9), (10) and the relation

$$p^2 + q^2 = (p + q)^2 - 2pq \quad \text{for all } p, q \in R,$$

we obtain

$$\begin{aligned} (e + \alpha(a + b + c))((p + q)^2 - 2pq) + d(q + p) &= 2(e(a + b + c) - \alpha)pq, \\ 2[e(a + b + c) + \alpha + e + \alpha(a + b + c)]pq &= (e + \alpha(a + b + c))(p + q)^2 + d(q + p), \end{aligned}$$

$$2(e + \alpha)(a + b + c + 1)pq = \left(\frac{d}{\alpha(a + b + c) - e} \right)^2 (e + \alpha(a + b + c) + \alpha(a + b + c) - e),$$

$$2(e + \alpha)(a + b + c + 1)pq = 2\alpha(a + b + c) \left(\frac{d}{\alpha(a + b + c) - e} \right)^2.$$

Then,

$$pq = \left(\frac{\alpha(a + b + c)}{(a + b + c + 1)(e + \alpha)} \right) \left(\frac{d}{\alpha(a + b + c) - e} \right)^2. \quad (11)$$

Now it is obvious from Eq.(9) and Eq.(11) that p and q are the two distinct roots of the quadratic equation

$$\begin{aligned} t^2 - \frac{d}{\alpha(a + b + c) - e}t + \left(\frac{\alpha(a + b + c)}{(a + b + c + 1)(e + \alpha)} \right) \left(\frac{d}{\alpha(a + b + c) - e} \right)^2 &= 0, \\ (\alpha(a + b + c) - e)t^2 - dt + \frac{d^2\alpha(a + b + c)}{(a + b + c + 1)(e + \alpha)(\alpha(a + b + c) - e)} &= 0, \end{aligned} \quad (12)$$

and so

$$\begin{aligned} d^2 - \frac{4d^2\alpha(a + b + c)(\alpha(a + b + c) - e)}{(a + b + c + 1)(e + \alpha)(\alpha(a + b + c) - e)} &> 0, \\ (a + b + c + 1)(e + \alpha) - 4\alpha(a + b + c) &> 0, \\ e(a + b + c + 1) + \alpha - 3\alpha(a + b + c) &> 0, \end{aligned}$$

or

$$(e - 3\alpha)(a + b + c) + e + \alpha > 0.$$

For $\alpha(a+b+c) > e$ and $e > 3\alpha$ then the Inequality (i) holds.

Second suppose that Inequality (i) is true. We will show that Equation (1) has a prime period two solution.

Suppose that

$$p = \frac{d(1+\zeta)}{2(\alpha A - e)} \text{ and } q = \frac{d(1-\zeta)}{2(\alpha A - e)},$$

where $\zeta = \sqrt{1 - \frac{4\alpha A}{(A+1)(e+\alpha)}}$ and $A = a+b+c$.

We see from the inequality (i) that

$$\begin{aligned} (e-3\alpha)(a+b+c) + e + \alpha &> 0, \\ (a+b+c+1)(e+\alpha) - 4\alpha(a+b+c) &> 0, \end{aligned}$$

which equivalents to

$$(A+1)(e+\alpha) - 4\alpha A > 0.$$

Therefore p and q are distinct real numbers.

Set

$$x_{-l} = q, \ x_{-k} = q, \ x_{-s} = p, \ x_{-t} = q, \dots, \ x_{-3} = p, \ x_{-2} = q, \ x_{-1} = p, \ x_0 = q.$$

We would like to show that

$$x_1 = x_{-1} = p \quad \text{and} \quad x_2 = x_0 = q.$$

It follows from Eq.(1) that

$$\begin{aligned} x_1 &= aq + bq + cq - \frac{dp}{ep - \alpha q}, \\ &= (a+b+c) \left(\frac{d(1-\zeta)}{2(\alpha A - e)} \right) - \frac{d \left(\frac{d(1+\zeta)}{2(\alpha A - e)} \right)}{e \left(\frac{d(1+\zeta)}{2(\alpha A - e)} \right) - \alpha \left(\frac{d(1-\zeta)}{2(\alpha A - e)} \right)}. \end{aligned}$$

Dividing the denominator and numerator by $2(\alpha A - e)$ we get

$$x_1 = (a+c+b) \left(\frac{d(1-\zeta)}{2(\alpha A - e)} \right) - \frac{d(1+\zeta)}{(e-\alpha) + (e+\alpha)\zeta}.$$

Multiplying the denominator and numerator of the right side by $(e-\alpha) - (e+\alpha)\zeta$

$$\begin{aligned} x_1 &= (a+c+b) \left(\frac{d(1-\zeta)}{2(\alpha A - e)} \right) - \frac{d(1+\zeta) ((e-\alpha) - (e+\alpha)\zeta)}{((e-\alpha) + (e+\alpha)\zeta) ((e-\alpha) - (e+\alpha)\zeta)}, \\ &= \frac{dA(1-\zeta)}{2(\alpha A - e)} - \frac{d((e-\alpha) - 2\alpha\zeta - (e+\alpha)\zeta^2)}{(e-\alpha)^2 - (e+\alpha)^2\zeta^2} \\ &= \frac{Ad(1-\zeta)}{2(\alpha A - e)} - \frac{d((A-1) - \zeta(A+1))}{2(A\alpha - e)}, \end{aligned}$$

$$x_1 = \frac{d(A - A\zeta - A + 1 + A\zeta + \zeta)}{2(\alpha A - e)} = \frac{d(1 + \zeta)}{2(\alpha A - e)} = p.$$

Similarly as before, it is easy to show that

$$x_2 = q.$$

Then by induction we get

$$x_{2n} = q \quad \text{and} \quad x_{2n+1} = p \quad \text{for all} \quad n \geq -2.$$

Thus Eq.(1) has the prime period two solution

$$\dots, p, q, p, q, \dots,$$

where p and q are the distinct roots of the quadratic equation (12) and the proof is complete.

Example 4. The solution of the difference equation (1) has a prime period two solution when $k = 4$, $l = 2$, $s = 3$, $t = 2$, $a = 0.3$, $b = 0.02$, $c = 0.01$, $d = 9$, $e = 3$ and $\alpha = 1.1$ and the initial conditions $x_{-5} = p$, $x_{-4} = q$, $x_{-3} = p$, $x_{-2} = q$, $x_{-1} = p$ and $x_0 = q$ since p and q as in the previous theorem (See Fig. 4).

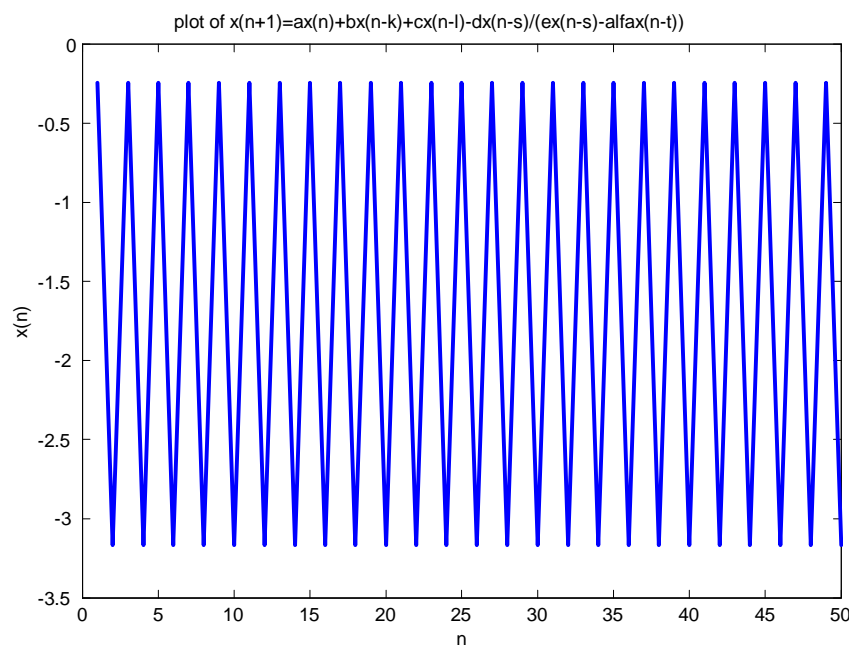


Figure 4. Plot the periodicity of the solution of equation (1).

Theorem 4 Equation (1) has no prime period two solutions if one

- (i) $1 + a + b + c \neq 0$, l, k, s, t – even.
- (ii) $1 + a - b - c \neq 0$, l, k, s, t – odd.
- (iii) $1 + a + c - b \neq 0$, l, s, t – even and k – odd.
- (iv) $1 + a + c + b \neq 0$, l, k – even and t, s – odd.
- (v) $1 + a + c - b \neq 0$, l – even and k, s, t – odd.
- (vi) $1 + a + b - c \neq 0$, l, s, t – odd and k – even.
- (vii) $1 + a - b - c \neq 0$, l, k – odd and s, t – even.
- (viii) $1 + a + b - c \neq 0$, l – odd and k, s, t – even.

Proof: We prove first case when l, k, s and t are both even positive integers (the other cases are similar and will be left to readers).

First suppose that there exists a prime period two solution

$$\dots p, q, p, q, \dots,$$

of Equation (1). We will prove that Inequality (i) holds.

We see from Equation (1) when l, k, s and t are both even positive integers that

$$p = aq + bq + cq - \frac{dq}{eq - \alpha q},$$

and

$$q = ap + bp + cp - \frac{dp}{ep - \alpha p}.$$

Therefore,

$$p - (-a - b - c)q = -\frac{d}{e - \alpha}, \quad (13)$$

and

$$q - (-a - b - c)p = -\frac{d}{e - \alpha}. \quad (14)$$

Subtracting (14) from (13) gives

$$(1 - a - b - c)(p - q) = 0.$$

Since $a + b + c \neq 1$, then $p = q$. This is a contradiction. Thus, the proof of (i) is now completed.

Example 5. Figure (5) shows the difference equation (1) has no period two solution when $k = 4$, $l = 2$, $s = 2$, $t = 4$, $a = 0.09$, $b = 0.2$, $c = 1$, $d = 9$, $e = 3$ and $\alpha = 2.1$

and the initial conditions $x_{-4} = 2$, $x_{-3} = 5$, $x_{-2} = 8$, $x_{-1} = 1.2$ and $x_0 = 5$.

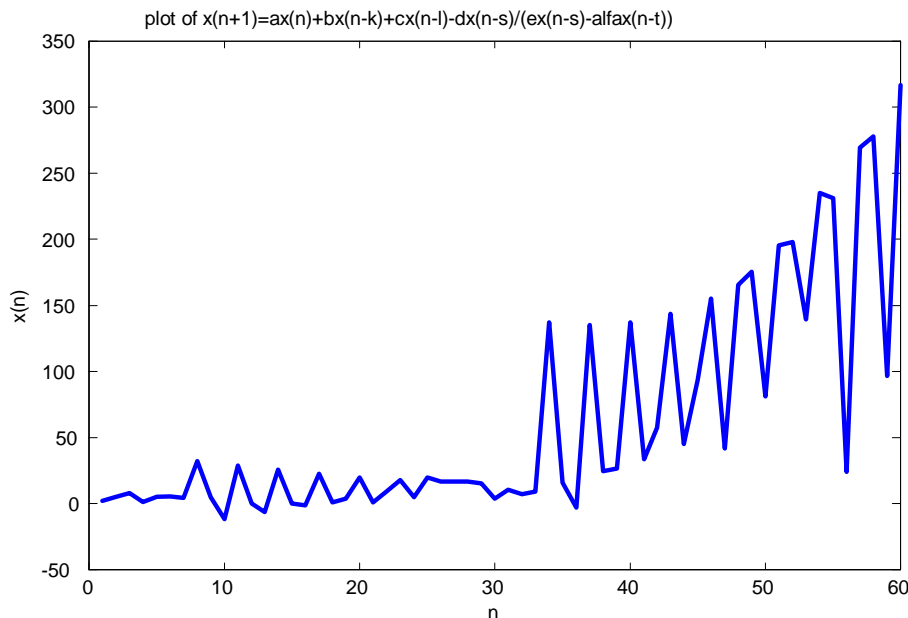


Figure 5. Plot of the solution of equation (1) has no periodic.

5 Existence of Bounded and Unbounded Solutions of Eq.(1)

In this section, we investigate the boundedness nature of the positive solutions of Eq.(1).

Theorem 5 Suppose $\{x_n\}$ be a solution of Eq.(1). Then the following statements are true:

(i) Let $d < e$ and for some $N \geq 0$, the initial conditions $x_{N-\sigma+1}$, $x_{N-\sigma+2}$, ..., x_{N-1} , $x_N \in [\frac{d}{e}, 1]$, are valid, then for $d \neq \alpha$ and $e^2 \neq d\alpha$, we have the inequality

$$\frac{d}{e}(a+b+c) - \frac{d}{d-\alpha} \leq x_n \leq a+b+c - \frac{d^2}{e^2 - \alpha d}, \quad \text{for all } n \geq N. \quad (15)$$

(ii) Let $d > e$ and for some $N \geq 0$, the initial conditions $x_{N-\sigma+1}$, $x_{N-\sigma+2}$, ..., x_{N-1} , $x_N \in [1, \frac{d}{e}]$, are valid, then for $d \neq \alpha$, $e^2 \neq d\alpha$ and $ex_{n-s} \neq \alpha x_{n-t}$, we have the inequality

$$a+b+c - \frac{d^2}{e^2 - \alpha d} \leq x_n \leq \frac{d}{e}(a+b+c) - \frac{d}{d-\alpha}, \quad \text{for all } n \geq N. \quad (16)$$

Proof: Let $\{x_n\}$ be a solution of Eq.(1). If for some $N \geq 0$, $\frac{d}{e} \leq x_n \leq 1$ and $d \neq \alpha$, we have

$$x_{n+1} = ax_n + bx_{n-k} + cx_{n-l} - \frac{dx_{n-s}}{ex_{n-s} - \alpha x_{n-t}} \leq a + b + c - \frac{dx_{n-s}}{ex_{n-s} - \alpha x_{n-t}}.$$

But, we can see that $ex_{n-s} - \alpha x_{n-t} \leq e - \alpha\left(\frac{d}{e}\right)$, $ex_{n-s} - \alpha x_{n-t} \leq \frac{e^2 - \alpha d}{e}$, $\frac{1}{ex_{n-s} - \alpha x_{n-t}} \geq \frac{e}{e^2 - \alpha d}$, $\frac{dx_{n-s}}{ex_{n-s} - \alpha x_{n-t}} \geq \frac{de\left(\frac{d}{e}\right)}{e^2 - \alpha d}$, $\frac{dx_{n-s}}{ex_{n-s} - \alpha x_{n-t}} \geq \frac{d^2}{e^2 - \alpha d}$, $-\frac{dx_{n-s}}{ex_{n-s} - \alpha x_{n-t}} \leq -\frac{d^2}{e^2 - \alpha d}$. Then for $\alpha d \neq e^2$, we get

$$x_{n+1} \leq a + b + c - \frac{d^2}{e^2 - \alpha d}. \quad (17)$$

Similarly, we can show that

$$x_{n+1} = ax_n + bx_{n-k} + cx_{n-l} - \frac{dx_{n-s}}{ex_{n-s} - \alpha x_{n-t}} \geq \frac{d}{e}(a + b + c) - \frac{dx_{n-s}}{ex_{n-s} - \alpha x_{n-t}}.$$

But, $ex_{n-s} - \alpha x_{n-t} \geq d - \alpha$, $\frac{1}{ex_{n-s} - \alpha x_{n-t}} \leq \frac{1}{d - \alpha}$, $\frac{dx_{n-s}}{ex_{n-s} - \alpha x_{n-t}} \leq \frac{d}{d - \alpha}$, $-\frac{dx_{n-s}}{ex_{n-s} - \alpha x_{n-t}} \geq -\frac{d}{d - \alpha}$. Then for $d \neq \alpha$, we get

$$x_{n+1} \geq \frac{d}{e}(a + b + c) - \frac{d}{d - \alpha}. \quad (18)$$

From (17) and (18), we get

$$\frac{d}{e}(a + b + c) - \frac{d}{d - \alpha} \leq x_{n+1} \leq a + b + c - \frac{d^2}{e^2 - \alpha d}, \quad \text{for all } n \geq N.$$

The proof of part (i) is completed.

Similarly, for some $N \geq 0$, $1 \leq x_n \leq \frac{d}{e}$, $d \neq \alpha$ and $e^2 \neq d\alpha$ we can prove part (ii) which is omitted here for convenience. Thus, the proof is now completed.

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A QUADRATIC FUNCTIONAL EQUATION IN INTUITIONISTIC FUZZY 2-BANACH SPACES

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ABSTRACT. In this paper, we define an intuitionistic fuzzy 2-normed space. Using the fixed point alternative approach, we investigate the Hyers-Ulam stability of the following quadratic functional equation

$$f(ax + by) + f(ax - by) = \frac{a}{2}f(x + y) + \frac{a}{2}f(x - y) + (2a^2 - a)f(x) + (2b^2 - a)f(y)$$

in intuitionistic fuzzy 2-Banach spaces.

1. INTRODUCTION

In 1940, Ulam [1] proposed the famous Ulam stability problem for a metric group homomorphism. In 1941, Hyers [2] solved this stability problem for additive mappings subject to the Hyers condition on approximately additive mappings in Banach spaces. In 1951, Bourgin [3] treated the Ulam stability problem for additive mappings. Subsequently the result of Hyers was generalized by Rassias [4] for linear mapping by considering an unbounded Cauchy difference.

The functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. The Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [5] for mappings $f : \mathcal{X} \rightarrow \mathcal{Y}$, where \mathcal{X} is a normed space and \mathcal{Y} is a Banach space.

In 1984, Katrasas [6] defined a fuzzy norm on a linear space to construct a fuzzy vector topological structure on the space. Later, some mathematicians have defined fuzzy norms on a linear space from various points of view [7, 8]. In particular, in 2003, Bag and Samanta [9], following Cheng and Mordeson [10], gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [11]. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces. Recently, considerable attention has been increasing to the problem of fuzzy stability of functional equations. Several various fuzzy stability results concerning Cauchy, Jensen, simple quadratic, and cubic functional equations have been investigated [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22].

Quite recently, the stability results in the setting of intuitionistic fuzzy normed space were studied in [23, 24, 25, 26]; respectively, while the idea of intuitionistic fuzzy normed space was introduced in [27].

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2. PRELIMINARIES

Definition 2.1. Let \mathcal{X} be a real linear space of dimension greater than one and let $\|\cdot, \cdot\|$ be a real-valued function on $\mathcal{X} \times \mathcal{X}$ satisfying the following condition:

1. $\|x, y\| = \|y, x\|$ for all $x, y \in \mathcal{X}$
2. $\|x, y\| = 0$ if and only if x, y are linearly dependent.
3. $\|\alpha x, y\| = |\alpha| \|x, y\|$ for all $x, y \in \mathcal{X}$ and $\alpha \in \mathbb{R}$.
4. $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for all $x, y, z \in \mathcal{X}$.

Then the function $\|\cdot, \cdot\|$ is called a 2-norm on X and pair $(X, \|\cdot, \cdot\|)$ is called a 2-normed linear space.

Definition 2.2. A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t -norm if $*$ satisfies the following conditions:

1. $*$ is commutative and associative;
2. $*$ is continuous;
3. $a * 1 = a$ for all $a \in [0, 1]$;
4. $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Example 2.1. An example of continuous t -norm is

$$a * b = \min\{a, b\}$$

Definition 2.3. A binary operation \diamond : $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t -conorm if \diamond satisfies the following conditions:

1. \diamond is commutative and associative;
2. \diamond is continuous;
3. $a \diamond 0 = a$ for all $a \in [0, 1]$;
4. $a \diamond b \leq c \diamond d$, whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Example 2.2. An example of continuous t -conorm is

$$a \diamond b = \max\{a, b\}$$

Definition 2.4. Let \mathcal{X} be a real linear space. A fuzzy subset μ of $\mathcal{X} \times \mathcal{X} \times \mathbb{R}$ is called a fuzzy 2-norm on \mathcal{X} if and only if for $x, y, z \in \mathcal{X}$, and $t, s, c \in \mathbb{R}$:

1. $\mu(x, y, t) = 0$ if $t \leq 0$.
2. $\mu(x, y, t) = 1$ if and only if x, y are linearly dependent, for all $t > 0$.
3. $\mu(x, y, t)$ is invariant under any permutation of x, y .
4. $\mu(x, cy, t) = \mu(x, y, \frac{t}{|c|})$ for all $t > 0$ and $c \neq 0$.
5. $\mu(x + z, y, t + s) \geq \mu(x, y, t) * \mu(z, y, s)$ for all $t, s > 0$.
6. $\mu(x, y, \cdot)$ is a non-decreasing function on \mathbb{R} and

$$\lim_{t \rightarrow \infty} \mu(x, y, t) = 1.$$

Then μ is said to be a fuzzy 2-norm on a linear space \mathcal{X} , and the pair (\mathcal{X}, μ) is called a fuzzy 2-normed linear space.

Example 2.3. Let $(\mathcal{X}, \|\cdot, \cdot\|)$ be a 2-normed linear space. Define

$$\mu(x, y, t) = \begin{cases} \frac{t}{t + \|x, y\|} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

where $x, y \in \mathcal{X}$ and $t \in \mathbb{R}$. Then (\mathcal{X}, μ) is a fuzzy 2-normed linear space.

Definition 2.5. Let (\mathcal{X}, μ) be a fuzzy 2-normed linear space. Let $\{x_n\}$ be a sequence in \mathcal{X} . Then $\{x_n\}$ is said to be convergent if there exists $x \in \mathcal{X}$ such that

$$\lim_{n \rightarrow \infty} \mu(x_n - x, y, t) = 1$$

for all $t > 0$.

Definition 2.6. Let (\mathcal{X}, μ) be a fuzzy 2-normed linear space. Let $\{x_n\}$ be a sequence in \mathcal{X} . Then $\{x_n\}$ is said to be a Cauchy sequence if

$$\lim_{n \rightarrow \infty} \mu(x_{n+p} - x_n, y, t) = 1$$

for all $t > 0$ and $p = 1, 2, 3, \dots$.

Let (\mathcal{X}, μ) be a fuzzy 2-normed linear space and $\{x_n\}$ be a Cauchy sequence in \mathcal{X} . If $\{x_n\}$ is convergent in \mathcal{X} then (\mathcal{X}, μ) is said to be a fuzzy 2-Banach space.

Definition 2.7. Let \mathcal{X} be a real linear space. A fuzzy subset ν of $\mathcal{X} \times \mathcal{X} \times \mathbb{R}$ such that for all $x, y, z \in \mathcal{X}$, and $t, s, c \in \mathbb{R}$

1. $\nu(x, y, t) = 1$, for all $t \leq 0$.
2. $\nu(x, y, t) = 0$ if and only if x, y are linearly dependent, for all $t > 0$.
3. $\nu(x, y, t)$ is invariant under any permutation of x, y .
4. $\nu(x, cy, t) = \nu(x, y, \frac{t}{|c|})$ for all $t > 0$, $c \neq 0$.
5. $\nu(x, y + z, t + s) \leq \nu(x, y, t) \diamond \nu(x, z, s)$ for all $s, t > 0$
6. $\nu(x, y, \cdot)$ is a nonincreasing function and

$$\lim_{t \rightarrow \infty} \nu(x, y, t) = 0$$

Then ν is said to be an anti fuzzy 2-norm on a linear space \mathcal{X} and the pair (\mathcal{X}, ν) is called an anti fuzzy 2-normed linear space.

Definition 2.8. Let (\mathcal{X}, ν) be an anti fuzzy 2-normed linear space and $\{x_n\}$ be a sequence in \mathcal{X} . Then $\{x_n\}$ is said to be convergent if there exists $x \in \mathcal{X}$ such that

$$\lim_{n \rightarrow \infty} \nu(x_n - x, y, t) = 0$$

for all $t > 0$.

Definition 2.9. Let (\mathcal{X}, ν) be an anti fuzzy 2-normed linear space and $\{x_n\}$ be a sequence in \mathcal{X} . Then $\{x_n\}$ is said to be a Cauchy sequence if

$$\lim_{n \rightarrow \infty} \nu(x_{n+p} - x_n, y, t) = 0$$

for all $t > 0$ and $p = 1, 2, 3, \dots$.

Let (\mathcal{X}, ν) be an anti fuzzy 2-normed linear space and $\{x_n\}$ be a Cauchy sequence in \mathcal{X} . If $\{x_n\}$ is convergent in \mathcal{X} then (\mathcal{X}, ν) is said to be an anti fuzzy 2-Banach space.

The following lemma is easy to prove and we will omit it.

Lemma 2.1. Consider the set L^* and operation \leq_{L^*} defined by

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 + x_2 \leq 1\}$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1, x_2 \geq y_2$$

for all $(x_1, x_2), (y_1, y_2) \in L^*$. Then (L^*, \leq_{L^*}) is a complete lattice.

Definition 2.10. A continuous t -norm τ on $L = [0, 1]^2$ is said to be continuous t -representable if there exist a continuous t -norm $*$ and a continuous t -conorm \diamond on $[0, 1]$ such that, for all $x = (x_1, x_2), y = (y_1, y_2) \in L$

$$\tau(x, y) = (x_1 * y_1, x_2 \diamond y_2).$$

Definition 2.11. Let \mathcal{X} be a set. A function $d : \mathcal{X} \times \mathcal{X} \rightarrow [0, \infty]$ is called a generalized metric on \mathcal{X} if and only if d satisfies:

- (M₁) $d(x, y) = 0 \Leftrightarrow x = y \quad \forall x, y \in \mathcal{X}$
- (M₂) $d(x, y) = d(y, x) \quad \forall x, y \in \mathcal{X}$
- (M₃) $d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in \mathcal{X}$

Theorem 2.1. ([28]) Let (\mathcal{X}, d) be a complete generalized metric space and $\mathcal{J} : \mathcal{X} \rightarrow \mathcal{X}$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in \mathcal{X}$, either

$$d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

- (a) $d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) < \infty$ for all $n \geq n_0$;
- (b) the sequence $\{\mathcal{J}^n x\}$ converges to a fixed point y^* of \mathcal{J} ;
- (c) y^* is the unique fixed point of \mathcal{J} in the set $\mathcal{Y} = \{y \in \mathcal{X} : d(\mathcal{J}^{n_0} x, y) < \infty\}$;
- (d) $d(y, y^*) \leq \frac{1}{1-L} d(y, \mathcal{J}y)$ for all $y \in \mathcal{Y}$.

3. MAIN RESULTS

3.1. Intuitionistic fuzzy 2-normed spaces. In this subsection we define an intuitionistic fuzzy 2-normed space. Then in next subsection by the fixed point technique we investigate the Hyers-Ulam stability of a generalized quadratic functional equation in intuitionistic fuzzy 2-normed spaces.

Definition 3.1. A 3-tuple $(\mathcal{X}, \rho_{\mu, \nu}, \tau)$ is said to be an intuitionistic fuzzy 2-normed space (for short, IF2NS) if \mathcal{X} is a real linear space, and μ and ν are a fuzzy 2-norm and an anti fuzzy 2-norm, respectively, such that $\nu(x, y, t) + \mu(x, y, t) \leq 1$. τ is continuous t -representable, and

$$\rho_{\mu, \nu} : \mathcal{X} \times \mathcal{X} \times \mathbb{R} \rightarrow L^*$$

$$\rho_{\mu, \nu}(x, y, t) = (\mu(x, y, t), \nu(x, y, t))$$

is a function satisfying the following conditions, for all $x, y, z \in \mathcal{X}$, and $t, s, \alpha \in \mathbb{R}$

- (1) $\rho_{\mu, \nu}(x, y, t) = (0, 1) = 0_{L^*}$ for all $t \leq 0$.
- (2) $\rho_{\mu, \nu}(x, y, t) = (1, 0) = 1_{L^*}$ if and only if x, y are linearly dependent, for all $t > 0$.
- (3) $\rho_{\mu, \nu}(\alpha x, y, t) = \rho_{\mu, \nu}(x, y, \frac{t}{|\alpha|})$ for all $t > 0$ and $\alpha \neq 0$
- (4) $\rho_{\mu, \nu}(x, y, t)$ is invariant under any permutation of x, y .
- (5) $\rho_{\mu, \nu}(x + z, y, t + s) \geq_{L^*} \tau(\rho_{\mu, \nu}(x, y, t), \rho_{\mu, \nu}(z, y, s))$ for all $t, s > 0$.
- (6) $\rho_{\mu, \nu}(x, y, \cdot)$ is continuous and

$$\lim_{t \rightarrow 0} \rho_{\mu, \nu}(x, y, t) = 0_{L^*} \text{ and } \lim_{t \rightarrow \infty} \rho_{\mu, \nu}(x, y, t) = 1_{L^*}$$

Then $\rho_{\mu, \nu}$ is said to be an intuitionistic fuzzy 2-norm on a real linear space \mathcal{X} .

Example 3.1. Let $(\mathcal{X}, \|\cdot, \cdot\|)$ be a 2-normed space,

$$\tau(a, b) = (a_1 b_1, \min(a_2 + b_2, 1))$$

be continuous t -representable for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ and μ, ν be a fuzzy and an anti fuzzy 2-norm, respectively. We define

$$\rho_{\mu,\nu}(x, y, t) = \left(\frac{t}{t + m\|x, y\|}, \frac{\|x, y\|}{t + m\|x, y\|} \right)$$

for all $t \in \mathbb{R}^+$ and $m > 1$. Then $(\mathcal{X}, \rho_{\mu,\nu}, \tau)$ is an IF2NS.

Definition 3.2. A sequence $\{x_n\}$ in an IF2NS $(\mathcal{X}, \rho_{\mu,\nu}, \tau)$ is said to be convergent to a point $x \in \mathcal{X}$ if

$$\lim_{n \rightarrow \infty} \rho_{\mu,\nu}(x_n - x, y, t) = 1_{L^*}$$

for every $t > 0$.

Definition 3.3. A sequence $\{x_n\}$ in an IF2NS $(\mathcal{X}, \rho_{\mu,\nu}, \tau)$ is said to be a Cauchy sequence if for any $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathcal{N}$ such that

$$\rho_{\mu,\nu}(x_n - x_m, y, t) \geq_{L^*} (1 - \epsilon, \epsilon)$$

for all $n, m \geq n_0$.

Definition 3.4. An IF2NS space $(\mathcal{X}, \rho_{\mu,\nu}, \tau)$ is said to be complete if every Cauchy sequence in $(\mathcal{X}, \rho_{\mu,\nu}, \tau)$ is convergent. A complete intuitionistic fuzzy 2-normed space is called an intuitionistic fuzzy 2-Banach space.

3.2. Hyers-Ulam stability of a generalized quadratic functional equation in IF2NS.

In this subsection, using the fixed point alternative approach, we prove the Hyers-Ulam stability of a generalized quadratic functional equation in intuitionistic fuzzy 2-Banach spaces.

Definition 3.5. Let \mathcal{X}, \mathcal{Y} be real linear spaces. For a given mapping $f : \mathcal{X} \rightarrow \mathcal{Y}$, we define

$$\begin{aligned} Df(x, y) := & f(ax + by) + f(ax - by) - \frac{a}{2}f(x + y) \\ & - \frac{a}{2}f(x - y) - (2a^2 - a)f(x) - (2b^2 - a)f(y) \end{aligned}$$

where $a, b \geq 1$, $a \neq 2b^2$ and $x, y \in \mathcal{X}$.

Theorem 3.1. Let \mathcal{X} be a real linear space, $(\mathcal{Z}, \rho'_{\mu,\nu}, \tau')$ an intuitionistic fuzzy 2-normed space and let $\phi : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{Z}, \varphi : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{Z}$ be mappings such that for some $0 < |\alpha| < a$

$$\rho'_{\mu,\nu}(\phi(ax, ay), \varphi(ax, ay), t) \geq_{L^*} \rho'_{\mu,\nu}\left(\frac{\alpha}{a^2}\phi(x, y), \varphi(x, y), t\right) \quad (3.1)$$

for all $x, y \in \mathcal{X}$ and $t \in \mathbb{R}^+$. Let $(\mathcal{Y}, \rho_{\mu,\nu}, \tau)$ be a complete intuitionistic fuzzy 2-normed space. If $\xi : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{Y}$ is a mapping such that $\xi(ax, ay) = \frac{1}{\alpha a^2}\xi(x, y)$ for all $x, y \in \mathcal{X}$ and $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a mapping satisfying $f(0) = 0$ and

$$\rho_{\mu,\nu}(Df(x, y), \xi(x, y), t) \geq_{L^*} \rho'_{\mu,\nu}(\phi(x, y), \varphi(x, y), t) \quad (3.2)$$

for all $x, y \in \mathcal{X}, t > 0$, then there is a unique quadratic mapping $Q : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$\rho_{\mu,\nu}(f(x) - Q(x), \xi(x, 0), t) \geq_{L^*} \rho'_{\mu,\nu}(\phi(x, 0), \varphi(x, 0), (2(a^2 - \alpha^2)t)) \quad (3.3)$$

Proof. Putting $y = 0$ in (3.2), we have

$$\rho_{\mu,\nu}(2f(ax) - a^2f(x), \xi(x, 0), t) \geq_{L^*} \rho'_{\mu,\nu}(\phi(x, 0), \varphi(x, 0), t)$$

and so

$$\rho_{\mu,\nu}\left(\frac{1}{a^2}f(ax) - f(x), \xi(x, 0), t\right) \geq_{L^*} \rho'_{\mu,\nu}\left(\frac{1}{2a^2}\phi(x, 0), \varphi(x, 0), t\right) \quad (3.4)$$

for all $x \in \mathcal{X}$ and $t > 0$. Consider the set $E = \{g : \mathcal{X} \rightarrow \mathcal{Y}\}$ and define a generalized metric d on E by

$$d(g, h) = \inf \{c \in \mathbb{R}^+ : \rho_{\mu, \nu}(g(x) - h(x), \xi(x, 0), t) \geq_{L^*} (c\phi(x, 0), \varphi(x, 0), t)\}$$

for all $x \in \mathcal{X}$ and $t > 0$ with $\inf \emptyset = \infty$. It is easy to show that (E, d) is complete (see [29]). Define $J : \mathcal{X} \rightarrow \mathcal{X}$ by $Jg(x) = \frac{1}{a^2}g(ax)$ for all $x \in \mathcal{X}$. Now, we prove that J is strictly contractive mapping of E with the Lipschitz constant $\frac{\alpha^2}{a^2}$. Let $g, h \in E$ be given such that $d(g, h) < \epsilon$. Then

$$\rho_{\mu, \nu}(g(x) - h(x), \xi(x, 0), t) \geq_{L^*} \rho'_{\mu, \nu}(\epsilon\phi(x, 0), \varphi(x, 0), t)$$

for all $x \in \mathcal{X}$ and $t > 0$. So

$$\begin{aligned} \rho_{\mu, \nu}(Jg(x) - Jh(x), \xi(x, 0), t) &= \rho_{\mu, \nu}(g(ax) - h(ax), \xi(x, 0), a^2t) \\ &\geq_{L^*} \rho'_{\mu, \nu}\left(c\phi(ax, 0), \varphi(ax, 0), \frac{t}{\alpha}\right) \\ &=_{L^*} \rho'_{\mu, \nu}\left(\frac{\alpha^2}{a^2}c\phi(x, 0), \varphi(x, 0), t\right). \end{aligned}$$

Then $d(Jg, Jh) < \frac{\alpha^2}{a^2} d(g, h)$ for all $g, h \in E$. It follows from (3.4) that

$$d(f, Jf) \leq \frac{1}{2a^2} < \infty$$

It follows from Theorem 2.1 that there exists a mapping $Q : \mathcal{X} \rightarrow \mathcal{Y}$ satisfying the following

- (1) Q is a fixed point of J , that is ;

$$Q(ax) = a^2Q(x) \quad (3.5)$$

- (2) The mapping Q is a unique fixed point of J in the set

$$\Delta = \{h \in E : d(g, h) < \infty\}$$

This implies that Q is a unique mapping satisfying (3.5).

- (3) $d(J^n f, Q) \rightarrow 0$ as $n \rightarrow \infty$. This implies that

$$\lim_{n \rightarrow \infty} \frac{f(a^n x)}{a^{2n}} = Q(x)$$

for all $x \in X$.

- (4) $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$ with $f \in \Delta$, which implies the inequality

$$d(f, Q) \leq \frac{1}{2(a^2 - \alpha^2)}$$

So

$$\rho_{\mu, \nu}(f(x) - Q(x), \xi(x, 0), t) \geq_{L^*} \rho'_{\mu, \nu}(\phi(x, 0), \varphi(x, 0), 2(a^2 - \alpha^2)t).$$

This implies that the inequality (3.2) holds.

It remains to show that Q is a quadratic mapping. Replacing x and y by $2^n x$ and $2^n y$ in (3.2), respectively, we get

$$\rho_{\mu, \nu}\left(\frac{1}{a^{2n}}Df(a^n x, a^n y), \xi(a^n x, a^n y), \frac{t}{a^{2n}}\right) \geq_{L^*} \rho'_{\mu, \nu}(\phi(a^n x, a^n y), \varphi(a^n x, a^n y), t).$$

By the property of $\xi(x, y)$, we have

$$\begin{aligned} \rho_{\mu, \nu} \left(\frac{1}{a^{2n}} Df(a^n x, a^n y), \frac{1}{(\alpha a^2)^n} \xi(x, y), \frac{t}{a^{2n}} \right) \\ \geq_{L^*} \rho'_{\mu, \nu} (\phi(a^n x, a^n y), \varphi(a^n x, a^n y), t). \end{aligned}$$

Thus

$$\rho_{\mu, \nu} \left(\frac{1}{a^{2n}} Df(a^n x, a^n y), \xi(x, y), t \right) \geq_{L^*} \rho'_{\mu, \nu} \left(\phi(a^n x, a^n y), \varphi(a^n x, a^n y), \frac{t}{\alpha^n} \right).$$

By (3.1), we obtain

$$\begin{aligned} \rho_{\mu, \nu} \left(\frac{1}{a^{2n}} Df(a^n x, a^n y), \xi(x, y), t \right) &\geq_{L^*} \rho'_{\mu, \nu} \left(\frac{\alpha^n}{a^{2n}} \phi(x, y), \varphi(x, y), \frac{t}{\alpha^n} \right) \\ &= \rho'_{\mu, \nu} \left(\frac{\alpha^{2n}}{a^{2n}} \phi(x, y), \varphi(x, y), t \right). \end{aligned}$$

As $n \rightarrow \infty$, we have

$$\rho_{\mu, \nu}(DQ(x, y), \xi(x, y), t) \geq_{L^*} 1_{L^*}.$$

Thus Q is a quadratic mapping, as desired. \square

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ON A q -ANALOGUE OF (h, q) -DAEHEE NUMBERS AND POLYNOMIALS OF HIGHER ORDER

JIN-WOO PARK

ABSTRACT. In this paper, we introduce a new q -analogue of the Daehee numbers and polynomials of the first kind and the second kind, and derive some new interesting identities.

1. INTRODUCTION

Let p be a fixed prime number. Throughout this paper, \mathbb{Z}_p , \mathbb{Q}_p , and \mathbb{C}_p will respectively denote the ring of p -adic rational integers, the field of p -adic rational numbers and the completions of algebraic closure of \mathbb{Q}_p . The p -adic norm is defined $|p|_p = \frac{1}{p}$.

When one talks of q -extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or p -adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. Throughout this paper, we use the notation :

$$[x]_q = \frac{1 - q^x}{1 - q}.$$

Note that $\lim_{q \rightarrow 1} [x]_q = x$ for each $x \in \mathbb{Z}_p$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on \mathbb{Z}_p . For $f \in UD(\mathbb{Z}_p)$, the p -adic invariant integral on \mathbb{Z}_p is defined by Kim as follows :

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \quad (\text{see [8, 9, 10]}). \quad (1.1)$$

Let f_1 be the translation of f with $f_1(x) = f(x+1)$. Then, by (1.1), we get

$$-qI_q(f_1) + I_q(f) = (1-q)f(0) + \frac{1-q}{\log q} f'(0), \quad \text{where } f'(0) = \left. \frac{df(x)}{dx} \right|_{x=0}. \quad (1.2)$$

As it is well-known fact, the *Stirling number of the first kind* is defined by

$$(x)_n = x(x-1) \cdots (x-n+1) = \sum_{l=0}^n S_1(n, l) x^l, \quad (1.3)$$

and the *Stirling number of the second kind* is given by the generating function to be

$$(e^t - 1)^m = m! \sum_{l=m}^{\infty} S_2(l, m) \frac{t^l}{l!}, \quad (\text{see [3, 17]}). \quad (1.4)$$

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Unsigned Stirling numbers of the first kind is given by

$$x^{(n)} = x(x+1)\cdots(x+n-1) = \sum_{l=0}^n |S_1(n, l)| x^l. \quad (1.5)$$

Note that if we replace x to $-x$ in (1.3), then

$$\begin{aligned} (-x)_n &= (-1)^n x^{(n)} = \sum_{l=0}^n S_1(n, l) (-1)^l x^l \\ &= (-1)^n \sum_{l=0}^n |S_1(n, l)| x^l. \end{aligned} \quad (1.6)$$

Hence $S_1(n, l) = |S_1(n, l)| (-1)^{n-l}$.

Recently, D. S. Kim and T. Kim introduced the *Daehee polynomials of the first kind of order r* are defined by the generating function to be

$$\left(\frac{\log(1+t)}{t} \right)^r (1+t)^x = \sum_{n=0}^{\infty} D_n^{(r)}(x) \frac{t^n}{n!}, \quad (1.7)$$

and the *Daehee polynomials of the second kind of order r* are given by

$$\left(\frac{\log(1+t)}{t+1} \right)^r (1+t)^x = \sum_{n=0}^{\infty} \widehat{D}_n^{(r)}(x) \frac{t^n}{n!} \quad (\text{see [5, 7, 9, 14, 16]}),$$

and Cho et. al. defined the *q -Daehee polynomials of order r* as follows.

$$\left(\frac{1-q + \frac{1-q}{\log q} \log(1+t)}{1-q-qt} \right)^r (1+t)^x = \sum_{n=0}^{\infty} D_{n,q}^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see [2]}).$$

In recent years, Kim et. al. have studies the various generalization of Daehee polynomials (see [2, 6, 12, 14, 15, 16]), and in [1], authors give new q -analogue of Changhee numbers and polynomials.

In this paper, we introduce a new q -analogue of the Daehee numbers and polynomials of the first kind and the second kind of order r , which are called the Witt-type formula for the q -analogue of Daehee polynomials of order r . We can derive some new interesting identities related to the q -Daehee polynomials of order r .

2. ON A q -ANALOGUE OF DAEHEE NUMBERS AND POLYNOMIALS OF ORDER r

In this section, we assume that $t, q \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$. First, we consider the following integral representation associated with the Pochhammer symbol :

$$\underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r\text{-times}} q^{\sum_{i=1}^r h_i y_i} (x + y_1 + \cdots + y_r)_n d\mu_q(y_1) \cdots d\mu_q(y_r), \quad (2.1)$$

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where $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $h_1, \dots, h_r \in \mathbb{Z}$ and $r \in \mathbb{N}$. By (2.1),

$$\begin{aligned} & \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} (x + y_1 + \dots + y_r)_n d\mu_q(y_1) \cdots d\mu_q(y_r) \frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} \left(\sum_{n=0}^{\infty} \binom{x + y_1 + \dots + y_r}{n} t^n \right) d\mu_q(y_1) \cdots d\mu_q(y_r) \quad (2.2) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} (1+t)^{x+y_1+\dots+y_r} d\mu_q(y_1) \cdots d\mu_q(y_r), \end{aligned}$$

where $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$. By (1.2) and (2.2), we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} (x + y_1 + \dots + y_r)_n d\mu_q(y_1) \cdots d\mu_q(y_r) \frac{t^n}{n!} \\ &= \prod_{i=1}^r \left(\frac{q-1 + \frac{q-1}{\log q} (h_i \log q + \log(1+t))}{q^{h_i+1}(1+t) - 1} \right) (1+t)^x. \end{aligned} \quad (2.3)$$

If we put

$$F_q^{(h_1, \dots, h_r)}(x, t) = \prod_{i=1}^r \left(\frac{q-1 + \frac{q-1}{\log q} (h_i \log q + \log(1+t))}{q^{h_i+1}(1+t) - 1} \right) (1+t)^x,$$

then

$$\lim_{q \rightarrow 1} F_q^{(-1, \dots, -1)}(x, t) = \left(\frac{\log(1+t)}{t} \right)^r (1+t)^x.$$

Note that $F_q^{(h_1, \dots, h_r)}(x, t)$ seems to be a new q -extension of the generating function for the Daehee polynomials of the first kind of order r . Thus, by (1.7) and (2.2), we obtain the following definition.

Definition 2.1. A q -analogue of the n th (h, q) -Daehee polynomials of the first kind is defined by the generating function to be

$$\sum_{n=0}^{\infty} D_n^{(h_1, \dots, h_r)}(x|q) \frac{t^n}{n!} = \prod_{i=1}^r \left(\frac{q-1 + \frac{q-1}{\log q} (h_i \log q + \log(1+t))}{q^{h_i+1}(1+t) - 1} \right) (1+t)^x.$$

Moreover,

$$D_n^{(h_1, \dots, h_r)}(x|q) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} (x + y_1 + \dots + y_r)_n d\mu_q(y_1) \cdots d\mu_q(y_r).$$

In the special case $x = 0$ in Definition 2.1, $D_n^{(h_1, \dots, h_r)}(0|q) = D_n^{(h_1, \dots, h_r)}(q)$ is called a q -analogue of the n th (h, q) -Daehee numbers of the first kind of order r . Note that, by (1.7) and Definition 2.1,

$$D_n^{(-1, \dots, -1)}(x|q) = \left(\frac{q-1}{\log q} \right)^r D_n^{(r)}(x). \quad (2.4)$$

The equation (2.4) shows that the q -analogue of the (h, q) -Daehee polynomials of the first kind of order r is closely related the n th Daehee polynomials of order r .

It is easy to show that

$$\begin{aligned} & \prod_{i=1}^r \left(\frac{q-1 + \frac{q-1}{\log q} (h_i \log q + \log(1+t))}{q^{h_i+1}(1+t) - 1} \right) (1+t)^x \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \binom{n}{m} D_{n-m}^{(h_1, \dots, h_r)}(q) (x)_m \right) \frac{t^n}{n!}. \end{aligned} \quad (2.5)$$

By Definition 2.1 and (2.5), we have

$$\begin{aligned} D_n^{(h_1, \dots, h_r)}(x|q) &= \sum_{m=0}^n \binom{x}{m} D_{n-m}^{(h_1, \dots, h_r)}(q) \frac{n!}{n-m!} \\ &= \sum_{m=0}^n \binom{x}{n-m} D_m^{(h_1, \dots, h_r)}(q) \frac{n!}{m!}. \end{aligned} \quad (2.6)$$

Since

$$\begin{aligned} (x + y_1 + \dots + y_r)_n &= \sum_{l=0}^n S_1(n, l) (x + y_1 + \dots + y_r)^l \\ &= \sum_{l=0}^n S_1(n, l) \sum_{l_1 + \dots + l_r = l} y_1^{l_1} y_2^{l_2} \dots (x + y_r)^{l_r}, \end{aligned} \quad (2.7)$$

by Definition 2.1 and (2.6), we have

$$\begin{aligned} & D_n^{(h_1, \dots, h_r)}(x|q) \\ &= \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} \sum_{l=0}^n S_1(n, l) \sum_{l_1 + \dots + l_r = l} y_1^{l_1} y_2^{l_2} \dots (x + y_r)^{l_r} \\ &= \sum_{l=0}^n S_1(n, l) \sum_{l_1 + \dots + l_r = l} B_{l_1, q}^{(h_1)} \dots B_{l_{r-1}, q}^{(h_{r-1})} B_{l_r, q}^{(h_r)}(x), \end{aligned} \quad (2.8)$$

where $B_{n, q}^{(h)}(x)$ are the (h, q) -Bernoulli polynomials derived from

$$B_{n, q}^{(h)}(x) = \int_{\mathbb{Z}_p} q^{hy} (x+y)^n d\mu_q(y), \quad (\text{see [18]}).$$

Thus, by (2.6) and (2.8), we obtain the following theorem.

Theorem 2.2. For $n \geq 0$, we have

$$\begin{aligned} D_n^{(h_1, \dots, h_r)}(x|q) &= \sum_{m=0}^n \binom{x}{n-m} D_m^{(h_1, \dots, h_r)}(q) \frac{n!}{m!} \\ &= \sum_{l=0}^n \sum_{l_1 + \dots + l_r = l} S_1(n, l) B_{l_1, q}^{(h_1)} \dots B_{l_{r-1}, q}^{(h_{r-1})} B_{l_r, q}^{(h_r)}(x) \end{aligned}$$

Note that, by (1.1), the generating function of (h, q) -Bernoulli polynomials are

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n, q}^{(h)}(x) \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} q^{hy} e^{(x+y)t} d\mu_q(y) \\ &= \frac{q-1 + \frac{q-1}{\log q} (h \log q + t)}{q^{h+1} e^t - 1} e^{xt}. \end{aligned} \quad (2.9)$$

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By replacing t by $e^t - 1$ in Definition 2.1,

$$\begin{aligned} & \sum_{n=0}^{\infty} D_n^{(h_1, \dots, h_r)}(x|q) \frac{1}{n!} (e^t - 1)^n \\ &= \sum_{n=0}^{\infty} D_n^{(h_1, \dots, h_r)}(x|q) \frac{1}{n!} \sum_{l=n}^{\infty} S_2(l, n) \frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n D_m^{(h_1, \dots, h_r)}(x|q) S_2(n, m) \frac{t^n}{n!}, \end{aligned} \quad (2.10)$$

and, by (2.9),

$$\begin{aligned} & \prod_{i=1}^r \left(\frac{q - 1 + \frac{q-1}{\log q} (h_i \log q + t)}{q^{h_i+1} e^t - 1} \right) e^{xt} \\ &= \left(\prod_{i=1}^{r-1} \left(\sum_{n=0}^{\infty} B_{n,q}^{(h_i)} \frac{t^n}{n!} \right) \right) \left(\sum_{n=0}^{\infty} B_{n,q}^{(h_r)}(x) \frac{t^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \sum_{l_1 + \dots + l_r = n} \binom{n}{l_1, \dots, l_r} B_{l_1,q}^{(h_1)} \cdots B_{l_{r-1},q}^{(h_{r-1})} B_{l_r,q}^{(h_r)}(x) \frac{t^n}{n!}. \end{aligned} \quad (2.11)$$

Thus, by (2.10) and (2.11), we obtain the following theorem.

Theorem 2.3. For $n \geq 0$, we have

$$\sum_{l_1 + \dots + l_r = n} \binom{n}{l_1, \dots, l_r} B_{l_1,q}^{(h_1)} \cdots B_{l_{r-1},q}^{(h_{r-1})} B_{l_r,q}^{(h_r)}(x) = \sum_{m=0}^n D_m^{(h_1, \dots, h_r)}(x|q) S_2(n, m).$$

Let us define the q -analogue of the n th (h, q) -Daehee polynomials of the second kind is defined as follows:

$$\widehat{D}_n^{(h_1, \dots, h_r)}(x|q) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} (-x - y_1 - \cdots - y_r)_n d\mu_q(y_1) \cdots d\mu_q(y_r) \quad (2.12)$$

where $n \in \mathbb{N} \cup \{0\}$. In particular, $\widehat{D}_n^{(h_1, \dots, h_r)}(0|q) = \widehat{D}_n^{(h_1, \dots, h_r)}(q)$ are called the q -analogue of the n th (h, q) -Daehee numbers of the second kind.

By (1.3) and (2.12), it leads to

$$\begin{aligned} & \widehat{D}_n^{(h_1, \dots, h_r)}(x|q) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} (-x - y_1 - \cdots - y_r)_n d\mu_q(y_1) \cdots d\mu_q(y_r) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} (-1)^n (x + y_1 + \cdots + y_r)^{(n)} d\mu_q(y_1) \cdots d\mu_q(y_r) \\ &= \sum_{l=0}^n |S_1(n, l)| (-1)^n \sum_{l_1 + \dots + l_r = l} B_{l_1,q}^{(h_1)} \cdots B_{l_r,q}^{(h_r)}(x). \end{aligned} \quad (2.13)$$

Thus, we state the following theorem.

Theorem 2.4. For $n \geq 0$, we have

$$\widehat{D}_n^{(h_1, \dots, h_r)}(x|q) = \sum_{l=0}^n \sum_{l_1 + \dots + l_r = l} |S_1(n, l)| (-1)^n B_{l_1,q}^{(h_1)} \cdots B_{l_r,q}^{(h_r)}(x).$$

Let us now consider the generating function of the q -analogue of the (h, q) -Daehee polynomials of the second kind as follows:

$$\begin{aligned}
& \sum_{n=0}^{\infty} \widehat{D}_n^{(h_1, \dots, h_r)}(x|q) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} (x - y_1 - \cdots - y_r)_n d\mu_q(y_1) \cdots d\mu_q(y_r) \frac{t^n}{n!} \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} \binom{x - y_1 - \cdots - y_r}{n} t^n d\mu_q(y_1) \cdots d\mu_q(y_r) \quad (2.14) \\
&= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} (1+t)^{-x-y_1-\cdots-y_r} d\mu_q(y_1) \cdots d\mu_q(y_r) \\
&= \left(\prod_{i=1}^r \frac{q-1 + \frac{q-1}{\log q} (h_i \log q - \log(1+t))}{q^{h_i+1} - 1 - t} \right) (1+t)^{r-x}.
\end{aligned}$$

By replacing t by $e^t - 1$, we have

$$\begin{aligned}
& \left(\prod_{i=1}^r \frac{q-1 + \frac{q-1}{\log q} (h_i \log q - t)}{q^{h_i+1} e^{-t} - 1} \right) e^{-xt} \\
&= \sum_{n=0}^{\infty} \widehat{D}_n^{(h_1, \dots, h_r)}(x|q) \frac{(e^t - 1)^n}{n!} \\
&= \sum_{n=0}^{\infty} \widehat{D}_n^{(h_1, \dots, h_r)}(x|q) \frac{1}{n!} \sum_{l=n}^{\infty} S_2(l, n) \frac{x^l}{l!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^n \widehat{D}_m^{(h_1, \dots, h_r)}(x|q) S_2(n, m) \right) \frac{x^n}{n!}, \quad (2.15)
\end{aligned}$$

and

$$\begin{aligned}
& \left(\prod_{i=1}^r \frac{q-1 + \frac{q-1}{\log q} (h_i \log q - t)}{q^{h_i+1} e^{-t} - 1} \right) e^{-xt} \\
&= \left(\prod_{i=1}^{r-1} \left(\sum_{n=0}^{\infty} (-1)^n B_{n,q}^{(h_i)} \frac{t^n}{n!} \right) \right) \left(\sum_{n=0}^{\infty} (-1)^n B_{n,q}^{(h_r)}(x) \frac{t^n}{n!} \right) \quad (2.16) \\
&= \sum_{n=0}^{\infty} (-1)^n \left(\sum_{l_1 + \cdots + l_r = n} \binom{n}{l_1, \dots, l_r} B_{l_1,q}^{(h_1)} \cdots B_{l_{r-1},q}^{(h_{r-1})} B_{l_r,q}^{(h_r)}(x) \right) \frac{t^n}{n!}.
\end{aligned}$$

By (2.15) and (2.16), we obtain the following theorem.

Theorem 2.5. For $n \geq 0$, we have

$$\begin{aligned}
& \sum_{m=0}^n \widehat{D}_m^{(h_1, \dots, h_r)}(x|q) S_2(n, m) \\
&= (-1)^n \sum_{l_1 + \cdots + l_r = n} \binom{n}{l_1, \dots, l_r} B_{l_1,q}^{(h_1)} \cdots B_{l_{r-1},q}^{(h_{r-1})} B_{l_r,q}^{(h_r)}(x).
\end{aligned}$$

By Theorem 2.3 and Theorem 2.5, we obtain the following corollary.

Corollary 2.6. For $n \geq 0$, we have

$$\sum_{m=0}^n D_m^{(h_1, \dots, h_r)}(x|q) S_2(n, m) = (-1)^n \sum_{m=0}^n \widehat{D}_m^{(h_1, \dots, h_r)}(x|q) S_2(n, m).$$

By Definition 2.1,

$$\begin{aligned} & (-1)^n \frac{D_n^{(h_1, \dots, h_r)}(x|q)}{n!} \\ &= (-1)^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} \binom{x + y_1 + \cdots + y_r}{n} d\mu_q(y_1) \cdots d\mu_q(y_r) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} \binom{-x - y_1 - \cdots - y_r + n - 1}{n} d\mu_q(y_1) \cdots d\mu_q(y_r) \\ &= \sum_{m=0}^n \binom{n-1}{n-m} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} \binom{-x - y_1 - \cdots - y_r}{m} d\mu_q(y_1) \cdots d\mu_q(y_r) \\ &= \sum_{m=1}^n \binom{n-1}{m-1} \frac{\widehat{D}_m^{(h_1, \dots, h_r)}(x|q)}{m!}, \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} & (-1)^n \frac{\widehat{D}_n^{(h_1, \dots, h_r)}(x|q)}{n!} \\ &= (-1)^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} \binom{-x - y_1 - \cdots - y_r}{n} d\mu_q(y_1) \cdots d\mu_q(y_r) \\ &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} \binom{x + y_1 + \cdots + y_r + n - 1}{n} d\mu_q(y_1) \cdots d\mu_q(y_r) \\ &= \sum_{m=0}^n \binom{n-1}{n-m} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{i=1}^r h_i y_i} \binom{x + y_1 + \cdots + y_r}{m} d\mu_q(y_1) \cdots d\mu_q(y_r) \\ &= \sum_{m=1}^n \binom{n-1}{m-1} \frac{D_m^{(h_1, \dots, h_r)}(x|q)}{m!}. \end{aligned} \tag{2.18}$$

Therefore, by (2.17) and (2.18), we obtain the following theorem.

Theorem 2.7. For $n \geq 0$, we have

$$(-1)^n \frac{D_n^{(h_1, \dots, h_r)}(x|q)}{n!} = \sum_{m=1}^n \binom{n-1}{m-1} \frac{\widehat{D}_m^{(h_1, \dots, h_r)}(x|q)}{m!},$$

and

$$(-1)^n \frac{\widehat{D}_n^{(h_1, \dots, h_r)}(x|q)}{n!} = \sum_{m=1}^n \binom{n-1}{m-1} \frac{D_m^{(h_1, \dots, h_r)}(x|q)}{m!}.$$

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On blow-up of solutions for a semilinear damped wave equation with nonlinear dynamic boundary conditions

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Abstract: In this paper, we study a semilinear damped wave equation with nonlinear dynamic boundary conditions. Under certain assumptions, we extend the earlier exponentially growth result in Gerbi and Said-Houari (Adv. Differential Equations 13: 1051-1074, 2008) to a blow-up in finite time result with positive initial energy.

Keywords: damped wave equation; dynamic boundary conditions; blow-up

AMS Subject Classification (2010): 35L20; 35L71; 35B44

1. INTRODUCTION

In this work, we investigate the following semilinear damped wave equation with dynamic boundary conditions

$$\begin{cases} u_{tt} - \Delta u - \alpha \Delta u_t = |u|^{p-2}u, & x \in \Omega, \quad t > 0, \\ u(x, t) = 0, & x \in \Gamma_0, \quad t \geq 0, \\ u_{tt}(x, t) = - \left[\frac{\partial u}{\partial \nu}(x, t) + \alpha \frac{\partial u_t}{\partial \nu}(x, t) + r|u_t|^{m-2}u_t \right], & x \in \Gamma_1, \quad t > 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega. \end{cases} \quad (1.1)$$

Here Ω is a regular and bounded domain in \mathbb{R}^N ($N \geq 1$) and $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\text{mes}(\Gamma_0) > 0$, $\Gamma_0 \cap \Gamma_1 = \emptyset$. We denote Δ the Laplacian operator with respect to the x variable and $\frac{\partial}{\partial \nu}$ the unit outer normal derivative, $m \geq 2$, $p > 2$, α, r are positive constants and u_0 and u_1 are given functions.

From the mathematical point of view, the boundary conditions that do not neglect the acceleration terms are usually called dynamic boundary conditions. Researches on these problems are very important in practical problems as well as in the theoretical fields.

For the cases of one dimension space, many results have been established (see [1, 2, 3, 11, 12, 13, 15, 24, 23, 35]). For example, Grobbelaar-van Dalsen [12] studied the following problem:

$$\begin{cases} u_{tt} - u_{xx} - u_{txx} = 0, & x \in (0, L), \quad t > 0, \\ u(0, t) = 0, & t > 0, \\ u_{tt}(L, t) = -[u_x + u_{tx}](L, t), & t > 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x), & x \in (0, L), \\ u(L, 0) = \eta, \quad u_t(L, 0) = \mu & t > 0. \end{cases} \quad (1.2)$$

By using the theory of B-evolutions and the theory of fractional powers, the author proved that problem (1.2) gives rise to an analytic semigroup in an appropriate functional space and obtained the existence and the uniqueness of solutions. For a problem related to (1.2), an exponential

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decay result was obtained in [13], which describes the weakly damped vibrations of an extensible beam. Later, Zhang and Hu [35] considered (1.2) in a more general form and an exponential and polynomial decay rates for the energy were obtained by using the Nakao inequality. Pellicer and Solà-Morales [24] considered the linear wave equation with strong damping and dynamical boundary conditions as an alternative model for the classical spring-mass-damper ODE:

$$m_1 u''(t) + d_1 u'(t) + k_1(t) = 0. \quad (1.3)$$

Based on the semigroup theory, spectral perturbation analysis and dominant eigenvalues, they compared analytically these two approaches to the same physical system. Then, Pellicer [23] considered the same problem with a control acceleration $\varepsilon f\left(u(1, t), \frac{u_t(1, t)}{\sqrt{\varepsilon}}\right)$ as a model for a controlled spring-mass-damper system and established some results concerning its large time behavior. By applying invariant manifold theory, the author proved that the infinite dimensional system admits a two-dimensional attracting manifold where the equation is well represented by a classical nonlinear oscillations ODE, which can be exhibited explicitly.

For the multi-dimensional cases, we can cite [5, 6, 14, 21, 22, 30] for problems with the Dirichlet boundary conditions and [27, 28, 29] for the Cauchy problems. Recently, Gerbi and Said-Houari [7, 8] studied problem (1.1), in which the strong damping term $-\Delta u_t$ is involved. They showed in [7] that if the initial data are large enough then the energy and the L^p norm of the solution of problem (1.1) is unbounded and grows up exponentially as time goes to infinity. Later, they established in [8] the global existence and asymptotic stability of solutions starting in a stable set by combining the potential well method and the energy method. A blow-up result for the case $m = 2$ with initial data in the unstable set was also obtained. However, as indicated in [8], the blow-up of solutions in the presence of a strong damping and a nonlinear boundary damping (i.e., $m > 2$) at the same time is still an open problem. For other related works, we refer the readers to [4, 10, 9, 17, 18, 19, 20, 25, 26, 31, 32, 33, 34] and the references therein.

Motivated by the above works, in this article, we intend to extend the exponentially growth result in [7] to a blow-up result with positive initial energy. The main difficulty here is the simultaneous appearance of the strong damping term Δu_t , the nonlinear boundary damping term $r|u_t|^{m-2}u_t$, and the nonlinear source term $|u|^{p-2}u$. For our purpose, the functional like $L(t) = H(t) + \varepsilon F(t)$ in [7] is modified to $L(t) = H^{1-\alpha}(t) + \varepsilon F(t)$ for some $\alpha > 0$ in this paper. We also give a modified manner to estimate the term $\left|\int_{\Gamma_1} |u_t|^{m-2}u_t u d\sigma\right|$ so that the appearance of the form like $\gamma = RH^{-\sigma}(t)$ (for constants γ , R and σ) which has been used in many earlier works (for example in [10, 16, 21, 22]) can be avoided.

The paper is organized as follows. In Section 2 we present some notations and assumptions and state the main result. Section 3 is devoted to proof of the blow-up result - Theorem 2.2.

2. PRELIMINARIES AND MAIN RESULT

In this section, we first recall some notations and assumptions given in [7]. We denote

$$H_{\Gamma_0}^1(\Omega) = \{u \in H^1(\Omega) \mid u_{\Gamma_0} = 0\}$$

with the scalar product (\cdot, \cdot) in $L^2(\Omega)$ and we also mean by $\|\cdot\|_q$ the $L^q(\Omega)$ norm for $1 \leq q \leq \infty$ and by $\|\cdot\|_{q,\Gamma_1}$ the $L^q(\Gamma_1)$ norm. We will use the following embedding

$$H_{\Gamma_0}^1(\Omega) \hookrightarrow L^q(\Gamma_1), \quad 2 \leq q \leq \bar{q},$$

where

$$\bar{q} = \begin{cases} \frac{2(N-1)}{N-2}, & \text{if } N \geq 3, \\ +\infty, & \text{if } N = 1, 2. \end{cases} \quad (2.1)$$

We state the following local existence and uniqueness theorem established in [7].

Theorem 2.1. ([7, theorem 2.1]) *Let $2 \leq p \leq \bar{q}$ and $\max\left\{2, \frac{\bar{q}}{\bar{q}+1-q}\right\} \leq m \leq \bar{q}$. Then given $u_0 \in H_{\Gamma_0}^1(\Omega)$ and $u_1 \in L^2(\Omega)$, there exists $T > 0$ and a unique solution $u(t)$ of the problem (1.1) on $[0, T)$ such that*

$$u \in C(0, T; H_{\Gamma_0}^1(\Omega)) \cap C^1(0, T; L^2(\Omega)), \quad u_t \in L^2(0, T; H_{\Gamma_0}^1(\Omega)) \cap L^m(\Gamma_1 \times [0, T)).$$

We define the energy functional

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p + \frac{1}{2}\|u_t\|_{2,\Gamma_1}^2 \quad (2.2)$$

and set

$$\alpha_1 = B^{-p/(p-2)}, \quad E_1 = \left(\frac{1}{2} - \frac{1}{p}\right)\alpha_1^2, \quad (2.3)$$

where B is the best constant of the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$. We can easily get

$$E'(t) = -\alpha\|\nabla u_t\|_2^2 - r\|u_t\|_{m,\Gamma_1}^m \leq 0. \quad (2.4)$$

Our main result reads as follows.

Theorem 2.2. *Suppose that $m < p$ with $2 < p \leq \bar{q}$ and that*

$$0 < \frac{N}{2} - \frac{N-1}{m} \leq \min\left\{\frac{p-2}{p}, \frac{2(p-m)}{mp}\right\} \quad (2.5)$$

holds. Assume that

$$E(0) < E_1, \quad \|\nabla u_0\|_2 > \alpha_1. \quad (2.6)$$

Then the solution of problem (1.1) blows up in a finite time T_0 , in the sense that

$$\lim_{t \rightarrow T_0^-} [\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 + \|\nabla u\|_2^2] = +\infty. \quad (2.7)$$

3. BLOW-UP OF SOLUTIONS

In this section, we prove our main result and use C to denote a generic positive constant. To this end, we need the following lemmas.

Lemma 3.1. ([7, Lemma 3.1]) *Let u be the solution of problem (1.1). Assume that $2 < p \leq \bar{q}$ and (2.6) holds. Then there exists a constant $\alpha_2 > \alpha_1$ such that*

$$\|\nabla u(\cdot, t)\|_2 \geq \alpha_2, \quad \forall t \geq 0, \quad (3.1)$$

and

$$\|u\|_p \geq B\alpha_2, \quad \forall t \geq 0. \quad (3.2)$$

Lemma 3.2. *Let u be the solution of problem (1.1). Assume that $2 < p \leq \bar{q}$ and (2.6) holds. Then we have*

$$E_1 < \frac{p-2}{2p} \|u\|_p^p, \quad \forall t \geq 0. \quad (3.3)$$

Proof. Exploiting (2.3) and (3.2), we get

$$E_1 = \frac{p-2}{2p} \alpha_1^{2-p} \alpha_1^p = \frac{p-2}{2p} B^p \alpha_1^p < \frac{p-2}{2p} B^p \alpha_2^p \leq \frac{p-2}{2p} \|u\|_p^p.$$

Set

$$H(t) = E_1 - E(t), \quad (3.4)$$

then we have

$$0 < H(0) \leq H(t) < \frac{1}{p} \|u\|_p^p + \frac{p-2}{2p} \|u\|_p^p \leq \frac{1}{2} \|u\|_p^p. \quad (3.5)$$

As a result of (2.2) and (3.5), we can deduce as in [21, 22] the following lemma.

Lemma 3.3. *Let u be the solution of problem (1.1). Assume that $2 < p \leq \bar{q}$ and (2.6) holds. Then*

$$\|u\|_p^\kappa \leq C(\|\nabla u\|_2^2 + \|u\|_p^p) \leq C[-H(t) - \|u_t\|_2^2 - \|u_t\|_{2,\Gamma_1}^2 + \|u\|_p^p] \quad (3.6)$$

for any $2 \leq \kappa \leq p$.

Now, we are ready to prove our result.

Proof of Theorem 2.2. We assume by contradiction that (2.7) does not hold true. Then for $\forall T^* < +\infty$ and all $t \in [0, T^*]$, we have

$$\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 + \|\nabla u\|_2^2 \leq C_1, \quad (3.7)$$

where C_1 is a positive constant. Set

$$L(t) = H^{1-\theta}(t) + \varepsilon \int_{\Omega} u_t u dx + \varepsilon \int_{\Gamma_1} u_t u d\sigma + \frac{\varepsilon \alpha}{2} \|\nabla u\|_2^2 \quad (3.8)$$

for ε small to be chosen later and

$$\frac{s}{2} \leq \theta \leq \min \left\{ \frac{p-2}{2p}, \quad \frac{1}{m} - \frac{1}{p} \right\}$$

with

$$0 < \frac{N}{2} - \frac{N-1}{m} \leq s \leq \min \left\{ \frac{p-2}{p}, \quad \frac{2(p-m)}{mp} \right\}.$$

Taking a derivative of $L(t)$ in (3.8) and use (1.1) and (2.2), we get

$$\begin{aligned} L'(t) &= (1-\theta)H^{-\theta}(t)H'(t) + \varepsilon(\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 - \|\nabla u\|_2^2) + \varepsilon\|u\|_p^p - \varepsilon r \int_{\Omega} |u_t|^{m-2} u_t u d\sigma \\ &= (1-\theta)H^{-\theta}(t)H'(t) + 2\varepsilon(\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2) - 2\varepsilon E_1 + 2\varepsilon H(t) \\ &\quad + \varepsilon \left(1 - \frac{2}{p}\right) \|u\|_p^p - \varepsilon r \int_{\Omega} |u_t|^{m-2} u_t u d\sigma. \end{aligned} \quad (3.9)$$

We now estimate the last term on the right-hand side of (3.9). By Hölder's inequality, we obtain

$$\left| \int_{\Gamma_1} |u_t|^{m-2} u_t u d\sigma \right| \leq \|u_t\|_{m,\Gamma_1}^{m-1} \|u\|_{m,\Gamma_1}. \quad (3.10)$$

As in [7], for $m \geq 1$ and $\frac{N}{2} - \frac{N-1}{m} \leq s \leq \min \left\{ \frac{p-2}{p}, \quad \frac{2(p-m)}{mp} \right\} < 1$,

$$\|u\|_{m,\Gamma_1} \leq C\|u\|_{H^s(\Omega)} \leq C\|u\|_2^{1-s} \|\nabla u\|_2^s \leq C\|u\|_p^{1-s} \|\nabla u\|_2^s \leq C\|u\|_p^{\frac{p}{m}} \|u\|_p^{1-s-\frac{p}{m}} \|\nabla u\|_2^s. \quad (3.11)$$

Combining (3.10), (3.11), (3.5) and (3.8), and using Young's inequality we have

$$\begin{aligned} \left| \int_{\Gamma_1} |u_t|^{m-2} u_t u d\sigma \right| &\leq C C_1^{\frac{s}{2}} \|u_t\|_{m,\Gamma_1}^{m-1} \|u\|_p^{\frac{p}{m}} \|u\|_p^{1-s-\frac{p}{m}} \leq C C_1^{\frac{s}{2}} \|u_t\|_{m,\Gamma_1}^{m-1} \|u\|_p^{\frac{p}{m}} H^{-\theta_1}(t) \\ &\leq C C_1^{\frac{s}{2}} H^{-\theta_1}(t) \left(\frac{\beta^m}{m} \|u\|_p^p + \frac{(m-1)\beta^{-m}}{m} \|u_t\|_{m,\Gamma_1}^m \right), \end{aligned} \quad (3.12)$$

where $\theta_1 = \frac{1}{m} - \frac{1-s}{p} \geq \theta$ and $\beta > 0$ will be chosen later. Substituting (3.12) in (3.9) yields

$$\begin{aligned} L'(t) &\geq (1-\theta)H^{-\theta}(t)H'(t) + 2\varepsilon(\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2) - 2\varepsilon E_1 + 2\varepsilon H(t) \\ &\quad + \varepsilon \left(1 - \frac{2}{p} - \frac{r C C_1^{\frac{s}{2}} \beta^m}{m} H^{-\theta_1}(t) \right) \|u\|_p^p - \frac{\varepsilon r C C_1^{\frac{s}{2}} (m-1)}{m \beta^m} H^{-\theta_1}(t) \|u_t\|_{m,\Gamma_1}^m. \end{aligned} \quad (3.13)$$

By virtue of (2.4), (3.4) and (3.5), we get

$$H'(t) \geq r \|u_t\|_{m,\Gamma_1}^m$$

and

$$H^{-\theta_1}(t) \leq H^{-\theta_1}(0), \quad H^{-\theta_1}(t) \leq H^{-(\theta_1-\theta)}(0) H^{-\theta}(t).$$

Furthermore, using (3.2), we have

$$-2\varepsilon E_1 \geq -2\varepsilon E_1 B^{-p} \alpha_2^{-p} \|u\|_p^p.$$

Therefore, (3.13) becomes

$$\begin{aligned} L'(t) &\geq r \left(1 - \theta - \frac{\varepsilon C C_1^{\frac{s}{2}} (m-1)}{m \beta^m} H^{-(\theta_1-\theta)}(0) \right) H^{-\theta}(t) \|u_t\|_{m,\Gamma_1}^m + 2\varepsilon(\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2) \\ &\quad + \varepsilon \left(1 - \frac{2}{p} - \frac{r C C_1^{\frac{s}{2}} \beta^m}{m} H^{-\theta_1}(0) - 2E_1 B^{-p} \alpha_2^{-p} \right) \|u\|_p^p + 2\varepsilon H(t). \end{aligned} \quad (3.14)$$

Since $\alpha_2 > \alpha_1$ and combining the definition of E_1 , we have

$$1 - \frac{2}{p} - 2E_1 B^{-p} \alpha_2^{-p} = \frac{p-2}{p} \left[1 - \left(\frac{\alpha_1}{\alpha_2} \right)^p \right] > 0.$$

So, we can choose β small enough so that

$$1 - \frac{2}{p} - \frac{r C C_1^{\frac{s}{2}} \beta^m}{m} H^{-\theta_1}(0) - 2E_1 B^{-p} \alpha_2^{-p} > 0.$$

Once β is fixed, we choose ε small enough such that

$$L(0) = H^{1-\theta}(0) + \varepsilon \int_{\Omega} u_0 u_1 dx + \varepsilon \int_{\Gamma_1} u_0 u_1 d\sigma + \frac{\varepsilon \alpha}{2} \|\nabla u_0\|_2^2 > 0$$

and

$$1 - \theta - \frac{\varepsilon C C_1^{\frac{s}{2}} (m-1)}{m \beta^m} H^{-(\theta_1-\theta)}(0) > 0.$$

Hence, we have

$$L'(t) \geq \Lambda \varepsilon (\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 + \|u\|_p^p + H(t)) \quad (3.15)$$

for some positive constant Λ .

On the other hand, we have

$$L^{\frac{1}{1-\theta}}(t) = \left(H^{1-\theta}(t) + \varepsilon \int_{\Omega} u_t u dx + \varepsilon \int_{\Gamma_1} u_t u d\sigma + \frac{\varepsilon \alpha}{2} \|\nabla u\|_2^2 \right)^{\frac{1}{1-\theta}}$$

$$\leq C \left(H(t) + \left| \int_{\Omega} u_t u dx \right|^{\frac{1}{1-\theta}} + \left| \int_{\Gamma_1} u_t u d\sigma \right|^{\frac{1}{1-\theta}} + \|\nabla u\|_2^{\frac{2}{1-\theta}} \right). \quad (3.16)$$

Using Hölder and Young inequalities, (3.7), (3.11) and Lemma 3.3, we get

$$\begin{aligned} \left| \int_{\Omega} u_t u dx \right|^{\frac{1}{1-\theta}} &\leq C (\|u\|_2 \|u_t\|_2)^{\frac{1}{1-\theta}} \leq C \|u\|_p^{\frac{1}{1-\theta}} \|u_t\|_2^{\frac{1}{1-\theta}} \leq C \left(\|u\|_p^{\frac{2}{1-2\theta}} + \|u_t\|_2^2 \right) \\ &\leq C (H(t) + \|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 + \|u\|_p^p), \end{aligned} \quad (3.17)$$

$$\begin{aligned} \left| \int_{\Gamma_1} u_t u d\sigma \right|^{\frac{1}{1-\theta}} &\leq C (\|u\|_{2,\Gamma_1} \|u_t\|_{2,\Gamma_1})^{\frac{1}{1-\theta}} \leq C \|u_t\|_{2,\Gamma_1}^{\frac{1}{1-\theta}} \|u\|_p^{\frac{1-s}{1-\theta}} \|\nabla u\|_2^{\frac{s}{1-\theta}} \\ &\leq C C_1^{\frac{s}{2(1-\theta)}} \left(\|u\|_p^{\frac{2(1-s)}{1-2\theta}} + \|u_t\|_{2,\Gamma_1}^2 \right) \leq C (H(t) + \|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 + \|u\|_p^p), \end{aligned} \quad (3.18)$$

and

$$\|\nabla u\|_2^{\frac{2}{1-\theta}} \leq C_1^{\frac{1}{1-\theta}}. \quad (3.19)$$

Using the Poincaré's inequality and (3.7), we have

$$\|u\|_p^p \leq B^p \|\nabla u\|_2^p \leq B^p C_1^{\frac{p}{2}}. \quad (3.20)$$

By virtue of (3.5) and (3.20), we know that $H(t)$ is bounded. There exists a positive constant C_2 such that

$$H(t) + C_1^{\frac{1}{1-\theta}} \leq C_2 H(t).$$

Therefore, we obtain

$$L^{\frac{1}{1-\theta}}(t) \leq C (\|u_t\|_2^2 + \|u_t\|_{2,\Gamma_1}^2 + \|u\|_p^p + H(t)). \quad (3.21)$$

A combining of (3.15) and (3.21) leads to

$$L'(t) \geq \frac{\varepsilon \Lambda}{C} L^{\frac{1}{1-\theta}}(t). \quad (3.22)$$

A simple integration of (3.22) over $[0, t]$ gives

$$L^{\frac{\theta}{1-\theta}}(t) \geq \frac{1}{L^{-\frac{\theta}{1-\theta}}(0) - \frac{\theta \Lambda \varepsilon}{C(1-\theta)} t}, \quad \forall t \geq 0. \quad (3.23)$$

This shows that $L(t)$ blows up in a finite time T_0 , where

$$T_0 \leq \frac{(1-\theta)C}{\Lambda \varepsilon \theta [L(0)]^{\theta/(1-\theta)}}.$$

If we choose $T^* \geq \frac{(1-\theta)C}{\Lambda \varepsilon \theta [L(0)]^{\theta/(1-\theta)}}$, then we obtain $T_0 \leq T^*$, which contradicts to our assumption. This completes the proof.

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UNIQUENESS OF MEROMORPHIC FUNCTIONS WITH THEIR DIFFERENCE OPERATORS

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ABSTRACT. This paper is devoted to considering sharing value problems for a meromorphic function $f(z)$ with its difference operator $\Delta_c f = f(z+c) - f(z)$, which improve some recent results in Chen and Yi in [2].

1. INTRODUCTION

In this paper a meromorphic function will mean meromorphic in the whole complex plane. We assume that the reader is familiar with the elementary Nevanlinna Theory, see, e.g. [8, 19]. In particular, we denote the order, exponent of convergence of zeros and poles of a meromorphic function $f(z)$ by $\sigma(f)$, $\lambda(f)$ and $\lambda(\frac{1}{f})$, respectively. As usual, the abbreviation CM stands for "counting multiplicities", while IM means "ignoring multiplicities".

The classical results in the uniqueness theory of meromorphic functions are the five-point, resp. four-point, theorems due to Nevanlinna [17]:

Theorem A. *If two meromorphic functions $f(z)$ and $g(z)$ share five distinct values $a_1, a_2, a_3, a_4, a_5 \in \mathbb{C} \cup \{\infty\}$ IM, then $f(z) = g(z)$.*

Theorem B. *If two meromorphic functions $f(z)$ and $g(z)$ share four distinct values $a_1, a_2, a_3, a_4 \in \mathbb{C} \cup \{\infty\}$ CM, then $f(z) = g(z)$ or $f(z) = T \circ g(z)$, where T is a Möbius transformation.*

It is well-known that 4 CM can not be improved to 4 IM, see [4]. Further, Gundersen [6, Theorem 1] has improved the assumption 4 CM to 2 CM+2 IM, while 1 CM+3 IM remains an open problem. Meanwhile, Gundersen [7], Mues and Stinmetz [16] got some uniqueness results on the case when $g(z)$ is the derivative of $f(z)$:

Theorem C. *If a meromorphic functions $f(z)$ and its derivative $f'(z)$ share two distinct values a_1, a_2 CM, then $f(z) = f'(z)$.*

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Gundersen [7] has given a counterexample to show that the conclusion of Theorem C is not valid if 2 CM is replaced by 1 CM + 1 IM. However, 2 CM can be replaced by 3 IM, see [5, 15].

In recent papers [9, 10], Heittokangas et al. started to consider the uniqueness of a finite order meromorphic function sharing values with its shift. They concluded that:

Theorem D. *Let $f(z)$ be a meromorphic function of finite order, let $c \in \mathbb{C}$, and let $a_1, a_2, a_3 \in S(f) \cup \{\infty\}$ be three distinct periodic functions with period c . If $f(z)$ and $f(z+c)$ share a_1, a_2 CM and a_3 IM, then $f(z) = f(z+c)$ for all $z \in \mathbb{C}$.*

Some improvements of Theorem D can be found in [2, 11, 12, 18]. The difference operator $\Delta_c f = f(z+c) - f(z)$ can be regarded as the difference counterpart of $f'(z)$. Therefore, some research results [9, 13] have been obtained for the problem that $\Delta_c f$ and $f(z)$ share one value a CM, which can be seen as difference analogues of Brück conjecture in [1]. Here, we just recall the following result in [2] as an example:

Theorem E. *Let $f(z)$ be a finite order transcendental entire function which has a finite Borel exceptional value a , and let $f(z)$ be not periodic of period c . If $\Delta_c f$ and $f(z)$ share a CM, then $a = 0$ and $\Delta_c f = \tau f(z)$, where τ is a non-zero constant.*

Zhang et al. gave some improvements of Theorem E, the reader is referred to [14, 20]. A natural question is: what is the uniqueness result on the case when $f(z)$ is meromorphic and $a(z)$ is a small function of $f(z)$ in Theorem E. Corresponding to this question, we get the following results:

Theorem 1.1. *Let $f(z)$ be a transcendental meromorphic function of finite order which has two Borel exceptional values a and ∞ , and let $f(z)$ be not periodic of period c . If $\Delta_c f$ and $f(z)$ share values a and ∞ CM, then $a = 0$ and $f(z) = Ae^{Bz}$, where A, B are non-zero constants.*

Theorem 1.2. *Let $f(z)$ be a transcendental meromorphic function of finite order which has a Borel exceptional value ∞ , and let $a(z)$ be a non-constant meromorphic function such that $\sigma(a) < \sigma(f)$ and $\lambda(f-a) < \sigma(f)$. If $\Delta_c f$ and $f(z)$ share values $a(z)$ and ∞ CM, then $f(z) = a(z) + Ce^{Dz}$ and $\sigma(a) < 1$, where C, D are non-zero constants, .*

2. SOME LEMMAS

Lemma 2.1. [3, Theorem 2.1] *Let $f(z)$ be a non-constant meromorphic function with finite order σ , and let c be a non-zero constant. Then, for each $\varepsilon > 0$, we have*

$$T(r, f(z+c)) = T(r, f(z)) + O(r^{\sigma-1+\varepsilon}) + O(\log r).$$

Lemma 2.2. [3, Theorem 8.2] *Let $f(z)$ be a meromorphic function of finite order σ , c be a non-zero constant. Let $\varepsilon > 0$ be a given real constant, then there exists a subset $E \subset (1, \infty)$ of finite logarithmic measure such that for all $|z| = r \notin [0, 1] \cup E$, we have*

$$\exp(-r^{\sigma-1+\varepsilon}) \leq \left| \frac{f(z+\eta)}{f(z)} \right| \leq \exp(r^{\sigma-1+\varepsilon}).$$

3. PROOF OF THEOREM 1.2

It follows by the assumption that

$$f(z) = a(z) + \frac{u(z)}{v(z)} e^{h(z)}, \quad (3.1)$$

where $u(z)$, $v(z)$ are two non-zero entire functions, $h(z)$ is a non-constant polynomial of degree m . Furthermore, we know $f(z)$ is of normal growth, and $a(z)$, $u(z)$, $v(z)$ satisfy:

$$\lambda(f-a) = \lambda(u) = \sigma(u) < \sigma(f) = m, \quad \lambda\left(\frac{1}{f}\right) = \lambda(v) = \sigma(v) < \sigma(f),$$

and

$$T(r, a) = S(r, f), \quad T(r, u) = S(r, e^{h(z)}), \quad T(r, v) = S(r, e^{h(z)}) = S(r, f).$$

From (3.1), we have

$$\begin{aligned} \Delta_c f &= \left(\frac{u(z+c)}{v(z+c)} e^{h(z+c)-h(z)} - \frac{u(z)}{v(z)} \right) e^{h(z)} + a(z+c) - a(z) \\ &= H(z) e^{h(z)} + a(z+c) - a(z). \end{aligned} \quad (3.2)$$

Applying Lemma 2.1 to equation (3.2), we conclude

$$\sigma(H) < m = \sigma(f),$$

which means that

$$T(r, H) = S(r, f).$$

By the sharing assumption, we obtain that

$$\frac{\Delta_c f - a(z)}{f(z) - a(z)} = e^{p(z)}, \quad (3.3)$$

where $p(z)$ is a polynomial. By combining Lemma 2.1 and (3.3), it follows that $\deg p(z) \leq \sigma(f) = m$. From (3.1), (3.2) and (3.3), we deduce that

$$H(z) e^{h(z)} + a(z+c) - 2a(z) = \frac{u(z)}{v(z)} e^{h(z)+p(z)}. \quad (3.4)$$

Case 1. Suppose that $a(z+c) - 2a(z) \not\equiv 0$. Then by (3.4) we get

$$N\left(r, \frac{1}{e^{h(z)} + \frac{a(z+c)-2a(z)}{H(z)}}\right) \leq N\left(r, \frac{1}{u(z)}\right) = S(r, e^{h(z)}),$$

when $H(z) \not\equiv 0$, which is a contradiction.

If $H(z) \equiv 0$, then it follows from (3.2) that

$$\frac{u(z+c)}{v(z+c)} e^{h(z+c)-h(z)} - \frac{u(z)}{v(z)} \equiv 0,$$

this gives

$$\frac{u(z+c)v(z)}{u(z)v(z+c)} e^{h(z+c)-h(z)} \equiv 1. \quad (3.5)$$

Denote

$$G(z) = \frac{u(z+c)v(z)}{u(z)v(z+c)}. \quad (3.6)$$

From equation (3.5), we know that $G(z)$ is a non-zero entire function. By Lemma 2.2, we see

$$\left| \frac{u(z+c)}{u(z)} \right| \leq \exp(r^{\sigma(u)-1+\varepsilon}), \quad \left| \frac{v(z)}{v(z+c)} \right| \leq \exp(r^{\sigma(v)-1+\varepsilon}),$$

which implies that

$$|G(z)| = \left| \frac{u(z+c)v(z)}{u(z)v(z+c)} \right| \leq \exp(2r^{\sigma-1+\varepsilon}),$$

where $\sigma = \max\{\sigma(u), \sigma(v)\} < \sigma(f) = m$, and $0 < \varepsilon < \frac{m-\sigma}{2}$. Hence,

$$T(r, G(z)) = m(r, G(z)) = 2r^{\sigma-1+\varepsilon},$$

that is

$$\sigma(G) \leq \sigma - 1 + \varepsilon < m - 1. \quad (3.7)$$

Consequently,

$$\sigma(f) = m = 1, \quad (3.8)$$

from equations (3.5) and (3.7). Thus,

$$\sigma(a) < 1, \quad \sigma(u) < 1, \quad \sigma(v) < 1. \quad (3.9)$$

Rewriting equation (3.5) as following

$$\frac{u(z+c)v(z)}{u(z)} e^{h(z+c)-h(z)} \equiv v(z+c), \quad (3.10)$$

$$\frac{u(z+c)v(z)}{v(z+c)} e^{h(z+c)-h(z)} \equiv u(z). \quad (3.11)$$

Next, we will prove that

$$u(z) \not\equiv 0, \quad v(z) \not\equiv 0. \quad (3.12)$$

In fact, suppose z_0 is a zero of $v(z)$, then from (3.10) and $u(z_0) \neq 0$, we get that $z_0 + c$ is also a zero of $v(z)$. By calculation, we know $v(z_0 + kc) = 0$ as well, where k is a positive integer. Hence, $\sigma(v) \geq 1$, which contradicts equation (3.9). Hence, $v(z) \not\equiv 0$. Similarly, we get $u(z) \not\equiv 0$. By combining equation (3.9) with (3.12), we affirm that $v(z)$ and $u(z)$ are two non-zero constants. Therefore, we conclude that $f(z) = a(z) + Ce^{Dz}$, where C, D are non-zero constants.

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Case 2. Assume that

$$a(z+c) - 2a(z) \equiv 0. \quad (3.13)$$

Then we affirm that $\sigma(a) \geq 1$. In fact, if $\sigma(a) < 1$, then from (3.13) and Lemma 2.2, for any given ε such that $0 < \varepsilon < \frac{1-\sigma(a)}{2}$, there exists a subset $E \subset (1, \infty)$ of finite logarithmic measure such that for all $|z| = r \notin [0, 1] \cup E$, we deduce that

$$2 = \left| \frac{a(z+c)}{a(z)} \right| \leq \exp(r^{\sigma(a)-1+\varepsilon}) \rightarrow 0, \quad r \rightarrow \infty,$$

which is impossible. Therefore, we conclude that

$$\sigma(a) \geq 1.$$

(a). If $\deg p(z) = 0$, then $p(z) = p$, where p is a constant. From equations (3.1) and (3.3), it follows that

$$G(z)e^{h(z+c)-h(z)} = 1 + e^p,$$

where $G(z)$ denotes as equation (3.6). Using a similar way as Case 1, we know that $\sigma(f) = m = 1$, which contradicts the assumption that $\sigma(a) < \sigma(f)$.

(b). If $\deg p(z) \geq 1$, then by equations (3.1), (3.3) and (3.6), it follows that

$$G(z)e^{h(z+c)-h(z)} - 1 = e^{p(z)}. \quad (3.14)$$

Similarly as Case 1, we get equation (3.7) hold as well. Therefore, by equation (3.7), we obtain

$$\lambda(G(z)e^{h(z+c)-h(z)}) = \lambda(G(z)) \leq \sigma(G(z)) < m-1 = \sigma(G(z)e^{h(z+c)-h(z)}),$$

which means 0 is a Borel exceptional value. Clearly, we obtain that 1 and ∞ are two Borel exceptional values of $G(z)e^{h(z+c)-h(z)}$. Hence, we get the function $G(z)e^{h(z+c)-h(z)}$ have three Borel exceptional values, which is a contradiction. This completes the proof of Theorem 1.2.

4. PROOF OF THEOREM 1.1

From the assumption that $f(z)$ has two Borel exceptional values a and ∞ , $f(z)$ can be expressed as in the following form:

$$f(z) = a + \frac{U(z)}{V(z)}e^{\phi(z)}, \quad (4.1)$$

where $U(z)$, $V(z)$ are two non-zero entire functions, $\phi(z)$ is a non-constant polynomial of degree n . Similarly as Theorem 1.2, we get $U(z)$, $V(z)$ satisfy:

$$\lambda(f-a) = \lambda(U) = \sigma(U) < \sigma(f) = n, \quad \lambda\left(\frac{1}{f}\right) = \lambda(V) = \sigma(V) < \sigma(f),$$

and

$$T(r, U) = S(r, e^{\phi(z)}) = S(r, f), \quad T(r, V) = S(r, e^{\phi(z)}) = S(r, f).$$

Moreover, we get

$$\Delta_c f = \left(\frac{U(z+c)}{V(z+c)} e^{\phi(z+c)-\phi(z)} - \frac{U(z)}{V(z)} \right) e^{\phi(z)} = \psi(z) e^{\phi(z)}, \quad (4.2)$$

$$\frac{\Delta_c f - a}{f(z) - a} = e^{q(z)}, \quad (4.3)$$

and

$$\psi(z) e^{\phi(z)} - a = \frac{U(z)}{V(z)} e^{\phi(z)+q(z)},$$

where $q(z)$ is a polynomial.

Case 1. If $a \neq 0$, then by the above equation we obtain

$$N \left(r, \frac{1}{e^{\phi(z)} - \frac{a}{\psi(z)}} \right) \leq N \left(r, \frac{1}{U(z)} \right) = S(r, e^{\phi(z)})$$

when $\psi(z) \not\equiv 0$, which is a contradiction. If $\psi(z) \equiv 0$, then we have $\Delta_c f \equiv 0$ by (4.2), which contradicts the assumption $\Delta_c f \not\equiv 0$.

Case 2. If $a = 0$, then it follows that

$$\frac{\Delta_c f}{f} = \frac{U(z+c)V(z)}{U(z)V(z+c)} e^{\phi(z+c)-\phi(z)} - 1 = \omega(z) e^{\phi(z+c)-\phi(z)} - 1 = e^{q(z)}. \quad (4.4)$$

Similarly as Theorem 1.2, we conclude that $\sigma(\omega) < n-1 = \deg \phi(z)-1$, which means that

$$T(r, \omega) = S(r, e^{\phi(z)}).$$

From equation (4.4), we have

$$N \left(r, \frac{1}{e^{q(z)} + 1} \right) = N \left(r, \frac{1}{\omega(z)} \right) = S(r, e^{\phi(z)}),$$

which is impossible when $q(z)$ is a non-constant polynomial. Hence, $q(z)$ is a constant. Let $q(z) = q$, then from equations (4.4), it follows that

$$\frac{U(z+c)V(z)}{U(z)V(z+c)} e^{\phi(z+c)-\phi(z)} = 1 + e^q.$$

Using a similar way as Case 1 of Theorem 1.2, we know that $\sigma(f) = 1$, and obtain $f(z) = Ae^{Bz}$ further. The conclusion follows.

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QUADRATIC ρ -FUNCTIONAL INEQUALITIES IN NON-ARCHIMEDEAN NORMED SPACES

SUNGSIK YUN AND CHOONKIL PARK*

ABSTRACT. In this paper, we solve the quadratic ρ -functional inequalities

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right) \right\|, \quad (0.1)$$

where ρ is a fixed non-Archimedean number with $|\rho| < 1$, and

$$\left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right\| \leq \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\|, \quad (0.2)$$

where ρ is a fixed non-Archimedean number with $|\rho| < \frac{1}{2}$.

Furthermore, we prove the Hyers-Ulam stability of the quadratic ρ -functional inequalities (0.1) and (0.2) in non-Archimedean Banach spaces and prove the Hyers-Ulam stability of quadratic ρ -functional equations associated with the quadratic ρ -functional inequalities (0.1) and (0.2) in non-Archimedean Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

A *valuation* is a function $|\cdot|$ from a field K into $[0, \infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r| \cdot |s|$ and the triangle inequality holds, i.e.,

$$|r+s| \leq |r| + |s|, \quad \forall r, s \in K.$$

A field K is called a *valued field* if K carries a valuation. The usual absolute values of \mathbb{R} and \mathbb{C} are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \leq \max\{|r|, |s|\}, \quad \forall r, s \in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

Definition 1.1. ([12]) Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \rightarrow [0, \infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:

- (i) $\|x\| = 0$ if and only if $x = 0$;
- (ii) $\|rx\| = |r|\|x\|$ ($r \in K, x \in X$);
- (iii) the strong triangle inequality

$$\|x+y\| \leq \max\{\|x\|, \|y\|\}, \quad \forall x, y \in X$$

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holds. Then $(X, \|\cdot\|)$ is called a *non-Archimedean normed space*.

Definition 1.2. (i) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X . Then the sequence $\{x_n\}$ is called *Cauchy* if for a given $\varepsilon > 0$ there is a positive integer N such that

$$\|x_n - x_m\| \leq \varepsilon$$

for all $n, m \geq N$.

(ii) Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X . Then the sequence $\{x_n\}$ is called *convergent* if for a given $\varepsilon > 0$ there are a positive integer N and an $x \in X$ such that

$$\|x_n - x\| \leq \varepsilon$$

for all $n \geq N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n \rightarrow \infty} x_n = x$.

(iii) If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a *non-Archimedean Banach space*.

The stability problem of functional equations originated from a question of Ulam [25] concerning the stability of group homomorphisms. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [17] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [24] for mappings $f : E_1 \rightarrow E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The functional equation $2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = f(x) + f(y)$ is called a *Jensen type quadratic equation*. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 3, 4, 15, 16, 19, 20, 21, 22, 23, 26, 27]).

In [9], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\| \quad (1.1)$$

then f satisfies the Jordan-von Neumann functional equation $2f(x) + 2f(y) = f(xy) + f(xy^{-1})$. See also [18]. Gilányi [10] and Fechner [7] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [14] proved the Hyers-Ulam stability of additive functional inequalities. The stability problems of functional equations and inequalities have also been treated by many authors ([6, 13]).

In Section 2, we solve the quadratic ρ -functional inequality (0.1) and prove the Hyers-Ulam stability of the quadratic ρ -functional inequality (0.1) in non-Archimedean Banach spaces. We moreover prove the Hyers-Ulam stability of a quadratic ρ -functional equation associated with the quadratic ρ -functional inequality (0.1) in non-Archimedean Banach spaces.

In Section 3, we solve the quadratic ρ -functional inequality (0.2) and prove the Hyers-Ulam stability of the quadratic ρ -functional inequality (0.2) in non-Archimedean Banach spaces. We moreover prove the Hyers-Ulam stability of a quadratic ρ -functional equation associated with the quadratic ρ -functional inequality (0.2) in non-Archimedean Banach spaces.

Throughout this paper, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space. Let $|2| \neq 1$.

2. QUADRATIC ρ -FUNCTIONAL INEQUALITY (0.1) IN NON-ARCHIMEDEAN NORMED SPACES

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < 1$.

QUADRATIC ρ -FUNCTIONAL INEQUALITIES

In this section, we solve the quadratic ρ -functional inequality (0.1) in non-Archimedean normed spaces.

Lemma 2.1. *A mapping $f : X \rightarrow Y$ satisfies*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right) \right\| \quad (2.1)$$

for all $x, y \in X$ if and only if $f : X \rightarrow Y$ is quadratic.

Proof. Assume that $f : X \rightarrow Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $\|2f(0)\| \leq |\rho| \|2f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.1), we get $\|f(2x) - 4f(x)\| \leq 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \quad (2.2)$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| &\leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right) \right\| \\ &= \frac{|\rho|}{2} \|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$.

The converse is obviously true. □

Corollary 2.2. *A mapping $f : X \rightarrow Y$ satisfies*

$$f(x+y) + f(x-y) - 2f(x) - 2f(y) = \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right) \quad (2.3)$$

for all $x, y \in X$ if and only if $f : X \rightarrow Y$ is quadratic.

The functional equation (2.3) is called a *quadratic ρ -functional equation*.

We prove the Hyers-Ulam stability of the quadratic ρ -functional inequality (2.1) in non-Archimedean Banach spaces.

Theorem 2.3. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be a mapping such that*

$$\begin{aligned} &\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \\ &\leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right) \right\| + \theta(\|x\|^r + \|y\|^r) \end{aligned} \quad (2.4)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \rightarrow Y$ such that

$$\|f(x) - h(x)\| \leq \frac{2\theta}{|2|^r} \|x\|^r \quad (2.5)$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (2.4), we get $\|2f(0)\| \leq |\rho| \|2f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.4), we get

$$\|f(2x) - 4f(x)\| \leq 2\theta \|x\|^r \quad (2.6)$$

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for all $x \in X$. So $\|f(x) - 4f\left(\frac{x}{2}\right)\| \leq \frac{2}{|2|^r} \theta \|x\|^r$ for all $x \in X$. Hence

$$\begin{aligned} & \left\| 4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| \\ & \leq \max \left\{ \left\| 4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right) \right\|, \dots, \left\| 4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| \right\} \\ & = \max \left\{ |4|^l \left\| f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{2^{l+1}}\right) \right\|, \dots, |4|^{m-1} \left\| f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right) \right\| \right\} \\ & \leq \max \left\{ \frac{|4|^l}{|2|^{rl+1}}, \dots, \frac{|4|^{m-1}}{|2|^{r(m-1)+1}} \right\} 2\theta \|x\|^r = \frac{2\theta}{|2|^{(r-2)l+1}} \|x\|^r \end{aligned} \quad (2.7)$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.7) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \rightarrow Y$ by

$$h(x) := \lim_{n \rightarrow \infty} 4^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (2.7), we get (2.5).

It follows from (2.4) that

$$\begin{aligned} & \|h(x+y) + h(x-y) - 2h(x) - 2h(y)\| \\ & = \lim_{n \rightarrow \infty} |4|^n \left\| f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right) \right\| \\ & \leq \lim_{n \rightarrow \infty} |4|^n |\rho| \left\| 2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) \right\| + \lim_{n \rightarrow \infty} \frac{|4|^n \theta}{|2|^{nr}} (\|x\|^r + \|y\|^r) \\ & = |\rho| \left\| 2h\left(\frac{x+y}{2}\right) + 2h\left(\frac{x-y}{2}\right) - h(x) - h(y) \right\| \end{aligned}$$

for all $x, y \in X$. So

$$\|h(x+y) + h(x-y) - 2h(x) - 2h(y)\| \leq \left\| \rho \left(2h\left(\frac{x+y}{2}\right) + 2h\left(\frac{x-y}{2}\right) - h(x) - h(y) \right) \right\|$$

for all $x, y \in X$. By Lemma 2.1, the mapping $h : X \rightarrow Y$ is quadratic.

Now, let $T : X \rightarrow Y$ be another quadratic mapping satisfying (2.5). Then we have

$$\begin{aligned} \|h(x) - T(x)\| & = \left\| 4^q h\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right) \right\| \\ & \leq \max \left\{ \left\| 4^q h\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right) \right\|, \left\| 4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right) \right\| \right\} \leq \frac{2\theta}{|2|^{(r-2)q+1}} \|x\|^r, \end{aligned}$$

which tends to zero as $q \rightarrow \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h . Thus the mapping $h : X \rightarrow Y$ is a unique quadratic mapping satisfying (2.5). \square

Theorem 2.4. Let $r > 2$ and θ be positive real numbers, and let $f : X \rightarrow Y$ be a mapping satisfying (2.4). Then there exists a unique quadratic mapping $h : X \rightarrow Y$ such that

$$\|f(x) - h(x)\| \leq \frac{2\theta}{|4|} \|x\|^r \quad (2.8)$$

for all $x \in X$.

QUADRATIC ρ -FUNCTIONAL INEQUALITIES

Proof. It follows from (2.6) that

$$\left\| f(x) - \frac{1}{4}f(2x) \right\| \leq \frac{2\theta}{|4|} \|x\|^r$$

for all $x \in X$. Hence

$$\begin{aligned} & \left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^m} f(2^m x) \right\| \\ & \leq \max \left\{ \left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^{l+1}} f(2^{l+1} x) \right\|, \dots, \left\| \frac{1}{4^{m-1}} f(2^{m-1} x) - \frac{1}{4^m} f(2^m x) \right\| \right\} \\ & = \max \left\{ \frac{1}{|4|^l} \left\| f(2^l x) - \frac{1}{4} f(2^{l+1} x) \right\|, \dots, \frac{1}{|4|^{m-1}} \left\| f(2^{m-1} x) - \frac{1}{4} f(2^m x) \right\| \right\} \\ & \leq \max \left\{ \frac{|2|^{rl}}{|4|^{l+1}}, \dots, \frac{|2|^{r(m-1)}}{|4|^{(m-1)+1}} \right\} 2\theta \|x\|^r = \frac{2\theta}{|2|^{(2-r)l+2}} \|x\|^r \end{aligned} \quad (2.9)$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.9) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ converges. So one can define the mapping $h : X \rightarrow Y$ by

$$h(x) := \lim_{n \rightarrow \infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (2.9), we get (2.8).

The rest of the proof is similar to the proof of Theorem 2.3. \square

Let $A(x, y) := f(x + y) + f(x - y) - 2f(x) - 2f(y)$ and

$$B(x, y) := \rho \left(2f \left(\frac{x+y}{2} \right) + 2f \left(\frac{x-y}{2} \right) - f(x) - f(y) \right)$$

for all $x, y \in X$.

For $x, y \in X$ with $\|A(x, y)\| \leq \|B(x, y)\|$,

$$\|A(x, y)\| - \|B(x, y)\| \leq \|A(x, y) - B(x, y)\|.$$

For $x, y \in X$ with $\|A(x, y)\| > \|B(x, y)\|$,

$$\begin{aligned} \|A(x, y)\| &= \|A(x, y) - B(x, y) + B(x, y)\| \\ &\leq \max\{\|A(x, y) - B(x, y)\|, \|B(x, y)\|\} \\ &= \|A(x, y) - B(x, y)\| \\ &\leq \|A(x, y) - B(x, y)\| + \|B(x, y)\|, \end{aligned}$$

since $\|A(x, y)\| > \|B(x, y)\|$. So we have

$$\begin{aligned} & \|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| - \left\| \rho \left(2f \left(\frac{x+y}{2} \right) + 2f \left(\frac{x-y}{2} \right) - f(x) - f(y) \right) \right\| \\ & \leq \left\| f(x + y) + f(x - y) - 2f(x) - 2f(y) - \rho \left(2f \left(\frac{x+y}{2} \right) + 2f \left(\frac{x-y}{2} \right) - f(x) - f(y) \right) \right\|. \end{aligned}$$

As corollaries of Theorems 2.3 and 2.4, we obtain the Hyers-Ulam stability results for the quadratic ρ -functional equation (2.3) in non-Archimedean Banach spaces.

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Corollary 2.5. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be a mapping such that*

$$\begin{aligned} & \|f(x+y) + f(x-y) - 2f(x) - 2f(y) \\ & - \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right) \| \leq \theta(\|x\|^r + \|y\|^r) \end{aligned} \quad (2.10)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \rightarrow Y$ satisfying (2.5).

Corollary 2.6. *Let $r > 2$ and θ be positive real numbers, and let $f : X \rightarrow Y$ be a mapping satisfying (2.10). Then there exists a unique quadratic mapping $h : X \rightarrow Y$ satisfying (2.8).*

3. QUADRATIC ρ -FUNCTIONAL INEQUALITY (0.2)

Throughout this section, assume that ρ is a fixed non-Archimedean number with $|\rho| < \frac{1}{2}$.

In this section, we solve the quadratic ρ -functional inequality (0.2) in non-Archimedean normed spaces.

Lemma 3.1. *A mapping $f : X \rightarrow Y$ satisfies*

$$\left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right\| \leq \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\| \quad (3.1)$$

for all $x, y \in X$ if and only if $f : X \rightarrow Y$ is quadratic.

Proof. Assume that $f : X \rightarrow Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $\|2f(0)\| \leq |\rho|\|2f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.1), we get

$$\left\| 4f\left(\frac{x}{2}\right) - f(x) \right\| \leq 0 \quad (3.2)$$

and so $f\left(\frac{x}{2}\right) = \frac{1}{4}f(x)$ for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} \frac{1}{2}\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| &= \left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right\| \\ &\leq |\rho|\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$.

The converse is obviously true. □

Corollary 3.2. *A mapping $f : X \rightarrow Y$ satisfies*

$$2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) = \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \quad (3.3)$$

for all $x, y \in X$ and only if $f : X \rightarrow Y$ is quadratic.

The functional equation (3.3) is called a *quadratic ρ -functional equation*.

We prove the Hyers-Ulam stability of the quadratic ρ -functional inequality (3.1) in non-Archimedean Banach spaces.

QUADRATIC ρ -FUNCTIONAL INEQUALITIES

Theorem 3.3. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be a mapping such that*

$$\begin{aligned} & \|2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)\| \\ & \leq \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\| + \theta(\|x\|^r + \|y\|^r) \end{aligned} \quad (3.4)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \rightarrow Y$ such that

$$\|f(x) - h(x)\| \leq \theta\|x\|^r \quad (3.5)$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (3.4), we get $\|2f(0)\| \leq |\rho|\|2f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.4), we get

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq \theta\|x\|^r \quad (3.6)$$

for all $x \in X$. So

$$\begin{aligned} & \left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| \\ & \leq \max \left\{ \left\|4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \dots, \left\|4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| \right\} \\ & = \max \left\{ |4|^l \left\|f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{2^{l+1}}\right)\right\|, \dots, |4|^{m-1} \left\|f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right)\right\| \right\} \\ & \leq \max \left\{ \frac{|4|^l}{|2|^{rl}}, \dots, \frac{|4|^{m-1}}{|2|^{r(m-1)}} \right\} \theta\|x\|^r = \frac{\theta}{|2|^{(r-2)l}} \|x\|^r \end{aligned} \quad (3.7)$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.7) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \rightarrow Y$ by

$$h(x) := \lim_{n \rightarrow \infty} 4^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (3.7), we get (3.5).

The rest of the proof is similar to the proof of Theorem 2.3. \square

Theorem 3.4. *Let $r > 2$ and θ be positive real numbers, and let $f : X \rightarrow Y$ be a mapping satisfying (3.4). Then there exists a unique quadratic mapping $h : X \rightarrow Y$ such that*

$$\|f(x) - h(x)\| \leq \frac{|2|^r \theta}{|4|} \|x\|^r \quad (3.8)$$

for all $x \in X$.

Proof. It follows from (3.6) that

$$\left\|f(x) - \frac{1}{4}f(2x)\right\| \leq \frac{|2|^r \theta}{|4|} \|x\|^r$$

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for all $x \in X$. Hence

$$\begin{aligned}
 & \left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^m} f(2^m x) \right\| \\
 & \leq \max \left\{ \left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^{l+1}} f(2^{l+1} x) \right\|, \dots, \left\| \frac{1}{4^{m-1}} f(2^{m-1} x) - \frac{1}{4^m} f(2^m x) \right\| \right\} \\
 & = \max \left\{ \frac{1}{|4|^l} \left\| f(2^l x) - \frac{1}{4} f(2^{l+1} x) \right\|, \dots, \frac{1}{|4|^{m-1}} \left\| f(2^{m-1} x) - \frac{1}{4} f(2^m x) \right\| \right\} \\
 & \leq \max \left\{ \frac{|2|^r l}{|4|^{l+1}}, \dots, \frac{|2|^{r(m-1)}}{|4|^{(m-1)+1}} \right\} |2|^r \theta \|x\|^r = \frac{|2|^r \theta}{|2|^{(2-r)l+2}} \|x\|^r
 \end{aligned} \tag{3.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.9) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ converges. So one can define the mapping $h : X \rightarrow Y$ by

$$h(x) := \lim_{n \rightarrow \infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (3.9), we get (3.8).

The rest of the proof is similar to the proofs of Theorems 2.3 and 3.3. \square

Let $A(x, y) := 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y)$ and

$$B(x, y) := \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))$$

for all $x, y \in X$.

For $x, y \in X$ with $\|A(x, y)\| \leq \|B(x, y)\|$,

$$\|A(x, y)\| - \|B(x, y)\| \leq \|A(x, y) - B(x, y)\|.$$

For $x, y \in X$ with $\|A(x, y)\| > \|B(x, y)\|$,

$$\begin{aligned}
 \|A(x, y)\| &= \|A(x, y) - B(x, y) + B(x, y)\| \\
 &\leq \max\{\|A(x, y) - B(x, y)\|, \|B(x, y)\|\} \\
 &= \|A(x, y) - B(x, y)\| \\
 &\leq \|A(x, y) - B(x, y)\| + \|B(x, y)\|,
 \end{aligned}$$

since $\|A(x, y)\| > \|B(x, y)\|$. So we have

$$\begin{aligned}
 & \left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right\| - \|\rho(f(x+y) + f(x-y) - 2f(x) - 2f(y))\| \\
 & \leq \left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) - \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \right\|.
 \end{aligned}$$

As corollaries of Theorems 3.3 and 3.4, we obtain the Hyers-Ulam stability results for the quadratic ρ -functional equation (3.3) in non-Archimedean Banach spaces.

Corollary 3.5. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be a mapping such that*

$$\begin{aligned}
 & \left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - f(x) - f(y) \right. \\
 & \quad \left. - \rho(f(x+y) + f(x-y) - 2f(x) - 2f(y)) \right\| \leq \theta(\|x\|^r + \|y\|^r)
 \end{aligned} \tag{3.10}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \rightarrow Y$ satisfying (3.5).

Corollary 3.6. *Let $r > 2$ and θ be positive real numbers, and let $f : X \rightarrow Y$ be a mapping satisfying (3.10). Then there exists a unique quadratic mapping $h : X \rightarrow Y$ satisfying (3.8).*

QUADRATIC ρ -FUNCTIONAL INEQUALITIES

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Mixed problems of fractional coupled systems of Riemann-Liouville differential equations and Hadamard integral conditions

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Abstract

In this paper we study existence and uniqueness of solutions for mixed problems consisting non-local Hadamard fractional integrals for coupled systems of Riemann-Liouville fractional differential equations. The existence and uniqueness of solutions is established by using the Banach's contraction principle, while the existence of solutions is derived by applying Leray-Schauder's alternative. Examples illustrating our results are also presented.

Key words and phrases: Riemann-Liouville fractional derivative; Hadamard fractional integral; coupled system; existence; uniqueness; fixed point theorems.

AMS (MOS) Subject Classifications: 34A08; 34A12; 34B15.

1 Introduction

The aim of this paper is to investigate the existence and uniqueness of solutions for nonlocal Hadamard fractional integrals for a coupled system of Riemann-Liouville fractional differential equations of the form:

$$\begin{cases} {}_{RL}D^p x(t) = f(t, x(t), y(t)), & t \in [0, T], \quad 1 < p \leq 2, \\ {}_{RL}D^q y(t) = g(t, x(t), y(t)), & t \in [0, T], \quad 1 < q \leq 2, \\ x(0) = 0, \quad \sum_{i=1}^{m_1} \mu_i {}_HI^{\alpha_i} x(\eta_i) = \sum_{j=1}^{n_1} \delta_j {}_HI^{\beta_j} y(\xi_j) + \lambda_1, \\ y(0) = 0, \quad \sum_{k=1}^{m_2} \tau_k {}_HI^{\sigma_k} x(\gamma_k) = \sum_{l=1}^{n_2} \omega_l {}_HI^{\nu_l} y(\theta_l) + \lambda_2, \end{cases} \quad (1)$$

where ${}_{RL}D^q$, ${}_{RL}D^p$ are the standard Riemann-Liouville fractional derivative of orders q, p , two continuous functions $f, g : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$, ${}_HI^{\alpha_i}$, ${}_HI^{\beta_j}$, ${}_HI^{\sigma_k}$ and ${}_HI^{\nu_l}$ are the Hadamard fractional integral of orders $\alpha_i, \beta_j, \sigma_k, \nu_l > 0$, $\lambda_1, \lambda_2 \in \mathbb{R}$ are given constants, $\eta_i, \xi_j, \gamma_k, \theta_l \in (0, T)$, and $\mu_i, \delta_j, \tau_k, \omega_l \in \mathbb{R}$, for $m_1, m_2, n_1, n_2 \in \mathbb{N}$, $i = 1, 2, \dots, m_1$, $j = 1, 2, \dots, n_1$, $k = 1, 2, \dots, m_2$, $l = 1, 2, \dots, n_2$ are real constants such that

$$\left(\sum_{i=1}^{m_1} \frac{\mu_i \eta_i^{p-1}}{(p-1)^{\alpha_i}} \right) \left(\sum_{l=1}^{n_2} \frac{\omega_l \theta_l^{q-1}}{(q-1)^{\nu_l}} \right) \neq \left(\sum_{j=1}^{n_1} \frac{\delta_j \xi_j^{q-1}}{(q-1)^{\beta_j}} \right) \left(\sum_{k=1}^{m_2} \frac{\tau_k \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \right).$$

Fractional calculus has a long history with more than three hundred years. Up to now, it has been proved that fractional calculus is very useful. Many mathematical models of real problems arising

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in various fields of science and engineering were established with the help of fractional calculus, such as viscoelastic systems, dielectric polarization, electrode-electrolyte polarization, and electromagnetic waves. For examples and recent development of the topic, see ([1, 2, 3, 4, 5, 6, 7, 14, 16, 17, 18, 19, 20, 21]). However, it has been observed that most of the work on the topic involves either Riemann-Liouville or Caputo type fractional derivative. Besides these derivatives, Hadamard fractional derivative is another kind of fractional derivatives that was introduced by Hadamard in 1892 [12]. This fractional derivative differs from the other ones in the sense that the kernel of the integral (in the definition of Hadamard derivative) contains logarithmic function of arbitrary exponent. For background material of Hadamard fractional derivative and integral, we refer to the papers [8, 9, 10, 13, 14, 15].

The paper is organized as follows: In Section 2 we will present some useful preliminaries and lemmas. The main results are given in Section 3, where existence and uniqueness results are obtained by using Banach's contraction principle and Leray-Schauder's alternative. Finally the uncoupled integral boundary conditions case is studied in Section 4. Examples illustrating our results are also presented.

2 Preliminaries

In this section, we introduce some notations and definitions of fractional calculus and present preliminary results needed in our proofs later [18, 14].

Definition 2.1 The Riemann-Liouville fractional derivative of order $q > 0$ of a continuous function $f : (0, \infty) \rightarrow \mathbb{R}$ is defined by

$${}_R L D^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt} \right)^n \int_0^t (t-s)^{n-q-1} f(s) ds, \quad n-1 < q < n,$$

where $n = [q] + 1$, $[q]$ denotes the integer part of a real number q , provided the right-hand side is point-wise defined on $(0, \infty)$, where Γ is the gamma function defined by $\Gamma(q) = \int_0^\infty e^{-s} s^{q-1} ds$.

Definition 2.2 The Riemann-Liouville fractional integral of order $q > 0$ of a continuous function $f : (0, \infty) \rightarrow \mathbb{R}$ is defined by

$${}_R L I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds,$$

provided the right-hand side is point-wise defined on $(0, \infty)$.

Definition 2.3 The Hadamard derivative of fractional order q for a function $f : (0, \infty) \rightarrow \mathbb{R}$ is defined as

$${}_H D^q f(t) = \frac{1}{\Gamma(n-q)} \left(t \frac{d}{dt} \right)^n \int_0^t \left(\log \frac{t}{s} \right)^{n-q-1} \frac{f(s)}{s} ds, \quad n-1 < q < n, \quad n = [q] + 1,$$

where $\log(\cdot) = \log_e(\cdot)$.

Definition 2.4 The Hadamard fractional integral of order $q \in \mathbb{R}^+$ of a function $f(t)$, for all $t > 0$, is defined as

$${}_H I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \left(\log \frac{t}{s} \right)^{q-1} f(s) \frac{ds}{s},$$

provided the integral exists.

Lemma 2.5 ([14], page 113) Let $q > 0$ and $\beta > 0$. Then the following formulas

$${}_H I^q t^\beta = \beta^{-q} t^\beta \quad \text{and} \quad {}_H D^q t^\beta = \beta^q t^\beta$$

hold.

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Lemma 2.6 Let $q > 0$ and $x \in C(0, T) \cap L(0, T)$. Then the fractional differential equation ${}_R L D^q x(t) = 0$ has a unique solution $x(t) = c_1 t^{q-1} + c_2 t^{q-2} + \dots + c_n t^{q-n}$, where $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n$, and $n-1 < q < n$.

Lemma 2.7 Let $q > 0$. Then for $x \in C(0, T) \cap L(0, T)$ it holds

$${}_R L I^q {}_R L D^q x(t) = x(t) + c_1 t^{q-1} + c_2 t^{q-2} + \dots + c_n t^{q-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n$, and $n-1 < q < n$.

Lemma 2.8 Given $\phi, \psi \in C([0, T], \mathbb{R})$, the unique solution of the problem

$$\begin{cases} {}_R L D^p x(t) = \phi(t), & t \in [0, T], & 1 < p \leq 2, \\ {}_R L D^q y(t) = \psi(t), & t \in [0, T], & 1 < q \leq 2, \\ x(0) = 0, & \sum_{i=1}^{m_1} \mu_i {}_H I^{\alpha_i} x(\eta_i) = \sum_{j=1}^{n_1} \delta_j {}_H I^{\beta_j} y(\xi_j) + \lambda_1, \\ y(0) = 0, & \sum_{k=1}^{m_2} \tau_k {}_H I^{\sigma_k} x(\gamma_k) = \sum_{l=1}^{n_2} \omega_l {}_H I^{\nu_l} y(\theta_l) + \lambda_2, \end{cases} \quad (2)$$

is

$$\begin{aligned} x(t) &= {}_R L I^p \phi(t) + \frac{t^{p-1}}{\Omega} \left\{ \sum_{l=1}^{n_2} \frac{\omega_l \theta_l^{q-1}}{(q-1)^{\nu_l}} \left(\sum_{j=1}^{n_1} \delta_j {}_H I^{\beta_j} {}_R L I^q \psi(\xi_j) - \sum_{i=1}^{m_1} \mu_i {}_H I^{\alpha_i} {}_R L I^p \phi(\eta_i) + \lambda_1 \right) \right. \\ &\quad \left. - \sum_{j=1}^{n_1} \frac{\delta_j \xi_j^{q-1}}{(q-1)^{\beta_j}} \left(\sum_{l=1}^{n_2} \omega_l {}_H I^{\nu_l} {}_R L I^q \psi(\theta_l) - \sum_{k=1}^{m_2} \tau_k {}_H I^{\sigma_k} {}_R L I^p \phi(\gamma_k) + \lambda_2 \right) \right\}, \end{aligned} \quad (3)$$

and

$$\begin{aligned} y(t) &= {}_R L I^q \psi(t) + \frac{t^{q-1}}{\Omega} \left\{ \sum_{k=1}^{m_2} \frac{\tau_k \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \left(\sum_{j=1}^{n_1} \delta_j {}_H I^{\beta_j} {}_R L I^q \psi(\xi_j) - \sum_{i=1}^{m_1} \mu_i {}_H I^{\alpha_i} {}_R L I^p \phi(\eta_i) + \lambda_1 \right) \right. \\ &\quad \left. - \sum_{i=1}^{m_1} \frac{\mu_i \eta_i^{p-1}}{(p-1)^{\alpha_i}} \left(\sum_{l=1}^{n_2} \omega_l {}_H I^{\nu_l} {}_R L I^q \psi(\theta_l) - \sum_{k=1}^{m_2} \tau_k {}_H I^{\sigma_k} {}_R L I^p \phi(\gamma_k) + \lambda_2 \right) \right\}, \end{aligned} \quad (4)$$

where

$$\Omega = \sum_{i=1}^{m_1} \frac{\mu_i \eta_i^{p-1}}{(p-1)^{\alpha_i}} \sum_{l=1}^{n_2} \frac{\omega_l \theta_l^{q-1}}{(q-1)^{\nu_l}} - \sum_{j=1}^{n_1} \frac{\delta_j \xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{k=1}^{m_2} \frac{\tau_k \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \neq 0. \quad (5)$$

Proof. Using Lemmas 2.6-2.7, the equations in (2) can be expressed as equivalent integral equations

$$x(t) = {}_R L I^p \phi(t) + c_1 t^{p-1} + c_2 t^{p-2}, \quad (6)$$

$$y(t) = {}_R L I^q \psi(t) + d_1 t^{q-1} + d_2 t^{q-2}, \quad (7)$$

for $c_1, c_2, d_1, d_2 \in \mathbb{R}$. The conditions $x(0) = 0, y(0) = 0$ imply that $c_2 = 0, d_2 = 0$. Taking the Hadamard fractional integral of order $\alpha_i > 0, \sigma_k > 0$ for (6) and $\beta_j > 0, \nu_l > 0$ for (7) and using the property of the Hadamard fractional integral given in Lemma 2.5 we get the system

$$\begin{aligned} \sum_{i=1}^{m_1} \mu_i {}_H I^{\alpha_i} {}_R L I^p \phi(\eta_i) + c_1 \sum_{i=1}^{m_1} \frac{\mu_i \eta_i^{p-1}}{(p-1)^{\alpha_i}} &= \sum_{j=1}^{n_1} \delta_j {}_H I^{\beta_j} {}_R L I^q \psi(\xi_j) + d_1 \sum_{j=1}^{n_1} \frac{\delta_j \xi_j^{q-1}}{(q-1)^{\beta_j}} + \lambda_1, \\ \sum_{k=1}^{m_2} \tau_k {}_H I^{\sigma_k} {}_R L I^p \phi(\gamma_k) + c_1 \sum_{k=1}^{m_2} \frac{\tau_k \gamma_k^{p-1}}{(p-1)^{\sigma_k}} &= \sum_{l=1}^{n_2} \omega_l {}_H I^{\nu_l} {}_R L I^q \psi(\theta_l) + d_1 \sum_{l=1}^{n_2} \frac{\omega_l \theta_l^{q-1}}{(q-1)^{\nu_l}} + \lambda_2, \end{aligned}$$

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from which we have

$$c_1 = \frac{1}{\Omega} \left\{ \sum_{l=1}^{n_2} \frac{\omega_l \theta_l^{q-1}}{(q-1)^{\nu_l}} \left(\sum_{j=1}^{n_1} \delta_{jH} I^{\beta_j} {}_{RL}I^q \psi(\xi_j) - \sum_{i=1}^{m_1} \mu_i {}_H I^{\alpha_i} {}_{RL}I^p \phi(\eta_i) + \lambda_1 \right) \right. \\ \left. - \sum_{j=1}^{n_1} \frac{\delta_j \xi_j^{q-1}}{(q-1)^{\beta_j}} \left(\sum_{l=1}^{n_2} \omega_l {}_H I^{\nu_l} {}_{RL}I^q \psi(\theta_l) - \sum_{k=1}^{m_2} \tau_k {}_H I^{\sigma_k} {}_{RL}I^p \phi(\gamma_k) + \lambda_2 \right) \right\}$$

and

$$d_1 = \frac{1}{\Omega} \left\{ \sum_{k=1}^{m_2} \frac{\tau_k \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \left(\sum_{j=1}^{n_1} \delta_{jH} I^{\beta_j} {}_{RL}I^q \psi(\xi_j) - \sum_{i=1}^{m_1} \mu_i {}_H I^{\alpha_i} {}_{RL}I^p \phi(\eta_i) + \lambda_1 \right) \right. \\ \left. - \sum_{i=1}^{m_1} \frac{\mu_i \eta_i^{p-1}}{(p-1)^{\alpha_i}} \left(\sum_{l=1}^{n_2} \omega_l {}_H I^{\nu_l} {}_{RL}I^q \psi(\theta_l) - \sum_{k=1}^{m_2} \tau_k {}_H I^{\sigma_k} {}_{RL}I^p \phi(\gamma_k) + \lambda_2 \right) \right\}.$$

Substituting the values of c_1, c_2, d_1 and d_2 in (6) and (7), we obtain the solutions (3) and (4). \square

3 Main Results

Throughout this paper, for convenience, we use the following expressions

$${}_{RL}I^w h(s, x(s), y(s))(v) = \frac{1}{\Gamma(w)} \int_0^v (v-s)^{w-1} h(s, x(s), y(s)) ds,$$

and

$${}_H I^u {}_{RL}I^w h(s, x(s), y(s))(v) = \frac{1}{\Gamma(u)\Gamma(w)} \int_0^v \int_0^t \left(\log \frac{v}{t} \right)^{u-1} (t-s)^{w-1} \frac{h(s, x(s), y(s))}{t} ds dt,$$

where $u \in \{\rho_i, \gamma_j\}$, $v \in \{t, T, \eta_i, \theta_j\}$, $w = \{p, q\}$ and $h = \{f, g\}$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$.

Let $\mathcal{C} = C([0, T], \mathbb{R})$ denotes the Banach space of all continuous functions from $[0, T]$ to \mathbb{R} . Let us introduce the space $X = \{x(t) | x(t) \in C([0, T])\}$ endowed with the norm $\|x\| = \max\{|x(t)|, t \in [0, T]\}$. Obviously $(X, \|\cdot\|)$ is a Banach space. Also let $Y = \{y(t) | y(t) \in C([0, T])\}$ be endowed with the norm $\|y\| = \max\{|y(t)|, t \in [0, T]\}$. Obviously the product space $(X \times Y, \|(x, y)\|)$ is a Banach space with norm $\|(x, y)\| = \|x\| + \|y\|$.

In view of Lemma 2.8, we define an operator $\mathcal{T} : X \times Y \rightarrow X \times Y$ by $\mathcal{T}(x, y)(t) = \begin{pmatrix} \mathcal{T}_1(x, y)(t) \\ \mathcal{T}_2(x, y)(t) \end{pmatrix}$, where

$$\mathcal{T}_1(x, y)(t) = {}_{RL}I^p f(s, x(s), y(s))(t) + \frac{t^{p-1}}{\Omega} \left\{ \sum_{l=1}^{n_2} \frac{\omega_l \theta_l^{q-1}}{(q-1)^{\nu_l}} \left(\sum_{j=1}^{n_1} \delta_{jH} I^{\beta_j} {}_{RL}I^q g(s, x(s), y(s))(\xi_j) \right. \right. \\ \left. \left. - \sum_{i=1}^{m_1} \mu_i {}_H I^{\alpha_i} {}_{RL}I^p f(s, x(s), y(s))(\eta_i) + \lambda_1 \right) - \sum_{j=1}^{n_1} \frac{\delta_j \xi_j^{q-1}}{(q-1)^{\beta_j}} \left(\sum_{l=1}^{n_2} \omega_l {}_H I^{\nu_l} {}_{RL}I^q g(s, x(s), y(s))(\theta_l) \right. \right. \\ \left. \left. - \sum_{k=1}^{m_2} \tau_k {}_H I^{\sigma_k} {}_{RL}I^p f(s, x(s), y(s))(\gamma_k) + \lambda_2 \right) \right\},$$

and

$$\mathcal{T}_2(x, y)(t) = {}_{RL}I^q g(s, x(s), y(s))(t) + \frac{t^{q-1}}{\Omega} \left\{ \sum_{k=1}^{m_2} \frac{\tau_k \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \left(\sum_{j=1}^{n_1} \delta_{jH} I^{\beta_j} {}_{RL}I^q g(s, x(s), y(s))(\xi_j) \right. \right. \\ \left. \left. - \sum_{i=1}^{m_1} \mu_i {}_H I^{\alpha_i} {}_{RL}I^p f(s, x(s), y(s))(\eta_i) + \lambda_1 \right) - \sum_{i=1}^{m_1} \frac{\mu_i \eta_i^{p-1}}{(p-1)^{\alpha_i}} \left(\sum_{l=1}^{n_2} \omega_l {}_H I^{\nu_l} {}_{RL}I^q g(s, x(s), y(s))(\theta_l) \right. \right. \\ \left. \left. - \sum_{k=1}^{m_2} \tau_k {}_H I^{\sigma_k} {}_{RL}I^p f(s, x(s), y(s))(\gamma_k) + \lambda_2 \right) \right\}.$$

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$$-\sum_{k=1}^{m_2} \tau_k {}_H I^{\sigma_k} {}_{RL} I^p f(s, x(s), y(s))(\gamma_k) + \lambda_2 \Bigg) \Bigg\}.$$

Let us introduce the following assumptions which are used hereafter.

(H₁) Assume that $f, g : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous functions and there exist constants $m_i, n_i, i = 1, 2$ such that for all $t \in [0, T]$ and $u_i, v_i \in \mathbb{R}, i = 1, 2$,

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq K_1 |u_1 - v_1| + K_2 |u_2 - v_2|$$

and

$$|g(t, u_1, u_2) - g(t, v_1, v_2)| \leq L_1 |u_1 - v_1| + L_2 |u_2 - v_2|.$$

(H₂) Assume that there exist real constants $k_i, l_i \geq 0$ ($i = 1, 2$) and $k_0 > 0, l_0 > 0$ such that $\forall x_i \in \mathbb{R}, (i = 1, 2)$ we have

$$|f(t, x_1, x_2)| \leq k_0 + k_1 |x_1| + k_2 |x_2|, \quad |g(t, x_1, x_2)| \leq l_0 + l_1 |x_1| + l_2 |x_2|.$$

For the sake of convenience, we set

$$M_1 = \frac{1}{\Gamma(p+1)} \left(T^p + \frac{T^{p-1}}{|\Omega|} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^p}{p^{\alpha_i}} + \frac{T^{p-1}}{|\Omega|} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^p}{p^{\sigma_k}} \right), \quad (8)$$

$$M_2 = \frac{T^{p-1}}{|\Omega| \Gamma(q+1)} \left(\sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^q}{q^{\beta_j}} + \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^q}{q^{\nu_l}} \right), \quad (9)$$

$$M_3 = \frac{T^{p-1}}{|\Omega|} \left(|\lambda_1| \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} + |\lambda_2| \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \right), \quad (10)$$

$$M_4 = \frac{1}{\Gamma(q+1)} \left(T^q + \frac{T^{q-1}}{|\Omega|} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^q}{q^{\beta_j}} + \frac{T^{q-1}}{|\Omega|} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^{p-1}}{(p-1)^{\alpha_i}} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^q}{q^{\nu_l}} \right), \quad (11)$$

$$M_5 = \frac{T^{q-1}}{|\Omega| \Gamma(p+1)} \left(\sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^p}{p^{\alpha_i}} + \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^{p-1}}{(p-1)^{\alpha_i}} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^p}{p^{\sigma_k}} \right), \quad (12)$$

$$M_6 = \frac{T^{q-1}}{|\Omega|} \left(|\lambda_1| \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^{p-1}}{(p-1)^{\sigma_k}} + |\lambda_2| \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^{p-1}}{(p-1)^{\alpha_i}} \right), \quad (13)$$

and

$$M_0 = \min\{1 - (M_1 + M_5)k_1 - (M_2 + M_4)l_1, 1 - (M_1 + M_5)k_2 - (M_2 + M_4)l_2\}, \quad (14)$$

$k_i, l_i \geq 0$ ($i = 1, 2$).

The first result is concerned with the existence and uniqueness of solutions for the problem (1) and is based on Banach's contraction mapping principle.

Theorem 3.1 Assume that (H₁) holds. In addition, suppose that

$$(M_1 + M_5)(K_1 + K_2) + (M_2 + M_4)(L_1 + L_2) < 1,$$

where $M_i, i = 1, 2, 4, 5$ are given by (3.1)-(3.2) and (3.4)-(3.5). Then the boundary value problem (1) has a unique solution.

Proof. Define $\sup_{t \in [0, T]} f(t, 0, 0) = N_1 < \infty$ and $\sup_{t \in [0, T]} g(t, 0, 0) = N_2 < \infty$ such that

$$r \geq \max \left\{ \frac{M_1 N_1 + M_2 N_2 + M_3}{1 - (M_1 K_1 + M_2 L_1 + M_1 K_2 + M_2 L_2)}, \frac{M_4 N_2 + M_5 N_1 + M_6}{1 - (M_4 L_1 + M_5 K_1 + M_4 L_2 + M_5 K_2)} \right\},$$

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where M_3 and M_6 are defined by (3.3) and (3.6), respectively.

We show that $\mathcal{T}B_r \subset B_r$, where $B_r = \{(x, y) \in X \times Y : \|(x, y)\| \leq r\}$.

For $(x, y) \in B_r$, we have

$$\begin{aligned}
|\mathcal{T}_1(x, y)(t)| &= \max_{t \in [0, T]} \left\{ {}_{RL}I^p f(s, x(s), y(s))(t) + \frac{t^{p-1}}{\Omega} \left[\sum_{l=1}^{n_2} \frac{\omega_l \theta_l^{q-1}}{(q-1)^{\nu_l}} \right. \right. \\
&\quad \times \left(\sum_{j=1}^{n_1} \delta_j {}_H I^{\beta_j} {}_{RL}I^q g(s, x(s), y(s))(\xi_j) - \sum_{i=1}^{m_1} \mu_i {}_H I^{\alpha_i} {}_{RL}I^p f(s, x(s), y(s))(\eta_i) + \lambda_1 \right) \\
&\quad - \sum_{j=1}^{n_1} \frac{\delta_j \xi_j^{q-1}}{(q-1)^{\beta_j}} \left(\sum_{l=1}^{n_2} \omega_l {}_H I^{\nu_l} {}_{RL}I^q g(s, x(s), y(s))(\theta_l) \right. \\
&\quad \left. \left. - \sum_{k=1}^{m_2} \tau_k {}_H I^{\sigma_k} {}_{RL}I^p f(s, x(s), y(s))(\gamma_k) + \lambda_2 \right) \right] \Big\} \\
&\leq {}_{RL}I^p (|f(s, x(s), y(s)) - f(s, 0, 0)| + |f(s, 0, 0)|)(T) + \frac{T^{p-1}}{|\Omega|} \left[\sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} \right. \\
&\quad \times \left(\sum_{j=1}^{n_1} |\delta_j| {}_H I^{\beta_j} {}_{RL}I^q (|g(s, x(s), y(s)) - g(s, 0, 0)| + |g(s, 0, 0)|)(\xi_j) \right. \\
&\quad \left. + \sum_{i=1}^{m_1} |\mu_i| {}_H I^{\alpha_i} {}_{RL}I^p (|f(s, x(s), y(s)) - f(s, 0, 0)| + |f(s, 0, 0)|)(\eta_i) + |\lambda_1| \right) \\
&\quad + \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \left(\sum_{l=1}^{n_2} |\omega_l| {}_H I^{\nu_l} {}_{RL}I^q (|g(s, x(s), y(s)) - g(s, 0, 0)| + |g(s, 0, 0)|)(\theta_l) \right. \\
&\quad \left. + \sum_{k=1}^{m_2} |\tau_k| {}_H I^{\sigma_k} {}_{RL}I^p (|f(s, x(s), y(s)) - f(s, 0, 0)| + |f(s, 0, 0)|)(\gamma_k) + |\lambda_2| \right) \Big] \\
&\leq {}_{RL}I^p (K_1 \|x\| + K_2 \|y\| + N_1)(T) + \frac{T^{p-1}}{|\Omega|} \left[\sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} \right. \\
&\quad \times \left(\sum_{j=1}^{n_1} |\delta_j| {}_H I^{\beta_j} {}_{RL}I^q (L_1 \|x\| + L_2 \|y\| + N_2)(\xi_j) \right. \\
&\quad \left. + \sum_{i=1}^{m_1} |\mu_i| {}_H I^{\alpha_i} {}_{RL}I^p (K_1 \|x\| + K_2 \|y\| + N_1)(\eta_i) + |\lambda_1| \right) \\
&\quad + \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \left(\sum_{l=1}^{n_2} |\omega_l| {}_H I^{\nu_l} {}_{RL}I^q (L_1 \|x\| + L_2 \|y\| + N_2)(\theta_l) \right. \\
&\quad \left. + \sum_{k=1}^{m_2} |\tau_k| {}_H I^{\sigma_k} {}_{RL}I^p (K_1 \|x\| + K_2 \|y\| + N_1)(\gamma_k) + |\lambda_2| \right) \Big] \\
&= (K_1 \|x\| + K_2 \|y\| + N_1) \left\{ {}_{RL}I^p(1)(T) + \frac{T^{p-1}}{|\Omega|} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} \sum_{i=1}^{m_1} |\mu_i| {}_H I^{\alpha_i} {}_{RL}I^p(1)(\eta_i) \right. \\
&\quad \left. + \frac{T^{p-1}}{|\Omega|} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{k=1}^{m_2} |\tau_k| {}_H I^{\sigma_k} {}_{RL}I^p(1)(\gamma_k) \right\} + (L_1 \|x\| + L_2 \|y\| + N_2) \left\{ \frac{T^{p-1}}{|\Omega|} \right. \\
&\quad \times \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} \sum_{j=1}^{n_1} |\delta_j| {}_H I^{\beta_j} {}_{RL}I^q(1)(\xi_j) + \frac{T^{p-1}}{|\Omega|} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{l=1}^{n_2} |\omega_l| {}_H I^{\nu_l} {}_{RL}I^q(1)(\theta_l) \Big\}
\end{aligned}$$

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$$\begin{aligned}
& + |\lambda_1| \frac{T^{p-1}}{|\Omega|} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} + |\lambda_2| \frac{T^{p-1}}{|\Omega|} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \\
= & (K_1 \|x\| + K_2 \|y\| + N_1) \left\{ \frac{T^p}{\Gamma(p+1)} + \frac{T^{p-1}}{|\Omega| \Gamma(p+1)} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^p}{p^{\alpha_i}} \right. \\
& + \left. \frac{T^{p-1}}{|\Omega| \Gamma(p+1)} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^p}{p^{\sigma_k}} \right\} + (L_1 \|x\| + L_2 \|y\| + N_2) \left\{ \frac{T^{p-1}}{|\Omega| \Gamma(q+1)} \right. \\
& \times \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^q}{q^{\beta_j}} + \frac{T^{p-1}}{|\Omega| \Gamma(q+1)} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^q}{q^{\nu_l}} \left. \right\} \\
& + |\lambda_1| \frac{T^{p-1}}{|\Omega|} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} + |\lambda_2| \frac{T^{p-1}}{|\Omega|} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \\
= & (K_1 \|x\| + K_2 \|y\| + N_1) M_1 + (L_1 \|x\| + L_2 \|y\| + N_2) M_2 + M_3 \\
= & (M_1 K_1 + M_2 L_1) \|x\| + (M_1 K_2 + M_2 L_2) \|y\| + M_1 N_1 + M_2 N_2 + M_3 \\
\leq & (M_1 K_1 + M_2 L_1 + M_1 K_2 + M_2 L_2) r + M_1 N_1 + M_2 N_2 + M_3 \leq r.
\end{aligned}$$

In the same way, we can obtain that

$$\begin{aligned}
|\mathcal{T}_2(x, y)(t)| \leq & (L_1 \|x\| + L_2 \|y\| + N_2) \left\{ \frac{T^q}{\Gamma(q+1)} + \frac{T^{q-1}}{|\Omega| \Gamma(q+1)} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^q}{q^{\beta_j}} \right. \\
& + \left. \frac{T^{q-1}}{|\Omega| \Gamma(q+1)} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^{p-1}}{(p-1)^{\alpha_i}} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^q}{q^{\nu_l}} \right\} + (K_1 \|x\| + K_2 \|y\| + N_1) \left\{ \frac{T^{q-1}}{|\Omega| \Gamma(p+1)} \right. \\
& \times \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^p}{p^{\alpha_i}} + \frac{T^{q-1}}{|\Omega| \Gamma(p+1)} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^{p-1}}{(p-1)^{\alpha_i}} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^p}{p^{\sigma_k}} \left. \right\} \\
& + |\lambda_1| \frac{T^{q-1}}{|\Omega|} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^{p-1}}{(p-1)^{\sigma_k}} + |\lambda_2| \frac{T^{q-1}}{|\Omega|} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^{p-1}}{(p-1)^{\alpha_i}} \\
= & (L_1 \|x\| + L_2 \|y\| + N_2) M_4 + (K_1 \|x\| + K_2 \|y\| + N_1) M_5 + M_6 \\
= & (M_4 L_1 + M_5 K_1) \|x\| + (M_4 L_2 + M_5 K_2) \|y\| + M_4 N_2 + M_5 N_1 + M_6 \\
\leq & (M_4 L_1 + M_5 K_1 + M_4 L_2 + M_5 K_2) r + M_4 N_2 + M_5 N_1 + M_6 \leq r.
\end{aligned}$$

Consequently, $\|\mathcal{T}(x, y)(t)\| \leq r$.

Now for $(x_2, y_2), (x_1, y_1) \in X \times Y$, and for any $t \in [0, T]$, we get

$$\begin{aligned}
& |\mathcal{T}_1(x_2, y_2)(t) - \mathcal{T}_1(x_1, y_1)(t)| \\
\leq & {}_{RL}I^p |f(s, x_2(s), y_2(s)) - f(s, x_1(s), y_1(s))|(T) + \frac{T^{p-1}}{|\Omega|} \left[\sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} \right. \\
& \times \left(\sum_{j=1}^{n_1} |\delta_j| {}_H I^{\beta_j} {}_{RL}I^q (|g(s, x_2(s), y_2(s)) - g(s, x_1(s), y_1(s))|)(\xi_j) \right. \\
& + \left. \sum_{i=1}^{m_1} |\mu_i| {}_H I^{\alpha_i} {}_{RL}I^p (|f(s, x_2(s), y_2(s)) - f(s, x_1(s), y_1(s))|)(\eta_i) \right) \\
& + \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \left(\sum_{l=1}^{n_2} |\omega_l| {}_H I^{\nu_l} {}_{RL}I^q (|g(s, x_2(s), y_2(s)) - g(s, x_1(s), y_1(s))|)(\theta_l) \right. \\
& + \left. \left. \sum_{k=1}^{m_2} |\tau_k| {}_H I^{\sigma_k} {}_{RL}I^p (|f(s, x_2(s), y_2(s)) - f(s, x_1(s), y_1(s))|)(\gamma_k) \right) \right]
\end{aligned}$$

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$$\begin{aligned}
&\leq (K_1\|x_2 - x_1\| + K_2\|y_2 - y_1\|) \left\{ \frac{T^p}{\Gamma(p+1)} + \frac{T^{p-1}}{|\Omega|\Gamma(p+1)} \sum_{l=1}^{n_2} \frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}} \sum_{i=1}^{m_1} \frac{|\mu_i|\eta_i^p}{p^{\alpha_i}} \right. \\
&\quad \left. + \frac{T^{p-1}}{|\Omega|\Gamma(p+1)} \sum_{j=1}^{n_1} \frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{k=1}^{m_2} \frac{|\tau_k|\gamma_k^p}{p^{\sigma_k}} \right\} + (L_1\|x_2 - x_1\| + L_2\|y_2 - y_1\|) \\
&\quad \times \left\{ \frac{T^{p-1}}{|\Omega|\Gamma(q+1)} \sum_{l=1}^{n_2} \frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}} \sum_{j=1}^{n_1} \frac{|\delta_j|\xi_j^q}{q^{\beta_j}} + \frac{T^{p-1}}{|\Omega|\Gamma(q+1)} \sum_{j=1}^{n_1} \frac{|\delta_j|\xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{l=1}^{n_2} \frac{|\omega_l|\theta_l^q}{q^{\nu_l}} \right\} \\
&= (K_1\|x_2 - x_1\| + K_2\|y_2 - y_1\|)M_1 + (L_1\|x_2 - x_1\| + L_2\|y_2 - y_1\|)M_2 \\
&= (M_1K_1 + M_2L_1)\|x_2 - x_1\| + (M_1K_2 + M_2L_2)\|y_2 - y_1\|,
\end{aligned}$$

and consequently we obtain

$$\|\mathcal{T}_1(x_2, y_2)(t) - \mathcal{T}_1(x_1, y_1)\| \leq (M_1K_1 + M_2L_1 + M_1K_2 + M_2L_2)(\|x_2 - x_1\| + \|y_2 - y_1\|). \quad (15)$$

Similarly,

$$\|\mathcal{T}_2(x_2, y_2)(t) - \mathcal{T}_2(x_1, y_1)\| \leq (M_4L_1 + M_5K_1 + M_4L_2 + M_5K_2)(\|x_2 - x_1\| + \|y_2 - y_1\|). \quad (16)$$

It follows from (15) and (16) that

$$\|\mathcal{T}(x_2, y_2)(t) - \mathcal{T}(x_1, y_1)(t)\| \leq [(M_1 + M_5)(K_1 + K_2) + (M_2 + M_4)(L_1 + L_2)](\|x_2 - x_1\| + \|y_2 - y_1\|).$$

Since $(M_1 + M_5)(K_1 + K_2) + (M_2 + M_4)(L_1 + L_2) < 1$, therefore, \mathcal{T} is a contraction operator. So, By Banach's fixed point theorem, the operator \mathcal{T} has a unique fixed point, which is the unique solution of problem (1). This completes the proof. \square

In the next result, we prove the existence of solutions for the problem (1) by applying Leray-Schauder alternative.

Lemma 3.2 (Leray-Schauder alternative) ([11], page.4.) Let $F : E \rightarrow E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in E is compact). Let

$$\mathcal{E}(F) = \{x \in E : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}.$$

Then either the set $\mathcal{E}(F)$ is unbounded, or F has at least one fixed point.

Theorem 3.3 Assume that (H_2) holds. In addition it is assumed that

$$(M_1 + M_5)k_1 + (M_2 + M_4)l_1 < 1 \quad \text{and} \quad (M_1 + M_5)k_2 + (M_2 + M_4)l_2 < 1,$$

where M_1, M_2, M_4, M_5 are given by (3.1)-(3.2) and (3.4)-(3.5). Then there exists at least one solution for the boundary value problem (1).

Proof. First we show that the operator $\mathcal{T} : X \times Y \rightarrow X \times Y$ is completely continuous. By continuity of functions f and g , the operator \mathcal{T} is continuous.

Let $\Theta \subset X \times Y$ be bounded. Then there exist positive constants P_1 and P_2 such that

$$|f(t, x(t), y(t))| \leq P_1, \quad |g(t, x(t), y(t))| \leq P_2, \quad \forall (x, y) \in \Theta.$$

Then for any $(x, y) \in \Theta$, we have

$$\begin{aligned}
\|\mathcal{T}_1(x, y)\| &\leq {}_{RL}I^p |f(s, x(s), y(s))|(T) + \frac{T^{p-1}}{|\Omega|} \left[\sum_{l=1}^{n_2} \frac{|\omega_l|\theta_l^{q-1}}{(q-1)^{\nu_l}} \left(\sum_{j=1}^{n_1} |\delta_j|_H I^{\beta_j} {}_{RL}I^q |g(s, x(s), y(s))|(\xi_j) \right. \right. \\
&\quad \left. \left. + \sum_{i=1}^{m_1} |\mu_i|_H I^{\alpha_i} {}_{RL}I^p |f(s, x(s), y(s))|(\eta_i) + |\lambda_1| \right) \right]
\end{aligned}$$

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$$\begin{aligned}
& + \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \left(\sum_{l=1}^{n_2} |\omega_l| {}_H I^{\nu_l} {}_{RL} I^q |g(s, x(s), y(s))| (\theta_l) \right. \\
& \left. + \sum_{k=1}^{m_2} |\tau_k| {}_H I^{\sigma_k} {}_{RL} I^p |f(s, x(s), y(s))| (\gamma_k) + |\lambda_2| \right) \Bigg] \\
\leq & \left(\frac{T^p}{\Gamma(p+1)} + \frac{T^{p-1}}{|\Omega| \Gamma(p+1)} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^p}{p^{\alpha_i}} \right. \\
& + \frac{T^{p-1}}{|\Omega| \Gamma(p+1)} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^p}{p^{\sigma_k}} \Bigg) P_1 + \left(\frac{T^{p-1}}{|\Omega| \Gamma(p+1)} \right. \\
& \times \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^q}{q^{\beta_j}} + \frac{T^{p-1}}{|\Omega| \Gamma(p+1)} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^q}{q^{\nu_l}} \Bigg) P_2 \\
& + |\lambda_1| \frac{T^{p-1}}{|\Omega|} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} + |\lambda_2| \frac{T^{p-1}}{|\Omega|} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \\
= & M_1 P_1 + M_2 P_2 + M_3.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\|\mathcal{T}_2(x, y)\| \leq & \left(\frac{T^q}{\Gamma(q+1)} + \frac{T^{q-1}}{|\Omega| \Gamma(q+1)} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^q}{q^{\beta_j}} \right. \\
& + \frac{T^{q-1}}{|\Omega| \Gamma(q+1)} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^{p-1}}{(p-1)^{\alpha_i}} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^q}{q^{\nu_l}} \Bigg) P_2 + \left(\frac{T^{q-1}}{|\Omega| \Gamma(p+1)} \right. \\
& \times \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^p}{p^{\alpha_i}} + \frac{T^{q-1}}{|\Omega| \Gamma(p+1)} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^{p-1}}{(p-1)^{\alpha_i}} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^p}{p^{\sigma_k}} \Bigg) P_1 \\
& + |\lambda_1| \frac{T^{q-1}}{|\Omega|} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^{p-1}}{(p-1)^{\sigma_k}} + |\lambda_2| \frac{T^{q-1}}{|\Omega|} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^{p-1}}{(p-1)^{\alpha_i}} \\
= & M_4 P_2 + M_5 P_1 + M_6.
\end{aligned}$$

Thus, it follows from the above inequalities that the operator \mathcal{T} is uniformly bounded.

Next, we show that \mathcal{T} is equicontinuous. Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
& |\mathcal{T}_1(x(t_2), y(t_2)) - \mathcal{T}_1(x(t_1), y(t_1))| \\
\leq & \frac{1}{\Gamma(p)} \int_0^{t_1} [(t_2 - s)^{p-1} - (t_1 - s)^{p-1}] |f(s, x(s), y(s))| ds \\
& + \frac{1}{\Gamma(p)} \int_{t_1}^{t_2} (t_2 - s)^{p-1} |f(s, x(s), y(s))| ds + \frac{t_2^{p-1} - t_1^{p-1}}{|\Omega|} \left[\sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} \right. \\
& \times \left(\sum_{j=1}^{n_1} |\delta_j| {}_H I^{\beta_j} {}_{RL} I^q |g(s, x(s), y(s))| (\xi_j) + \sum_{i=1}^{m_1} |\mu_i| {}_H I^{\alpha_i} {}_{RL} I^p |f(s, x(s), y(s))| (\eta_i) + |\lambda_1| \right) \\
& + \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \left(\sum_{l=1}^{n_2} |\omega_l| {}_H I^{\nu_l} {}_{RL} I^q |g(s, x(s), y(s))| (\theta_l) \right. \\
& \left. + \sum_{k=1}^{m_2} |\tau_k| {}_H I^{\sigma_k} {}_{RL} I^p |f(s, x(s), y(s))| (\gamma_k) + |\lambda_2| \right) \Bigg] \\
\leq & \frac{P_1}{\Gamma(p)} \int_0^{t_1} [(t_2 - s)^{p-1} - (t_1 - s)^{p-1}] ds + \frac{P_1}{\Gamma(p)} \int_{t_1}^{t_2} (t_2 - s)^{p-1} ds
\end{aligned}$$

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$$\begin{aligned}
& + \frac{t_2^{p-1} - t_1^{p-1}}{|\Omega|} \left[\sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} \left(P_2 \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^q}{q^{\beta_j} \Gamma(q+1)} \right) + P_1 \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^p}{p^{\alpha_i} \Gamma(p+1)} + |\lambda_1| \right) \\
& + \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \left(P_2 \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^q}{q^{\nu_l} \Gamma(q+1)} + P_1 \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^p}{p^{\sigma_k} \Gamma(p+1)} + |\lambda_2| \right) \Big].
\end{aligned}$$

Analogously, we can obtain

$$\begin{aligned}
& |\mathcal{T}_2(x(t_2), y(t_2)) - \mathcal{T}_2(x(t_1), y(t_1))| \\
& \leq \frac{P_2}{\Gamma(q)} \int_0^{t_1} [(t_2 - s)^{q-1} - (t_1 - s)^{q-1}] ds + \frac{P_2}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} ds \\
& + \frac{t_2^{q-1} - t_1^{q-1}}{|\Omega|} \left[\sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \left(P_2 \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^q}{q^{\beta_j} \Gamma(q+1)} \right) + P_1 \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^p}{p^{\alpha_i} \Gamma(p+1)} + |\lambda_1| \right) \\
& + \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^{p-1}}{(p-1)^{\alpha_i}} \left(P_2 \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^q}{q^{\nu_l} \Gamma(q+1)} + P_1 \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^p}{p^{\sigma_k} \Gamma(p+1)} + |\lambda_2| \right) \Big].
\end{aligned}$$

Therefore, the operator $\mathcal{T}(x, y)$ is equicontinuous, and thus the operator $\mathcal{T}(x, y)$ is completely continuous.

Finally, it will be verified that the set $\mathcal{E} = \{(x, y) \in X \times Y | (x, y) = \lambda \mathcal{T}(x, y), 0 \leq \lambda \leq 1\}$ is bounded. Let $(x, y) \in \mathcal{E}$, then $(x, y) = \lambda \mathcal{T}(x, y)$. For any $t \in [0, T]$, we have

$$x(t) = \lambda \mathcal{T}_1(x, y)(t), \quad y(t) = \lambda \mathcal{T}_2(x, y)(t).$$

Then

$$\begin{aligned}
|x(t)| & \leq (k_0 + k_1 \|x\| + k_2 \|y\|) \left(\frac{T^p}{\Gamma(p+1)} + \frac{T^{p-1}}{|\Omega| \Gamma(p+1)} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^p}{p^{\alpha_i}} \right. \\
& + \frac{T^{p-1}}{|\Omega| \Gamma(p+1)} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^p}{p^{\sigma_k}} \Big) + (l_0 + l_1 \|x\| + l_2 \|y\|) \\
& \times \left(\frac{T^{p-1}}{|\Omega| \Gamma(q+1)} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^q}{q^{\beta_j}} + \frac{T^{p-1}}{|\Omega| \Gamma(q+1)} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^q}{q^{\nu_l}} \right) \\
& + |\lambda_1| \frac{T^{p-1}}{|\Omega|} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^{q-1}}{(q-1)^{\nu_l}} + |\lambda_2| \frac{T^{p-1}}{|\Omega|} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^{q-1}}{(q-1)^{\beta_j}}
\end{aligned}$$

and

$$\begin{aligned}
|y(t)| & \leq (l_0 + l_1 \|x\| + l_2 \|y\|) \left(\frac{T^q}{\Gamma(q+1)} + \frac{T^{q-1}}{|\Omega| \Gamma(q+1)} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^q}{q^{\beta_j}} \right. \\
& + \frac{T^{q-1}}{|\Omega| \Gamma(q+1)} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^{p-1}}{(p-1)^{\alpha_i}} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^q}{q^{\nu_l}} \Big) + (k_0 + k_1 \|x\| + k_2 \|y\|) \\
& \times \left(\frac{T^{q-1}}{|\Omega| \Gamma(p+1)} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^p}{p^{\alpha_i}} + \frac{T^{q-1}}{|\Omega| \Gamma(p+1)} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^{p-1}}{(p-1)^{\alpha_i}} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^p}{p^{\sigma_k}} \right) \\
& + |\lambda_1| \frac{T^{q-1}}{|\Omega|} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^{p-1}}{(p-1)^{\sigma_k}} + |\lambda_2| \frac{T^{q-1}}{|\Omega|} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^{p-1}}{(p-1)^{\alpha_i}}.
\end{aligned}$$

Hence we have

$$\|x\| \leq (k_0 + k_1 \|x\| + k_2 \|y\|) M_1 + (l_0 + l_1 \|x\| + l_2 \|y\|) M_2 + M_3$$

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and

$$\|y\| \leq (l_0 + l_1\|x\| + l_2\|y\|)M_4 + (k_0 + k_1\|x\| + k_2\|y\|)M_5 + M_6,$$

which imply that

$$\begin{aligned} \|x\| + \|y\| &\leq (M_1 + M_5)k_0 + (M_2 + M_4)l_0 + [(M_1 + M_5)k_1 + (M_2 + M_4)l_1]\|x\| \\ &\quad + [(M_1 + M_5)k_2 + (M_2 + M_4)l_2]\|y\| + M_3 + M_6. \end{aligned}$$

Consequently,

$$\|(x, y)\| \leq \frac{(M_1 + M_5)k_0 + (M_2 + M_4)l_0 + M_3 + M_6}{M_0},$$

for any $t \in [0, T]$, where M_0 is defined by (14), which proves that \mathcal{E} is bounded. Thus, by Lemma 3.2, the operator \mathcal{T} has at least one fixed point. Hence the boundary value problem (1) has at least one solution. The proof is complete. \square

3.1 Examples

Example 3.4 Consider the following system of coupled Riemann-Liouville fractional differential equations with Hadamard type fractional integral boundary conditions

$$\left\{ \begin{array}{l} {}_{RL}D^{4/3}x(t) = \frac{t}{(t+6)^2} \frac{|x(t)|}{(1+|x(t)|)} + \frac{e^{-t}}{(t^2+3)^3} \frac{|y(t)|}{(1+|y(t)|)} + \frac{3}{4}, \quad t \in [0, 2], \\ {}_{RL}D^{3/2}y(t) = \frac{1}{18} \sin x(t) + \frac{1}{2^{2t}+19} \cos y(t) + \frac{5}{4}, \quad t \in [0, 2], \\ x(0) = 0, \quad {}_2HI^{2/3}x(3/5) + \pi {}_HI^{7/5}x(1) = \sqrt{2} {}_HI^{3/2}y(1/3) + e^2 {}_HI^{5/4}y(\sqrt{3}) + 4, \\ y(0) = 0, \quad -3 {}_HI^{9/5}x(2/3) + 4 {}_HI^{7/4}x(9/7) + \frac{2}{5} {}_HI^{1/3}x(\sqrt{2}) \\ \quad = \frac{e}{2} {}_HI^{11/6}y(8/5) - 2 {}_HI^{12/11}y(1/4) - 10. \end{array} \right. \quad (17)$$

Here $p = 4/3$, $q = 3/2$, $T = 2$, $\lambda_1 = 4$, $\lambda_2 = -10$, $m_1 = 2$, $n_1 = 2$, $m_2 = 3$, $n_2 = 2$, $\mu_1 = 2$, $\mu_2 = \pi$, $\alpha_1 = 2/3$, $\alpha_2 = 7/5$, $\eta_1 = 3/5$, $\eta_2 = 1$, $\delta_1 = \sqrt{2}$, $\delta_2 = e^2$, $\beta_1 = 3/2$, $\beta_2 = 5/4$, $\xi_1 = 1/3$, $\xi_2 = \sqrt{3}$, $\tau_1 = -3$, $\tau_2 = 4$, $\tau_3 = 2/5$, $\sigma_1 = 9/5$, $\sigma_2 = 7/4$, $\sigma_3 = 1/3$, $\gamma_1 = 2/3$, $\gamma_2 = 9/7$, $\gamma_3 = \sqrt{2}$, $\omega_1 = e/2$, $\omega_2 = -2$, $\nu_1 = 11/6$, $\nu_2 = 12/11$, $\theta_1 = 8/5$, $\theta_2 = 1/4$ and $f(t, x, y) = (t|x|)/((t+6)^2(1+|x|)) + (e^{-t}|y|)/((t^2+3)^3(1+|y|)) + (3/4)$ and $g(t, x, y) = (\sin x/18) + (\cos y)/(2^{2t}+19) + (5/4)$. Since $|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq ((1/18)|x_1 - x_2| + (1/27)|y_1 - y_2|)$ and $|g(t, x_1, y_1) - g(t, x_2, y_2)| \leq ((1/18)|x_1 - x_2| + (1/20)|y_1 - y_2|)$. By using the Maple program, we can find

$$\Omega = \sum_{i=1}^{m_1} \frac{\mu_i \eta_i^{p-1}}{(p-1)^{\alpha_i}} \sum_{l=1}^{n_2} \frac{\omega_l \theta_l^{q-1}}{(q-1)^{\nu_l}} - \sum_{j=1}^{n_1} \frac{\delta_j \xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{k=1}^{m_2} \frac{\tau_k \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \approx -218.9954766 \neq 0.$$

With the given values, it is found that $K_1 = 1/18$, $K_2 = 1/27$, $L_1 = 1/18$, $L_2 = 1/20$, $M_1 \simeq 2.847852451$, $M_2 \simeq 0.5295490231$, $M_4 \simeq 4.723846069$, $M_5 \simeq 1.276954854$, and

$$(M_1 + M_5)(K_1 + K_2) + (M_2 + M_4)(L_1 + L_2) \simeq 0.9364516398 < 1.$$

Thus all the conditions of Theorem 3.1 are satisfied. Therefore, by the conclusion of Theorem 3.1, the problem (17) has a unique solution on $[0, 2]$.

Example 3.5 Consider the following system of coupled Riemann-Liouville fractional differential equations

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tions with Hadamard type fractional integral boundary conditions

$$\left\{ \begin{array}{l} {}_{RL}D^{\pi/2}x(t) = \frac{2}{5} + \frac{1}{(t+6)^2} \tan^{-1}x(t) + \frac{1}{20e}y(t), \quad t \in [0, 3], \\ {}_{RL}D^{7/4}y(t) = \frac{\sqrt{\pi}}{2} + \frac{1}{42} \sin x(t) + \frac{1}{t+20}y(t) \cos x(t), \quad t \in [0, 3], \\ x(0) = 0, \quad {}_3H I^{1/4}x(5/2) + \sqrt{5} {}_H I^{\sqrt{2}}x(7/8) + \tan(4) {}_H I^{\sqrt{3}}x(9/4) \\ \quad = \frac{\sqrt{8}\pi}{3} {}_H I^{5/3}y(5/4) - 2 {}_H I^{6/11}y(\pi/3) + 2, \\ y(0) = 0, \quad -\frac{2}{3} {}_H I^{2/3}x(\pi/2) + 3 {}_H I^{6/5}x(5/3) + \frac{\sqrt{2}}{\pi} {}_H I^{1/3}x(\sqrt{2}) \\ \quad + \frac{7}{9} {}_H I^{11/9}x(\sqrt{5}) = e {}_H I^{7/6}y(\pi/6) - \log(9) {}_H I^{3/4}y(7/4) - 1. \end{array} \right. \quad (18)$$

Here $p = \pi/2$, $q = 7/4$, $T = 3$, $\lambda_1 = 2$, $\lambda_2 = -1$, $m_1 = 3$, $n_1 = 2$, $m_2 = 4$, $n_2 = 2$, $\mu_1 = 3$, $\mu_2 = \sqrt{5}$, $\mu_3 = \tan(4)$, $\alpha_1 = 1/4$, $\alpha_2 = \sqrt{2}$, $\alpha_3 = \sqrt{3}$, $\eta_1 = 5/2$, $\eta_2 = 7/8$, $\eta_3 = 9/4$, $\delta_1 = \sqrt{8}\pi/3$, $\delta_2 = -2$, $\beta_1 = 5/3$, $\beta_2 = 6/11$, $\xi_1 = 5/4$, $\xi_2 = \pi/3$, $\tau_1 = -2/3$, $\tau_2 = 3$, $\tau_3 = \sqrt{2}/\pi$, $\tau_4 = 7/9$, $\sigma_1 = 2/3$, $\sigma_2 = 6/5$, $\sigma_3 = 1/3$, $\sigma_4 = 11/9$, $\gamma_1 = \pi/2$, $\gamma_2 = 5/3$, $\gamma_3 = \sqrt{2}$, $\gamma_4 = \sqrt{5}$, $\omega_1 = e$, $\omega_2 = -\log(9)$, $\nu_1 = 7/6$, $\nu_2 = 3/4$, $\theta_1 = \pi/6$, $\theta_2 = 7/4$, $f(t, x, y) = (2/5) + (\tan^{-1}x)/((t+6)^2) + (y)/(20e)$ and $g(t, x, y) = (\sqrt{\pi}/2) + (\sin x)/(42) + (y \cos x)/(t+20)$. By using the Maple program, we get

$$\Omega = \sum_{i=1}^{m_1} \frac{\mu_i \eta_i^{p-1}}{(p-1)^{\alpha_i}} \sum_{l=1}^{n_2} \frac{\omega_l \theta_l^{q-1}}{(q-1)^{\nu_l}} - \sum_{j=1}^{n_1} \frac{\delta_j \xi_j^{q-1}}{(q-1)^{\beta_j}} \sum_{k=1}^{m_2} \frac{\tau_k \gamma_k^{p-1}}{(p-1)^{\sigma_k}} \approx -59.01857601 \neq 0.$$

Since $|f(t, x, y)| \leq k_0 + k_1|x| + k_2|y|$ and $|g(t, x, y)| \leq l_0 + l_1|x| + l_2|y|$, where $k_0 = 2/5$, $k_1 = 1/36$, $k_2 = 1/(20e)$, $l_0 = \sqrt{\pi}/2$, $l_1 = 1/42$, $l_2 = 1/20$, it is found that $M_1 \simeq 7.406711671$, $M_2 \simeq 1.110132269$, $M_4 \simeq 6.802999724$, $M_5 \simeq 7.790182643$. Furthermore, we can find that

$$(M_1 + M_5)k_1 + (M_2 + M_4)l_1 \approx 0.6105438577 < 1,$$

and

$$(M_1 + M_5)k_2 + (M_2 + M_4)l_2 \approx 0.6751878489 < 1.$$

Thus all the conditions of Theorem 3.3 holds true and consequently the conclusion of Theorem 3.3, the problem (18) has at least one solution on $[0, 3]$.

4 Uncoupled integral boundary conditions case

In this section we consider the following system

$$\left\{ \begin{array}{l} {}_{RL}D^p x(t) = f(t, x(t), y(t)), \quad t \in [0, T], \quad 1 < p \leq 2, \\ {}_{RL}D^q y(t) = g(t, x(t), y(t)), \quad t \in [0, T], \quad 1 < q \leq 2, \\ x(0) = 0, \quad \sum_{i=1}^{m_1} \mu_i {}_H I^{\alpha_i} x(\eta_i) = \sum_{j=1}^{n_1} \delta_j {}_H I^{\beta_j} x(\xi_j) + \lambda_1, \\ y(0) = 0, \quad \sum_{k=1}^{m_2} \tau_k {}_H I^{\sigma_k} y(\gamma_k) = \sum_{l=1}^{n_2} \omega_l {}_H I^{\nu_l} y(\theta_l) + \lambda_2. \end{array} \right. \quad (19)$$

Lemma 4.1 (Auxiliary Lemma) For $h \in C([0, T], \mathbb{R})$, the unique solution of the problem

$$\left\{ \begin{array}{l} {}_{RL}D^p x(t) = h(t), \quad 1 < p \leq 2, \quad t \in [0, T] \\ x(0) = 0, \quad \sum_{i=1}^{m_1} \mu_i {}_H I^{\alpha_i} x(\eta_i) = \sum_{j=1}^{n_1} \delta_j {}_H I^{\beta_j} x(\xi_j) + \lambda_1, \end{array} \right. \quad (20)$$

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is given by

$$x(t) = {}_{RL}I^p h(t) + \frac{t^{p-1}}{\Lambda} \left(\sum_{j=1}^{n_1} \delta_{jH} I^{\beta_j} {}_{RL}I^p h(\xi_j) - \sum_{i=1}^{m_1} \mu_{iH} I^{\alpha_i} {}_{RL}I^p h(\eta_i) + \lambda_1 \right), \quad (21)$$

where

$$\Lambda := \sum_{i=1}^{m_1} \frac{\mu_i \eta_i^{p-1}}{(p-1)^{\alpha_i}} - \sum_{j=1}^{n_1} \frac{\delta_j \xi_j^{p-1}}{(p-1)^{\beta_j}} \neq 0. \quad (22)$$

4.1 Existence results for uncoupled case

In view of Lemma 4.1, we define an operator $\mathfrak{T} : X \times Y \rightarrow X \times Y$ by $\mathfrak{T}(u, v)(t) = \begin{pmatrix} \mathfrak{T}_1(u, v)(t) \\ \mathfrak{T}_2(u, v)(t) \end{pmatrix}$ where

$$\begin{aligned} \mathfrak{T}_1(u, v)(t) &= {}_{RL}I^p f(s, u(s), v(s))(t) + \frac{t^{p-1}}{\Lambda} \left(\sum_{j=1}^{n_1} \delta_{jH} I^{\beta_j} {}_{RL}I^p f(s, u(s), v(s))(\xi_j) \right. \\ &\quad \left. - \sum_{i=1}^{m_1} \mu_{iH} I^{\alpha_i} {}_{RL}I^p f(s, u(s), v(s))(\eta_i) + \lambda_1 \right), \end{aligned}$$

and

$$\begin{aligned} \mathfrak{T}_2(u, v)(t) &= {}_{RL}I^q g(s, u(s), v(s))(t) + \frac{t^{q-1}}{\Phi} \left(\sum_{l=1}^{n_2} \omega_{lH} I^{\nu_l} {}_{RL}I^q g(s, u(s), v(s))(\theta_l) \right. \\ &\quad \left. - \sum_{k=1}^{m_2} \tau_{kH} I^{\sigma_k} {}_{RL}I^q g(s, u(s), v(s))(\gamma_k) + \lambda_2 \right), \end{aligned}$$

where

$$\Phi = \sum_{k=1}^{m_2} \frac{\tau_k \gamma_k^{q-1}}{(q-1)^{\sigma_k}} - \sum_{l=1}^{n_2} \frac{\omega_l \theta_l^{q-1}}{(q-1)^{\nu_l}} \neq 0.$$

In the sequel, we set

$$\overline{M}_1 = \frac{1}{\Gamma(p+1)} \left(T^p + \frac{T^{p-1}}{|\Lambda|} \sum_{j=1}^{n_1} \frac{|\delta_j| \xi_j^p}{p^{\beta_j}} + \frac{T^{p-1}}{|\Lambda|} \sum_{i=1}^{m_1} \frac{|\mu_i| \eta_i^p}{p^{\alpha_i}} \right), \quad (23)$$

$$\overline{M}_2 = \frac{T^{p-1} \lambda_1}{|\Lambda|}, \quad (24)$$

$$\overline{M}_3 = \frac{1}{\Gamma(q+1)} \left(T^q + \frac{T^{q-1}}{|\Phi|} \sum_{l=1}^{n_2} \frac{|\omega_l| \theta_l^q}{q^{\nu_l}} + \frac{T^{q-1}}{|\Phi|} \sum_{k=1}^{m_2} \frac{|\tau_k| \gamma_k^q}{q^{\sigma_k}} \right), \quad (25)$$

$$\overline{M}_4 = \frac{T^{q-1} \lambda_2}{|\Phi|}. \quad (26)$$

Now we present the existence and uniqueness result for the problem (19). We do not provide the proof of this result as it is similar to the one for Theorem 3.1.

Theorem 4.2 Assume that $f, g : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous functions and there exist constants $\overline{K}_i, \overline{L}_i, i = 1, 2$ such that for all $t \in [0, T]$ and $u_i, v_i \in \mathbb{R}, i = 1, 2$,

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq \overline{K}_1 |u_1 - v_1| + \overline{K}_2 |u_2 - v_2|$$

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and

$$|g(t, u_1, u_2) - g(t, v_1, v_2)| \leq \bar{L}_1 |u_1 - v_1| + \bar{L}_2 |u_2 - v_2|.$$

In addition, assume that

$$\bar{M}_1(\bar{K}_1 + \bar{K}_2) + \bar{M}_3(\bar{L}_1 + \bar{L}_2) < 1,$$

where \bar{M}_1 and \bar{M}_3 are given by (23) and (25) respectively. Then the boundary value problem (19) has a unique solution.

Example 4.3 Consider the following system of coupled Riemann-Liouville fractional differential equations with uncoupled Hadamard type fractional integral boundary conditions

$$\left\{ \begin{array}{l} {}_{RL}D^{e/2}x(t) = \frac{\cos(\pi t)}{(\pi^t + 4)^2} \frac{|x(t)|}{|x(t)| + 2} + \frac{3e^{t/2}}{(t+5)^3} \frac{|y(t)|}{|y(t)| + 3} + \frac{2}{e}, \quad t \in [0, 4], \\ {}_{RL}D^{\sqrt{3}}y(t) = \frac{\sin x(t)}{15(e^t + 3)} + \frac{2\sqrt{|y(t)| + 1}}{7\pi(t+3)} + 5, \quad t \in [0, 4], \\ x(0) = 0, \quad \sqrt{11} {}_H I^{5/2}x(2/3) + \frac{\tan^2(5)}{20} {}_H I^{10/3}x(\pi) = \frac{5}{e} {}_H I^{3/7}x(e) \\ \quad - \frac{7}{6} {}_H I^{\sqrt{5}}x(\sqrt{2}) + \frac{\pi}{2} {}_H I^{2/5}x(12/7) + 11, \\ y(0) = 0, \quad \frac{\log(15)}{9} {}_H I^{7/4}y(1/4) + 2 {}_H I^{5/6}y(\sqrt{7}) \\ \quad = \frac{\pi^2}{15} {}_H I^{4/3}y(1/e) + \sqrt{5} {}_H I^{9/7}y(7/2) + \sqrt{8}/3. \end{array} \right. \quad (27)$$

Here $p = e/2$, $q = \sqrt{3}$, $T = 4$, $\lambda_1 = 11$, $\lambda_2 = \sqrt{8}/3$, $m_1 = 2$, $n_1 = 3$, $m_2 = 2$, $n_2 = 2$, $\mu_1 = \sqrt{11}$, $\mu_2 = \tan^2(5)/20$, $\alpha_1 = 5/2$, $\alpha_2 = 10/3$, $\eta_1 = 2/3$, $\eta_2 = \pi$, $\delta_1 = 5/e$, $\delta_2 = -7/6$, $\delta_3 = \pi/2$, $\beta_1 = 3/7$, $\beta_2 = \sqrt{5}$, $\beta_3 = 2/5$, $\xi_1 = e$, $\xi_2 = \sqrt{2}$, $\xi_3 = 12/7$, $\tau_1 = \log(15)/9$, $\tau_2 = 2$, $\sigma_1 = 7/4$, $\sigma_2 = 5/6$, $\gamma_1 = 1/4$, $\gamma_2 = \sqrt{7}$, $\omega_1 = \pi^2/15$, $\omega_2 = \sqrt{5}$, $\nu_1 = 4/3$, $\nu_2 = 9/7$, $\theta_1 = 1/e$, $\theta_2 = 7/2$, $f(t, x, y) = (\cos(\pi t)|x|)/(((\pi^t + 4)^2)(|x| + 2)) + (3e^{t/2}|y|)/(((t+5)^3)(|y| + 3)) + (2/e)$ and $g(t, x, y) = (\sin x(t))/(15(e^t + 3)) + (2\sqrt{|y| + 1})/(7\pi(t+3)) + 5$. Since $|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq ((1/50)|x_1 - x_2| + (e^2/125)|y_1 - y_2|)$ and $|g(t, x_1, y_1) - g(t, x_2, y_2)| \leq ((1/60)|x_1 - x_2| + (1/(21\pi))|y_1 - y_2|)$. By using the Maple program, we can find

$$\Lambda := \sum_{i=1}^{m_1} \frac{\mu_i \eta_i^{p-1}}{(p-1)^{\alpha_i}} - \sum_{j=1}^{n_1} \frac{\delta_j \xi_j^{p-1}}{(p-1)^{\beta_j}} \approx 69.35947949 \neq 0$$

and

$$\Phi = \sum_{k=1}^{m_2} \frac{\tau_k \gamma_k^{q-1}}{(q-1)^{\sigma_k}} - \sum_{l=1}^{n_2} \frac{\omega_l \theta_l^{q-1}}{(q-1)^{\nu_l}} \approx -3.358717154 \neq 0.$$

With the given values, it is found that $\bar{K}_1 = 1/50$, $\bar{K}_2 = e^2/125$, $\bar{L}_1 = 1/60$, $\bar{L}_2 = 1/(21\pi)$, $\bar{M}_1 \simeq 5.673444294$, $\bar{M}_3 \simeq 15.54186374$. In consequence,

$$\bar{M}_1(\bar{K}_1 + \bar{K}_2) + \bar{M}_3(\bar{L}_1 + \bar{L}_2) \approx 0.9434486991 < 1.$$

Thus all the conditions of Theorem 4.2 are satisfied. Therefore, there exists a unique solution for the problem (27) on $[0, 4]$.

The second result dealing with the existence of solutions for the problem (19) is analogous to Theorem 3.3 and is given below.

Theorem 4.4 Assume that there exist real constants \bar{k}_i , $\bar{l}_i \geq 0$ ($i = 1, 2$) and $\bar{k}_0 > 0, \bar{l}_0 > 0$ such that $\forall x_i \in \mathbb{R}$, ($i = 1, 2$) we have

$$\begin{aligned} |f(t, x_1, x_2)| &\leq \bar{k}_0 + \bar{k}_1 |x_1| + \bar{k}_2 |x_2|, \\ |g(t, x_1, x_2)| &\leq \bar{l}_0 + \bar{l}_1 |x_1| + \bar{l}_2 |x_2|. \end{aligned}$$

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In addition it is assumed that

$$\bar{k}_1 \bar{M}_1 + \bar{l}_1 \bar{M}_3 < 1 \quad \text{and} \quad \bar{k}_2 \bar{M}_1 + \bar{l}_2 \bar{M}_3 < 1,$$

where \bar{M}_1 and \bar{M}_3 are given by (23) and (25) respectively. Then the boundary value problem (19) has at least one solution.

Proof. Setting

$$\bar{M}_0 = \min\{1 - \bar{k}_1 \bar{M}_1 - \bar{l}_1 \bar{M}_3, 1 - \bar{k}_2 \bar{M}_1 - \bar{l}_2 \bar{M}_3\}, \quad \bar{k}_i, \bar{l}_i \geq 0 \quad (i = 1, 2),$$

the proof is similar to that of Theorem 3.3. So we omit it. \square

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Ternary Jordan ring derivations on Banach ternary algebras: A fixed point approach

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Abstract. Let A be a Banach ternary algebra. An additive mapping $D : (A, [\]) \rightarrow (A, [\])$ is called a ternary Jordan ring derivation if $D([xxx]) = [D(x)xx] + [xD(x)x] + [xxD(x)]$ for all $x \in A$.

In this paper, we prove the Hyers-Ulam stability of ternary Jordan ring derivations on Banach ternary algebras.

1. INTRODUCTION

We say that a functional equation (Q) is stable if any function g satisfying the equation (Q) approximately is near to true solution of (Q). Also, we say that a functional equation is superstable if every approximately solution is an exact solution of it.

Recently, Bavand Savadkouhi et al. [4] investigate the stability of ternary Jordan derivations on Banach ternary algebras by direct methods.

Ternary algebraic operations were considered in the 19th century by several mathematicians. Cayley [7] introduced the notion of cubic matrix, which in turn was generalized by Kapranov, Gelfand and Zelevinskii [17].

The comments on physical applications of ternary structures can be found in [3, 12, 13, 14, 22, 23, 26, 28, 31, 32].

Let A be a Banach ternary algebra. An additive mapping $D : (A, [\]) \rightarrow (A, [\])$ is called a ternary ring derivation if

$$D([xyz]) = [D(x)yz] + [xD(y)z] + [xyD(z)]$$

for all $x, y, z \in A$.

An additive mapping $D : (A, [\]) \rightarrow (A, [\])$ is called a ternary Jordan ring derivation if

$$D([xxx]) = [D(x)xx] + [xD(x)x] + [xxD(x)]$$

for all $x \in A$.

Theorem 1.1. ([11]) *Suppose that (Ω, d) is a complete generalized metric space and $T : \Omega \rightarrow \Omega$ is a strictly contractive mapping with the Lipschitz constant L . Then, for any $x \in \Omega$, either*

$$d(T^n x, T^{n+1} x) = \infty, \quad \forall n \geq 0,$$

or there exists a positive integer n_0 such that

- (1) $d(T^n x, T^{n+1} x) < \infty$ for all $n \geq n_0$;
- (2) the sequence $\{T^n x\}$ is convergent to a fixed point y^* of T ;
- (3) y^* is the unique fixed point of T in $\Lambda = \{y \in \Omega : d(T^{n_0} x, y) < \infty\}$;
- (4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Lambda$.

The study of stability problems originated from a famous talk given by Ulam [30] in 1940: "Under what condition does there exist a homomorphism near an approximate homomorphism?" In the next year 1941, Hyers [15] answered affirmatively the question of Ulam for additive mappings between Banach spaces. A generalized version of the theorem of Hyers for approximately additive mappings was given by Rassias [24] in 1978. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 5, 8, 10, 18, 19, 20, 21, 25, 27, 29, 33, 34]).

In this paper, we prove the Hyers-Ulam stability and superstability of ternary Jordan ring derivations on Banach ternary algebras by the fixed point method.

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2. HYERS-ULAM STABILITY OF TERNARY JORDAN RING DERIVATIONS

In this section, we prove the Hyers-Ulam stability of ternary Jordan ring derivations on Banach ternary algebras. Throughout this section, assume that A is a Banach ternary algebra.

Lemma 2.1. *Let $f : A \rightarrow A$ be an additive mapping. Then the following assertions are equivalent.*

$$f([a, a, a]) = [f(a), a, a] + [a, f(a), a] + [a, a, f(a)] \quad (2.1)$$

for all $a \in A$, and

$$\begin{aligned} f([a, b, c] + [b, c, a] + [c, a, b]) = & [f(a), b, c] + [a, f(b), c] + [a, b, f(c)] + [f(b), c, a] + [b, f(c), a] \\ & + [b, c, f(a)] + [f(c), a, b] + [c, f(a), b] + [c, a, f(b)] \end{aligned} \quad (2.2)$$

for all $a, b, c \in A$.

Proof. Replacing a by $a + b + c$ in (2.1), we have

$$\begin{aligned} f([(a + b + c), (a + b + c), (a + b + c)]) = & [f(a + b + c), (a + b + c), (a + b + c)] + [(a + b + c), f(a + b + c), (a + b + c)] \\ & + [(a + b + c), (a + b + c), f(a + b + c)] \end{aligned}$$

and so

$$\begin{aligned} & f([(a + b + c), (a + b + c), (a + b + c)]) \\ &= f([a, a, a] + [a, b, a] + [a, c, a] + [b, a, a] + [b, b, a] + [b, c, a] + [c, a, a] + [c, b, a] + [c, c, a] \\ &+ [a, a, b] + [a, b, b] + [a, c, b] + [b, a, b] + [b, b, b] + [b, c, b] + [c, a, b] + [c, b, b] + [c, c, b] \\ &+ [a, a, c] + [a, b, c] + [a, c, c] + [b, a, c] + [b, b, c] + [b, c, c] + [c, a, c] + [c, b, c] + [c, c, c]) \\ &= f([a, a, a]) + f([a, b, a]) + f([a, c, a]) + f([b, a, a]) + f([b, b, a]) + f([b, c, a]) + f([c, a, a]) + f([c, b, a]) + f([c, c, a]) \\ &+ f([a, a, b]) + f([a, b, b]) + f([a, c, b]) + f([b, a, b]) + f([b, b, b]) + f([b, c, b]) + f([c, a, b]) + f([c, b, b]) + f([c, c, b]) \\ &+ f([a, a, c]) + f([a, b, c]) + f([a, c, c]) + f([b, a, c]) + f([b, b, c]) + f([b, c, c]) + f([c, a, c]) + f([c, b, c]) + f([c, c, c]) \\ &= [f(a), a, a] + [a, f(a), a] + [a, a, f(a)] + [f(a), b, a] + [a, f(b), a] + [a, b, f(a)] + [f(a), c, a] + [a, f(c), a] + [a, c, f(a)] \\ &+ [f(b), a, a] + [b, f(a), a] + [b, a, f(a)] + [f(b), b, a] + [b, f(b), a] + [b, b, f(a)] + [f(b), c, a] + [b, f(c), a] + [b, c, f(a)] \\ &+ [f(c), a, a] + [c, f(a), a] + [c, a, f(a)] + [f(c), b, a] + [c, f(b), a] + [c, b, f(a)] + [f(c), c, a] + [c, f(c), a] + [c, c, f(a)] \\ &+ [f(a), a, b] + [a, f(a), b] + [a, a, f(b)] + [f(a), b, b] + [a, f(b), b] + [a, b, f(b)] + [f(a), c, b] + [a, f(c), b] + [a, c, f(b)] \\ &+ [f(b), a, b] + [b, f(a), b] + [b, a, f(b)] + [f(b), b, b] + [b, f(b), b] + [b, b, f(b)] + [f(b), c, b] + [b, f(c), b] + [b, c, f(b)] \\ &+ [f(c), a, b] + [c, f(a), b] + [c, a, f(b)] + [f(c), b, b] + [c, f(b), b] + [c, b, f(b)] + [f(c), c, b] + [c, f(c), b] + [c, c, f(b)] \\ &+ [f(a), a, c] + [a, f(a), c] + [a, a, f(c)] + [f(a), b, c] + [a, f(b), c] + [a, b, f(c)] + [f(a), c, c] + [a, f(c), c] + [a, c, f(c)] \\ &+ [f(b), a, c] + [b, f(a), c] + [b, a, f(c)] + [f(b), b, c] + [b, f(b), c] + [b, b, f(c)] + [f(b), c, c] + [b, f(c), c] + [b, c, f(c)] \\ &+ [f(c), a, c] + [c, f(a), c] + [c, a, f(c)] + [f(c), b, c] + [c, f(b), c] + [c, b, f(c)] + [f(c), c, c] + [c, f(c), c] + [c, c, f(c)] \end{aligned}$$

for all $a, b, c \in A$.

On the other hand, we have

$$\begin{aligned} & f([(a + b + c), (a + b + c), (a + b + c)]) \\ &= [f(a), a, a] + [f(a), a, b] + [f(a), a, c] + [f(a), b, a] + [f(a), b, b] + [f(a), b, c] + [f(a), c, a] + [f(a), c, b] + [f(a), c, c] \\ &+ [f(b), a, a] + [f(b), a, b] + [f(b), a, c] + [f(b), b, a] + [f(b), b, b] + [f(b), b, c] + [f(b), c, a] + [f(b), c, b] + [f(b), c, c] \\ &+ [f(c), a, a] + [f(c), a, b] + [f(c), a, c] + [f(c), b, a] + [f(c), b, b] + [f(c), b, c] + [f(c), c, a] + [f(c), c, b] + [f(c), c, c] \\ &+ [a, f(a), a] + [a, f(a), b] + [a, f(a), c] + [b, f(a), a] + [b, f(a), b] + [b, f(a), c] + [c, f(a), a] + [c, f(a), b] + [c, f(a), c] \\ &+ [a, f(b), a] + [a, f(b), b] + [a, f(b), c] + [b, f(b), a] + [b, f(b), b] + [b, f(b), c] + [c, f(b), a] + [c, f(b), b] + [c, f(b), c] \\ &+ [a, f(c), a] + [a, f(c), b] + [a, f(c), c] + [b, f(c), a] + [b, f(c), b] + [b, f(c), c] + [c, f(c), a] + [c, f(c), b] + [c, f(c), c] \\ &+ [a, a, f(a)] + [a, b, f(a)] + [a, c, f(a)] + [b, a, f(a)] + [b, b, f(a)] + [b, c, f(a)] + [c, a, f(a)] + [c, b, f(a)] + [c, c, f(a)] \\ &+ [a, a, f(b)] + [a, b, f(b)] + [a, c, f(b)] + [b, a, f(b)] + [b, b, f(b)] + [b, c, f(b)] + [c, a, f(b)] + [c, b, f(b)] + [c, c, f(b)] \\ &+ [a, a, f(c)] + [a, b, f(c)] + [a, c, f(c)] + [b, a, f(c)] + [b, b, f(c)] + [b, c, f(c)] + [c, a, f(c)] + [c, b, f(c)] + [c, c, f(c)] \end{aligned}$$

for all $a, b, c \in A$.

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It follows that

$$([f(b), c, a] + [b, f(c), a] + [b, c, f(a)]) + ([f(c), a, b] + [c, f(a), b] + [c, a, f(b)]) + ([f(a), b, c] + [a, f(b), c] + [a, b, f(c)]) \\ = f([b, c, a]) + f([c, a, b]) + f([a, b, c]) = f([b, c, a] + [c, a, b] + [a, b, c]) = f([a, b, c] + [b, c, a] + [c, a, b])$$

and

$$[f(a), b, c] + [f(b), c, a] + [f(c), a, b] + [c, f(a), b] + [a, f(b), c] + [b, f(c), a] + [b, c, f(a)] + [c, a, f(b)] + [a, b, f(c)] \\ = ([f(a), b, c] + [a, f(b), c] + [a, b, f(c)]) + ([f(b), c, a] + [b, f(c), a] + [b, c, f(a)]) + ([f(c), a, b] + [c, f(a), b] + [c, a, f(b)])$$

for all $a, b, c \in A$. Then

$$f([a, b, c] + [b, c, a] + [c, a, b]) = ([f(a), b, c] + [a, f(b), c] + [a, b, f(c)]) + ([f(b), c, a] + [b, f(c), a] \\ + [b, c, f(a)]) + ([f(c), a, b] + [c, f(a), b] + [c, a, f(b)])$$

for all $a, b, c \in A$. Hence (2.2) holds true.

For the converse, replacing b and c by a in (2.2), we have

$$f([a, a, a] + [a, a, a] + [a, a, a]) = [f(a), a, a] + [a, f(a), a] + [a, a, f(a)] + [f(a), a, a] + [a, f(a), a] \\ + [a, a, f(a)] + [f(a), a, a] + [a, f(a), a] + [a, a, f(a)]$$

and so

$$f(3[a, a, a]) = 3([f(a), a, a] + [a, f(a), a] + [a, a, f(a)])$$

for all $a \in A$. Thus

$$f([a, a, a]) = [f(a), a, a] + [a, f(a), a] + [a, a, f(a)]$$

for all $a \in A$. This completes the proof. \square

Theorem 2.2. Let $f : A \rightarrow A$ be a mapping for which there exists function $\varphi : A \times A \times A \rightarrow [0, \infty)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \varphi(x, y, 0), \quad (2.3)$$

$$\|f([x, y, z] + [y, z, x] + [z, x, y]) - [f(x), y, z] - [x, f(y), z] - [x, y, f(z)] - [f(y), z, x] \\ - [y, f(z), x] - [y, z, f(x)] - [f(z), x, y] - [z, f(x), y] - [z, x, f(y)]\| \leq \varphi(x, y, z) \quad (2.4)$$

for all $x, y, z \in A$. If there exists a constant $0 < L < 1$ such that

$$\varphi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{8} \varphi(x, y, z) \quad (2.5)$$

for all $x, y, z \in A$, then there exists a unique ternary Jordan ring derivation $D : A \rightarrow A$ such that

$$\|f(x) - D(x)\| \leq \frac{L}{8-2L} \varphi(x, x, 0) \quad (2.6)$$

for all $x \in A$.

Proof. It follows from (2.5) that

$$\lim_{n \rightarrow \infty} 2^{3n} \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0 \quad (2.7)$$

for all $x, y, z \in A$. By (2.5), $\varphi(0, 0, 0) = 0$. Letting $x = y = 0$ in (2.3), we get $\|f(0)\| \leq \varphi(0, 0, 0) = 0$ and so $f(0) = 0$. Let $\Omega = \{g : A \rightarrow X, g(0) = 0\}$. We introduce a generalized metric on Ω as follows:

$$d(g, h) = d_\varphi(g, h) = \inf\{C \in (0, \infty) : \|g(x) - h(x)\| \leq C\varphi(x, x, 0), \forall x \in A\}$$

It is easy to show that (Ω, d) is a generalized complete metric space [16]. Now, we consider the mapping $T : \Omega \rightarrow \Omega$ defined by $Tg(x) = 2g(\frac{x}{2})$ for all $x \in A$ and $g \in \Omega$. Note that, for all $g, h \in \Omega$ and $x \in A$,

$$d(g, h) < C \Rightarrow \|g(x) - h(x)\| \leq C\varphi(x, x, 0) \\ \Rightarrow \|2g(\frac{x}{2}) - 2h(\frac{x}{2})\| \leq 2C\varphi(\frac{x}{2}, \frac{x}{2}, 0) \\ \Rightarrow \|2g(\frac{x}{2}) - 2h(\frac{x}{2})\| \leq \frac{L}{4} C\varphi(x, x, 0) \\ \Rightarrow d(Tg, Th) \leq \frac{L}{4} C.$$

Hence we obtain that

$$d(Tg, Th) \leq \frac{L}{4} d(g, h)$$

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for all $g, h \in \Omega$, that is, T is a strictly contractive mapping of Ω with the Lipschitz constant L . Putting $y = x$ in (2.3), we have

$$\|f(2x) - 2f(x)\| \leq \varphi(x, x, 0), \quad (2.8)$$

and so

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}, 0\right) \leq \frac{L}{8} \varphi(x, x, 0)$$

for all $x \in A$. Let us denote

$$D(x) = \lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right) \quad (2.9)$$

for all $x \in A$. By the result in ([2, 6]), D is an additive mapping and so it follows from the definition of D , (2.4) and (2.7) that

$$\begin{aligned} & \|D([x, y, z] + [y, z, x] + [z, x, y]) - [D(x), y, z] - [x, D(y), z] - [x, y, D(z)] - [D(y), z, x] - [y, D(z), x] \\ & - [y, z, D(x)] - [D(z), x, y] - [z, D(x), y] - [z, x, D(y)]\| \\ &= \lim_{n \rightarrow \infty} 8^n \|f\left(\left[\frac{x, y, z}{2^{3n}}\right] + \left[\frac{y, z, x}{2^{3n}}\right] + \left[\frac{z, x, y}{2^{3n}}\right]\right) - \left[f\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) - \left[\frac{x}{2^n}, f\left(\frac{y}{2^n}, \frac{z}{2^n}\right) - \left[\frac{x}{2^n}, \frac{y}{2^n}, f\left(\frac{z}{2^n}\right)\right] \right. \\ & \quad \left. - \left[f\left(\frac{y}{2^n}, \frac{z}{2^n}, \frac{x}{2^n}\right) - \left[\frac{y}{2^n}, f\left(\frac{z}{2^n}, \frac{x}{2^n}\right) - \left[\frac{y}{2^n}, \frac{z}{2^n}, f\left(\frac{x}{2^n}\right)\right] - \left[f\left(\frac{z}{2^n}, \frac{x}{2^n}, \frac{y}{2^n}\right) - \left[\frac{z}{2^n}, f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - \left[\frac{z}{2^n}, \frac{x}{2^n}, f\left(\frac{y}{2^n}\right)\right]\right]\right] \right\| \\ & \leq \lim_{n \rightarrow \infty} 8^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0 \end{aligned}$$

for all $x, y, z \in A$ and so $D([x, y, z] + [y, z, x] + [z, x, y]) = [D(x), y, z] + [x, D(y), z] + [x, y, D(z)] + [D(y), z, x] + [y, D(z), x] + [y, z, D(x)] + [D(z), x, y] + [z, D(x), y] + [z, x, D(y)]$, which implies that D is a ternary Jordan ring derivation, by Lemma 2.1. According to Theorem 1.1, since D is the unique fixed point of T in the set $\Lambda = \{g \in \Omega : d(f, g) < \infty\}$, D is the unique mapping such that

$$\|f(x) - D(x)\| \leq C \varphi(x, x, 0)$$

for all $x \in A$ and $C > 0$. By Theorem 1.1, we have

$$d(f, D) \leq \frac{1}{1 - \frac{L}{4}} d(f, Tf) \leq \frac{4L}{8(4-L)}$$

and so

$$\|f(x) - D(x)\| \leq \frac{L}{8-2L} \varphi(x, x, 0)$$

for all $x \in A$. This completes the proof. \square

Corollary 2.3. Let θ, r be nonnegative real numbers with $r > 1$. Suppose that $f : A \rightarrow A$ is a mapping such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^r + \|y\|^r), \quad (2.10)$$

$$\begin{aligned} & \|f([x, y, z] + [y, z, x] + [z, x, y]) - [f(x), y, z] - [x, f(y), z] - [x, y, f(z)] - [f(y), z, x] \\ & - [y, f(z), x] - [y, z, f(x)] - [f(z), x, y] - [z, f(x), y] - [z, x, f(y)]\| \leq \theta(\|x\|^r + \|y\|^r + \|z\|^r) \end{aligned} \quad (2.11)$$

for all $x, y, z \in A$. Then there exists a unique ternary Jordan ring derivation $D : A \rightarrow A$ satisfying

$$\|f(x) - D(x)\| \leq \frac{\theta}{2^{r+1} - 1} \|x\|^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.2 by taking

$$\varphi(x, y, z) := \theta(\|x\|^r + \|y\|^r + \|z\|^r)$$

for all $x, y, z \in A$. Then we can choose $L = 2^{1-r}$ and so we obtain the desired conclusion. \square

Remark 2.4. Let $f : A \rightarrow A$ be a mapping with $f(0) = 0$ such that there exists a function $\varphi : A \times A \times A \rightarrow [0, \infty)$ satisfying (2.3) and (2.4). Let $0 < L < 1$ be a constant such that

$$\varphi(2x, 2y, 2z) \leq 2L\varphi(x, y, z)$$

for all $x, y, z \in A$. By a similar method as in the proof of Theorem 2.2, one can show that there exists a unique ternary Jordan ring derivation $D : A \rightarrow A$ satisfying

$$\|f(x) - D(x)\| \leq \frac{2}{4-L} \varphi(x, x, 0)$$

for all $x \in A$. For the case

$$\varphi(x, y, z) := \delta + \theta(\|x\|^r + \|y\|^r + \|z\|^r),$$

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(where θ, δ are nonnegative real numbers and $0 < r < 1$, there exists a unique ternary Jordan ring derivation $D : A \rightarrow X$ satisfying

$$\|f(x) - D(x)\| \leq \frac{4\delta}{8-2^r} + \frac{8\theta}{8-2^r} \|x\|^r$$

for all $x \in A$.

Now, we formulate a theorem for the superstability of ternary Jordan ring derivations.

Theorem 2.5. Suppose that there exist a function $\varphi : A \times A \times A \rightarrow [0, \infty)$ and a constant $0 < L < 1$ such that

$$\psi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{8} \varphi(x, y, z)$$

for all $x, y, z \in A$. Moreover, if $f : A \rightarrow A$ is an additive mapping such that

$$\begin{aligned} & \|f([x, y, z] + [y, z, x] + [z, x, y]) - [f(x), y, z] - [x, f(y), z] - [x, y, f(z)] - [f(y), z, x] \\ & - [y, f(z), x] - [y, z, f(x)] - [f(z), x, y] - [z, f(x), y] - [z, x, f(y)]\| \leq \varphi(x, y, z) \end{aligned}$$

for all $x, y, z \in A$, then f is a ternary Jordan ring derivation.

Proof. The proof is similar to the proof of Theorem 2.2. We will omit it. \square

Corollary 2.6. Let θ, s be nonnegative real numbers and $s > 3$. If $f : A \rightarrow A$ is an additive mapping such that

$$\begin{aligned} & \|f([x, y, z] + [y, z, x] + [z, x, y]) - [f(x), y, z] - [x, f(y), z] - [x, y, f(z)] - [f(y), z, x] \\ & - [y, f(z), x] - [y, z, f(x)] - [f(z), x, y] - [z, f(x), y] - [z, x, f(y)]\| \leq \theta(\|x\|^s + \|y\|^s + \|z\|^s) \end{aligned}$$

for all $x, y, z \in A$, then f is a ternary Jordan ring derivation.

Remark 2.7. Suppose that there exist a function $\psi : A \times A \times A \rightarrow [0, \infty)$ and a constant $0 < L < 1$ such that

$$\varphi(2x, 2y, 2z) \leq 2L\varphi(x, y, z)$$

for all $x, y, z \in A$. Moreover, if $f : A \rightarrow A$ is an additive mapping such that

$$\begin{aligned} & \|f([x, y, z] + [y, z, x] + [z, x, y]) - [f(x), y, z] - [x, f(y), z] - [x, y, f(z)] - [f(y), z, x] \\ & - [y, f(z), x] - [y, z, f(x)] - [f(z), x, y] - [z, f(x), y] - [z, x, f(y)]\| \leq \varphi(x, y, z) \end{aligned}$$

for all $x, y, z \in A$, then f is a ternary Jordan ring derivation.

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Initial value problems for a nonlinear integro-differential equation of mixed type in Banach spaces*

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Abstract

In this paper, we discuss the following initial value problem for first order nonlinear integro-differential equations of mixed type in a Banach space:

$$\begin{cases} u' = f(t, u, Tu, Su) \\ u(t_0) = u_0. \end{cases}$$

In the case of the integral kernel $k(t, s)$ of the operator $(Tu)(t) = \int_{t_0}^t k(t, s)u(s)ds$ being unbounded, we obtain the existence of maximal and minimal solutions for the above problem by establishing a new comparison theorem.

Keywords: noncompactness measure, unbounded integral kernel, maximal and minimal solutions, integro-differential equations.

1 Introduction and Preliminaries

Suppose that E is a Banach space. In this paper, We consider the following initial value problem for first order nonlinear integro-differential equations of mixed type in E :

$$\begin{cases} u = f(t, u, Tu, Su) \\ u(t_0) = u_0, \end{cases} \quad (1.1)$$

where $f \in C[J \times E \times E \times E, E]$, $J = [t_0, t_0 + a]$ ($a > 0$), $u_0 \in E$, and

$$(Tu)(t) = \int_{t_0}^t k(t, s)u(s)ds, \quad (Su)(t) = \int_{t_0}^{t_0+a} h(t, s)u(s)ds. \quad (1.2)$$

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In (1.2), $k(t, s) = \frac{\rho(t, s)}{(t-s)^\alpha}$ ($0 < \alpha < 1$), $\rho(t, s) \in C[D, R^+]$, and $h(t, s) \in C[D_0, R^+]$, where $R^+ = [0, +\infty)$, $D = \{(t, s) \in R^2 | t_0 \leq s \leq t \leq t_0 + a\}$, $D_0 = \{(t, s) \in R^2 | (t, s) \in J \times J\}$. Here, $k(t, s)$ is unbounded on D , $\rho(t, s)$ is bounded on D , and $h(t, s)$ is bounded on D_0 . Set $R_0 = \max\{\rho(t, s) | (t, s) \in D\}$, $h_0 = \max\{h(t, s) | (t, s) \in D_0\}$.

The study of initial value problems for nonlinear integro-differential equations has been of great interest for many researchers for its physical backgrounds and applications in mathematical models. We refer the reader to [1, 5–12] and references therein for some recent results on equation (1.1). However, in many earlier results, the kernel $k(t, s)$ of the operator T is bounded. In this paper, we will make further study on the initial value problem (1.1) in the case of $k(t, s)$ being unbounded. By establishing a comparison theorem, we achieve an existence theorem about minimal and maximal solutions for equation (1.1).

Throughout the rest of this paper, let $(E, \|\cdot\|)$ be a real Banach space and P be a cone in E which defines a partial ordering in E denoted by " \leq ".

Suppose that E^* is the dual space of E , the dual cone of the cone P is $P^* = \{\varphi \in E^* | \varphi(x) \geq 0, \forall x \in P\}$. A cone $P \subset E$ is said to be normal there exists a constant $\gamma > 0$ such that

$$\theta \leq x \leq y \implies \|x\| \leq \gamma\|y\|, \forall x, y \in E.$$

The cone P is normal if and only if any order interval $[x, y] = \{z \in E | x \leq z \leq y\}$ is bounded in norm (see [3]). Set

$$C[J, E] = \left\{ u(t) : J \rightarrow E \mid u(t) \text{ is continuous on } J \right\},$$

$$C^1[J, E] = \left\{ u(t) : J \rightarrow E \mid u(t) \text{ and } u'(t) \text{ are continuous on } J \right\}.$$

Let $\|u\|_c = \max_{t \in J} \|u(t)\|$ be a norm for $u \in C[J, E]$, then $C[J, E]$ will be a Banach space with norm $\|\cdot\|_c$. It is easy to know $P_c = \{u \in C[J, E] | u(t) \geq \theta, \forall t \in J\}$ is a cone in $C[J, E]$. The cone P_c defines an ordering in $C[J, E]$ which also denoted by " \leq " here. Obviously, when the cone P is normal, P_c is a normal cone in $C[J, E]$.

Assume that V is a bounded set in E . The Kuratowski measure of noncompactness $\alpha(V)$ and the Hausdorff measure of noncompactness $\beta(V)$ are defined respectively as follow:

$$\alpha(V) = \inf\{\delta > 0 | V \text{ can be expressed as the union } S = \bigcup_{i=1}^m V_i \text{ of a finite number of sets } V_i \text{ with diameter } \text{diam}(V_i) \leq \delta\},$$

$$\beta(V) = \inf\left\{\delta > 0 \mid V \text{ can be covered by a finite number of closed balls } V_i \text{ with diameter } \text{diam}(V_i) \leq \delta\right\}.$$

The relationship of the two noncompactness measures is

$$\beta(V) \leq \alpha(V) \leq 2\beta(V). \quad (1.3)$$

For the basic properties of cones and noncompactness measures, we refer the reader to [2–4]. For convenience, the Kuratowski measure of noncompactness for bounded sets in E and $C[J, E]$ are all denoted by $\alpha(\cdot)$. In the sequel, we denote $B(t) = \{u(t) | u \in B\}$, $(TB)(t) = \{(Tu)(t) | u \in B\}$, $(SB)(t) = \{(Su)(t) | u \in B\}$ for all $B \subset C[J, E]$ with $t \in J$.

Lemma 1.1. *Let $m \in C^1[J, R^1]$ be such that*

$$m'(t) \geq -Mm(t) - N \int_{t_0}^t k(t, s)m(s)ds, \quad m(t_0) \geq 0, \quad t \in J, \quad (1.4)$$

where $M \geq 0$ and $N \geq 0$ are two constants satisfying one of the following conditions:

(i)

$$NR_0 e^{Ma} \frac{a^{2-\alpha}}{1-\alpha} \leq 1; \quad (1.5)$$

(ii)

$$aM + \frac{NR_0 a^{2-\alpha}}{1-\alpha} \leq 1. \quad (1.6)$$

Then $m(t) \geq 0$ for all $t \in J$.

Proof. **Case 1.** If the condition (i) is established, let $v(t) = m(t)e^{Mt}$. From (1.4), we have

$$v'(t) \geq -N \int_{t_0}^t k^*(t, s)v(s)ds, \quad \forall t \in J, \quad v(t_0) \geq 0, \quad (1.7)$$

where $k^*(t, s) = k(t, s)e^{M(t-s)}$. Now, we prove that

$$v(t) \geq 0, \quad \forall t \in J. \quad (1.8)$$

In fact, if there exists $t_0 \leq t_1 \leq t_0 + a$ such that $v(t_1) < 0$ and let $\max\{v(t) : t_0 \leq t \leq t_1\} = b$, then $b \geq 0$. If $b = 0$, then $v(t) \leq 0$ for all $t_0 \leq t \leq t_1$ and so (1.7) implies that

$$v'(t) \geq 0, \quad \forall t_0 \leq t \leq t_1.$$

Hence we have $v(t_1) \geq v(t_0) = m(t_0)e^{Mt_0} \geq 0$, which contradicts $v(t_1) < 0$.

If $b > 0$, then there exists $t_0 \leq t_2 < t_1$ such that $v(t_2) = b > 0$ and so there exists $t_2 < t_3 < t_1$ such that $v(t_3) = 0$. Then, by the mean value theorem, there exists $t_2 < t_4 < t_3$ such that

$$v'(t_4) = \frac{v(t_3) - v(t_2)}{t_3 - t_2} = \frac{-v(t_2)}{t_3 - t_2} = \frac{-b}{t_3 - t_2} < -\frac{b}{a}. \quad (1.9)$$

On the other hand, from (1.7), we have

$$v'(t_4) \geq -N \int_{t_0}^{t_4} k^*(t_4, s)v(s)ds$$

$$\begin{aligned}
&\geq -Nb \int_{t_0}^{t_4} k^*(t_4, s) ds \\
&= -Nb \int_{t_0}^{t_4} \frac{\rho(t_4, s)}{(t_4 - s)^\alpha} e^{M(t_4 - s)} ds \\
&\geq -NbR_0 \int_{t_0}^{t_4} (t_4 - s)^{-\alpha} e^{M(t_4 - s)} ds \\
&\geq -NbR_0 e^{Ma} \int_{t_0}^{t_4} (t_4 - s)^{-\alpha} ds \\
&= -NbR_0 e^{Ma} \frac{(t_4 - t_0)^{1-\alpha}}{1-\alpha} \\
&\geq -NbR_0 e^{Ma} \frac{a^{1-\alpha}}{1-\alpha}.
\end{aligned}$$

Then from (1.9), we have $NbR_0 e^{Ma} \frac{a^{2-\alpha}}{1-\alpha} > 1$ which contradicts (1.5). Therefore, (1.8) is true and so $m(t) \geq 0$ for all $t \in J$.

Case 2. If the assumption (ii) holds, but the conclusion does not hold, then there exists $t_1 \in (t_0, t_0 + a]$ such that

$$m(t_1) = \min_{t \in J} m(t) < 0,$$

and so $m'(t_1) \leq 0$. If $\max_{t_0 \leq t \leq t_1} m(t) \leq 0$, from (1.4), we have

$$0 \geq m'(t_1) \geq -Mm(t_1) - N \int_{t_0}^{t_1} k(t_1, s)m(s)ds \geq -Mm(t_1) > 0,$$

which is a contradictory statement. Therefore, there exists $t_2 \in [t_0, t_1)$ such that $m(t_2) = \max_{t_0 \leq t \leq t_1} m(t) = \mu > 0$. Then, by the mean value theorem, there exists $t_3 \in (t_2, t_1)$ such that

$$m'(t_3) = \frac{m(t_1) - m(t_2)}{t_1 - t_2} < -\frac{\mu}{a}.$$

It follows from (1.4) that

$$\begin{aligned}
-\frac{\mu}{a} > m'(t_3) &\geq -Mm(t_3) - N \int_{t_0}^{t_3} \frac{\rho(t_3, s)}{(t_3 - s)^\alpha} m(s) ds \\
&\geq -M\mu - NR_0\mu \int_{t_0}^{t_3} \frac{1}{(t_3 - s)^\alpha} ds \\
&= -M\mu - NR_0\mu \frac{(t_3 - t_0)^{1-\alpha}}{1-\alpha} \\
&\geq -M\mu - NR_0\mu \frac{a^{1-\alpha}}{1-\alpha},
\end{aligned}$$

i.e. $aM + NR_0 \frac{a^{2-\alpha}}{1-\alpha} > 1$ which contradicts (1.6). The Lemma is proved. \square

Lemma 1.2. Let $m \in C[J, R^+]$ be such that

$$m(t) \leq M_1 \int_{t_0}^t m(s)ds + M_2(t - t_0) \int_{t_0}^{t_0+a} m(s)ds, \quad t \in J \quad (1.10)$$

where $M_1 > 0$, $M_2 \geq 0$, are constants for satisfying one of the following conditions: (i) $aM_2(e^{aM_1} - 1) < M_1$, (ii) $a(2M_1 + aM_2) < 2$. Then $m(t) \equiv 0$, $t \in J$.

Proof. Case 1. If the condition (i) holds, letting $v(t) = m(t)e^{Mt}$, then $m_1(t_0) = 0$, $m'_1(t) = m(t)$, $t \in J$. If $m_1(t_0 + a) \neq 0$, it follows from (1.10) that

$$m'_1(t) \leq M_1 m_1(t) + aM_2 m_1(t_0 + a), \quad t \in J$$

and from $e^{-M_1(t-t_0)} > 0$ we have

$$\left(m_1(t)e^{-M_1(t-t_0)} \right)' \leq aM_2 m_1(t_0 + a)e^{-M_1(t-t_0)}, \quad t \in J.$$

Now, we integrate the above inequality between t_0 and t with noticing $m_1(t_0) = 0$, we can obtain

$$\begin{aligned} m_1(t)e^{-M_1(t-t_0)} &\leq aM_2 m_1(t_0 + a) \int_{t_0}^t e^{-M_1(s-t_0)} ds \\ &\leq \frac{aM_2}{M_1} m_1(t_0 + a) \left(1 - e^{-M_1(t-t_0)} \right), \quad t \in J. \end{aligned}$$

By choosing $t = t_0 + a$, we can get

$$aM_2(e^{aM_1} - 1) \geq M_1$$

which contradicts (i). Consequently, $m_1(t_0 + a) = \int_{t_0}^{t_0+a} m(s)ds = 0$ which implies $m(t) \equiv 0$, $t \in J$.

Case 2. If the condition (ii) is established, it follows from (1.10) that

$$m(t) \leq [M_1 + M_2(t - t_0)] \int_{t_0}^{t_0+a} m(s)ds.$$

Integrating the above inequality between t_0 and $t_0 + a$, we get

$$\int_{t_0}^{t_0+a} m(t)dt \leq \left[aM_1 + \frac{a^2 M_2}{2} \right] \int_{t_0}^{t_0+a} m(s)ds.$$

From the above inequality and condition (ii), it follows that $\int_{t_0}^{t_0+a} m(t)dt = 0$, so $m(t) \equiv 0$, $t \in J$. This completes the proof. \square

Lemma 1.3. If B is a equicontinuous bounded set in $C[J, E]$, then $\alpha(B) = \max_{t \in J} \alpha(B(t))$.

Lemma 1.4. *If B is a equicontinuous bounded set in $C[J, E]$ with $J = [a, b]$, then $\alpha(\{u(t)|u \in B\})$ is continuous with respect to $t \in J$ and*

$$\alpha\left(\left\{\int_a^b u(t)dt \middle| u \in B\right\}\right) \leq \int_a^b \alpha\{u(t)|u \in B\} dt.$$

Lemma 1.5. *(see [2]) Let E be a separable Banach space, $J = [a, b]$ and $\{u_n\} : J \rightarrow E$ be continuous abstract function sequences. If there exists a function $\phi \in L[a, b]$ such that $\|u_n(t)\| \leq \phi(t)$, $t \in J$, $n = 1, 2, 3, \dots$, then $\beta(\{u_n(t)|n = 1, 2, 3, \dots\})$ is integrable on J and*

$$\beta\left(\left\{\int_a^b u_n(t)dt \middle| n = 1, 2, 3, \dots\right\}\right) \leq \int_a^b \beta(\{u_n(t)|n = 1, 2, 3, \dots\}) dt.$$

Now, we give our assumptions:

(H_1) There exist $v_0, \omega_0 \in C^1[J, E]$ such that $v_0(t) \leq \omega_0(t)$ ($t \in J$) and v_0, ω_0 are a lower solution and an upper solution respectively for the initial value problem (1.1), that is

$$v'_0 \leq f(t, v_0, Tv_0, Sv_0), \quad \forall t \in J; \quad v_0(t_0) \leq u_0,$$

$$\omega'_0 \geq f(t, \omega_0, T\omega_0, S\omega_0), \quad \forall t \in J; \quad \omega_0(t_0) \geq u_0.$$

(H_2) For any $t \in J$, any $u, v \in [v_0, \omega_0] = \{u \in C[J, E] | v_0 \leq u \leq \omega_0\}$ and $u \leq v$, we have

$$f(t, v, Tv, Sv) - f(t, u, Tu, Su) \geq -M(v - u) - NT(v - u),$$

where $M > 0$, $N \geq 0$ are constants satisfying the condition (i) or (ii) in Lemma 1.1.

(H_3) For any $t \in J$ and equicontinuous bounded monotone sequences $B = \{u_n\} \subset [v_0, \omega_0]$, we have

$$\alpha(f(t, B(t), (TB)(t), (SB)(t))) \leq c_1\alpha(B(t)) + c_2\alpha((TB)(t)) + c_3\alpha((SB)(t)),$$

where c_i ($i = 1, 2, 3$) are constants satisfying one of the following two conditions:

$$(i) ah_0c_3 \left(e^{2a(c_1+M+\frac{c_2R_0a^{1-\alpha}}{1-\alpha}+\frac{2NR_0a^{1-\alpha}}{1-\alpha})} - 1 \right) < c_1 + M + \frac{c_2R_0a^{1-\alpha}}{1-\alpha} + \frac{2NR_0a^{1-\alpha}}{1-\alpha};$$

$$(ii) a \left(2c_1 + 2M + \frac{2c_2R_0a^{1-\alpha}}{1-\alpha} + \frac{4NR_0a^{1-\alpha}}{1-\alpha} + ah_0c_3 \right) < 1.$$

2 Main results

Theorem 2.1. *Let E be a real Banach space, $P \subset E$ be a normal cone and the conditions (H_1), (H_2), (H_3) be satisfied. Then the initial value problem (1.1) has a minimal solution and a maximal solution $\bar{u}, u^* \in C^1[J, E]$ in $[v_0, \omega_0]$, and for the initial value v_0 and ω_0 , the iterative sequences $\{v_n(t)\}$ and $\{\omega_n(t)\}$ defined by the following formulas converge uniformly to $\bar{u}(t), u^*(t)$ on J according to the norm in E respectively:*

$$v_n(t) = u_0 e^{-M(t-t_0)} + \int_{t_0}^t e^{M(s-t)} [f(s, v_{n-1}(s), (Tv_{n-1})(s), (Sv_{n-1})(s))$$

$$+Mv_{n-1}(s) - NT(v_n - v_{n-1})(s)]ds, \quad \forall t \in J, \quad (2.1)$$

$$\begin{aligned} \omega_n(t) = & u_0 e^{-M(t-t_0)} + \int_{t_0}^t e^{M(s-t)} [f(s, \omega_{n-1}(s), (T\omega_{n-1})(s), (S\omega_{n-1})(s)) \\ & + M\omega_{n-1}(s) - NT(\omega_n - \omega_{n-1})(s)]ds, \quad \forall t \in J \quad (n = 1, 2, 3, \dots). \end{aligned} \quad (2.2)$$

Moreover, there holds

$$v_0 \leq v_1 \leq \dots \leq v_n \leq \dots \leq \bar{u} \leq u^* \leq \dots \leq \omega_1 \leq \omega_0. \quad (2.3)$$

Proof. For any $\eta \in [v_0, \omega_0]$, we consider the initial value problem of linear integro-differential equation in Banach space E :

$$u' = g(t) - Mu - NTu, \quad u(t_0) = u_0, \quad (2.4)$$

where $g(t) = f(t, \eta(t), (T\eta)(t), (S\eta)(t)) + M\eta(t) + N(T\eta)(t)$. It is easy to show that u is a solution of the linear initial value problem (2.4) if and only if u is the fixed point in $C[J, E]$ of the following operator

$$(Au)(t) = u_0 e^{-M(t-t_0)} + \int_{t_0}^t e^{M(s-t)} [g(s) - N(Tu)(s)]ds. \quad (2.5)$$

In the following, we will prove there exists n_0 such that A^{n_0} is a contraction operator. For any $u, v \in C[J, E]$, $t \in J$, it follows from (2.5) that

$$\begin{aligned} \|(Au)(t) - (Av)(t)\| & \leq N \int_{t_0}^t \|T(u-v)(s)\|ds \\ & \leq N \int_{t_0}^t \left[\int_{t_0}^s k(s, \tau) \|u(\tau) - v(\tau)\|d\tau \right] ds \\ & = N \int_{t_0}^t \left[\int_{t_0}^s \frac{\rho(s, \tau)}{(s-\tau)^\alpha} \|u(\tau) - v(\tau)\|d\tau \right] ds \\ & \leq NR_0 \|u - v\|_c \int_{t_0}^t \int_{t_0}^s \frac{1}{(s-\tau)^\alpha} d\tau ds \\ & = \frac{NR_0(t-t_0)^{2-\alpha}}{(1-\alpha)(2-\alpha)} \|u - v\|_c. \end{aligned} \quad (2.6)$$

In the same way, by (2.5) and (2.6), we have

$$\begin{aligned} \|(A^2u)(t) - (A^2v)(t)\| & \leq N \int_{t_0}^t \|T(Au - Av)(s)\|ds \\ & \leq N \int_{t_0}^t \left[\int_{t_0}^s k(s, \tau) \|(Au)(\tau) - (Av)(\tau)\|d\tau \right] ds \end{aligned}$$

$$\begin{aligned}
&\leq NR_0 \int_{t_0}^t \left[\int_{t_0}^s \frac{1}{(s-\tau)^\alpha} \frac{NR_0(\tau-t_0)^{2-\alpha}}{(1-\alpha)(2-\alpha)} \|u-v\|_c d\tau \right] ds \\
&= \frac{(NR_0)^2}{(1-\alpha)(2-\alpha)} \|u-v\|_c \int_{t_0}^t \left[\int_{t_0}^s \frac{(\tau-t_0)^{2-\alpha}}{(s-\tau)^\alpha} d\tau \right] ds \\
&\leq \frac{(NR_0)^2 \|u-v\|_c}{(1-\alpha)(2-\alpha)} \int_{t_0}^t \int_{t_0}^s (s-\tau)^{-\alpha} (s-t_0)^{2-\alpha} d\tau ds \\
&= \frac{(NR_0)^2 \|u-v\|_c}{(1-\alpha)(2-\alpha)} \int_{t_0}^t \frac{(s-t_0)^{3-2\alpha}}{1-\alpha} ds \\
&= \frac{(NR_0)^2}{(1-\alpha)^2 2(2-\alpha)^2} \|u-v\|_c (t-t_0)^{4-2\alpha}.
\end{aligned}$$

It is easy to prove that by mathematical induction

$$\|(A^n u)(t) - (A^n v)(t)\| \leq \frac{(NR_0)^n}{n![(1-\alpha)(2-\alpha)]^n} (t-t_0)^{n(2-\alpha)} \|u-v\|_c, \quad t \in J, \quad n = 1, 2, 3, \dots$$

Thus

$$\|A^n u - A^n v\|_c \leq \frac{(NR_0 a^{2-\alpha})^n}{n![(1-\alpha)(2-\alpha)]^n} \|u-v\|_c, \quad n = 1, 2, 3, \dots$$

We can choose $n_0 \in \{1, 2, 3, \dots\}$ such that $\frac{(NR_0 a^{2-\alpha})^{n_0}}{n_0![(1-\alpha)(2-\alpha)]^{n_0}} < 1$, and so A^{n_0} a contraction operator in $C[J, E]$. Therefore, it follows from the principle of contraction mapping that A^{n_0} , that is, A has a unique fixed point u_η in $C[J, E]$ which implies the linear initial value problem (2.4) has a unique solution u_η in $C[J, E]$. Now, we define a operator

$$B\eta = u_\eta \tag{2.7}$$

where u_η is a unique solution for η of the linear initial value problem (2.4), and satisfies

$$u'_\eta = f(t, \eta(t), (T\eta)(t), (S\eta)(t)) - M(u_\eta(t) - \eta(t)) - NT(u_\eta - \eta)(t), \quad u_\eta(t_0) = u_0.$$

Then $B : [v_0, \omega_0] \longrightarrow C[J, E]$, and the iterative sequences (2.1)(2.2) can be written

$$v_n = Bv_{n-1}, \quad \omega_n = B\omega_{n-1}, \quad n = 1, 2, 3, \dots \tag{2.8}$$

Moreover, we claim that the operator B defined by (2.7) satisfies

i)

$$v_0 \leq Bv_0, \quad B\omega_0 \leq \omega_0; \tag{2.9}$$

ii)

$$B\eta_1 \leq B\eta_2, \quad \forall \eta_1, \eta_2 \in [v_0, \omega_0], \quad \eta_1 \leq \eta_2. \tag{2.10}$$

Next, we will prove i) and ii). Firstly, we prove the result i). Set $v_1 = Bv_0$, it follows from the definition of B that

$$v'_1 = f(t, v_0, Tv_0, Sv_0) - M(v_1 - v_0) - NT(v_1 - v_0), \quad v_1(t_0) = u_0. \quad (2.11)$$

For any $\varphi \in P^*$, let $m(t) = \varphi(v_1(t) - v_0(t))$, it follows from (2.11) and the assumption (H_1) that

$$m'(t) \geq -Mm(t) - N \int_{t_0}^t k(t, s)m(s)ds, \quad m(t_0) \geq 0.$$

Thus, by lemma 1.1, it follows that $m(t) \geq 0$ for all $t \in J$, which implies $v_1(t) - v_0(t) \geq 0$ for all $t \in J$. It follows theorem 2.4.3 in [3] that $v_0 \leq Bv_0$. Similarly, we can prove that $B\omega_0 \leq \omega_0$. Consequently, the result i) is proved.

Next, we prove ii). Let $u_{\eta_1} = B\eta_1$, $u_{\eta_2} = B\eta_2$, it follows from the hypothesis (H_2) and the definition of B that

$$\begin{aligned} u'_{\eta_1} - u'_{\eta_2} &= f(t, \eta_2, T\eta_2, S\eta_2) - M(u_{\eta_2} - \eta_2) - NT(u_{\eta_2} - \eta_2) \\ &\quad - f(t, \eta_1, T\eta_1, S\eta_1) + M(u_{\eta_1} - \eta_1) + NT(u_{\eta_1} - \eta_1) \\ &\geq -M(u_{\eta_2} - u_{\eta_1}) - NT(u_{\eta_2} - u_{\eta_1}) \end{aligned} \quad (2.12)$$

and

$$u_{\eta_2}(t_0) - u_{\eta_1}(t_0) = u_0 - u_0 = \theta. \quad (2.13)$$

For any $\varphi \in P^*$, let $m(t) = \varphi(u_{\eta_2}(t) - u_{\eta_1}(t))$. From (2.12) and (2.13), it follows that

$$m'(t) \geq -Mm(t) - N \int_{t_0}^t k(t, s)m(s)ds, \quad m(t_0) = 0$$

Thus, by lemma 1.1, it follows that $m(t) \geq 0$ for all $t \in J$, which implies $u_{\eta_2}(t) - u_{\eta_1}(t) \geq \theta$, $t \in J$, that is, $B\eta_1 \leq B\eta_2$. So the result ii) is proved.

Form (2.8)-(2.10) and observing that $v_0 \leq \omega_0$, it follows that

$$v_0 \leq v_1 \leq \cdots \leq v_n \leq \cdots \leq \omega_n \leq \cdots \leq \omega_1 \leq \omega_0. \quad (2.14)$$

and B is a mapping with $[v_0, \omega_0]$ into $[v_0, \omega_0]$.

In the following, we prove that $\{v_n(t)\}$ converges uniformly to some element $\bar{u} \in C[J, E]$ in J . By the normality of P , the cone P_c is normal in $C[J, E]$ which implies the order interval $[v_0, \omega_0]$ is a bounded set in $C[J, E]$. Then, it follows from (2.14) that $\{v_n\}$ is a bounded set in $C[J, E]$. On the one hand, for any $\eta \in [v_0, \omega_0]$, by the conditions (H_1) and (H_2) , we have

$$\begin{aligned} v'_0 + Mv_0 + NTv_0 &\leq f(t, v_0, Tv_0, Sv_0) + Mv_0 + NTv_0 \\ &\leq f(t, \eta, T\eta, S\eta) + M\eta + NT\eta \\ &\leq f(t, \omega_0, T\omega_0, S\omega_0) + M\omega_0 + NT\omega_0 \\ &\leq \omega'_0 + M\omega_0 + NT\omega_0. \end{aligned}$$

Then, by the normality of P_c , the set $\{f(t, \eta, T\eta, S\eta) + M\eta + NT\eta | \eta \in [v_0, \omega_0]\}$ is a bounded set in $C[J, E]$. On the other hand, the set $\{T\eta | \eta \in [v_0, \omega_0]\}$ is also a bounded set in $C[J, E]$, because it follows from the boundedness of $[v_0, \omega_0]$ that for any $\eta \in [v_0, \omega_0]$,

$$\begin{aligned} \|T\eta(t)\| &\leq \int_{t_0}^t k(t, s) \|\eta(s)\| ds \\ &\leq \|\eta\|_c \int_{t_0}^t \frac{\rho(t, s)}{(t-s)^\alpha} ds \\ &\leq R_0 \|\eta\|_c \int_{t_0}^t \frac{1}{(t-s)^\alpha} ds \\ &= R_0 \|\eta\|_c \frac{(t-t_0)^{1-\alpha}}{1-\alpha}. \end{aligned}$$

Therefore, $\{f(t, \eta, T\eta, S\eta) | \eta \in [v_0, \omega_0]\}$ is a bounded set in $C[J, E]$. Thus, from

$$v'_n = f(t, v_{n-1}, Tv_{n-1}, Sv_{n-1}) - M(v_n - v_{n-1}) - NT(v_n - v_{n-1}), t \in J, n = 1, 2, 3, \dots, \quad (2.15)$$

it follows that $\{v'_n | n = 1, 2, 3, \dots\}$ is a bounded set in $C[J, E]$. Applying the mean value theorem, we see that all the functions $\{v_n(t) | n = 1, 2, 3, \dots\}$ is equicontinuous on J . From Lemma 1.3, we have

$$\alpha(\{v_n | n = 1, 2, 3, \dots\}) = \max_{t \in J} \alpha(\{v_n(t) | n = 1, 2, 3, \dots\}). \quad (2.16)$$

Now, we prove $\alpha(\{v_n | n = 1, 2, 3, \dots\}) = 0$. From (2.4), (2.5), (2.7) and (2.8), it follows that

$$\begin{aligned} v_n(t) &= u_0 e^{-M(t-t_0)} + \int_{t_0}^t e^{M(s-t)} [f(s, v_{n-1}(s), (Tv_{n-1})(s), (Sv_{n-1})(s)) \\ &\quad + Mv_{n-1}(s) - NT(v_n - v_{n-1})(s)] ds. \end{aligned} \quad (2.17)$$

Let $m(t) = \alpha(\{v_n(t) | n = 1, 2, 3, \dots\})$, then $m(t_0) = \alpha(\{u_0\}) = 0$, $m \in C[J, R^+]$. For every n , by the continuity of $v_n(t)$, $\{v_n(t) | t \in J\}$ is a separable set in E , so $\{v_n(t) | t \in J, n = 1, 2, 3, \dots\}$ is a separable set in E . Thus, we can assume that E is a separable Banach space without loss of generality (otherwise, the closed subspace in E is spanned by $\{v_n(t) | t \in J, n = 1, 2, 3, \dots\}$ can be used in place of E). By (2.17), (1.3) and Lemma 1.5 and observing $0 < e^{M(s-t)} \leq 1$, $(t, s) \in D$, we can obtain

$$\begin{aligned} m(t) &\leq \alpha \left(\int_{t_0}^t e^{M(s-t)} [f(s, B(s), (TB)(s), (SB)(s)) + MB(s) - NT(B_1 - B)(s)] ds \right) \\ &\leq 2\beta \left(\int_{t_0}^t e^{M(s-t)} [f(s, B(s), (TB)(s), (SB)(s)) + MB(s) - NT(B_1 - B)(s)] ds \right) \\ &\leq 2 \int_{t_0}^t \beta [f(s, B(s), (TB)(s), (SB)(s)) + MB(s) - NT(B_1 - B)(s)] ds \end{aligned}$$

$$\begin{aligned} &\leq 2 \int_{t_0}^t [\beta(f(s, B(s), (TB)(s), (SB)(s))) \\ &\quad + M\beta(B(s)) + N\beta(T(B_1 - B)(s))] ds. \end{aligned} \quad (2.18)$$

where $B(s) = \{v_n(s) | n = 0, 1, 2, \dots\}$, $B_1(s) = \{v_n(s) | n = 1, 2, 3, \dots\}$. By the condition (H_3) and (1.3), we have

$$\begin{aligned} &\beta(f(s, B(s), (TB)(s), (SB)(s))) \\ &\leq \alpha(f(s, B(s), (TB)(s), (SB)(s))) \\ &\leq c_1\alpha(B(s)) + c_2\alpha((TB)(s)) + c_3\alpha((SB)(s)). \end{aligned} \quad (2.19)$$

From the uniform boundedness of $B(s)$ and uniform continuity of $h(t, s)$, it easy to prove $(SB)(s)$ is a equicontinuous bounded set, so it follows from Lemma 1.4 that

$$\alpha((SB)(s)) = \alpha\left(\int_{t_0}^{t_0+a} h(s, \tau) B(\tau) d\tau\right) \leq h_0 \int_{t_0}^{t_0+a} m(\tau) d\tau. \quad (2.20)$$

Now, we consider dealing with $\alpha((TB)(s))$. Firstly,

$$\int_{t_0}^s k(s, \tau) d\tau = \int_{t_0}^s \frac{\rho(s, \tau)}{(s - \tau)^\alpha} d\tau \leq R_0 \int_{t_0}^s \frac{1}{(s - t)^\alpha} d\tau \leq \frac{R_0 a^{1-\alpha}}{1 - \alpha}.$$

Since $B(s)$ is equicontinuous bounded sequences and $\alpha(B(s)) = m(s)$, there exists a partition $B(s) = \bigcup_{i=1}^l B_i$ such that the partition $(TB)(s) = \bigcup_{i=1}^l TB_i$ exists, where $TB_i = \left\{ \int_{t_0}^s k(s, \tau) v_i(\tau) d\tau \mid v_i \in B_i \right\}$, so we have

$$\begin{aligned} diam(TB_i) &= \sup_{\forall v_i^1, v_i^2 \in B_i} \left\| \int_{t_0}^s k(s, \tau) [v_i^1(\tau) - v_i^2(\tau)] d\tau \right\| \\ &\leq \frac{R_0 a^{1-\alpha}}{1 - \alpha} \sup_{\forall v_i^1, v_i^2 \in B_i} \|v_i^1(\tau) - v_i^2(\tau)\| \\ &= \frac{R_0 a^{1-\alpha}}{1 - \alpha} diam(B_i) \\ &< \frac{R_0 a^{1-\alpha}}{1 - \alpha} \alpha(B(s)) + \frac{R_0 a^{1-\alpha}}{1 - \alpha} \cdot \varepsilon. \end{aligned}$$

By using the arbitrariness of ε , we have

$$\alpha(TB(s)) \leq \frac{R_0 a^{1-\alpha}}{1 - \alpha} \alpha(B(s)) = \frac{R_0 a^{1-\alpha}}{1 - \alpha} m(s), \quad (2.21)$$

and by (1.3), we have

$$\beta(T(B_1 - B)(s)) \leq \alpha(T(B_1 - B)(s)) \leq \frac{2R_0 a^{1-\alpha}}{1 - \alpha} m(s). \quad (2.22)$$

Thus, it follows from (2.18)-(2.22) that

$$\begin{aligned} m(t) &\leq 2 \int_{t_0}^t \left[c_1 m(s) + \frac{c_2 R_0 a^{1-\alpha}}{1-\alpha} m(s) + c_3 h_0 \int_{t_0}^{t_0+a} m(\tau) d\tau \right. \\ &\quad \left. + M m(s) + \frac{2N R_0 a^{1-\alpha}}{1-\alpha} m(s) \right] ds \\ &= 2 \left(c_1 + M + \frac{c_2 R_0 a^{1-\alpha}}{1-\alpha} + \frac{2N R_0 a^{1-\alpha}}{1-\alpha} \right) \int_{t_0}^t m(s) ds \\ &\quad + 2h_0 c_3 (t - t_0) \int_{t_0}^{t_0+a} m(s) ds. \end{aligned}$$

Therefore, from Lemma 1.2 and the conditions (i)(ii) of the assumption (H_3) , we have $m(t) \equiv 0$, $t \in J$ which implies $\alpha\{v_n | n = 1, 2, 3, \dots\} = 0$ from (2.16), that is, $v_n \subset [v_0, \omega_0]$ is a relatively compact set in $C[J, E]$. Thus there exists a subsequence $\{v_{n_k}\} \subset \{v_n\}$ and some $\bar{u} \in [v_0, \omega_0]$ such that $\{v_{n_k}\}$ converges to \bar{u} in norm $\|\cdot\|_c$. Further, from (2.14) and the normality of P_c , it is easy to prove that $\{v_n\}$ converges to \bar{u} in norm $\|\cdot\|_c$, that is, $\{v_n(t)\}$ converges uniformly to $\bar{u}(t)$ on J according to the norm in E . Similarly, we can prove that $\{\omega_n(t)\}$ converges uniformly to some $u^* \in [v_0, \omega_0]$ on J according to the norm in E . Clearly, the result (2.3) is true.

Finally, we prove that \bar{u} and u^* are a minimal solution and a maximal solution respectively of the initial value problem (1.1). Let

$$u_n(t) = -M(v_n(t) - v_{n-1}(t)) - NT(v_n - v_{n-1})(t), \quad t \in J, \quad n = 1, 2, 3, \dots$$

Since $\{v_n(t)\}$ converges uniformly to $\bar{u}(t)$ on J , it is easy to prove $\|u_n\|_c \rightarrow 0 (n \rightarrow \infty)$. Setting $\varepsilon_n = \|u_n\|_c$, from (2.15), we get

$$v'_n = f(t, v_{n-1}, Tv_{n-1}, Sv_{n-1}) + u_n(t), \quad v_n(t_0) = u_0, \quad \|u_n(t)\| \leq \varepsilon_n, \quad t \in J.$$

Applying Corollary 2.1.1 in [4], we know \bar{u} is a solution of the initial value problem (1.1). Similarly, we can prove that u^* is also a solution of the initial value problem (1.1). If u is a solution in $[v_0, \omega_0]$ of the initial value problem (1.1), then $Bu = u$, so by $v_0 \leq u \leq \omega_0$ and (2.8)-(2.10), it is easy to obtain

$$v_n \leq u \leq \omega_n, \quad n = 1, 2, 3, \dots$$

Letting $n \rightarrow \infty$ in above formula, we get $\bar{u} \leq u \leq u^*$. Consequently, \bar{u} , u^* are the minimal solution and maximal solution of the initial value problem (1.1) respectively. This completes the proof. \square

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Solving the multicriteria transportation equilibrium system problem with nonlinear path costs

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Abstract. In this paper, we present an self-adaptive algorithm for solving the multicriteria transportation equilibrium system problem with variable demand and nonlinear path costs. The path cost function considered is comprised of three attributes, travel time, toll and travel fares, that are combined into a nonlinear generalized cost. Travel demand is determined endogenously according to a travel disutility function. Travelers choose routes with the minimum overall generalized costs. Numerical experiments are conducted to demonstrate the feasibility of the algorithm to this class of transportation equilibrium system problems.

Key Words and Phrases: multicriteria, general networks, nonlinear path costs, transportation equilibrium system

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1 Introduction

Usually, there are more than one kind of goods transported through the traffic network, in reality. As we know, the transportation cost of one kind of goods can be affected by other kinds of goods under the same traffic network. In detail, the flows of different kinds of goods are not independent. Generally, in 2010, He et. al . [1] called this problem as dynamic traffic network equilibrium system. Several authors (see, for instance, [2-5]) study the model with elastic demands and develop some results in this context theoretical features and numerical procedures. For example, in the general economic case, the equilibrium cost will affect to the market demand of goods, so the O-D pair demand of these goods depends on the equilibrium cost and the equilibrium distribution. Therefore, it is reasonable to consider the traffic equilibrium problem with elastic demand when there are many kinds of goods transported through the same traffic network. At the same time, the travel cost function is considered widely and deeply. It is generally accepted that travelers consider a number of criteria (e.g., time, money, distance, safety, route complexity, etc.) when selecting routes. Presumably, these criteria are then combined in some manner to form a generalized cost for each particular route or path under consideration, and a route selected based on minimization of the generalized cost of the trip. Most commonly, it is assumed that travelers select the 'best' route based on either a single criterion, such as travel time, or several criteria using a linear (or additive) path cost function. However, as pointed out by Gabriel and Bernstein [6], there are many situations in which the linear path cost function is inadequate for addressing factors affecting a variety of transportation policies. Such factors include: (i) Nonlinear valuation of travel time—small amounts of time are valued proportionately less than larger amounts of time. (ii) Emissions fees—emissions of hydrocarbons and carbon monoxide are a nonlinear function of travel times. (iii) Path-specific tolls and fares—most existing fare and toll pricing structures are not directly proportional to either travel time or distance. These, and other such factors, are generally difficult to accommodate without explicitly using path flows in the formulation and solution, particularly for traffic equilibrium problems involving multi-dimensional nonlinear path costs. Despite the obvious usefulness of incorporating multiple criteria and relaxing the assumption of linear path costs for an important class of traffic equilibrium problems, there have been relatively few attempts to incorporate multiple criteria within route choice modeling. Under the assumption that the nonlinear path cost function is known a priori, Scott and Bernstein [7] solved a constrained shortest path problem (CSPP) to generate a set of Pareto optimal paths and then identify the best path by evaluating the cost values of the alternative paths. In a later study, Scott and Bernstein [8] embedded the CSPP into the gradient projection method to solve the non-additive

traffic equilibrium problem. Using a new gap function recently proposed by Facchinei and Soares [9], Lo and Chen [10] reformulated the nonadditive traffic equilibrium problem as an equivalent unconstrained optimization and solved a special case involving fixed demand and route-specific costs. Chen et al. [11] provided a projection and contraction algorithm for solving the elastic traffic equilibrium problem with route-specific costs. Recently, some formulations and properties of the non-additive traffic equilibrium models were also explored, such as the nonlinear time/money relation [12], the uniqueness and convexity of the bicriteria traffic equilibrium problem [13]. Furthermore, Altman and Wynter [14] discussed the non-additive cost structures in both transportation and telecommunication networks. However, there are few results to discuss the problem related to the transportation network system for the nonlinear multicriteria transportation cost functions. On the other hand, Verma [15] investigated the approximation solvability of a new system of nonlinear variational inequalities involving strongly monotone mappings. In 2005, Bnouhachem [16] presented a new self-adaptive method for solving general mixed variational inequalities. In 2007, Shi [17] proposed a new self-adaptive iterative method for solving nonlinear variational inequality system (SNVI) and proved the convergence of the proposed method. The numerical examples were given to illustrate the efficiency of the proposed method. In this paper, we consider the traffic equilibrium problem with variable demand, fixed tolls, and a nonlinear path cost function. We first discuss the multicriteria traffic equilibrium problem and its equivalent nonlinear variational inequality formulation, and present the associated multicriteria shortest path problem and solution algorithm. We then explore a new self-adaptive iterative method (SI) developed by Shi [17] to solve SNVI that characterizes this class of traffic equilibrium system problem. The SI method is simple and can handle a general monotone mapping. Unlike the non-smooth equations/sequential quadratic programming (NE/SQP) method proposed by Gabriel and Bernstein [6] to solve the non-additive traffic equilibrium problem, the SI method does not require the mapping to be differentiable.

2 Preliminaries

Without loss of generality, we consider the case that there are only two kinds of goods transported through the network. Suppose that a traffic network consists of a set N of nodes, a set Ω of origin-destination (O/D) pairs, and a set R of routes. Each route $r \in R$ links one given origin-destination pair $\omega \in \Omega$. The set of all $r \in R$ which links the same origin-destination pair $\omega \in \Omega$ is denoted by $R(\omega)$. Assume that n is the number of the route in R and m is the number of origin-destination (O/D) pairs in Ω . Let vector $H^i = (H_1^i, H_2^i, \dots, H_r^i, \dots, H_n^i)^T \in R^n$ $i = 1, 2$ denote the flow vector for the two kinds of goods, where $H_r^i, r \in R$, denotes the flow in route $r \in R$. A feasible flow has to satisfy the capacity restriction principle: $\lambda_r^i \leq H_r^i \leq \mu_r^i$, for all $r \in R$, and a traffic conservation law: $\sum_{r \in R(\omega)} H_r^i = \rho_\omega^i(H^1, H^2)$, for all $\omega \in \Omega$, where λ and μ are given in R^n , is the travel demand related to the given pair $\omega \in \Omega$, and $\rho_\omega^i(H^1, H^2) \geq 0$ denotes the travel demand vector, which generally depends on equilibrium cost and, essentially, on the equilibrium distribution H^1 and H^2 . Thus the set of all feasible flows is given by

$$K_i(H^1, H^2) := \{H \in R^n \mid \lambda^i \leq H \leq \mu^i, \Phi H = \rho^i(H^1, H^2)\}, \quad (2.1)$$

where $\Phi = (\delta_{\omega r})_m \times n$ is defined as

$$\delta_{\omega r} = \begin{cases} 1 & \text{if } r \in R(\omega) \\ 0 & \text{Otherwise} \end{cases}$$

Thus the set of feasible flows is given by $K_1(H^1, H^2) \times K_2(H^1, H^2)$. We call that is a flow of the traffic network system with elastic demands. As pointed out by Gabriel and Bernstein [6], the linear assumption is rather restrictive and cannot adequately model certain important applications.

Let mapping $C^i : K \rightarrow R^n$ be the cost function of the i th kinds of goods for $i = 1, 2$. $C_r^i(H^1, H^2)$ gives the marginal cost of transporting one additional unit of the i th kind of goods under the r th route. For the multicriteria traffic equilibrium problem with nonlinear path costs based on travel time, toll and transportation fares, a possible nonlinear path cost function can be the following form:

$$C_r^i(H^1, H^2) = g_r(\sum_{a \in A} \delta_{pa}^{rs} t_a^i(H^1, H^2)) + \sum_{a \in A} \delta_{pa}^{rs} \tau_a + \sum_{a \in A} \delta_{pa}^{rs} f_a^i(H^1, H^2), \quad (2.2)$$

Where g_r is a nonlinear function describing the value-of-time for path r , τ_a is the toll on link a , and f_a is the transportation fares function on link a .

Definition 2.1. $(H^1, H^2) \in K_1(H^1, H^2) \times K_2(H^1, H^2)$ is an equilibrium flow if and only if for all $\omega \in \Omega$ and $q, s, p, r \in R(\omega)$ there holds

$$\begin{aligned} C_q^1(H^1, H^2) < C_s^1(H^1, H^2) &\Rightarrow H_q^1 = \mu_q^1 \quad \text{or} \quad H_s^1 = \lambda_s^1, \\ C_p^2(H^1, H^2) < C_r^2(H^1, H^2) &\Rightarrow H_p^2 = \mu_p^2 \quad \text{or} \quad H_r^2 = \lambda_r^2, \end{aligned} \quad (2.3)$$

3 Existence and Uniqueness of the solution for the multicriteria transportation equilibrium system problem

The following result establishes relationship between the system of dynamic traffic equilibrium problem and a system of variational inequalities.

Theorem 3.1. $(H^1, H^2) \in K_1(H^1, H^2) \times K_2(H^1, H^2)$ is an equilibrium flow if and only if ,

$$\begin{aligned} < C^1(H^1, H^2), F^1 - H^1 > \geq 0 \quad \forall F^1 \in K_1(H^1, H^2), \\ < C^2(H^1, H^2), F^2 - H^2 > \geq 0 \quad \forall F^2 \in K_2(H^1, H^2), \end{aligned} \quad (3.1)$$

Proof. First assume that (3.1) holds and (2.3) does not hold. Then there exist $\omega \in \Omega$ and $q, s \in R(\omega)$ such that

$$C_q^i(H^1, H^2) < C_s^i(H^1, H^2), \quad H_q^i < \mu_s^i, \quad H_q^i > \lambda_s^i, \quad i = 1, 2. \quad (3.2)$$

Let $\delta_i = \min\{\mu_q^i - H_q^i, H_s^i - \lambda_s^i\}$, $i = 1, 2$.

Then $\delta_i > 0$, $i = 1, 2$.

We define a vector $F_i \in K_i(H^1, H^2)$, $i = 1, 2$, whose components are

$$F_q^i(t) = H_q^i + \delta_i, \quad F_s^i(t) = H_s^i - \delta_i, \quad F_r^i = H_r^i, \quad (3.3)$$

when $r \neq q, s$.

Thus,

$$< C^i(H^1, H^2), F^i - H^i > = \sum_{j=1}^n C_j^i(H^1, H^2)(F_j^i - H_j^i) = \delta_i(C_q^i(H^1, H^2) - C_s^i(H^1, H^2)) < 0, \quad (3.4)$$

and so (3.1) is not satisfied. Therefore, it is proved that (3.1) implies (2.4).

Next, assume that (2.4) holds. That is

$$C_q^i(H^1, H^2) < C_s^i(H^1, H^2) \Rightarrow H_q^i = \mu_q^i, \quad \text{or} \quad H_s^i = \mu_s^i, \quad i = 1, 2. \quad (3.5)$$

Let $F^i \in K_i(H^1, H^2)$ for $i = 1, 2$. Then (3.1) holds from Definition 2.1. The proof is completed.

Furthermore, we discuss the existence and uniqueness of the solution for the dynamic traffic equilibrium system (3.1). In order to get our main results, the following definitions will be employed.

Definition 3.2. $C^i(x, y)$ ($i = 1, 2$) is said to be θ -strictly monotone with respect to x on $K_1(H^1, H^2) \times K_2(H^1, H^2)$ if there exists $\theta > 0$ such that

$$< C^i(x_1, y) - C^i(x_2, y), x_1 - x_2 > \geq \theta \|x_1 - x_2\|_2^2, \quad (3.6)$$

$\forall x_1, x_2 \in K_1(H^1, H^2), \forall y \in K_2(H^1, H^2)$.

Definition 3.3. $C^i(x, y)$ ($i = 1, 2$) is said to be L -Lipschitz continuous with respect to x on $K_1(H^1, H^2) \times K_2(H^1, H^2)$ if there exists $\theta > 0$ such that

$$\|C^i(x_1, y) - C^i(x_2, y)\|_2 \leq L\|x_1 - x_2\|_2, \quad (3.7)$$

$\forall x_1, x_2 \in K_1(H^1, H^2), \forall y \in K_2(H^1, H^2)$.

Remark 3.4. Based on Definitions 3.2 and 3.3, we can similarly define the θ -strict monotonicity and L -Lipschitz continuity of $C^i(x, y)$ with respect to y on $K_1(H^1, H^2) \times K_2(H^1, H^2)$, for $i = 1, 2$.

Theorem 3.5. $(H^1, H^2) \in K_1(H^1, H^2) \times K_2(H^1, H^2)$ is an equilibrium flow if and only if there exist $\alpha > 0$ and $\beta > 0$ such that

$$\begin{aligned} H^1 &= P_{K_1}(H^2 - \alpha C^1(H^1, H^2)), \\ H^2 &= P_{K_2}(H^1 - \beta C^2(H^1, H^2)), \end{aligned} \quad (3.8)$$

where $P_{k_i} : R^n \rightarrow K_i(H^1, H^2)$ is a projection operator for $i = 1, 2$.

Proof. The proof is analogous to that of Theorem 5.2.4 of [18].

Let $\|(x, y)_1\|$ be the norm on space $K_1(H^1, H^2) \times K_2(H^1, H^2)$ defined as follows:

$$\|(x, y)_1\| = \|x\|_2 + \|y\|_2, \quad \forall x \in K_1(H^1, H^2), y \in K_2(H^1, H^2). \quad (3.9)$$

It is easy to see that $(K_1(H^1, H^2) \times K_2(H^1, H^2), \|\cdot\|_1)$ is a Banach space. Similar to Theorem 3.9 in He et. al. [1], one can easily obtain the following theorem, the proof is omitted.

Theorem 3.6. Suppose that $C^1(H^1, H^2)$ is θ_1 -strictly monotone and L_{11} -Lipschitz continuous with respect to H^1 , and L_{12} -Lipschitz continuous with respect to H^2 on $K_1(H^1, H^2) \times K_2(H^1, H^2)$. Suppose that $C^2(H^1, H^2)$ is L_{21} -Lipschitz continuous with respect to H^1 , θ_2 -strictly monotone, and L_{22} -Lipschitz continuous with respect to H^2 on $K_1(H^1, H^2) \times K_2(H^1, H^2)$. If there exist $\alpha > 0$ and $\beta > 0$ such that

$$\begin{aligned} \sqrt{1 - 2\gamma\theta_1 + \alpha^2 L_{11}^2} + \beta L_{21} &< 1, \\ \sqrt{1 - 2\eta\theta_2 + \beta^2 L_{22}^2} + \alpha L_{12} &< 1, \end{aligned} \quad (3.10)$$

then problem (3.1) admits unique solution.

Remark 3.7. If $f_j^1(H^1, H^2)$ is $\hat{\theta}_j^1$ -strictly monotone with respect to H^1 and $g_j^1 \circ \sum_{j=1}^n \delta_{pj}^{rs} t_j^1$ is $\bar{\theta}_j^1$ -strictly monotone with respect to H^1 , then

$$\theta_1 = \sum_{j=1}^n (\bar{\theta}_j^1 + \delta_{pj}^{rs} \hat{\theta}_j^1).$$

In fact,

$$\begin{aligned} &< C_j^1(H^1, H^2) - C_j^2(\hat{H}^1, H^2), H^1 - \hat{H}^1 > \\ &= < g_r(\sum_{j=1}^n \delta_{pj}^{rs} t_j^1(H^1, H^2)) + \sum_{j=1}^n \delta_{pj}^{rs} \tau_j + \sum_{j=1}^n \delta_{pj}^{rs} f_j^1(H^1, H^2) - g_r(\sum_{j=1}^n \delta_{pj}^{rs} t_j^1(\hat{H}^1, H^2)) \\ &\quad + \sum_{j=1}^n \delta_{pj}^{rs} \tau_j + \sum_{j=1}^n \delta_{pj}^{rs} f_j^1(\hat{H}^1, H^2), H^1 - \hat{H}^1 > \\ &\geq \sum_{j=1}^n (\bar{\theta}_j^1 + \delta_{pj}^{rs} \hat{\theta}_j^1) \|H^1 - \hat{H}^1\|_2^2 \end{aligned} \quad (3.11)$$

So,

$$< C_j(H^1, H^2) - C_j(\hat{H}^1, H^2), H^1 - \hat{H}^1 > \geq \theta_1 \|H^1 - \hat{H}^1\|.$$

4 Algorithms for solving the multicriteria transportation equilibrium system problem

Here, we describe an iterative algorithm with fixed step-sizes, and also describe a self-adaptive algorithm, which uses a self-adaptive strategy of step-size choice.

Algorithm 4.1 Iterative Method with fixed step-sizes

Step 1. Given $\epsilon > 0, \alpha, \beta \in [0, 1]$, and $(H_1^0, H_2^0) \in K_1(H^1, H^2) \times K_2(H^1, H^2)$, set $k = 0$.

Step 2. Get the next iterate:

$$H^{1,k+1} = P_{K_1}(H^{2,k} - \alpha C^1(H^{1,k}, H^{2,k}),$$

$$H^{2,k+1} = P_{K_2}(H^{2,k} - \beta C^1(H^{1,k}, H^{2,k}),$$

Step 3. Compute $r_1 = \|H_1^{(k+1)} - H_1^{(k)}\|, r_2 = \|H_2^{(k+1)} - H_2^{(k)}\|$, if $r_1, r_2 < \epsilon$, then stop; Otherwise, $k = k+1$, go to step 2.

Algorithm 4.2 SI method

Step 1. Given $\epsilon > 0, \gamma \in [1, 2], \mu \in (0, 1), \rho > 0, \delta \in (0, 1), \delta_0 \in (0, 1)$, and $\mu_0 \in H$, set $k = 0$.

Step 2. Set $\rho_k = \rho$, if $\|r^1(H^{1K}, \rho)\| < \epsilon$ and $\|r^1(H^{1K}, \rho)\| < \epsilon$, then stop; otherwise, find the smallest nonnegative integer m_k , such that $\rho_k = \rho \mu^{m_k}$ satisfying

$$\|\rho_k(C^1(H^{1k}, H^{2k}) - C^1(w^k, H^{2k}))\| \leq \delta \|r(x^k, \rho_k)\|, \quad (4.1)$$

where $w^k = P_K[H^{1k} - \rho_k C^1(H^{1k}, H^{2k})]$.

Step 3. Compute

$$d(H^{1k}, \rho_k) = r(H^{1k}, \rho_k) - \rho_k C^2(H^{1k}, H^{2k}) + \rho_k C^2(P_K[H^{1k} - \rho_k C(H^{1k}, H^{2k})], H^{2k}), \quad (4.2)$$

where $r(H^{1k}, \rho) = H^{1k} - P_K[H^{1k} - \rho C^2(H^{1k}, H^{2k})]$.

Step 4. Get the next iterate:

$$\begin{aligned} H^{2k} &= P_K[H^{1k} - \gamma d(H^{1k}, \rho_k) - \gamma C^2(H^{1k}, H^{2k})]; \\ H^{1,k+1} &= P_K[H^{1k} - \rho C^1(H^{1k}, H^{2k})] \end{aligned} \quad (4.3)$$

Step 5. If $\|\rho_k(C(H^{1k}, H^{2k}) - C(w^k, H^{2k}))\| \leq \delta_0 \|r(x^k, \rho_k)\|$, then set $\rho = \rho_k / \mu$, else set $\rho = \rho_k$. Set $k = k + 1$, and go to Step 2.

Remark 4.2. Note that Algorithm 4.2 is obviously a modification of the standard procedure. In Algorithm 4.2, the searching direction is taken as $H^{1k} - \gamma d(H^{1k}, \rho_k) - \gamma C(H^{1k}, H^{2k})$, which is closely related to the projection residue, and differs from the standard procedure. In addition, the self-adaptive strategy of step-size choice is used. The numerical results show that these modifications can introduce computational efficiency substantially.

Theorem 4.3. Suppose that $C^1(H^1, H^2)$ is θ_1 -strictly monotone and L_{11} -Lipschitz continuous with respect to H^1 , and L_{12} -Lipschitz continuous with respect to H_2 on $K_1(H^1, H^2) \times K_2(H^1, H^2)$. Suppose that $C^2(H^1, H^2)$ is L_{21} -Lipschitz continuous with respect to H^1 , θ_2 -strictly monotone, and L_{22} -Lipschitz continuous with respect to H^2 on $K_1(H^1, H^2) \times K_2(H^1, H^2)$. Let $H^{1*}, H^{2*} \in K$ form a solution set for the SNVI (2.1) and let the sequences $\{H^{1k}\}$ and $\{H^{2k}\}$ be generated by Algorithm 4.2. If $0 < \bar{\theta} < \sqrt{1 - 2\rho\theta_1 + 2\rho^2 L_{12}^2} / (1 + \gamma L_{21}) + \sqrt{2\rho L_{11}^2 + 2\rho L_{11}} < 1$, then the sequence $\{H^{1k}\}$ converges to H^{1*} and the sequence $\{H^{2k}\}$ converges to H^{2*} , for $0 < \rho < 2r/s^2$.

Proof. Since (H^{1*}, H^{2*}) is a solution of transportation equilibrium system (3.2), it follows from Theorem 3.5 that

$$\begin{aligned} H^{1*} &= P_{K_1}[H^{2*} - \rho C^1(H^{1*}, H^{2*})], \\ H^{2*} &= P_{K_2}[H^{1*} - \gamma C^2(H^{1*}, H^{2*})] \end{aligned} \quad (4.4)$$

Applying Algorithm 4.2, we know

$$\begin{aligned}\|H^{1,k+1} - H^{1*}\| &= \|P_{K_1}[H^{2k} - \rho C^1(H^{1k}, H^{2k})] - P_{K_1}[H^{2*} - \rho C^1(H^{1*}, H^{2*})]\| \\ &\leq \|H^{2k} - H^{2*} - \rho C^1(H^{1k}, H^{2k}) + \rho C^1(H^{1*}, H^{2*})\|\end{aligned}$$

Since T is r -strongly monotone and s -Lipschitz continuous, we know

$$\begin{aligned}&\|H^{2k} - H^{2*} - \rho C^1(H^{1*}, H^{2*})\|^2 \\ &\leq \|H^{2k} - H^{2*}\|^2 - 2\rho \langle C^1(H^{1k}, H^{2k}) - C^1(H^{1*}, H^{2*}), H^{2k} - H^{2*} \rangle + \rho^2 \|C^1(H^{1k}, H^{2k}) - C^1(H^{1*}, H^{2*})\|^2 \\ &\leq \|H^{2k} - H^{2*}\|^2 - 2\rho\theta_1 \|H^{2k} - H^{2*}\|^2 + 2\rho L_{11} \|H^{1k} - H^{1*}\|^2 + \rho^2 \|C^1(H^{1k}, H^{2k}) - C^1(H^{1*}, H^{2*})\|^2 \\ &\leq \|H^{2k} - H^{2*}\|^2 - 2\rho\theta_1 \|H^{2k} - H^{2*}\|^2 + 2\rho L_{11} \|H^{1k} - H^{1*}\|^2 + 2\rho^2 L_{11}^2 \|H^{1k} - H^{1*}\|^2 + 2\rho^2 L_{12}^2 \|H^{2k} - H^{2*}\|^2 \\ &\leq (1 - 2\rho\theta_1 + 2\rho^2 L_{12}^2) \|H^{2k} - H^{2*}\|^2 + (2\rho^2 L_{11}^2 + 2\rho L_{11}) \|H^{1k} - H^{1*}\|^2\end{aligned}$$

It follows that

$$\|H^{1,k+1} - H^{1*}\| \leq \sqrt{1 - 2\rho\theta_1 + 2\rho^2 L_{12}^2} \|H^{2k} - H^{2*}\| + \sqrt{2\rho^2 L_{11}^2 + 2\rho L_{11}} \|H^{1k} - H^{1*}\|. \quad (4.5)$$

Next, we consider

$$\begin{aligned}\|H^{2k} - H^{2*}\| &= \|P_{K_2}[H^{1k} - \gamma d(H^{1k}, \rho_k) - \gamma C^2(H^{1k}, H^{2k})] - P_{K_2}[H^{1*} - \gamma C^2(H^{1*}, H^{2*})]\| \\ &\leq \|H^{1k} - \gamma d(H^{1k}, \rho_k) - \gamma C^2(H^{1k}, H^{2k}) - H^{1*} + \gamma C^2(H^{1*}, H^{2*})\| \\ &\leq \|H^{1k} - \gamma d(H^{1k}, \rho_k) - H^{1*}\| + \gamma \|C^2(H^{1k}, H^{2k}) - C^2(H^{1*}, H^{2*})\|\end{aligned} \quad (4.6)$$

where we use the property of the operator P_K . Now, we consider

$$\begin{aligned}&\|H^{1k} - H^{1*} - \gamma d(H^{1k}, \rho_k)\|^2 \\ &\leq \|H^{1k} - H^{1*}\|^2 - 2\gamma \langle H^{1k} - H^{1*}, d(H^{1k}, \rho_k) \rangle + \gamma^2 \|d(H^{1k}, \rho_k)\|^2 \\ &\leq \|H^{1k} - H^{1*}\|^2,\end{aligned} \quad (4.7)$$

where we use the definition of $d(H^{2k}, \rho_k)$.

It follows that

$$\|H^{2,k} - H^{2*}\| \leq (1 + \gamma L_{21}) \|H^{1k} - H^{1*}\| + \gamma L_{22} \|H^{2k} - H^{2*}\|. \quad (4.8)$$

From (4.5) to (4.8), we know

$$\|H^{1,k+1} - H^{1*}\| \leq (\sqrt{1 - 2\rho\theta_1 + 2\rho^2 L_{12}^2} (1 + \gamma L_{21}) / (1 - \gamma L_{22}) + \sqrt{2\rho L_{11}^2 + 2\rho L_{11}}) \|H^{1k} - H^{1*}\|. \quad (4.9)$$

Since $0 < \bar{\theta} < 1$, from (4.9), we know $H^{1k} \rightarrow H^{1*}$. Thus from (4.8), we know $H^{2k} \rightarrow H^{2*}$.

5 Numerical results

In this section, we presented some numerical results for the proposed method. we consider a simple traffic network consisting of two nodes, only a origin-destination (O/D) pair, and a set R of routes. Each route $r \in R$ links the origin-destination pair in parallel. Assume that n is the number of the route in R .

Let $C^1(H_1, H_2) = DH_1(t) + c_1^T H_2(t)$, $C^2(H_1, H_2) = DH_1(t) + c_2^T H_2(t)$, where

$$D = \begin{bmatrix} 4 & -2 & \cdots & \cdots \\ 1 & 4 & \cdots & \cdots \\ \cdots & \cdots & 4 & -2 \\ \cdots & \cdots & 1 & 4 \end{bmatrix},$$

$c_1 = (-1, -1, \dots, -1)^T$, $c_2 = (1, 1, \dots, 1)^T$, $H_1(t) = H_1 \in R^n$, $H_2(t) = H_2 \in R^n$. let

$$K_1(H_1, H_2) = \{H_1 | H_1 \in [l, u], H_1^i + H_2^i \leq 2000, i = 1, 2, \dots, n\},$$

$$K_2(H_1, H_2) = \{H_1 | H_1 \in [l, u], H_1^i + H_2^i \leq 2000, i = 1, 2, \dots, n\}.$$

where $l = (0, 0, \dots, 0)^T$, $u = (1000, 1000, \dots, 1000)^T$. The calculations are started with vectors $H_1 = (0, 0, \dots, 0)^T$, $H_2 = (5, 5, \dots, 5)^T$ and stopped whenever $r_1, r_2 < 10^{-5}$. Table 1 gives the numerical results of Algorithms 4. 1. Table 2 gives the numerical results of Algorithms 4. 2.

Comparing Table 2 and Table 1, it show that Algorithm 4.2 is very effective for the problem tested. In addition, it seems that the computational time and the iteration numbers are not very sensitive to the problem size.

Table 1: Computation performance with different scales by Algorithm 4.1

n	Iteration	CPU(s)
50	366	29.5469
100	183	28.5469
200	93	27.2813
300	63	30.4375

Table 2: Computation performance with different scales by Algorithm 4.2

n	Iteration	CPU(s)
50	220	17.7282
100	110	16.1281
200	56	15.3687
300	38	18.2625

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BARNES' MULTIPLE FROBENIUS-EULER AND HERMITE MIXED-TYPE POLYNOMIALS

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ABSTRACT. In this paper, we consider the Barnes' multiple Frobenius-Euler and Hermite mixed-type polynomials. Using the umbral calculus, we derive several explicit formulas and recurrence relations for these polynomials. Also, we establish connections between our polynomials and several known families of polynomials.

1. INTRODUCTION

For $\lambda \neq 1$, $s \in \mathbb{N}$, the Frobenius-Euler polynomials of order s are defined by the generating function

$$(1.1) \quad \left(\frac{1-\lambda}{e^t-\lambda} \right)^s e^{xt} = \sum_{n=0}^{\infty} \mathbb{H}_n^{(s)}(x | \lambda) \frac{t^n}{n!}, \quad (\text{see [7, 12, 19]}).$$

Let $a_1, a_2, \dots, a_r, \lambda_1, \lambda_2, \dots, \lambda_r \in \mathbb{C}$ with $a_1, \dots, a_r \neq 0$, $\lambda_1, \dots, \lambda_r \neq 1$. Then the Barnes' multiple Frobenius-Euler polynomials $H_n(x | a_1, \dots, a_r; \lambda_1, \dots, \lambda_r)$ are given by the generating function

$$(1.2) \quad \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t}-\lambda_j} \right) e^{xt} = \sum_{n=0}^{\infty} H_n(x | a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \frac{t^n}{n!}, \quad (\text{see [13, 15]}).$$

When $x = 0$, $H_n(a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) = H_n(0 | a_1, \dots, a_r; \lambda_1, \dots, \lambda_r)$ are called the Barnes' multiple Frobenius-Euler numbers (see [13]).

For $a_1 = a_2 = \dots = a_r = 1$, $\lambda_1 = \lambda_2 = \dots = \lambda_r = \lambda$, we have $H_n(x | \underbrace{1, 1, \dots, 1}_{r\text{-times}}; \underbrace{\lambda, \lambda, \dots, \lambda}_{r\text{-times}}) = \mathbb{H}_n^{(r)}(x | \lambda)$. When $x = 0$, $\mathbb{H}_n^{(r)}(\lambda) = \mathbb{H}_n^{(r)}(0 | \lambda)$ are called the Frobenius-Euler numbers of order r .

The Hermite polynomials $H_n^{(\nu)}(x)$ of variance ν ($0 \neq \nu \in \mathbb{R}$) are given by the generating function

$$(1.3) \quad e^{-\nu t^2/2} e^{xt} = \sum_{n=0}^{\infty} H_n^{(\nu)}(x) \frac{t^n}{n!}, \quad (\text{see [24]}).$$

When $x = 0$, $H_n^{(\nu)} = H_n^{(\nu)}(0)$ are called the Hermite numbers of variance ν . It is well known that the Bernoulli polynomials of order r ($r \in \mathbb{N}$) are defined by the generating function

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$$(1.4) \quad \left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \quad (\text{see [1-24, 26]}).$$

When $x = 0$, $B_n^{(r)} = B_n^{(r)}(0)$ are called the Bernoulli numbers of order r . For $n \geq 0$, the Stirling numbers of the first kind are given by

$$(1.5) \quad (x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^n S_1(n, l) x^l, \quad (\text{see [24]}).$$

The Stirling numbers of the second kind are defined by the generating function

$$(1.6) \quad (e^t - 1)^n = n! \sum_{l=n}^{\infty} S_2(l, n) \frac{x^l}{l!}, \quad (\text{see [24]}).$$

Let \mathbb{C} be the complex field and let \mathcal{F} be the set of all formal power series in the variable t :

$$(1.7) \quad \mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!} \mid a_k \in \mathbb{C} \right\}.$$

Let $\mathbb{P} = \mathbb{C}[x]$ and let \mathbb{P}^* be the vector space of all linear functionals on \mathbb{P} . We use the notation $\langle L \mid p(x) \rangle$ to denote the action of the linear functional L on the polynomial $p(x)$, and we recall that the vector space operations on \mathbb{P}^* are defined by $\langle L + M \mid p(x) \rangle = \langle L \mid p(x) \rangle + \langle M \mid p(x) \rangle$, and $\langle cL \mid p(x) \rangle = c \langle L \mid p(x) \rangle$, where c is a complex constant in \mathbb{C} . The linear functional $\langle f(t) \mid \cdot \rangle$ on \mathbb{P} is defined by

$$(1.8) \quad \langle f(t) \mid x^n \rangle = a_n, \quad (n \geq 0), \quad \text{where } f(t) \in \mathcal{F}, \quad (\text{see [17, 21, 24]}).$$

By (1.8), we easily get

$$(1.9) \quad \langle t^k \mid x^n \rangle = n! \delta_{n,k}, \quad (n, k \geq 0), \quad (\text{see [8, 21, 24]}),$$

where $\delta_{n,k}$ is the Kronecker's symbol.

Let $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L \mid x^k \rangle}{k!} t^k$. Then, by (1.9), we get $\langle f_L(t) \mid x^n \rangle = \langle L \mid x^n \rangle$. So, the map $L \mapsto f_L(t)$ is a vector space isomorphism from \mathbb{P}^* onto \mathcal{F} . Henceforth, \mathcal{F} denotes both the algebra of formal power series in t and the vector space of all linear functionals on \mathbb{P} , and so an element $f(t)$ of \mathcal{F} will be thought of as both a formal power series and a linear functional. We call \mathcal{F} the umbral algebra and the umbral calculus is the study of umbral algebra. The order $o(f(t))$ of a power series $f(t) \neq 0$ is the smallest integer k for which the coefficient of t^k does not vanish. If the order of $f(t)$ is 1, then $f(t)$ is called a delta series; if the order $g(t)$ is 0, then $g(t)$ is called an invertible series. Let $f(t), g(t) \in \mathcal{F}$ with $o(f(t)) = 1$ and $o(g(t)) = 0$. Then there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) such that $\langle g(t) f(t)^k \mid s_n(x) \rangle = n! \delta_{n,k}$ for $n, k \geq 0$. Such a sequence $s_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$ which is denoted by $s_n(x) \sim (g(t), f(t))$ (see [21, 24]). In particular, if $s_n(x) \sim (g(t), t)$, then $s_n(x)$ is called an Appell sequence for $g(t)$. For $f(t), g(t) \in \mathcal{F}$, we have

$$(1.10) \quad \langle f(t) g(t) \mid p(x) \rangle = \langle f(t) \mid g(t) p(x) \rangle = \langle g(t) \mid f(t) p(x) \rangle = \langle 1 \mid f(t) g(t) p(x) \rangle,$$

$$(1.11) \quad f(t) = \sum_{k=0}^{\infty} \langle f(t) \mid x^k \rangle \frac{t^k}{k!}, \quad p(x) = \sum_{k=0}^{\infty} \langle t^k \mid p(x) \rangle \frac{x^k}{k!}, \quad (\text{see [24]}).$$

Thus, by (1.11), we get
(1.12)

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \quad e^{yt} p(x) = p(x+y), \quad \text{and} \quad \langle e^{yt} | p(x) \rangle = p(y).$$

The sequence $s_n(x)$ is Sheffer for $(g(t), f(t))$ if and only if

$$(1.13) \quad \frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{k=0}^{\infty} \frac{s_k(x)}{k!} t^k, \quad (y \in \mathbb{C}), \quad (\text{see [17, 21, 24]}),$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ with $\bar{f}(f(t)) = f(\bar{f}(t)) = t$. It is well known that the Sheffer identity is given by

$$(1.14) \quad s_n(x+y) = \sum_{j=0}^{\infty} \binom{n}{j} s_j(x) p_{n-j}(y), \quad \text{where } p_n(x) = g(t) s_n(x), \quad (\text{see [17, 24]}).$$

For $s_n(x) \sim (g(t), f(t))$, we have

$$(1.15) \quad s_{n+1}(x) = \left(x - \frac{g'(x)}{g(x)} \right) \frac{1}{f'(x)} s_n(x), \quad (n \geq 0),$$

$$(1.16) \quad s_n(x) = \sum_{j=0}^n \frac{1}{j!} \left\langle g(\bar{f}(t))^{-1} \bar{f}(t)^j \middle| x^n \right\rangle x^j,$$

and

$$(1.17) \quad \langle f(t) | xp(x) \rangle = \langle \partial_t f(t) | p(x) \rangle, \quad f(t) s_n(x) = n s_{n-1}(x), \quad (n \geq 1).$$

Let $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), l(t))$, $(n \geq 0)$. Then we have

$$(1.18) \quad s_n(x) = \sum_{m=0}^n C_{n,m} r_m(x), \quad (n \geq 0),$$

where

$$(1.19) \quad C_{n,m} = \frac{1}{m!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l(\bar{f}(t))^m \middle| x^n \right\rangle, \quad (\text{see [17, 21, 24]}).$$

In this paper, we consider the polynomials $FH_n^{(\nu)}(x | a_1, \dots, a_r; \lambda_1, \dots, \lambda_r)$ whose generating function is given by

$$(1.20) \quad \prod_{j=1}^r \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} e^{xt} = \prod_{j=1}^r \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) e^{xt - \nu t^2/2} \\ = \sum_{n=0}^{\infty} FH_n^{(\nu)}(x | a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \frac{t^n}{n!},$$

where $r \in \mathbb{Z}_{>0}$, $a_1, \dots, a_r, \lambda_1, \dots, \lambda_r \in \mathbb{C}$ with $a_1, \dots, a_r \neq 0$, $\lambda_1, \dots, \lambda_r \neq 1$, and $\nu \in \mathbb{R}$ with $\nu \neq 0$. $FH_n^{(\nu)}(x | a_1, \dots, a_r; \lambda_1, \dots, \lambda_r)$ are called Barnes' multiple Frobenius-Euler and Hermite mixed-type polynomials.

When $x = 0$, $FH_n^{(\nu)}(a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) = FH_n^{(\nu)}(0 | a_1, \dots, a_r; \lambda_1, \dots, \lambda_r)$ are called the Barnes' multiple Frobenius-Euler and Hermite mixed-type numbers. We observe here that $FH_n^{(\nu)}(x | a_1, \dots, a_r; \lambda_1, \dots, \lambda_r)$, $H_n(x | a_1, \dots, a_r; \lambda_1, \dots, \lambda_r)$,

and $H_n^{(\nu)}(x)$ are respectively Appell sequences for $\prod_{j=1}^r \left(\frac{e^{a_j t} - \lambda_j}{1 - \lambda_j} \right) e^{\nu t^2/2}$, $\prod_{j=1}^r \left(\frac{e^{a_j t^n} - \lambda_j}{1 - \lambda_j} \right)$, and $e^{\nu t^2/2}$. That is,

$$(1.21) \quad FH_n^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \sim \left(\prod_{j=1}^r \left(\frac{e^{a_j t} - \lambda_j}{1 - \lambda_j} \right) e^{\nu t^2/2}, t \right),$$

$$(1.22) \quad H_n(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \sim \left(\prod_{j=1}^r \left(\frac{e^{a_j t} - \lambda_j}{1 - \lambda_j} \right), t \right),$$

and

$$(1.23) \quad H_n^{(\nu)}(x) \sim (e^{\nu t^2/2}, t).$$

From the Barnes' multiple Frobenius-Euler and Hermite mixed-type polynomials, we investigate some properties of those polynomials. Finally, we give some new and interesting identities which are derived from umbral calculus.

2. BARNES' MULTIPLE FROBENIUS-EULER AND HERMITE MIXED-TYPE POLYNOMIALS

From (1.21), (1.22) and (1.23), we note that

$$(2.1) \quad tFH_n^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) = \frac{d}{dx} FH_n^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \\ = nFH_{n-1}^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r),$$

$$(2.2) \quad tH_n(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) = \frac{d}{dx} H_n(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \\ = nH_{n-1}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r),$$

and

$$(2.3) \quad tH_n^{(\nu)}(x) = \frac{d}{dx} H_n^{(\nu)}(x) = nH_{n-1}^{(\nu)}(x).$$

Now, we give explicit expressions related to the Barnes' multiple Frobenius-Euler and Hermite mixed-type polynomials.

From (1.13), we note that

$$(2.4) \quad FH_n^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) = e^{-\nu t^2/2} \prod_{j=1}^r \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) x^n \\ = e^{-\nu t^2/2} H_n(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \\ = \sum_{m=0}^{\infty} \frac{1}{m!} \left(-\frac{\nu}{2} \right)^m t^{2m} H_n(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \\ = \sum_{m=0}^{\left[\frac{n}{2} \right]} \frac{1}{m!} \left(-\frac{\nu}{2} \right)^m (n)_{2m} H_{n-2m}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \\ = \sum_{m=0}^{\left[\frac{n}{2} \right]} \binom{n}{2m} \frac{(2m)!}{m!} \left(-\frac{\nu}{2} \right)^m H_{n-2m}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r).$$

By (1.9), we get

$$\begin{aligned}
 (2.5) \quad FH_n^{(\nu)}(y \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) &= \left\langle \sum_{i=0}^{\infty} FH_i^{(\nu)}(y \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \frac{t^i}{i!} \middle| x^n \right\rangle \\
 &= \left\langle \prod_{j=1}^r \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} e^{yt} \middle| x^n \right\rangle \\
 &= \left\langle \prod_{j=1}^r \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) \middle| e^{-\nu t^2/2} e^{yt} x^n \right\rangle \\
 &= \left\langle \prod_{j=1}^r \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) \middle| \sum_{l=0}^{\infty} H_l^{(\nu)}(y) \frac{t^l}{l!} x^n \right\rangle \\
 &= \sum_{l=0}^n \binom{n}{l} H_l^{(\nu)}(y) \left\langle \prod_{j=1}^r \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) \middle| x^{n-l} \right\rangle \\
 &= \sum_{l=0}^n \binom{n}{l} H_l^{(\nu)}(y) \left\langle \sum_{i=0}^{\infty} H_i(a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \frac{t^i}{i!} \middle| x^{n-l} \right\rangle \\
 &= \sum_{l=0}^n \binom{n}{l} H_{n-l}(a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) H_l^{(\nu)}(y).
 \end{aligned}$$

Thus, by (2.5), we get

$$(2.6) \quad FH_n^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) = \sum_{l=0}^n \binom{n}{l} H_{n-l}(a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) H_l^{(\nu)}(x).$$

Therefore, by (2.4) and (2.6), we obtain the following theorem.

Theorem 2.1. For $n \geq 0$, we have

$$\begin{aligned}
 FH_n^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) &= \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2m} \frac{(2m)!}{m!} \left(-\frac{\nu}{2} \right)^m H_{n-2m}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \\
 &= \sum_{l=0}^n \binom{n}{l} H_{n-l}(a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) H_l^{(\nu)}(x).
 \end{aligned}$$

From (1.9), we have

$$\begin{aligned}
 (2.7) \quad FH_n^{(\nu)}(y \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) &= \left\langle \sum_{i=0}^{\infty} FH_i^{(\nu)}(y \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \frac{t^i}{i!} \middle| x^n \right\rangle \\
 &= \left\langle \prod_{j=1}^r \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} e^{yt} \middle| x^n \right\rangle
 \end{aligned}$$

$$\begin{aligned}
 &= \left\langle e^{-\nu t^2/2} \left| \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{y t} x^n \right\rangle \right. \\
 &= \left\langle e^{-\nu t^2/2} \left| \sum_{l=0}^{\infty} H_l(y \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \frac{t^l}{l!} x^n \right\rangle \right. \\
 &= \sum_{l=0}^n \binom{n}{l} H_l(y \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \left\langle \sum_{i=0}^{\infty} H_i^{(\nu)} \frac{t^i}{i!} \left| x^{n-l} \right\rangle \right. \\
 &= \sum_{l=0}^n \binom{n}{l} H_l(y \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) H_{n-l}^{(\nu)}.
 \end{aligned}$$

Thus, by (2.7), we get

(2.8)

$$FH_n^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) = \sum_{l=0}^n \binom{n}{l} H_l(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) H_{n-l}^{(\nu)}.$$

Now, we will use the conjugation representation in (1.16). For $FH_n^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \sim (g(t) = \prod_{j=1}^r \left(\frac{e^{a_j t} - \lambda_j}{1 - \lambda_j} \right) e^{\nu t^2/2}, f(t) = t)$, we observe that

(2.9)

$$\begin{aligned}
 &\left\langle g(\bar{f}(t))^{-1} \bar{f}(t)^j \left| x^n \right\rangle \right. \\
 &= \left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} t^j \left| x^n \right\rangle \right. \\
 &= \left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} \left| t^j x^n \right\rangle \right. \\
 &= (n)_j \left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} \left| x^{n-j} \right\rangle \right. \\
 &= (n)_j \left\langle e^{-\nu t^2/2} \left| H_{n-j}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \right\rangle \right. \\
 &= (n)_j \left\langle \sum_{m=0}^{\infty} \frac{1}{m!} \left(-\frac{\nu}{2} \right)^m t^{2m} \left| H_{n-j}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \right\rangle \right. \\
 &= (n)_j \sum_{m=0}^{\left[\frac{n-j}{2} \right]} \frac{1}{m!} \left(-\frac{\nu}{2} \right)^m (n-j)_{2m} H_{n-j-2m}(a_1, \dots, a_r; \lambda_1, \dots, \lambda_r).
 \end{aligned}$$

From (1.16) and (2.9), we can derive the following equation:

(2.10) $FH_n^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r)$

$$= \sum_{j=0}^n \binom{n}{j} \sum_{m=0}^{\left[\frac{n-j}{2} \right]} \frac{1}{m!} \left(-\frac{\nu}{2} \right)^m (n-j)_{2m} H_{n-j-2m}(a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) x^j.$$

Therefore, by (2.8) and (2.10), we obtain the following theorem.

Theorem 2.2. For $n \geq 0$, we have

$$\begin{aligned} FH_n^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \\ = \sum_{l=0}^n \binom{n}{l} H_{n-l}^{(\nu)} H_l(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \\ = \sum_{j=0}^n \binom{n}{j} \sum_{m=0}^{\lfloor \frac{n-j}{2} \rfloor} \frac{1}{m!} \left(-\frac{\nu}{2}\right)^m (n-j)_{2m} H_{n-j-2m}(a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) x^j. \end{aligned}$$

Remark. From (1.14), we have

$$\begin{aligned} (2.11) \quad FH_n^{(\nu)}(x+y \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \\ = \sum_{j=0}^n \binom{n}{j} FH_j^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) y^{n-j}. \end{aligned}$$

By (1.15) and (1.21), we get

$$\begin{aligned} (2.12) \quad FH_{n+1}^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \\ = \left(x - \frac{g'(t)}{g(t)}\right) FH_n^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r), \end{aligned}$$

where $g(t) = \prod_{j=1}^r \left(\frac{e^{a_j t} - \lambda_j}{1 - \lambda_j}\right) e^{\nu t^2/2}$.

Now, we compute that

$$\begin{aligned} (2.13) \quad \frac{g'(t)}{g(t)} &= (\log g(t))' \\ &= \left(\sum_{j=1}^r \log(e^{a_j t} - \lambda_j) - \sum_{j=1}^r \log(1 - \lambda_j) + \frac{1}{2} \nu t^2 \right)' \\ &= \sum_{j=1}^r \frac{a_j e^{a_j t}}{e^{a_j t} - \lambda_j} + \nu t. \end{aligned}$$

So

$$\begin{aligned} (2.14) \quad \frac{g'(t)}{g(t)} FH_n^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \\ = \sum_{j=1}^r \frac{a_j e^{a_j t}}{1 - \lambda_j} \cdot \frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \prod_{i=1}^r \left(\frac{1 - \lambda_i}{e^{a_i t} - \lambda_i}\right) e^{-\nu t^2/2} x^n \\ + \nu t FH_n^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \\ = \sum_{j=1}^r \frac{a_j}{1 - \lambda_j} FH_n^{(\nu)}(x + a_j \mid a_1, \dots, a_r, a_j; \lambda_1, \dots, \lambda_r, \lambda_j) \\ + n \nu FH_{n-1}^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \\ = \sum_{j=1}^r \frac{a_j}{1 - \lambda_j} FH_n^{(\nu)}(x + a_j \mid a_1, \dots, a_r, a_j; \lambda_1, \dots, \lambda_r, \lambda_j) \\ + n \nu FH_{n-1}^{(\nu)}(x \mid a_1, \dots, a_r, a_j; \lambda_1, \dots, \lambda_r). \end{aligned}$$

By (2.12) and (2.14), we get

$$\begin{aligned}
 (2.15) \quad & FH_{n+1}^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \\
 &= x FH_n^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \\
 &\quad - \sum_{j=1}^r \frac{a_j}{1-\lambda_j} FH_n^{(\nu)}(x+a_j \mid a_1, \dots, a_r, a_j; \lambda_1, \dots, \lambda_r, \lambda_j) \\
 &\quad - n\nu FH_{n-1}^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r).
 \end{aligned}$$

For $n \geq 2$, by (1.9), we get

$$\begin{aligned}
 (2.16) \quad & FH_n^{(\nu)}(y \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \\
 &= \left\langle \sum_{i=0}^{\infty} FH_i^{(\nu)}(y \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \frac{t^i}{i!} \middle| x^n \right\rangle \\
 &= \left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} e^{y t} \middle| x^n \right\rangle \\
 &= \left\langle \partial_t \left(\prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} e^{y t} \right) \middle| x^{n-1} \right\rangle \\
 &= \left\langle \left(\partial_t \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) \right) e^{-\nu t^2/2} e^{y t} \middle| x^{n-1} \right\rangle \\
 &\quad + \left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) (\partial_t e^{-\nu t^2/2}) e^{y t} \middle| x^{n-1} \right\rangle \\
 &\quad + \left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} (\partial_t e^{y t}) \middle| x^{n-1} \right\rangle.
 \end{aligned}$$

The third term is

$$\begin{aligned}
 (2.17) \quad & y \left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} e^{y t} \middle| x^{n-1} \right\rangle \\
 &= y FH_{n-1}^{(\nu)}(y \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r).
 \end{aligned}$$

The second term is

$$\begin{aligned}
 (2.18) \quad & -\nu \left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} e^{y t} \middle| t x^{n-1} \right\rangle \\
 &= -\nu(n-1) \left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} e^{y t} \middle| x^{n-2} \right\rangle \\
 &= -\nu(n-1) FH_{n-2}^{(\nu)}(y \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r).
 \end{aligned}$$

We observe that

$$\begin{aligned}
 (2.19) \quad & \partial_t \left(\prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) \right) \\
 &= \sum_{i=1}^r \left(\frac{1-\lambda_i}{e^{a_i t} - \lambda_i} \right)' \prod_{j \neq i} \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) \\
 &= - \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) \sum_{i=1}^r \frac{a_i e^{a_i t}}{e^{a_i t} - \lambda_i},
 \end{aligned}$$

where

$$(2.20) \quad \sum_{i=1}^r \frac{a_i e^{a_i t}}{e^{a_i t} - \lambda_i} = \sum_{i=1}^r \frac{a_i}{1-\lambda_i} e^{a_i t} \frac{1-\lambda_i}{e^{a_i t} - \lambda_i} = \sum_{i=1}^r \frac{a_i}{1-\lambda_i} e^{a_i t} \sum_{m=0}^{\infty} \mathbb{H}_m(\lambda_i) \frac{a_i^m}{m!} t^m.$$

So, by (2.19) and (2.20), we get

$$(2.21) \quad \partial_t \left(\prod_{j=1}^r \frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) = - \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) \sum_{i=1}^r \frac{a_i}{1-\lambda_i} e^{a_i t} \sum_{m=0}^{\infty} \mathbb{H}_m(\lambda_i) \frac{a_i^m}{m!} t^m.$$

Now, the first term is

$$\begin{aligned}
 (2.22) \quad & - \sum_{i=1}^r \frac{a_i}{1-\lambda_i} \left\langle e^{(y+a_i)t} \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} \left| \sum_{m=0}^{n-1} \mathbb{H}_m(\lambda_i) \frac{a_i^m}{m!} t^m x^{n-1} \right| \right\rangle \\
 &= - \sum_{i=1}^r \frac{a_i}{1-\lambda_i} \sum_{m=0}^{n-1} \binom{n-1}{m} \mathbb{H}_m(\lambda_i) a_i^m \\
 &\quad \times \left\langle e^{(y+a_i)t} \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} \left| x^{n-1-m} \right| \right\rangle \\
 &= - \sum_{i=1}^r \frac{a_i}{1-\lambda_i} \sum_{m=0}^{n-1} \binom{n-1}{m} \mathbb{H}_m(\lambda_i) a_i^m \\
 &\quad \times \left\langle \sum_{l=0}^{\infty} FH_l^{(\nu)}(y+a_i \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \frac{t^l}{l!} \left| x^{n-1-m} \right| \right\rangle \\
 &= - \sum_{i=1}^r \sum_{m=0}^{n-1} \binom{n-1}{m} \frac{a_i^{m+1}}{1-\lambda_i} \mathbb{H}_m(\lambda_i) FH_{n-1-m}^{(\nu)}(y+a_i \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r).
 \end{aligned}$$

Therefore, by (2.16), (2.17), (2.18) and (2.22), we obtain the following theorem.

Theorem 2.3. For $n \geq 2$, we have

$$\begin{aligned}
 & FH_n^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \\
 &= x FH_{n-1}^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) - \nu(n-1) FH_{n-2}^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r)
 \end{aligned}$$

$$- \sum_{i=1}^r \sum_{m=0}^{n-1} \binom{n-1}{m} \frac{a_i^{m+1}}{1-\lambda_i} \mathbb{H}_m(\lambda_i) FH_{n-1-m}^{(\nu)}(x+a_i \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r).$$

Remark. We compute the following in two different ways in order to derive an identity:

$$\left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} t^m \middle| x^n \right\rangle, \quad (m, n \geq 0).$$

On one hand, it is

$$\begin{aligned} (2.23) \quad & \left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} t^m \middle| x^n \right\rangle \\ &= (n)_m \left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} \middle| x^{n-m} \right\rangle \\ &= (n)_m FH_{n-m}^{(\nu)}(a_1, \dots, a_r; \lambda_1, \dots, \lambda_r). \end{aligned}$$

On the other hand, it is

$$\begin{aligned} (2.24) \quad & \left\langle \partial_t \left(\prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} t^m \right) \middle| x^{n-1} \right\rangle \\ &= \left\langle \left(\partial_t \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) \right) e^{-\nu t^2/2} t^m \middle| x^{n-1} \right\rangle \\ &+ \left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) \left(\partial_t e^{-\nu t^2/2} \right) t^m \middle| x^{n-1} \right\rangle \\ &+ \left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2} (\partial_t t^m) \middle| x^{n-1} \right\rangle. \end{aligned}$$

From (2.23) and (2.24), we can derive the following equation: for $n \geq m+2$,

$$\begin{aligned} (2.25) \quad & FH_{n-m}^{(\nu)}(a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \\ &= -\nu(n-m-1) FH_{n-m-2}^{(\nu)}(a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \\ &- \sum_{i=1}^r \sum_{l=0}^{n-m-1} \binom{n-m-1}{l} \frac{a_i^{l+1}}{1-\lambda_i} \mathbb{H}_l(\lambda_i) FH_{n-1-l-m}^{(\nu)}(a_i; a_1, \dots, a_r; \lambda_1, \dots, \lambda_r). \end{aligned}$$

For $FH_n^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \sim \left(\prod_{j=1}^r \left(\frac{e^{a_j t} - \lambda_j}{1-\lambda_j} \right) e^{\nu t^2/2}, t \right)$, $(x)_n \sim (1, e^t - 1)$, we have

$$(2.26) \quad FH_n^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) = \sum_{m=0}^n C_{n,m}(x)_m,$$

$$(2.27) \quad C_{n,m}$$

$$\begin{aligned}
&= \frac{1}{m!} \left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} (e^t - 1)^m \middle| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} \middle| \frac{1}{m!} (e^t - 1)^m x^n \right\rangle \\
&= \left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} \middle| \sum_{l=m}^{\infty} S_2(l, m) \frac{t^l}{l!} x^n \right\rangle \\
&= \sum_{l=m}^n \binom{n}{l} S_2(l, m) \left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} \middle| x^{n-l} \right\rangle \\
&= \sum_{l=m}^n \binom{n}{l} S_2(l, m) FH_{n-l}^{(\nu)}(a_1, \dots, a_r; \lambda_1, \dots, \lambda_r).
\end{aligned}$$

Therefore, by (2.26) and (2.27), we obtain the following theorem.

Theorem 2.4. For $n \geq 0$, we have

$$FH_n^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) = \sum_{m=0}^n \sum_{l=m}^n \binom{n}{l} S_2(l, m) FH_{n-l}^{(\nu)}(a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) (x)_m.$$

It is easy to show that

$$x^{(n)} = x(x+1) \cdots (x+n-1) \sim (1, 1-e^{-t}).$$

From (1.18) and (1.19), we have

$$(2.28) \quad FH_n^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) = \sum_{m=0}^n C_{n,m} x^{(m)},$$

where

$$\begin{aligned}
(2.29) \quad C_{n,m} &= \frac{1}{m!} \left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} (1 - e^{-t})^m \middle| x^n \right\rangle \\
&= \frac{1}{m!} \left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} e^{-mt} (e^t - 1)^m \middle| x^n \right\rangle \\
&= \left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} e^{-mt} \middle| \frac{1}{m!} (e^t - 1)^m x^n \right\rangle \\
&= \left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} e^{-mt} \middle| \sum_{l=m}^{\infty} S_2(l, m) \frac{t^l}{l!} x^n \right\rangle \\
&= \sum_{l=m}^n \binom{n}{l} S_2(l, m) \left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} e^{-mt} \middle| x^{n-l} \right\rangle
\end{aligned}$$

$$\begin{aligned}
&= \sum_{l=m}^n \binom{n}{l} S_2(l, m) \left\langle \sum_{i=0}^{\infty} FH_i^{(\nu)}(-m \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \frac{t^i}{i!} \middle| x^{n-l} \right\rangle \\
&= \sum_{l=m}^n \binom{n}{l} S_2(l, m) FH_{n-l}^{(\nu)}(-m \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r).
\end{aligned}$$

Therefore, by (2.28) and (2.29), we obtain the following theorem.

Theorem 2.5. *For $n \geq 0$, we have*

$$\begin{aligned}
&FH_n^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \\
&= \sum_{m=0}^n \sum_{l=m}^n \binom{n}{l} S_2(l, m) FH_{n-l}^{(\nu)}(-m \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) x^{(m)}.
\end{aligned}$$

From (1.4), (1.13), (1.18), (1.19) and (1.21), we have

$$(2.30) \quad FH_n^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) = \sum_{m=0}^n C_{n,m} B_m^{(s)}(x), \quad (s \in \mathbb{N}),$$

where

$$(2.31)$$

$$\begin{aligned}
&C_{n,m} \\
&= \frac{1}{m!} \left\langle \prod_{j=1}^r \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} \left(\frac{e^t - 1}{t} \right)^s t^m \middle| x^n \right\rangle \\
&= \binom{n}{m} \left\langle \prod_{j=1}^r \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} \left(\frac{e^t - 1}{t} \right)^s x^{n-m} \right\rangle \\
&= \binom{n}{m} \left\langle \prod_{j=1}^r \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} \sum_{l=0}^{\infty} \frac{s!}{(l+s)!} S_2(l+s, s) t^l x^{n-m} \right\rangle \\
&= \binom{n}{m} \sum_{l=0}^{n-m} \frac{s!}{(l+s)!} S_2(l+s, s) (n-m)_l \left\langle \prod_{j=1}^r \left(\frac{1 - \lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} \middle| x^{n-m-l} \right\rangle \\
&= \binom{n}{m} \sum_{l=0}^{n-m} \frac{\binom{n-m}{l}}{\binom{l+s}{s}} S_2(l+s, s) FH_{n-m-l}^{(\nu)}(a_1, \dots, a_r; \lambda_1, \dots, \lambda_r).
\end{aligned}$$

Therefore, by (2.30) and (2.31), we obtain the following theorem.

Theorem 2.6. *For $n \geq 0$, and $s \in \mathbb{N}$, we have*

$$\begin{aligned}
&FH_n^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \\
&= \sum_{m=0}^n \binom{n}{m} \sum_{l=0}^{n-m} \frac{\binom{n-m}{l}}{\binom{l+s}{s}} S_2(l+s, s) FH_{n-m-l}^{(\nu)}(a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) B_m^{(s)}(x).
\end{aligned}$$

From (1.1), (1.18), (1.19) and (1.21), we have

$$(2.32) \quad FH_n^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) = \sum_{m=0}^n C_{n,m} \mathbb{H}_m^{(s)}(x \mid \lambda), \quad (s \in \mathbb{N}),$$

where

$$(2.33) \quad \begin{aligned} C_{n,m} &= \frac{1}{m!} \left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} \left(\frac{e^t - \lambda}{1-\lambda} \right)^s t^m \middle| x^n \right\rangle \\ &= \frac{1}{m! (1-\lambda)^s} \left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} (e^t - \lambda)^s \middle| t^m x^n \right\rangle \\ &= \frac{\binom{n}{m}}{(1-\lambda)^s} \left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} \left| \sum_{j=0}^s \binom{s}{j} (-\lambda)^{s-j} e^{j t} x^{n-m} \right. \right\rangle \\ &= \frac{\binom{n}{m}}{(1-\lambda)^s} \sum_{j=0}^s \binom{s}{j} (-\lambda)^{s-j} \left\langle \prod_{j=1}^r \left(\frac{1-\lambda_j}{e^{a_j t} - \lambda_j} \right) e^{-\nu t^2/2} e^{j t} \middle| x^{n-m} \right\rangle \\ &= \frac{\binom{n}{m}}{(1-\lambda)^s} \sum_{j=0}^s \binom{s}{j} (-\lambda)^{s-j} FH_{n-m}^{(\nu)}(j \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r). \end{aligned}$$

Therefore, by (2.32) and (2.33), we obtain the following theorem.

Theorem 2.7. For $n \geq 0$, we have

$$\begin{aligned} FH_n^{(\nu)}(x \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) &= \frac{1}{(1-\lambda)^s} \sum_{m=0}^n \binom{n}{m} \sum_{j=0}^s \binom{s}{j} (-\lambda)^{s-j} FH_{n-m}^{(\nu)}(j \mid a_1, \dots, a_r; \lambda_1, \dots, \lambda_r) \mathbb{H}_m^{(s)}(x \mid \lambda). \end{aligned}$$

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Robust stability and stabilization of linear uncertain stochastic systems with Markovian switching

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Abstract. This paper is concerned with robust stability and stabilization problem for a class of linear uncertain stochastic systems with Markovian switching. The uncertain system under consideration involves parameter uncertainties both in the drift part and in the diffusion part. New criteria for testing the robust stability of such systems are established in terms of bi-linear matrix inequalities (BLMIs), and sufficient conditions are proposed for the design of robust state-feedback controllers. An example illustrates the proposed techniques.

Keywords: Bi-linear matrix inequalities (BLMIs); Robust stabilization; Stochastic system with Markovian switching; Uncertainty

1 Introduction

Stochastic systems with Markovian switching have been used to model many practical systems where they may experience abrupt changes in their structure and parameters. Such systems have played a crucial role in many applications, such as hierarchical control of manufacturing systems ([4, 5, 16]), financial engineering ([19]) and wireless communications ([6]).

In the past decades, the stability and control of Markovian jump systems have recently received a lot of attention. For example, [3] and [15] systematically studied stochastic stability properties of jump linear systems. [1] discussed the stability of a semi-linear stochastic differential equation with Markovian switching. [7, 9, 10, 12] discussed the exponential stability of general nonlinear stochastic systems with Markovian switching of the form

$$dx(t) = f(x(t), t, r(t))dt + g(x(t), t, r(t))dB(t). \quad (1.1)$$

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Over the last decade, stochastic control problems governed by stochastic differential equation with Markovian switching have attracted considerable research interest, and we here mention [2, 11, 20, 23, 24]. It is well known that uncertainty occurs in many dynamic systems and is frequently a cause of instability and performance degradation. In the past few years, considerable attention has been given to the problem of designing robust controllers for linear systems with parameter uncertainty, such as [8, 13, 17, 21, 22]. However, a literature search reveals that the issue of stabilization of uncertain system under consideration involves parameter uncertainties both in the drift part and in the diffusion part has not been fully investigated and remains important and challenging. This situation motivates the present study on the robust stabilization of linear uncertain stochastic systems with Markovian switching. We aim at designing a robust state-feedback controller such that, for all admissible uncertainties, the closed-loop system is exponentially stable in mean square.

The structure of this paper is as follows. In Section 2, we introduce notations, definitions and results required from the literature. In Section 3, we shall discuss the problem of mean square exponential stabilization for a linear jump stochastic system. In Section 4, sufficient conditions are proposed for the design of robust state-feedback controllers. An example is discussed for illustrating our main results in Section 5.

2 Preliminaries

In this paper, we will employ the following notation. Let $|\cdot|$ be the Euclidean norm in \mathcal{R}^n . The interval $[0, \infty)$ be denoted by \mathcal{R}_+ . If A is a vector or matrix, its transpose is denoted by A^T . I_n denotes the $n \times n$ identity matrix. If A is a symmetric matrix $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ mean the smallest and largest eigenvalue, respectively. If A and B are symmetric matrices, by $A > B$ and $A \geq B$ we mean that $A - B$ is positive definite and nonnegative definite, respectively. And $\mathcal{C}^{2,1}(\mathcal{R}^n \times \mathcal{R}^+ \times S; \mathcal{R}^+)$ denotes the family of all \mathcal{R}^+ -valued functions on $\mathcal{R}^n \times \mathcal{R}^+ \times S$ which are continuously twice differentiable in x and once differentiable in t . We write $\text{diag}(a_1, \dots, a_n)$ for a diagonal matrix whose diagonal entries starting in the upper left corner are a_1, \dots, a_n .

Let $(\Omega, \mathcal{F}, (\mathcal{F})_t, P)$ be a complete probability space with a filtration $(\mathcal{F})_t$ satisfying the usual conditions. Let $r(t), t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \dots, N\}$ with generator $Q = (q_{ij})_{N \times N}$ given by

$$P(r(t + \Delta) = j \mid r(t) = i) = \begin{cases} q_{ij}\Delta + o(\Delta), & \text{if } i \neq j \\ 1 + q_{ii}\Delta + o(\Delta), & \text{if } i = j \end{cases}$$

where $\Delta > 0$, and $q_{ij} \geq 0$ denotes the switching rate from i to j if $i \neq j$ while $q_{ii} = -\sum_{i \neq j} q_{ij}$.

Definition 1 ([9]) The trivial solution of system (1), or simply system (1) is said to be exponentially stable in mean square if

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(E|x(t; t_0, x_0, r_0)|^2) < 0,$$

for all $(t_0, x_0, r_0) \in \mathcal{R}_+ \times \mathcal{R}^n \times S$.

If $V \in \mathcal{C}^{2,1}(\mathcal{R}^n \times \mathcal{R}^+ \times S; \mathcal{R}^+)$, define operator $\mathcal{L}V(x, t, i)$ associated with system (1) by

$$\begin{aligned} \mathcal{L}V(x, t, i) &= \frac{\partial V(x, t, i)}{\partial t} + \frac{\partial V(x, t, i)}{\partial x} f(x, t, i) \\ &+ \frac{1}{2} \text{tr}[g^T(x, t, i) \frac{\partial^2 V(x, t, i)}{\partial x^2} g(x, t, i)] + \sum_{j=1}^N q_{ij} V(x, t, i). \end{aligned}$$

We have the following lemma.

Lemma 2.1 ([9]) Let λ, c_1, c_2 be positive numbers. Assume that there exists a function $V(x, t, i) \in \mathcal{C}^{2,1}(\mathcal{R}^n \times \mathcal{R}^+ \times S; \mathcal{R}^+)$ such that

$$c_1 |x(t)|^2 \leq V(x, t, i) \leq c_2 |x(t)|^2$$

and

$$\mathcal{L}V(x, t, i) \leq -\lambda |x(t)|^2$$

for all $(x, t, i) \in \mathcal{R}^n \times \mathcal{R}^+ \times S$, then system (1) is exponentially stable in mean square.

In this note, we consider the following linear uncertain stochastic systems with Markovian switching:

$$\begin{aligned} dx(t) &= \tilde{A}(r(t))x(t)dt + \sum_{k=1}^d \tilde{B}_k(r(t))x(t)dw_k(t), \\ x(t_0) &= x_0 \in \mathcal{R}^n, t \geq t_0, \end{aligned} \tag{2.2}$$

where $w(t) = (w_1(t), w_2(t), \dots, w_d(t))^T$ denotes a d -dimensional Brownian motion or Wiener process, $x(t) \in \mathcal{R}^n$ is the system state, we assume that $w(t)$ and $r(t)$ are independent. For any $i \in S, 1 \leq k \leq d$, $\tilde{A}_i = \tilde{A}(r(t) = i)$ and $\tilde{B}_{ki} = \tilde{B}_k(r(t) = i)$ are not precisely known a priori, but belong to the following admissible uncertainty domains:

$$\mathcal{D}_a = \{A_i + D_{0i}F_{0i}(t)E_{0i} : F_{0i}(t)^T F_{0i}(t) \leq I, i \in S\},$$

$$\mathcal{D}_{bk} = \{B_{ki} + D_{ki}F_{ki}(t)E_{ki} : F_{ki}(t)^T F_{ki}(t) \leq I, i \in S\},$$

where $A_i, B_{ki}, D_{0i}, E_{0i}, D_{ki}, E_{ki}$ are known constant real matrices with appropriate dimensions, while $F_{0i}(t)$ and $F_{ki}(t)$ denotes the uncertainties in the system matrices, for all $i \in S$.

Lemma 2.2 ([14, 18]) Let A, D, E, W and $F(t)$ be real matrices of appropriate dimensions such that $F^T(t)F(t) \leq I$ and $W > 0$, then ,

- 1) For scalar $\varepsilon > 0$, $DF(t)E + (DF(t)E)^T \leq \varepsilon DD^T + \frac{1}{\varepsilon} E^T E$
- 2) For any scalar $\varepsilon > 0$ such that $W - \varepsilon DD^T > 0$,
 $(A + DF(t)E)^T W^{-1} (A + DF(t)E) \leq A^T (W - \varepsilon DD^T)^{-1} A + \frac{1}{\varepsilon} E^T E.$

3 Robust stability analysis

This section, we discuss the robust stability for system (2). For convenience, we will let the initial values x_0 and r_0 be non-random, namely $x_0 \in \mathcal{R}^n$ and $r_0 \in S$, but the theory developed in this paper can be generalized without any difficulty to cope with the case of random initial values, and we write $x(t; t_0, x_0, r_0) = x(t)$ simply.

Theorem 3.1 Suppose that there exist N symmetric positive-definite matrices P_i and positive scalars ε_i , γ_i , and λ_i , such that $\forall i \in S$, the following BLMI holds:

$$\begin{pmatrix} \Pi_{11} & * & * & * & * \\ E_{0i}P_i & -\gamma_i I & * & * & * \\ \Pi_{31} & 0 & \Pi_{33} & * & * \\ \Pi_{41} & 0 & 0 & -\varepsilon_i I & * \\ \Pi_{51} & 0 & 0 & 0 & \Pi_{55} \end{pmatrix} < 0, \quad (3.3)$$

where the symbol ‘*’ denotes the transposed element at the symmetric position, and

$$\begin{aligned} \Pi_{11} &= A_i P_i + P_i A_i^T + q_{ii} P_i + \lambda_i P_i + \gamma_i D_{0i} D_{0i}^T, \\ \Pi_{31} &= [P_i B_{1i}^T, P_i B_{2i}^T, \dots, P_i B_{di}^T]^T, \\ \Pi_{41} &= [P_i E_{1i}^T, P_i E_{2i}^T, \dots, P_i E_{di}^T]^T, \\ \Pi_{33} &= \text{diag}[\varepsilon_i D_{1i} D_{1i}^T - P_i, \dots, \varepsilon_i D_{di} D_{di}^T - P_i], \\ \Pi_{51} &= \underbrace{[P_i, P_i, \dots, P_i]^T}_{N-1}, \\ \Pi_{55} &= \text{diag}\left[\frac{-1}{q_{i1}} P_1, \dots, \frac{-1}{q_{i(i-1)}} P_{i-1}, \frac{-1}{q_{i(i+1)}} P_{i+1}, \dots, \frac{-1}{q_{iN}} P_N\right], \end{aligned}$$

then system (2) is exponentially stable in mean square.

Proof Let $X_i = P_i^{-1}$ and define $V(x, i) = x^T X_i x$ for all $i \in S$. And let $c_1 = \min\{\lambda_{\min}(X_i) : i \in S\}$, $c_2 = \max\{\lambda_{\max}(X_i) : i \in S\}$, it is clear that

$$c_1 |x(t)|^2 \leq V(x, i) \leq c_2 |x(t)|^2. \quad (3.4)$$

On the other hand, a calculation shows that

$$\begin{aligned}\mathcal{L}V(x, i) &= x(t)^T [X_i(A_i + D_{0i}F_{0i}E_{0i}) + (A_i + D_{0i}F_{0i}E_{0i})^T X_i + \sum_{j=1}^N q_{ij}X_j \\ &+ \sum_{k=1}^d (B_{ki} + D_{ki}F_{ki}E_{ki})^T X_i (B_{ki} + D_{ki}F_{ki}E_{ki})] x(t),\end{aligned}$$

by Lemma 2.2, for all $i \in S$, if there exist positive scalars ε_i and γ_i such that $\varepsilon_i D_{ki} D_{ki}^T - P_i < 0$, $1 \leq k \leq d$, then we have

$$\begin{aligned}\mathcal{L}V(x, i) &\leq x(t)^T [X_i A_i + A_i^T X_i + \gamma_i X_i D_{0i} D_{0i}^T X_i + \frac{1}{\gamma_i} E_{0i}^T E_{0i} \\ &+ \sum_{k=1}^d B_{ki}^T (P_i - \varepsilon_i D_{ki} D_{ki}^T)^{-1} B_{ki} + \sum_{k=1}^d \frac{1}{\varepsilon_i} E_{ki}^T E_{ki} + \sum_{j=1}^N q_{ij} X_j] x(t).\end{aligned}$$

Thus, there exists a $\lambda > 0$ such that

$$\mathcal{L}V(x, i) \leq -\lambda |x(t)|^2$$

will hold if for any $i \in S$ there exists a $\lambda_i > 0$ such that

$$\begin{aligned}&X_i A_i + A_i^T X_i + \gamma_i X_i D_{0i} D_{0i}^T X_i + \frac{1}{\gamma_i} E_{0i}^T E_{0i} \\ &+ \sum_{k=1}^d B_{ki}^T (P_i - \varepsilon_i D_{ki} D_{ki}^T)^{-1} B_{ki} + \sum_{k=1}^d \frac{1}{\varepsilon_i} E_{ki}^T E_{ki} + \sum_{j=1}^N q_{ij} X_j + \lambda_i X_i \\ &< 0.\end{aligned}\tag{3.5}$$

Pre- and post-multiplying (5) by P_i yields

$$\begin{aligned}&A_i P_i + P_i A_i^T + \gamma_i D_{0i} D_{0i}^T + \frac{1}{\gamma_i} P_i E_{0i}^T E_{0i} P_i \\ &+ \sum_{k=1}^d P_i B_{ki}^T (P_i - \varepsilon_i D_{ki} D_{ki}^T)^{-1} B_{ki} P_i \\ &+ \sum_{k=1}^d \frac{1}{\varepsilon_i} P_i E_{ki}^T E_{ki} P_i + \sum_{j \neq i} q_{ij} P_i P_j^{-1} P_i + q_{ii} P_i + \lambda_i P_i \\ &< 0,\end{aligned}$$

which is equivalent to inequality (3) in view of Schur complement equivalence. The assertion of this theorem follows from Lemma 2.1 immediately.

Remark 1 Theorem 3.1 provides the analysis results for the exponential stability of the system (2). It can be seen from (3) that we need to check whether there exist N symmetric positive-definite matrices P_i and positive scalars ε_i , γ_i , and λ_i meeting the N coupled matrix inequalities. It is clear that inequality (3) is BLMI, and it is LMI for a prescribed λ_i , then we are able to determine exponential stability of the system (3) readily by checking the solvability of the LMIs.

4 Robust stabilization synthesis

This section deals with the robust stabilization problem for linear uncertain stochastic systems with Markovian switching. Let us consider the uncertain stochastic control system of the form

$$dx(t) = [\tilde{A}(r(t))x(t) + C(r(t))u(t)]dt + \sum_{k=1}^d [\tilde{B}_k(r(t))x(t) + C_k(r(t))u(t)]dw_k(t), \quad (4.6)$$

$$x(t_0) = x_0 \in \mathcal{R}^n, t \geq t_0.$$

We aim to design a state-feedback controller $u(t) = K(r(t))x(t)$ such that the resulting closed-loop system

$$dx(t) = [\tilde{A}(r(t)) + C(r(t))K(r(t))]x(t)dt + \sum_{k=1}^d [\tilde{B}_k(r(t)) + C_k(r(t))K(r(t))]x(t)dw_k(t), \quad (4.7)$$

$$x(t_0) = x_0 \in \mathcal{R}^n, t \geq t_0.$$

is exponentially stable in mean square over all admissible uncertainty domains \mathcal{D}_a and \mathcal{D}_{bk} , where $K_i = K(r(t) = i)$ ($i \in S$) is the controller to be determined.

The following results solve the robust stabilization problem for system (6).

Theorem 4.1 The closed-loop system (7) is exponentially stable in mean square with respect to state-feedback gain $K_i = Y_i P_i^{-1}$, if there exist N symmetric positive-definite matrices P_i , N matrices Y_i and positive scalars ε_i , γ_i , and λ_i , such that $\forall i \in S$, the following BLMI holds:

$$\begin{pmatrix} \Pi_{11} & * & * & * & * \\ E_{0i}P_i & -\gamma_i I & * & * & * \\ \Pi_{31} & 0 & \Pi_{33} & * & * \\ \Pi_{41} & 0 & 0 & -\varepsilon_i I & * \\ \Pi_{51} & 0 & 0 & 0 & \Pi_{55} \end{pmatrix} < 0, \quad (4.8)$$

where

$$\begin{aligned} \Pi_{11} &= (A_i P_i + C_i Y_i) + (A_i P_i + C_i Y_i)^T + q_{ii} P_i \\ &\quad + \lambda_i P_i + \gamma_i D_{0i} D_{0i}^T, \end{aligned}$$

$$\Pi_{31} = [(B_{1i} P_i + C_{1i} Y_i)^T, \dots, (B_{di} P_i + C_{di} Y_i)^T]^T,$$

$$\Pi_{41} = [P_i E_{1i}^T, P_i E_{2i}^T, \dots, P_i E_{di}^T]^T,$$

$$\Pi_{33} = \text{diag}[\varepsilon_i D_{1i} D_{1i}^T - P_i, \dots, \varepsilon_i D_{di} D_{di}^T - P_i],$$

$$\Pi_{51} = [\underbrace{P_i, P_i, \dots, P_i}_{N-1}]^T,$$

$$\Pi_{55} = \text{diag}[\frac{-1}{q_{i1}} P_1, \dots, \frac{-1}{q_{i(i-1)}} P_{i-1}, \frac{-1}{q_{i(i+1)}} P_{i+1}, \dots, \frac{-1}{q_{iN}} P_N].$$

Proof The proof is similar to that of Theorem 3.1, so we only give an outlined one. Let $X_i = P_i^{-1}$ and define $V(x, i) = x^T X_i x$. There exists a $\lambda > 0$ such that $\mathcal{L}V(x, i) \leq -\lambda |x(t)|^2$ will hold if for any $i \in S$ there exist positive scalars ε_i , γ_i and λ_i , where $\varepsilon_i D_{ki} D_{ki}^T - P_i < 0$, $1 \leq k \leq d$, such that

$$\begin{aligned} & X_i(A_i + C_i K_i) + (A_i + C_i K_i)^T X_i + \gamma_i X_i D_{0i} D_{0i}^T X_i + \frac{1}{\gamma_i} E_{0i}^T E_{0i} \\ & + \sum_{k=1}^d (B_{ki} + C_{ki} K_i)^T (P_i - \varepsilon_i D_{ki} D_{ki}^T)^{-1} (B_{ki} + C_{ki} K_i) \\ & + \sum_{k=1}^d \frac{1}{\varepsilon_i} E_{ki}^T E_{ki} + \sum_{j=1}^N q_{ij} X_j + \lambda_i X_i < 0. \end{aligned} \quad (4.9)$$

Noting that $Y_i = K_i P_i$, and Pre- and post-multiplying (9) by P_i yields

$$\begin{aligned} & (A_i P_i + C_i Y_i) + (A_i P_i + C_i Y_i)^T + \gamma_i D_{0i} D_{0i}^T + \frac{1}{\gamma_i} P_i E_{0i}^T E_{0i} P_i \\ & + \sum_{k=1}^d (B_{ki} P_i + C_{ki} Y_i)^T (P_i - \varepsilon_i D_{ki} D_{ki}^T)^{-1} (B_{ki} P_i + C_{ki} Y_i) \\ & + \sum_{k=1}^d \frac{1}{\varepsilon_i} P_i E_{ki}^T E_{ki} P_i + \sum_{j \neq i} q_{ij} P_i P_j^{-1} P_i + q_{ii} P_i + \lambda_i P_i < 0, \end{aligned}$$

which is equivalent to (8) in view of Schur complement equivalence. The assertion of this theorem follows from Lemma 2.1 immediately.

Remark 2 It is shown in Theorem 4.1 that the robust exponentially stabilization of system (6)-(7) is guaranteed if the inequalities (8) are valid. And the inequality (8) is linear in Y_i and P_i for a prescribed λ_i , thus the standard LMI techniques can be exploited to check the exponential stability of the closed-loop system (7).

5 Example

Let $w(t)$ be a one-dimensional Brownian motion, let $r(t)$ be a right-continuous Markov chain

taking values in $S = \{1, 2\}$ with generator $Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$, consider a two-dimensional stochastic systems with Markovian switching of the form

$$\begin{aligned} dx(t) = & \left[(A(r(t)) + D_0(r(t))F_0(r(t), t)E_0(r(t)))x(t) + C(r(t))u(t) \right] dt \\ & + \left[(B(r(t)) + D_1(r(t))F_1(r(t), t)E_1(r(t)))x(t) + C_1(r(t))u(t) \right] dw(t), \end{aligned} \quad (5.10)$$

where

$$A_1 = \begin{pmatrix} 0.5 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0.1 \\ 0.2 & 2 \end{pmatrix}, B_1 = \begin{pmatrix} -1 & 0.5 \\ 0.5 & -1 \end{pmatrix}, B_2 = \begin{pmatrix} -2 & 0.1 \\ 0.1 & 1 \end{pmatrix},$$

$$D_{01} = \text{diag}(-1, -2), D_{02} = \text{diag}(0.2, 0.3), D_{11} = \text{diag}(-1, -1), D_{12} = \text{diag}(5, -0.5),$$

$$E_{01} = \text{diag}(0.2, 0.2), E_{02} = \text{diag}(-3, -5), E_{11} = \text{diag}(-0.9, -0.9), E_{12} = \text{diag}(0.5, 1),$$

$$C_1 = \begin{pmatrix} -8 & 0.1 \\ 0.05 & -10 \end{pmatrix}, C_2 = \begin{pmatrix} -20 & 0 \\ 0 & -30 \end{pmatrix}, C_{11} = \begin{pmatrix} -1 & 0.5 \\ 2 & 3 \end{pmatrix}, C_{12} = \begin{pmatrix} -2 & 1 \\ 0.5 & -4 \end{pmatrix},$$

for $i = 1, 2$, $F_{0i}(t)$ and $F_{1i}(t)$ denote the uncertainties of system (10). Let $\lambda_1 = 1, \lambda_2 = 2$, by solving LMIs (8), we find the feasible solution:

$$P_1 = \begin{pmatrix} 98.708 & 4.383 \\ 4.383 & 85.385 \end{pmatrix}, P_2 = \begin{pmatrix} 233.108 & -0.786 \\ -0.786 & 180.327 \end{pmatrix}, Y_1 = \begin{pmatrix} 93.468 & -16.376 \\ -20.698 & 70.862 \end{pmatrix},$$

$$Y_2 = \begin{pmatrix} 171.947 & -64.056 \\ 75.520 & 82.513 \end{pmatrix}, \gamma_1 = 0.082, \gamma_2 = 1.170, \varepsilon_1 = 0.034, \varepsilon_2 = 0.004,$$

therefore, by Theorem 4.1, closed-loop system (10) is exponentially stable in mean square with respect to state-feedback gain $K_i = Y_i P_i^{-1}$.

6 Conclusions

Based on the exponential stability theory, we have investigated the robust stochastic stability of the uncertain stochastic system with Markovian switching, sufficient stability conditions were developed. The robust stability of such systems can be tested based on the feasibility of bi-linear matrix inequalities. An example has been presented to illustrate the effectiveness of the main results. It is believed that this approach is one step further toward the descriptions of the uncertain stochastic systems.

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On interval valued functions and Mangasarian type duality involving Hukuhara derivative

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Abstract

In this paper, we introduce twice weakly differentiable and twice H -differentiable interval valued functions. The existence of twice H -differentiable interval-valued function and its relation with twice weakly differentiable functions are presented. Interval valued bonvex and generalized bonvex functions involving twice H -differentiability are proposed. Under the proposed settings, necessary conditions are elicited naturally in order to achieve LU -efficient solution. Mangasarian type dual is discussed for a nondifferentiable multiobjective programming problem and appropriate duality results are also derived. The theoretical developments are illustrated through non-trivial numerical examples.

Keyword: Interval valued functions; twice weak differentiability; twice H -differentiability, LU -efficient solution; generalized bonvexity; duality.

Mathematics Subject Classification: 90C25, 90C29, 90C30.

1 Introduction

The study of uncertain programming is always challenging in its modern face. Several attempts to achieve optimal in the same have been made in several directions. However optimality conditions still needs to be optimized. In this direction interval valued programming is one of the several techniques which has got attention of researchers in the recent past. Existing literature [2, 4, 5, 7, 8, 9, 10, 11, 12, 13, 17, 19, 20, 21, 22] contains many interesting results on the study of interval valued programming involving different types of differentiability concepts and various types of convexity concepts of interval valued functions.

Second order duality gives tighter bounds for the value of the objective function when approximations are used. For more information, authors may see ([11], pp

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93). One more advantage is that if a feasible point in the primal problem is given and first order duality does not use, then we can apply second order duality to provide a lower bound of the value of the primal problem.

Note that the study of nondifferentiable interval valued programming problems has not been studied extensively as quoted in Sun and Wang [18] therefore to study the second order duals of the aforesaid problem is an interesting move, we consider the following nondifferentiable vector programming problem with interval valued objective functions and constraint conditions and study its second order dual of Mangasarian type.

(IP)

$$\min f(x) + (x^T Bx)^{\frac{1}{2}} = \left(f_1(x) + (x^T Bx)^{\frac{1}{2}}, \dots, f_k(x) + (x^T Bx)^{\frac{1}{2}} \right)$$

$$\text{subject to } g_j(x) \preceq_{LU} [0, 0], j \in \Lambda_m$$

where $f_i = [f_i^L, f_i^U]$, $i \in \Lambda_k$ and $g_j = [g_j^L, g_j^U]$, $j \in \Lambda_m$ are interval valued functions with $f_i^L, f_i^U, g_j^L, g_j^U : R^n \rightarrow R$, $i \in \Lambda_k, j \in \Lambda_m$ be twice differentiable functions.

The remaining paper is designed as: section 2 is devoted to preliminaries. Section 3 represents the differentiation of interval valued functions with the introduction of twice weakly differentiable and twice H -differentiable interval valued functions. Some properties of these functions are also presented. Section 4 highlights the concept of so-called bonvexity and its quasi and pseudo forms of interval valued functions and their properties. In section 5, the necessary conditions for proposed solution concept are elicited naturally by considering above settings. Finally with the proposed settings the section 6 is devoted to study the Mangasarian type dual of primal problem (IP). Lastly we conclude in section 7.

2 Preliminaries

Let \mathcal{I}_c denote the class of all closed and bounded intervals in R . i.e.,

$$\mathcal{I}_c = \{[a, b] : a, b \in R \text{ and } a \leq b\}.$$

And $b - a$ is the width of the interval $[a, b] \in \mathcal{I}_c$. Then for $A \in \mathcal{I}_c$ we adopt the notation $A = [a^L, a^U]$, where a^L and a^U are respectively the lower and upper bounds of A . Let $A = [a^L, a^U]$, $B = [b^L, b^U] \in \mathcal{I}_c$ and $\lambda \in R$, we have the following operations.

$$(i) \quad A + B = \{a + b : a \in A \text{ and } b \in B\} = [a^L + b^L, a^U + b^U]$$

(ii)

$$\lambda A = \lambda[a^L, a^U] = \begin{cases} [\lambda a^L, \lambda a^U] & \text{if } \lambda \geq 0 \\ [\lambda a^U, \lambda a^L] & \text{if } \lambda < 0; \end{cases}$$

(iii)

$$A \times B = [\min_{ab}, \max_{ab}],$$

where

$$\min_{ab} = \min\{a^L b^L, a^L b^U, a^U b^L, a^U b^U\}$$

and

$$\max_{ab} = \max\{a^L b^L, a^L b^U, a^U b^L, a^U b^U\}$$

In view of (i) and (ii) we see that

$$-B = -[b^L, b^U] = [-b^U, -b^L] \text{ and } A - B = A + (-B) = [a^L - b^L, a^U - b^L].$$

Also the real number $a \in R$ can be regarded as a closed interval $A_a = [a, a]$, then we have for $B \in \mathcal{I}_c$

$$a + B = A_a + B = [a + b^L, a + b^U].$$

Note that the space \mathcal{I}_c is not a linear space with respect to the operations (i) and (ii), since it does not contain inverse elements.

3 Differentiation of interval valued functions

Definition 1. [20] Let X be open set in R . An interval-valued function $f : X \rightarrow \mathcal{I}_c$ is called weakly differentiable at x^* if the real-valued functions f^L and f^U are differentiable at x^* (in the usual sense).

Given $A, B \in \mathcal{I}_c$, if there exists $C \in \mathcal{I}_c$ such that $A = B + C$, then C is called the Hukuhara difference of A and B . We also write $C = A \ominus_H B$ when the Hukuhara difference C exists, which means that $a^L - b^L \leq a^U - b^U$ and $C = [a^L - b^L, a^U - b^U]$.

Proposition 1. [20] Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be two closed intervals in R . If $a^L - b^L \leq a^U - b^U$, then the Hukuhara difference C exists and $C = [a^L - b^L, a^U - b^U]$.

Definition 2. [20] Let X be an open set in R . An interval-valued function $f : X \rightarrow \mathcal{I}_c$ is called H -differentiable at x^* if there exists a closed interval $A(x^*) \in \mathcal{I}_c$ such that

$$\lim_{h \rightarrow 0+} \frac{f(x^* + h) \ominus_H f(x^*)}{h} \text{ and } \lim_{h \rightarrow 0+} \frac{f(x^*) \ominus_H f(x^* + h)}{h}$$

both exist and are equal to $A(x^*)$. In this case, $A(x^*)$ is called the H -derivative of f at x^* .

Proposition 2. [20] Let f be an interval-valued function defined on $X \subseteq R^n$. If f is H -differentiable at $x^* \in X$, then f is weakly differentiable at x^* .

Next we introduce twice differentiable interval valued functions and study some properties.

Definition 3. Let X be an open set in R^n , and let $x^* = (x_1^*, \dots, x_n^*) \in X$ be fixed. Then we say that f is twice weakly differentiable interval valued function at x^* if

f^L and f^U are twice differentiable functions at x^* (in usual sense). We denote by $\nabla^2 f$ the second differential of f , then we have

$$\begin{aligned}\nabla^2 f(x^*) &= \nabla(\nabla f(x))_{x=x^*} \\ &= \nabla(\nabla[f^L(x), f^U(x)])_{x=x^*} \\ &= \nabla([\nabla f^L(x), \nabla f^U(x)])_{x=x^*} \\ &= [\nabla^2 f^L(x), \nabla^2 f^U(x)]_{x=x^*} \\ &= \left[\left(\frac{\partial^2 f^L}{\partial x_i \partial x_j} \right)_{i,j}(x), \left(\frac{\partial^2 f^U}{\partial x_i \partial x_j} \right)_{i,j}(x) \right]_{x=x^*}.\end{aligned}$$

Definition 3 is illustrated by the following example.

Example 1. Consider the interval valued function

$$f(x_1, x_2) = [f^L = 2x_1 + x_2^2, f^U = x_1^2 + x_2^2 + 1]. \quad (1)$$

Therefore we have

$$\nabla f(x) = [(2, 2x_2), (2x_1, 2x_2)]^T$$

and

$$\nabla^2 f(x) = \left[\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right].$$

Definition 4. Let X be an open set in R^n , and let $x^* = (x_1^*, \dots, x_n^*) \in X$ be fixed. Then we say that f is twice H -differentiable interval valued function if f' is H -differentiable at x^* , where f' is H -derivative of f . We denote by $\nabla_H^2 f$ the second order H -differential of f , then we have

$$\begin{aligned}\nabla_H^2 f(x^*) &= \nabla_H(\nabla_H f(x))_{x=x^*} \\ &= \nabla_H \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x) \right)_{x=x^*}^T \\ &= \left(\nabla_H \left[\frac{\partial f^L}{\partial x_1}(x), \frac{\partial f^U}{\partial x_1}(x) \right], \dots, \nabla_H \left[\frac{\partial f^L}{\partial x_n}(x), \frac{\partial f^U}{\partial x_n}(x) \right] \right)_{x=x^*}^T \\ &= \begin{pmatrix} \left[\frac{\partial^2 f^L}{\partial^2 x_1}(x), \frac{\partial^2 f^U}{\partial^2 x_1}(x) \right] & \dots & \left[\frac{\partial^2 f^L}{\partial x_1 \partial x_n}(x), \frac{\partial^2 f^U}{\partial x_1 \partial x_n}(x) \right] \\ \vdots & \ddots & \vdots \\ \left[\frac{\partial^2 f^L}{\partial x_n \partial x_1}(x), \frac{\partial^2 f^U}{\partial x_n \partial x_1}(x) \right] & \dots & \left[\frac{\partial^2 f^L}{\partial^2 x_n}(x), \frac{\partial^2 f^U}{\partial^2 x_n}(x) \right] \end{pmatrix}_{n \times n, x=x^*}.\end{aligned} \quad (2)$$

Following example justifies the existence of twice H -differentiable interval valued function.

Example 2. Consider the interval valued function (1), then by definition we have

$$\nabla_H f(x) = ([2, 2x_1], [2x_2, 2x_2])^T$$

which exist for $x_1 \geq 1$. Therefore we have

$$\begin{aligned}\nabla_H^2 f(x) &= \nabla_H([2, 2x_1], [2x_2, 2x_2])^T \\ &= \begin{pmatrix} [0, 2] & [0, 0] \\ [0, 0] & [2, 2] \end{pmatrix}.\end{aligned}$$

The relation between twice weakly differentiable and twice H -differentiable interval valued functions is furnished as follows.

Proposition 3. *Let f be an interval-valued function defined on $X \subseteq R^n$. If f is twice H -differentiable at $x^* \in X$, then f is twice weakly differentiable at x^* .*

Proof. From (2) we have

$$\begin{aligned}\nabla_H^2 f(x^*) &= \left[\begin{pmatrix} \frac{\partial^2 f^L}{\partial^2 x_1}(x) & \dots & \frac{\partial^2 f^L}{\partial x_1 \partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f^L}{\partial x_n \partial x_1}(x) & \dots & \frac{\partial^2 f^L}{\partial^2 x_n}(x) \end{pmatrix}_{n \times n}, \begin{pmatrix} \frac{\partial^2 f^U}{\partial^2 x_1}(x) & \dots & \frac{\partial^2 f^U}{\partial x_1 \partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f^U}{\partial x_n \partial x_1}(x) & \dots & \frac{\partial^2 f^U}{\partial^2 x_n}(x) \end{pmatrix}_{n \times n} \right]_{x=x^*} \\ &= [\nabla^2 f^L(x), \nabla^2 f^U(x)]_{x=x^*} \\ &= \nabla^2 f(x^*).\end{aligned}$$

□

We authenticate Proposition 3 by following example.

Example 3. *From Example 2 we have*

$$\begin{aligned}\nabla_H^2 f(x) &= \begin{pmatrix} [0, 2] & [0, 0] \\ [0, 0] & [2, 2] \end{pmatrix} \\ &= \left[\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right] \\ &= [\nabla^2 f^L(x), \nabla^2 f^U(x)] \\ &= \nabla^2 f(x). \text{ (see Example 1).}\end{aligned}$$

The converse of Proposition 3 is not true in general, however we have the following result.

Proposition 4. *Let $f \in T$, be twice weakly differentiable function at x^* , with $(f^L)''(x^*) = a^{L'}(x^*)$ and $(f^U)''(x^*) = a^{U'}(x^*)$.*

1. *if $(f^L)'(x^*+h) - (f^L)'(x^*) \leq (f^U)'(x^*+h) - (f^U)'(x^*)$ and $(f^L)'(x^*) - (f^L)'(x^*-h) \leq (f^U)'(x^*) - (f^U)'(x^*-h)$ for every $h > 0$, then f is twice H -differentiable with second H -derivative $[a^{L'}(x^*), a^{U'}(x^*)]$.*

2. *if $a^{L'}(x^*) > a^{U'}(x^*)$, then f is not twice H -differentiable at x^* .*

Proof. The proof is similar as that of Proposition 4.3 of [20]. □

The existence of twice weakly differentiable interval valued functions which are not twice H -differentiable is proved by following example.

Example 4. *Consider $f : [0, 2] \rightarrow [x^3 + x^2 + 1, x^3 + 2x + 2]$ be an interval valued function defined on $[0, 2]$. Then f is twice weakly differentiable on $(0, 2)$ but f is not twice H -differentiable on $(0, 2)$ as $a^{L'}(x^*) > a^{U'}(x^*)$.*

4 Interval valued bonvex functions

Convexity is an important concept in studying the theory and methods of mathematical programming, which has been generalized in several ways. For differentiable functions numerous generalizations of convexity exist in the literature. An important concept so-called second order convexity for twice differentiable real valued functions was introduced in Mond [14], however Bector and Chandra [6] named them as bonvex functions. Now consider f to be real valued twice differentiable function, then for the definitions of (strictly) bonvexity, (strictly) pseudobonvexity and (strictly) quasibonvexity, one is referred to [3].

In this section, we introduce LU -bonvex, LU -pseudobonvex and LU -quasibonvex interval valued functions and their strict conditions. We consider T to be the set of all interval valued functions defined on $X \subseteq R^n$.

Definition 5. Let $f \in T$ be twice H -differentiable function at $x^* \in X$. If we have for every $x \in X$ and $P = (P_1, \dots, P_n)$ with $P_i \in \mathcal{I}_c$ such that $P_i^L \geq 0, i \in \Lambda_k$.

1.

$$f(x) \ominus_H f(x^*) + \frac{1}{2} P^T \nabla_H^2 f(x^*) P \succeq_{LU} \{ \nabla_H f(x^*) + \nabla_H^2 f(x^*) P \} (x - x^*)$$

then we say that f is LU -bonvex at x^* . We also say that f is strictly LU -bonvex at $x^* (\neq x)$ if the inequality is strict.

2. If

$$\begin{aligned} f(x) \ominus_H f(x^*) + \frac{1}{2} P^T \nabla_H^2 f(x^*) P &\preceq_{LU} [0, 0], \\ \Rightarrow \{ \nabla_H f(x^*) + \nabla_H^2 f(x^*) P \} (x - x^*) &\preceq_{LU} [0, 0] \end{aligned}$$

then we say that f is LU -quasibonvex at x^* . We also say that f is strictly LU -quasibonvex $x^* (\neq x)$ if the inequality is strict.

3. If

$$\begin{aligned} \{ \nabla_H f(x^*) + \nabla_H^2 f(x^*) P \} (x - x^*) &\succeq_{LU} [0, 0], \\ \Rightarrow f(x) \ominus_H f(x^*) + \frac{1}{2} P^T \nabla_H^2 f(x^*) P &\succeq_{LU} [0, 0] \end{aligned}$$

then we say that f is LU -pseudobonvex at x^* . We also say that f is strictly LU -pseudobonvex at $x^* (\neq x)$ if the inequality is strict.

Now we present some non-trivial examples which authenticates that the class of interval valued functions introduced in this section is non-empty.

Example 5. Consider an interval valued function $f(x) = [x^2 + 3x + 2, x^2 + 3x + 5], x \geq 0$. Then we have

$$\nabla_H f(x) = ([2x + 3, 2x + 3])$$

$$= ([3, 3])_{x=0}$$

and

$$\nabla_H^2 f(x) = ([2, 2])$$

we have

$$\begin{aligned} [x^2 + 3x + 2, x^2 + 3x + 5] \ominus_H [2, 5] + \frac{1}{2}([0, 1])^T [2, 2] [0, 1] &= [x^2 + 3x + 2, x^2 + 3x + 2] \\ &\succeq_{LU} ([3, 3] + [2, 2][0, 1])(x) \\ &= [3x, 5x] \end{aligned}$$

therefore f is LU -bonvex at $x = 0$.

Next consider another interval valued functions defined as

$$f(x_1, x_2) = [x_1^2 + x_2^2 + 3, x_1^2 + x_2^2 + 5], x \geq 0.$$

Then we have

$$\begin{aligned} \nabla_H f(x_1, x_2) &= ([2x_1, 2x_1], [2x_2, 2x_2])^T \\ &= ([4, 4], [4, 4])_{(x_1, x_2)=(2, 2)}^T \end{aligned}$$

and

$$\nabla_H^2 f(x) = \begin{pmatrix} [2, 2] & [0, 0] \\ [0, 0] & [2, 2] \end{pmatrix}$$

Now let

$$\begin{aligned} [x_1^2 + x_2^2 + 3, x_1^2 + x_2^2 + 5] \ominus_H [11, 13] + \frac{1}{2} \begin{pmatrix} [1, 1] \\ [1, 1] \end{pmatrix}^T \begin{pmatrix} [2, 2] & [0, 0] \\ [0, 0] & [2, 2] \end{pmatrix} \begin{pmatrix} [1, 1] \\ [1, 1] \end{pmatrix} \\ \preceq_{LU} [0, 0] \end{aligned}$$

then

$$\left(\begin{pmatrix} [4, 4] \\ [4, 4] \end{pmatrix} + \begin{pmatrix} [2, 2] & [0, 0] \\ [0, 0] & [2, 2] \end{pmatrix} \begin{pmatrix} [1, 1] \\ [1, 1] \end{pmatrix} \right) \begin{pmatrix} x_1 - 2 \\ x_2 - 2 \end{pmatrix} \preceq_{LU} [0, 0].$$

this shows that f is LU -quasibonvex at $(2, 2)$.

However if

$$\left(\begin{pmatrix} [4, 4] \\ [4, 4] \end{pmatrix} + \begin{pmatrix} [2, 2] & [0, 0] \\ [0, 0] & [2, 2] \end{pmatrix} \begin{pmatrix} [1, 1] \\ [1, 1] \end{pmatrix} \right) \begin{pmatrix} x_1 - 2 \\ x_2 - 2 \end{pmatrix} \succeq_{LU} [0, 0].$$

then

$$\begin{aligned} [x_1^2 + x_2^2 + 3, x_1^2 + x_2^2 + 5] \ominus_H [11, 13] + \frac{1}{2} \begin{pmatrix} [1, 1] \\ [1, 1] \end{pmatrix}^T \begin{pmatrix} [2, 2] & [0, 0] \\ [0, 0] & [2, 2] \end{pmatrix} \begin{pmatrix} [1, 1] \\ [1, 1] \end{pmatrix} \\ \succeq_{LU} [0, 0] \end{aligned}$$

this shows that f is LU -pseudobonvex at $(2, 2)$.

Proposition 5. Let $f \in T$ be twice H -differentiable function at x^* and $P = (P_1, \dots, P_n)$ with $P_i \in \mathcal{I}_c$ such that $P_i^L \geq 0, i \in \Lambda_n$.

1. if f is LU -bonvex at x^* then f^L and f^U are bonvex functions at x^* .
2. if f is LU -quasibonvex at x^* then f^L and f^U are quasibonvex functions at x^* .
3. if f is LU -pseudobonvex at x^* then f^L and f^U are pseudobonvex functions at x^* .

Proof. (i) Let f is LU -bonvex at x^* , then by definition we have

$$f(x) \ominus_H f(x^*) + \frac{1}{2} P^T \nabla_H^2 f(x^*) P \succeq_{LU} \{ \nabla_H f(x^*) + \nabla_H^2 f(x^*) P \} (x - x^*)$$

Since f is twice H -differentiable at x^* , then by Proposition 3 and Definition 3 f^L and f^U are twice differentiable at x^* . Also since $P_i^L \geq 0$, therefore we have

$$f^L(x) - f^L(x^*) + \frac{1}{2} P^{LT} \nabla^2 f^L(x^*) P^L \geq \{ \nabla f^L(x^*) + \nabla^2 f^L(x^*) P^L \} (x - x^*),$$

and

$$f^U(x) - f^U(x^*) + \frac{1}{2} P^{UT} \nabla^2 f^U(x^*) P^U \geq \{ \nabla f^U(x^*) + \nabla^2 f^U(x^*) P^U \} (x - x^*).$$

Therefore f^L and f^U are bonvex functions at x^* .

(ii) and (iii) follows by similar treatment. □

Note that the converse of Proposition 5 follows in the light of Proposition 4.

Proposition 6. Let $f \in T$ be twice H -differentiable function at x^* and $P = (P_1, \dots, P_n)$ with $P_i \in \mathcal{I}_c$ such that $P_i^L \geq 0, i \in \Lambda_n$.

1. if f is strictly LU -bonvex at x^* then either f^L or f^U or both are strictly bonvex functions at x^* .
2. if f is strictly LU -quasibonvex at x^* then either f^L or f^U or both are strictly quasibonvex functions at x^* .
3. if f is strictly LU -pseudobonvex at x^* then either f^L or f^U or both are strictly pseudobonvex functions at x^* .

Proof. Proof is same as that of Proposition 5. □

Remark 1. If we assume that $f^L = f^U$, then bonvexity comes as a sub-case of LU -bonvexity, and similarly for quasi and pseudobonvexity.

5 Solution concept and necessary conditions

In this section we shall propose solution concept and derive some necessary conditions for problem (IP). We define by S_{IP} the set of feasible solutions of (IP).

Definition 6. Let $x^* \in S_{IP}$. We say that x^* is an efficient solution of (IP) if there exist no $\hat{x} \in S_{IP}$, such that

$$f_i(\hat{x}) \preceq_{LU} f_i(x^*), i \in \Lambda_k \text{ and } f_h(\hat{x}) \prec_{LU} f_h(x^*), \text{ for at least one index } h.$$

An efficient solution x^* is said to be properly efficient solution of (IP) if there exist scalar $M > 0$, such that for all $i \in \Lambda_k, f_i(x) \prec_{LU} f_i(x^*)$ and $x \in S_{IP}$ imply that

$$f_i(x^*) \ominus_H f_i(x) \preceq_{LU} M\{f_h(x) \ominus_H f_h(x^*)\}$$

for atleast one index $h \in \Lambda_k - i$ such that $f_h(x^*) \prec_{LU} f_h(x)$.

Theorem 1. (Mond et al. [16]) Let x^* be a properly efficient solution of (P) (see, [3]) at which constraint qualification [15] is satisfied. Then there exist $\lambda^* \in R^k, u^* \in R^m$ and $v_i^* \in R^n, i \in \Lambda_K$ such that

$$\begin{aligned} \sum_{i=1}^k \lambda_i^* (f_i(x^*) + B_i v_i^*) + \nabla u^{*T} g(x^*) &= 0, \\ u^{*T} g(x^*) &= 0, \\ (x^{*T} B_i x^*)^{\frac{1}{2}} &= x^{*T} B_i v_i^*, i \in \Lambda_k, \\ v_i^{*T} B_i v_i^* &\leq 1, i \in \Lambda_k, \\ \lambda^* > 0, \sum_{i=1}^k \lambda_i^* &= 1, u^* \geq 0. \end{aligned}$$

Now we present the necessary conditions for problem(IP). Consider the following **constraint qualification CQ1**

$$\begin{aligned} d^T \nabla_H g_j(x^*) &\succeq_{LU} [0, 0], j \in J_0(x^*) \\ d^T \nabla_H f_i(x^*) + d^T B_i x^* / (x^{*T} B_i x^*)^{\frac{1}{2}} &\preceq_{LU} [0, 0], \text{ if } x^{*T} B_i x^* > 0 \\ d^T \nabla_H f_i(x^*) + (d^T B_i d)^{\frac{1}{2}} &\preceq_{LU} [0, 0], \text{ if } x^{*T} B_i x^* = 0 \end{aligned}$$

Theorem 2. Let x^* be a properly efficient solution of (IP) at which a constraint qualification CQ1 is satisfied. Then there exist $\lambda^* \in R^k, u^* \in R^m$ and $v_i^* \in R^n, i \in \Lambda_K$ such that

$$\begin{aligned} \sum_{i=1}^k \lambda_i^* (\nabla_H f_i(x^*) + B_i v_i^*) + \nabla_H u^{*T} g(x^*) &= [0, 0], \\ u^{*T} g(x^*) &= [0, 0], \\ (x^{*T} B_i x^*)^{\frac{1}{2}} &= x^{*T} B_i v_i^*, i \in \Lambda_k, \\ v_i^{*T} B_i v_i^* &\leq 1, i \in \Lambda_k, \\ \lambda^* > 0, \sum_{i=1}^k \lambda_i^* &= 1, u^* \geq 0. \end{aligned}$$

Proof. Since x^* is properly efficient solution of (IP) at which a constraint qualification $CQ1$ is satisfied. Then using the property of intervals and twice H -derivative, for $0 < \xi_i^L, \xi_i^U \in R, i \in \Lambda_k$ with $\xi_i^L + \xi_i^U = 1, i \in \Lambda_k$, we have

$CQ2$

$$\begin{aligned} d^T \nabla g_j^L(x^*) &> 0, j \in J_0(x^*) \\ d^T \nabla g_j^U(x^*) &> 0, j \in J_0(x^*) \\ d^T (\xi_i^L \nabla f_i^L(x^*) + \xi_i^U \nabla f_i^U(x^*)) + d^T B_i x^* / (x^{*T} B_i x^*)^{\frac{1}{2}} &< 0, \text{ if } x^{*T} B_i x^* > 0 \\ d^T (\xi_i^L \nabla f_i^L(x^*) + \xi_i^U \nabla f_i^U(x^*)) + (d^T B_i d)^{\frac{1}{2}} &< 0, \text{ if } x^{*T} B_i x^* = 0 \end{aligned}$$

Further using the property of intervals and twice H -derivative, for $0 < \xi_i^L, \xi_i^U \in R, i \in \Lambda_k$ with $\xi_i^L + \xi_i^U = 1, i \in \Lambda_k$ we have new conditions as

$$\begin{aligned} \sum_{i=1}^k \lambda_i^* ((\xi_i^L \nabla f_i^L(x^*) + \xi_i^U \nabla f_i^U(x^*)) + B_i v_i^*) + \nabla u^{*T} (g^L(x^*) + g^U(x^*)) &= 0, \\ u^{*T} g^L(x^*) &= 0, \\ u^{*T} g^U(x^*) &= 0, \\ (x^{*T} B_i x^*)^{\frac{1}{2}} = x^{*T} B_i v_i^*, i \in \Lambda_k, \\ v_i^{*T} B_i v_i^* &\leq 1, i \in \Lambda_k, \\ \lambda^* > 0, \sum_{i=1}^k \lambda_i^* &= 1, u^* \geq 0. \end{aligned}$$

Now using constraint qualification $CQ2$ the above conditions are justified by Theorem 1 for the problem (say $(IP1)$) heaving objective function $(\xi_1^L f_1^L(x) + \xi_1^U f_1^U(x), \dots, \xi_k^L f_k^L(x) + \xi_k^U f_k^U(x))$ and constraint functions $g_j^L(x), g_j^U(x) \leq 0, j \in \Lambda_m$. Now it is easy to see that the optimal solutions of (IP) and $(IP1)$ are same. This completes the proof. \square

6 Mangasarian type duality

In this section, we propose the following Mangasarian type dual of primal problem (IP) .

(MSD) V-maximize

$$\begin{aligned} &\left(f_1(y) + u^T g(y) + y^T B_1 v_1 \ominus_H \frac{1}{2} P^T \nabla_H^2 \{f_1(y) + u^T g(y)\} P, \dots, \right. \\ &\left. f_k(y) + u^T g(y) + y^T B_k v_k \ominus_H \frac{1}{2} P^T \nabla_H^2 \{f_k(y) + u^T g(y)\} P \right) \end{aligned}$$

subject to

$$\sum_{i=1}^k \lambda_i (\nabla_H f_i(y) + \nabla_H^2 f_i(y) P + B_i v_i) + \nabla_H u^T g(y) + \nabla_H^2 u^T g(y) P = [0, 0] \quad (3)$$

$$v_i^T B_i v_i \leq 1, i \in \Lambda_k \quad (4)$$

$$\lambda > 0, \sum_{i=1}^r \lambda_i = 1 \quad (5)$$

$u = (u_1, \dots, u_m)^T \geq 0, g = (g_1, \dots, g_m)$ such that $g_j = [g_j^L, g_j^U], j = 1, \dots, m, P = (P_1, \dots, P_n)$ with $P_i \in \mathcal{I}_c$ such that $P_i^L \geq 0, i \in \Lambda_k$. and $y, v_i \in R^n$.

We define by S_{MSD} the set of all feasible solutions of (MSD) , therefore if $z \in S_{MSD}$ then $z = (y, u, v, \lambda, P)$, such that $v \in R^k$ with $v_i \in R^n$, and $P_i \in \mathcal{I}_c$ such that $P_i^L \geq 0, i \in \Lambda_k$. We shall use the following generalized Schwartz inequality:

$$x^T A z \leq (x^T A x)^{1/2} (z^T A z)^{1/2},$$

where $x, z \in R^n$ and A is positive semidefinite symmetric matrix of order n .

Theorem 3. (weak duality) Let $x \in S_{IP}$ and $z \in S_{MSD}$. Assume that $f_i(\cdot) + (\cdot)^T B_i v_i, i \in \Lambda_k$ and $g_j(\cdot), j \in \Lambda_m$ are LU -bonvex at y , then the following can not hold.

$$f_i(x) + (x^T B_i x)^{\frac{1}{2}} \preceq_{LU} f_i(y) + u^T g(y) + y^T B_i v_i \ominus_H \frac{1}{2} P^T \{ \nabla_H^2 f_i(y) + u^T g(y) \} P, i \in \Lambda_k. \quad (6)$$

and

$$f_h(x) + (x^T B_h x)^{\frac{1}{2}} \prec_{LU} f_h(y) + u^T g(y) + y^T B_h v_h \ominus_H \frac{1}{2} P^T \{ \nabla_H^2 f_h(y) + u^T g(y) \} P, \quad (7)$$

for at least one index h .

Proof. From (3) we have

$$\begin{aligned} \sum_{i=1}^k \lambda_i (\nabla f_i^L(y) + \nabla^2 f_i^L(y) P^L + B_i v_i) + \nabla u^T g^L(y) + \nabla^2 u^T g^L(y) P^L &= 0. \\ \sum_{i=1}^k \lambda_i (\nabla f_i^U(y) + \nabla^2 f_i^U(y) P^U + B_i v_i) + \nabla u^T g^U(y) + \nabla^2 u^T g^U(y) P^U &= 0. \end{aligned}$$

Adding we get,

$$\begin{aligned} \sum_{i=1}^k \lambda_i (\nabla f_i^L(y) + \nabla f_i^U(y) + \nabla^2 f_i^L(y) P^L + \nabla^2 f_i^U(y) P^U + 2B_i v_i) + \nabla u^T g^L(y) \\ + \nabla u^T g^U(y) + \nabla^2 u^T g^L(y) P^L + \nabla^2 u^T g^U(y) P^U = 0. \end{aligned} \quad (8)$$

If possible let (6) and (7) holds then by definition we have

$$\begin{cases} f_i^L(x) + (x^T B_i x)^{\frac{1}{2}} \leq f_i^L(y) + u^T g^L(y) + y^T B_i v_i - \frac{1}{2} P^L T \nabla^2 \{ f_i^L(y) + u^T g^L(y) \} P^L. \\ f_i^U(x) + (x^T B_i x)^{\frac{1}{2}} \leq f_i^U(y) + u^T g^U(y) + y^T B_i v_i - \frac{1}{2} P^U T \nabla^2 \{ f_i^U(y) + u^T g^U(y) \} P^U. \end{cases}$$

for $i \in \Lambda_k$, and

$$\begin{cases} f_h^L(x) + (x^T B_h x)^{\frac{1}{2}} < f_h^L(y) + u^T g^L(y) + y^T B_h v_h - \frac{1}{2} P^L T \nabla^2 \{ f_h^L(y) + u^T g^L(y) \} P^L. \\ f_h^U(x) + (x^T B_h x)^{\frac{1}{2}} \leq f_h^U(y) + u^T g^U(y) + y^T B_h v_h - \frac{1}{2} P^U T \nabla^2 \{ f_h^U(y) + u^T g^U(y) \} P^U. \end{cases}$$

or

$$\begin{cases} f_h^L(x) + (x^T B_h x)^{\frac{1}{2}} \leq f_h^L(y) + u^T g^L(y) + y^T B_h v_h - \frac{1}{2} P^{LT} \nabla^2 \{f_h^L(y) + u^T g^L(y)\} P^L. \\ f_h^U(x) + (x^T B_h x)^{\frac{1}{2}} < f_h^U(y) + u^T g^U(y) + y^T B_h v_h - \frac{1}{2} P^{UT} \nabla^2 \{f_h^U(y) + u^T g^U(y)\} P^U. \end{cases}$$

or

$$\begin{cases} f_h^L(x) + (x^T B_h x)^{\frac{1}{2}} < f_h^L(y) + u^T g^L(y) + y^T B_h v_h - \frac{1}{2} P^{LT} \nabla^2 \{f_h^L(y) + u^T g^L(y)\} P^L. \\ f_h^U(x) + (x^T B_h x)^{\frac{1}{2}} < f_h^U(y) + u^T g^U(y) + y^T B_h v_h - \frac{1}{2} P^{UT} \nabla^2 \{f_h^U(y) + u^T g^U(y)\} P^U. \end{cases}$$

for atleast one index h .

This yields for $\lambda = (\lambda_1, \dots, \lambda_r); \lambda_i > 0$

$$\begin{aligned} & \sum_{i=1}^k \lambda_i \left\{ \left(f_i^L(x) + (x^T B_i x)^{\frac{1}{2}} \right) + \left(f_i^U(x) + (x^T B_i x)^{\frac{1}{2}} \right) \right\} < \\ & \sum_{i=1}^k \lambda_i \left\{ f_i^L(y) + y^T B_i v_i - \frac{1}{2} P^{LT} \nabla^2 f_i^L(y) P^L \right\} + u^T g^L(y) - \frac{1}{2} P^{LT} \nabla^2 u^T g^L(y) P^L + \\ & \sum_{i=1}^k \lambda_i \left\{ f_i^U(y) + y^T B_i v_i - \frac{1}{2} P^{UT} \nabla^2 f_i^U(y) P^U \right\} + u^T g^U(y) - \frac{1}{2} P^{UT} \nabla^2 u^T g^U(y) P^U. \end{aligned} \quad (9)$$

From the hypothesis that $f_i(\cdot) + (\cdot)^T B_i x, i \in \Lambda_k$ and $g_j, j \in \Lambda_m$ are LU -bonvex at y , we have

$$\begin{aligned} f_i(x) + x^T B_i v_i \ominus_H (f_i(y) + y^T B_i v_i) + \frac{1}{2} P^T \nabla_H^2 f_i(y) P \succ_{LU} \\ (\nabla_H f_i(y) + \nabla_H^2 f_i(y) P + B_i v_i) (x - y), i \in \Lambda_k \end{aligned} \quad (10)$$

and

$$g_j(x) \ominus_H g_j(y) + \frac{1}{2} P^T \nabla_H^2 g_j(y) P \succ_{LU} (\nabla_H g_j(y) + \nabla_H^2 g_j(y) P) (x - y), j \in \Lambda_m. \quad (11)$$

After multiplying (10) by $\lambda_i, i \in \Lambda_k$ and (11) by $u_j, j \in \Lambda_m$ and adding, yields

$$\begin{aligned} & \sum_{i=1}^k \lambda_i \left\{ f_i^L(x) + x^T B_i v_i - f_i^L(y) - y^T B_i v_i + \frac{1}{2} P^{LT} \nabla^2 f_i^L(y) P^L \right\} + u^T g^L(x) - u^T g^L(y) + \\ & \frac{1}{2} P^{LT} \nabla^2 u^T g^L(y) P^L + \sum_{i=1}^k \lambda_i \left\{ f_i^U(x) + x^T B_i v_i - f_i^U(y) - y^T B_i v_i + \frac{1}{2} P^{UT} \nabla^2 f_i^U(y) P^U \right\} + \\ & u^T g^U(x) - u^T g^U(y) + \frac{1}{2} P^{UT} \nabla^2 u^T g^U(y) P^U \geq \\ & \left\{ \sum_{i=1}^k \lambda_i (\nabla f_i^L(y) + \nabla^2 f_i^L(y) P^L + B_i v_i) + \nabla u^T g^L(y) + \nabla^2 u^T g^L(y) P^L \right\} (x - y) + \\ & \left\{ \sum_{i=1}^k \lambda_i (\nabla f_i^U(y) + \nabla^2 f_i^U(y) P^U + B_i v_i) + \nabla u^T g^U(y) + \nabla^2 u^T g^U(y) P^U \right\} (x - y). \end{aligned}$$

Now by (4), (9), Schewartz inequality and $u^T g(x) \preceq_{LU} [0, 0]$, we get

$$\sum_{i=1}^k \lambda_i \left\{ \nabla f_i^L(y) + \nabla^2 f_i^L(y) P^L + \nabla f_i^U(y) + \nabla^2 f_i^U(y) P^U + 2Bv_i \right\} + \nabla u^T g^L(y) + \nabla u^T g^U(y) + \nabla^2 u^T g^L(y) P^L + \nabla^2 u^T g^U(y) P^U < 0.$$

which is a contradiction to (8). This completes the proof. \square

Theorem 4. (Strong duality theorem) Assume that x^* is properly efficient solution of problem (IP) at which constraint qualification CQ1 is satisfied. Then there exist $\lambda^* \in R^k, u^* \in R^m$ and $v_i^* \in R^n, i \in \Lambda_k$, such that $(x^*, u^*, v_i^*, \lambda^*, P^{*T} = ([0, 0], \dots, [0, 0]))$ is feasible for (MSD) and the corresponding objective values of (IP) and (MSD) are equal. Moreover assume that the weak duality between (IP) and (MSD) in Theorem are satisfied, then $(x^*, u^*, v_i^*, \lambda^*, P^{*T} = ([0, 0], \dots, [0, 0]))$ is an efficient solution of (MSD).

Proof. Since x^* is efficient solution of problem (IP) at which constraint qualification CQ1 is satisfied. Then by Theorem 2 there exist $\lambda^* \in R^k, u^* \in R^m$ and $v_i^* \in R^n, i \in \Lambda_k$, such that

$$\sum_{i=1}^k \lambda_i^* (\nabla_H f_i(x^*) + B_i v_i^*) + \nabla_H u^{*T} g(x^*) = [0, 0],$$

$$u^{*T} g(x^*) = [0, 0],$$

$$(x^{*T} B_i x^*)^{\frac{1}{2}} = x^{*T} B_i v_i^*, i \in \Lambda_k,$$

$$v_i^{*T} B_i v_i^* \leq 1, i \in \Lambda_k,$$

$$\lambda^* > 0, \sum_{i=1}^k \lambda_i^* = 1, u^* \geq 0.$$

Which yields that $(x^*, u^*, v_i^*, \lambda^*, P^{*T} = ([0, 0], \dots, [0, 0])) \in S_{MSD}$ and the corresponding objective values of (IP) and (MSD) are equal.

Now let $(x^*, u^*, v_i^*, \lambda^*, P^{*T} = ([0, 0], \dots, [0, 0]))$ is not efficient solution of dual problem (MSD), then by Definition there exist $(y^*, u^*, v_i^*, \lambda^*, P^*) \in S_{MSD}$, such that

$$f_i(x^*) + x^{*T} B_i v_i^* + u^{*T} g(x^*) \preceq_{LU} f_i(y^*) + u^{*T} g(y^*) + y^{*T} B_i v_i^* \\ \ominus_H \frac{1}{2} P^{*T} \nabla_H^2 \{f_i(y^*) + u^{*T} g(y^*)\} P^*, i \in \Lambda_k$$

and

$$f_i(x^*) + x^{*T} B_i v_i^* + u^{*T} g(x^*) \prec_{LU} f_i(y^*) + u^{*T} g(y^*) + y^{*T} B_i v_i^* \\ \ominus_H \frac{1}{2} P^{*T} \nabla_H^2 \{f_i(y^*) + u^{*T} g(y^*)\} P^*,$$

for atleast one index h .

Now using $(x^{*T} B_i x^*)^{\frac{1}{2}} = x^{*T} B_i v_i^*, i \in \Lambda_k$ and $u^{*T} g(y^*) = [0, 0]$, we get a contradiction to weak duality theorem. Therefore $(x^*, u^*, v_i^*, \lambda^*, P^{*T} = ([0, 0], \dots, [0, 0]))$ is an efficient solution of dual problem (MSD). \square

Theorem 5. (Strict converse duality) Let $x^* \in S_{IP}$ and $z^* \in S_{MSP}$ such that

$$\sum_{i=1}^k \lambda_i^* \{f_i(x^*) + x^{*T} B_i v_i^*\} \preceq_{LU} \sum_{i=1}^k \lambda_i^* \left\{ f_i(y^*) + y^{*T} B_i v_i^* \ominus_H \frac{1}{2} P^{*T} \nabla_H^2 f_i(y^*) P^* \right\} \\ + u^{*T} g(y^*) \ominus_H \frac{1}{2} P^{*T} \nabla_H^2 u^{*T} g(y^*) P^*. \quad (12)$$

Assume that $f_i(\cdot) + (\cdot)^T B_i v_i^*, i \in \Lambda_k$ are strictly LU -bonvex at y^* and $g_j(\cdot), j \in \Lambda_m$ is LU -bonvex at y^* then $x^* = y^*$.

Proof. If possible let $x^* \neq y^*$. Now since $f_i(\cdot) + (\cdot)^T B_i v_i^*, i \in \Lambda_k$ are strictly LU -bonvex at y^* , we have

$$f_i(x^*) + x^{*T} B_i v_i^* \ominus_H (f_i(y^*) + y^{*T} B_i v_i^*) + \frac{1}{2} P^{*T} \nabla_H^2 f_i(y^*) P^* \succ_{LU} \\ (\nabla_H f_i(y^*) + \nabla_H^2 f_i(y^*) P^* + B_i v_i^*) (x^* - y^*), i \in \Lambda_k. \quad (13)$$

and

$$g_j(x^*) \ominus_H g_j(y^*) + \frac{1}{2} P^{*T} \nabla_H^2 g_j(y^*) P^* \succeq_{LU} (\nabla_H g_j(y^*) + \nabla_H^2 g_j(y^*) P^*) (x^* - y^*), j \in \Lambda_m. \quad (14)$$

Now multiplying (13) by $\lambda_i^*, i \in \Lambda_k$ and (14) by $u_j^*, j \in \Lambda_m$ and then summing up we get

$$\sum_{i=1}^k \lambda_i^* \{f_i(x^*) + x^{*T} B_i v_i^*\} + u^{*T} g(x^*) \ominus_H \sum_{i=1}^k \lambda_i^* \left\{ f_i(y^*) + y^{*T} B_i v_i^* \ominus_H \right. \\ \left. \frac{1}{2} P^{*T} \nabla_H^2 f_i(y^*) P^* \right\} \ominus_H u^{*T} g(y^*) + \frac{1}{2} P^{*T} \nabla_H^2 u^{*T} g(y^*) P^* \succ_{LU} \\ \left\{ \sum_{i=1}^k \lambda_i^* (\nabla_H f_i(y^*) + \nabla_H^2 f_i(y^*) P^* + B_i v_i^*) + \nabla_H u^{*T} g(y^*) + \nabla_H^2 u^{*T} g(y^*) P^* \right\} (x^* - y^*).$$

The above inequality on using (3) and $u^{*T} g(x^*) \preceq_{LU} [0, 0]$ gives

$$\sum_{i=1}^k \lambda_i^* \{f_i(x^*) + x^{*T} B_i v_i^*\} \succ_{LU} \sum_{i=1}^k \lambda_i^* \left\{ f_i(y^*) + y^{*T} B_i v_i^* \ominus_H \right. \\ \left. \frac{1}{2} P^{*T} \nabla_H^2 f_i(y^*) P^* \right\} + u^{*T} g(y^*) \ominus_H \frac{1}{2} P^{*T} \nabla_H^2 u^{*T} g(y^*) P^*.$$

which is a contradiction to (12). Hence $x^* = y^*$ □

7 Conclusions

This paper represents the study of nondifferentiable vector problem in which objective functions and constraints are interval valued. Firstly the twice H -differentiable interval valued functions are introduced, secondly the concepts of LU -bonvexity, LU -quasibonvexity and LU -pseudobonvexity are introduced, thirdly the necessary conditions for proposed solution concept are obtained. And lastly the Mangasarian

type dual is proposed and the corresponding duality results are obtained. Although the interval valued equality constraints are not considered in this paper, the similar methodology proposed in this paper can also be used to handle the interval valued equality constraints. However it will be interesting to study the Mond-Weir type duality results [1] for the problem (IP). Future research is oriented to consider the uncertain environment in order to study the optimality conditions involving Fuzzy parameters.

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ADDITIVE-QUADRATIC ρ -FUNCTIONAL INEQUALITIES IN β -HOMOGENEOUS NORMED SPACES

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ABSTRACT. In this paper, we solve the following additive-quadratic ρ -functional inequalities

$$\begin{aligned} & \|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \\ & \leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right) \right\|, \end{aligned} \quad (0.1)$$

where ρ is a fixed complex number with $|\rho| < 1$, and

$$\begin{aligned} & \left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right\| \\ & \leq \|\rho(f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y))\|, \end{aligned} \quad (0.2)$$

where ρ is a fixed complex number with $|\rho| < \frac{1}{2}$, and prove the Hyers-Ulam stability of the additive-quadratic ρ -functional inequalities (0.1) and (0.2) in β -homogeneous complex Banach spaces and prove the Hyers-Ulam stability of additive-quadratic ρ -functional equations associated with the additive-quadratic ρ -functional inequalities (0.1) and (0.2) in β -homogeneous complex Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

The stability problem of functional equations originated from a question of Ulam [24] concerning the stability of group homomorphisms. The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [11] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [15] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. The functional equation $f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called the *Jensen equation*.

The functional equation $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [23] for mappings $f: E_1 \rightarrow E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. The functional equation $2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) = f(x) + f(y)$ is called a *Jensen type quadratic equation*. The stability problems of various functional equations have been extensively investigated by a number of authors (see [1, 4, 5, 13, 14, 18, 19, 20, 21, 22]).

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In [9], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\| \quad (1.1)$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(xy) + f(xy^{-1}).$$

See also [16]. Gilányi [10] and Fechner [7] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [12] proved the Hyers-Ulam stability of additive functional inequalities.

Definition 1.1. Let X be a linear space. A nonnegative valued function $\|\cdot\|$ is an F -norm if it satisfies the following conditions:

(FN₁) $\|x\| = 0$ if and only if $x = 0$;

(FN₂) $\|\lambda x\| = \|x\|$ for all $x \in X$ and all λ with $|\lambda| = 1$;

(FN₃) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$;

(FN₄) $\|\lambda_n x\| \rightarrow 0$ provided $\lambda_n \rightarrow 0$;

(FN₅) $\|\lambda x_n\| \rightarrow 0$ provided $x_n \rightarrow 0$.

Then $(X, \|\cdot\|)$ is called an F^* -space. An F -space is a complete F^* -space.

An F -norm is called β -homogeneous ($\beta > 0$) if $\|tx\| = |t|^\beta \|x\|$ for all $x \in X$ and all $t \in \mathbb{C}$ (see [17]).

In Section 2, we solve the additive-quadratic ρ -functional inequality (0.1) and prove the Hyers-Ulam stability of the additive-quadratic ρ -functional inequality (0.1) in β -homogeneous complex Banach spaces. We moreover prove the Hyers-Ulam stability of an additive-quadratic ρ -functional equation associated with the additive-quadratic ρ -functional inequality (0.1) in β -homogeneous complex Banach spaces.

In Section 3, we solve the additive-quadratic ρ -functional inequality (0.2) and prove the Hyers-Ulam stability of the additive-quadratic ρ -functional inequality (0.2) in β -homogeneous complex Banach spaces. We moreover prove the Hyers-Ulam stability of an additive-quadratic ρ -functional equation associated with the additive-quadratic ρ -functional inequality (0.2) in β -homogeneous complex Banach spaces.

Throughout this paper, let β_1, β_2 be positive real numbers with $\beta_1 \leq 1$ and $\beta_2 \leq 1$. Assume that X is a β_1 -homogeneous real or complex normed space with norm $\|\cdot\|$ and that Y is a β_2 -homogeneous complex Banach space with norm $\|\cdot\|$.

2. ADDITIVE-QUADRATIC ρ -FUNCTIONAL INEQUALITY (0.1)

Throughout this section, assume that ρ is a fixed complex number with $|\rho| < 1$.

In this section, we investigate the additive-quadratic ρ -functional inequality (0.1) in β -homogeneous complex Banach spaces.

Lemma 2.1. *An even mapping $f : X \rightarrow Y$ satisfies*

$$\begin{aligned} & \|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \\ & \leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right) \right\| \end{aligned} \quad (2.1)$$

for all $x, y \in X$ if and only if $f : X \rightarrow Y$ is quadratic.

Proof. Assume that $f : X \rightarrow Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $\|2f(0)\| \leq |\rho|^{\beta_2} \|2f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.1), we get $\|f(2x) - 4f(x)\| \leq 0$ and so $f(2x) = 4f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{4}f(x) \quad (2.2)$$

ADDITIVE-QUADRATIC ρ -FUNCTIONAL INEQUALITIES

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} & \|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \\ & \leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right) \right\| \\ & = \frac{|\rho|^{\beta_2}}{2^{\beta_2}} \|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$.

The converse is obviously true. \square

Corollary 2.2. *An even mapping $f : X \rightarrow Y$ satisfies*

$$\begin{aligned} & f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y) \\ & = \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right) \end{aligned} \quad (2.3)$$

for all $x, y \in X$ if and only if $f : X \rightarrow Y$ is quadratic.

The functional equation (2.3) is called an *additive-quadratic ρ -functional equation*.

We prove the Hyers-Ulam stability of the additive-quadratic ρ -functional inequality (2.1) in β -homogeneous complex Banach spaces for an even mapping case.

Theorem 2.3. *Let $r > \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be an even mapping such that*

$$\begin{aligned} & \|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \\ & \leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right) \right\| + \theta(\|x\|^r + \|y\|^r) \end{aligned} \quad (2.4)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{2^{\beta_1 r} - 4^{\beta_2}} \|x\|^r \quad (2.5)$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (2.4), we get $\|2f(0)\| \leq |\rho|^{\beta_2} \|2f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.4), we get

$$\|f(2x) - 4f(x)\| \leq 2\theta \|x\|^r \quad (2.6)$$

for all $x \in X$. So $\|f(x) - 4f(\frac{x}{2})\| \leq \frac{2}{2^{\beta_1 r}} \theta \|x\|^r$ for all $x \in X$. Hence

$$\left\| 4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| \leq \sum_{j=l}^{m-1} \left\| 4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \leq \frac{2}{2^{\beta_1 r}} \sum_{j=l}^{m-1} \frac{4^{\beta_2 j}}{2^{\beta_1 r j}} \theta \|x\|^r \quad (2.7)$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.7) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \rightarrow Y$ by

$$Q(x) := \lim_{n \rightarrow \infty} 4^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (2.7), we get (2.5).

Since $f : X \rightarrow Y$ is even, the mapping $Q : X \rightarrow Y$ is even.

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It follows from (2.4) that

$$\begin{aligned}
& \|Q(x+y) + Q(x-y) - 2Q(x) - Q(y) - Q(-y)\| \\
&= \lim_{n \rightarrow \infty} 4^{\beta_2 n} \left\| f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{-y}{2^n}\right) \right\| \\
&\leq \lim_{n \rightarrow \infty} 4^{\beta_2 n} |\rho|^{\beta_2} \left(\left\| 2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - \frac{3}{2}f\left(\frac{x}{2^n}\right) + \frac{1}{2}f\left(\frac{-x}{2^n}\right) \right. \right. \\
&\quad \left. \left. - \frac{1}{2}f\left(\frac{y}{2^n}\right) - \frac{1}{2}f\left(\frac{-y}{2^n}\right) \right\| \right) + \lim_{n \rightarrow \infty} \frac{4^{\beta_2 n} \theta}{2^{\beta_1 n r}} (\|x\|^r + \|y\|^r) \\
&= |\rho|^{\beta_2} \left\| 2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - \frac{3}{2}Q(x) + \frac{1}{2}Q(-x) - \frac{1}{2}Q(y) - \frac{1}{2}Q(-y) \right\|
\end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned}
& \|Q(x+y) + Q(x-y) - 2Q(x) - Q(y) - Q(-y)\| \\
&\leq \left\| \rho \left(2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - \frac{3}{2}Q(x) + \frac{1}{2}Q(-x) - \frac{1}{2}Q(y) - \frac{1}{2}Q(-y) \right) \right\|
\end{aligned}$$

for all $x, y \in X$. By Lemma 2.1, the mapping $Q : X \rightarrow Y$ is quadratic.

Now, let $T : X \rightarrow Y$ be another quadratic mapping satisfying (2.5). Then we have

$$\begin{aligned}
\|Q(x) - T(x)\| &= 4^{\beta_2 n} \left\| Q\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right) \right\| \\
&\leq 4^{\beta_2 n} \left(\left\| Q\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| + \left\| T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| \right) \\
&\leq \frac{4 \cdot 4^{\beta_2 n}}{(2^{\beta_1 r} - 4^{\beta_2}) 2^{\beta_1 n r}} \theta \|x\|^r,
\end{aligned}$$

which tends to zero as $n \rightarrow \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q . Thus the mapping $Q : X \rightarrow Y$ is a unique quadratic mapping satisfying (2.5). \square

Theorem 2.4. Let $r < \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be an even mapping satisfying (2.4). Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{4^{\beta_2} - 2^{\beta_1 r}} \|x\|^r \quad (2.8)$$

for all $x \in X$.

Proof. It follows from (2.6) that $\left\| f(x) - \frac{1}{4}f(2x) \right\| \leq \frac{2\theta}{4^{\beta_2}} \|x\|^r$ for all $x \in X$. Hence

$$\left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^m} f(2^m x) \right\| \leq \sum_{j=l}^{m-1} \left\| \frac{1}{4^j} f(2^j x) - \frac{1}{4^{j+1}} f(2^{j+1} x) \right\| \leq \sum_{j=l}^{m-1} \frac{2^{\beta_1 r j}}{4^{\beta_2 j}} \frac{2\theta}{4^{\beta_2}} \|x\|^r \quad (2.9)$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.9) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ converges. So one can define the mapping $Q : X \rightarrow Y$ by

$$Q(x) := \lim_{n \rightarrow \infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (2.9), we get (2.8).

The rest of the proof is similar to the proof of Theorem 2.3. \square

ADDITIVE-QUADRATIC ρ -FUNCTIONAL INEQUALITIES

Lemma 2.5. *An odd mapping $f : X \rightarrow Y$ satisfies (2.1) if and only if $f : X \rightarrow Y$ is additive.*

Proof. Since $f : X \rightarrow Y$ is an odd mapping, $f(0) = 0$.

Assume that $f : X \rightarrow Y$ satisfies (2.1).

Letting $y = x$ in (2.1), we get $\|f(2x) - 2f(x)\| \leq 0$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2}f(x) \quad (2.10)$$

for all $x \in X$.

It follows from (2.1) and (2.10) that

$$\begin{aligned} & \|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \\ & \leq \left\| \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right) \right\| \\ & = |\rho|^{\beta_2} \|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) \quad (2.11)$$

for all $x, y \in X$. Letting $z = x + y$ and $w = z - y$ in (2.11), we get

$$f(z) + f(w) = 2f\left(\frac{z+w}{2}\right) = f(z+w)$$

for all $z, w \in X$.

The converse is obviously true. \square

Corollary 2.6. *An odd mapping $f : X \rightarrow Y$ satisfies (2.3) if and only if $f : X \rightarrow Y$ is additive.*

We prove the Hyers-Ulam stability of the additive-quadratic ρ -functional inequality (2.1) in β -homogeneous complex Banach spaces for an odd mapping case.

Theorem 2.7. *Let $r > \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be an odd mapping satisfying (2.4). Then there exists a unique additive mapping $A : X \rightarrow Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2^{\beta_1 r} - 2^{\beta_2}} \|x\|^r \quad (2.12)$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (2.4), we get $\|2f(0)\| \leq |\rho|^{\beta_2} \|2f(0)\|$. So $f(0) = 0$.

Letting $y = x$ in (2.4), we get

$$\|f(2x) - 2f(x)\| \leq 2\theta \|x\|^r \quad (2.13)$$

for all $x \in X$. So $\|f(x) - 2f(\frac{x}{2})\| \leq \frac{2}{2^{\beta_1 r}} \theta \|x\|^r$ for all $x \in X$. Hence

$$\left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\| \leq \sum_{j=l}^{m-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \leq \frac{2}{2^{\beta_1 r}} \sum_{j=l}^{m-1} \frac{2^{\beta_2 j}}{2^{\beta_1 r j}} \theta \|x\|^r \quad (2.14)$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.14) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \rightarrow Y$ by

$$A(x) := \lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (2.14), we get (2.12).

Since $f : X \rightarrow Y$ is odd, the mapping $A : X \rightarrow Y$ is odd.

It follows from (2.4) that

$$\begin{aligned}
& \|A(x+y) + A(x-y) - 2A(x) - A(y) - A(-y)\| \\
&= \lim_{n \rightarrow \infty} 2^{\beta_2 n} \left\| f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{-y}{2^n}\right) \right\| \\
&\leq \lim_{n \rightarrow \infty} 2^{\beta_2 n} |\rho|^{\beta_2} \left(\left\| 2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - \frac{3}{2}f\left(\frac{x}{2^n}\right) + \frac{1}{2}f\left(\frac{-x}{2^n}\right) \right. \right. \\
&\quad \left. \left. - \frac{1}{2}f\left(\frac{y}{2^n}\right) - \frac{1}{2}f\left(\frac{-y}{2^n}\right) \right\| \right) + \lim_{n \rightarrow \infty} \frac{2^{\beta_2 n} \theta}{2^{\beta_1 n r}} (\|x\|^r + \|y\|^r) \\
&= |\rho|^{\beta_2} \left\| 2A\left(\frac{x+y}{2}\right) + 2A\left(\frac{x-y}{2}\right) - \frac{3}{2}A(x) + \frac{1}{2}A(-x) - \frac{1}{2}A(y) - \frac{1}{2}A(-y) \right\|
\end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned}
& \|A(x+y) + A(x-y) - 2A(x) - A(y) - A(-y)\| \\
&\leq \left\| \rho \left(2A\left(\frac{x+y}{2}\right) + 2A\left(\frac{x-y}{2}\right) - \frac{3}{2}A(x) + \frac{1}{2}A(-x) - \frac{1}{2}A(y) - \frac{1}{2}A(-y) \right) \right\|
\end{aligned}$$

for all $x, y \in X$. By Lemma 2.5, the mapping $A : X \rightarrow Y$ is additive.

Now, let $T : X \rightarrow Y$ be another additive mapping satisfying (2.12). Then we have

$$\begin{aligned}
\|A(x) - T(x)\| &= 2^{\beta_2 n} \left\| A\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right) \right\| \\
&\leq 2^{\beta_2 n} \left(\left\| A\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| + \left\| T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| \right) \\
&\leq \frac{4 \cdot 2^{\beta_2 n}}{(2^{\beta_1 r} - 2^{\beta_2}) 2^{\beta_1 n r}} \theta \|x\|^r,
\end{aligned}$$

which tends to zero as $n \rightarrow \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A . Thus the mapping $A : X \rightarrow Y$ is a unique additive mapping satisfying (2.12). \square

Theorem 2.8. Let $r < \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be an odd mapping satisfying (2.4). Then there exists a unique additive mapping $A : X \rightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2\theta}{2^{\beta_2} - 2^{\beta_1 r}} \|x\|^r \quad (2.15)$$

for all $x \in X$.

Proof. It follows from (2.13) that $\left\| f(x) - \frac{1}{2}f(2x) \right\| \leq \frac{2\theta}{2^{\beta_2}} \|x\|^r$ for all $x \in X$. Hence

$$\left\| \frac{1}{2^l} f(2^l x) - \frac{1}{2^m} f(2^m x) \right\| \leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f(2^j x) - \frac{1}{2^{j+1}} f(2^{j+1} x) \right\| \leq \sum_{j=l}^{m-1} \frac{2^{\beta_1 r j}}{2^{\beta_2 j}} \frac{2\theta}{2^{\beta_2}} \|x\|^r \quad (2.16)$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.16) that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ converges. So one can define the mapping $A : X \rightarrow Y$ by

$$A(x) := \lim_{n \rightarrow \infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (2.16), we get (2.15).

The rest of the proof is similar to the proof of Theorem 2.7. \square

ADDITIVE-QUADRATIC ρ -FUNCTIONAL INEQUALITIES

By the triangle inequality, we have

$$\begin{aligned} & \|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \\ & - \left\| \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right) \right\| \\ & \leq \|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y) \\ & - \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right)\|. \end{aligned}$$

As corollaries of Theorems 2.3, 2.4, 2.7 and 2.8, we obtain the Hyers-Ulam stability results for the additive-quadratic ρ -functional equation (2.3) in β -homogeneous complex Banach spaces.

Corollary 2.9. Let $r > \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be an even mapping such that

$$\begin{aligned} & \|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y) \\ & - \rho \left(2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right)\| \leq \theta(\|x\|^r + \|y\|^r) \end{aligned} \quad (2.17)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ satisfying (2.5).

Corollary 2.10. Let $r < \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be an even mapping satisfying (2.17). Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ satisfying (2.8).

Corollary 2.11. Let $r > \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be an odd mapping satisfying (2.17). Then there exists a unique additive mapping $A : X \rightarrow Y$ satisfying (2.12).

Corollary 2.12. Let $r < \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be an odd mapping satisfying (2.17). Then there exists a unique additive mapping $A : X \rightarrow Y$ satisfying (2.15).

Remark 2.13. If ρ is a real number such that $-1 < \rho < 1$ and Y is a β_2 -homogeneous real Banach space, then all the assertions in this section remain valid.

3. ADDITIVE-QUADRATIC ρ -FUNCTIONAL INEQUALITY (0.2)

Throughout this section, assume that ρ is a fixed complex number with $|\rho| < \frac{1}{2}$.

In this section, we investigate the additive-quadratic ρ -functional inequality (0.2) in β -homogeneous complex Banach spaces.

Lemma 3.1. An even mapping $f : X \rightarrow Y$ satisfies

$$\begin{aligned} & \left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right\| \\ & \leq \|\rho(f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y))\| \end{aligned} \quad (3.1)$$

for all $x, y \in X$ if and only if $f : X \rightarrow Y$ is quadratic.

Proof. Assume that $f : X \rightarrow Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $\|2f(0)\| \leq |\rho|^{\beta_2} \|2f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.1), we get

$$\left\| 4f\left(\frac{x}{2}\right) - f(x) \right\| \leq 0 \quad (3.2)$$

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for all $x \in X$. So $f\left(\frac{x}{2}\right) = \frac{1}{4}f(x)$ for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} & \frac{1}{2^{\beta_2}} \|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \\ &= \left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right\| \\ &\leq |\rho|^{\beta_2} \|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in X$.

The converse is obviously true. □

Corollary 3.2. *An even mapping $f : X \rightarrow Y$ satisfies*

$$\begin{aligned} & 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \\ &= \rho(f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)) \end{aligned} \quad (3.3)$$

for all $x, y \in X$ if and only if $f : X \rightarrow Y$ is quadratic.

The functional equation (3.3) is called an *additive-quadratic ρ -functional equation*.

We prove the Hyers-Ulam stability of the additive-quadratic ρ -functional inequality (3.1) in β -homogeneous complex Banach spaces for an even mapping case.

Theorem 3.3. *Let $r > \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be an even mapping such that*

$$\begin{aligned} & \left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right\| \\ &\leq \|\rho(f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y))\| + \theta(\|x\|^r + \|y\|^r) \end{aligned} \quad (3.4)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{2^{\beta_1 r} \theta}{2^{\beta_1 r} - 4^{\beta_2}} \|x\|^r \quad (3.5)$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (3.4), we get $\|2f(0)\| \leq |\rho|^{\beta_2} \|2f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.4), we get

$$\left\| 4f\left(\frac{x}{2}\right) - f(x) \right\| \leq \theta \|x\|^r \quad (3.6)$$

for all $x \in X$. So

$$\left\| 4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right) \right\| \leq \sum_{j=l}^{m-1} \left\| 4^j f\left(\frac{x}{2^j}\right) - 4^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \leq \sum_{j=l}^{m-1} \frac{4^{\beta_2 j}}{2^{\beta_1 r j}} \theta \|x\|^r \quad (3.7)$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.7) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \rightarrow Y$ by

$$Q(x) := \lim_{n \rightarrow \infty} 4^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (3.7), we get (3.5).

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Since $f : X \rightarrow Y$ is even, the mapping $Q : X \rightarrow Y$ is even.

It follows from (3.4) that

$$\begin{aligned} & \left\| 2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - \frac{3}{2}Q(x) + \frac{1}{2}Q(-x) - \frac{1}{2}Q(y) - \frac{1}{2}Q(-y) \right\| \\ &= \lim_{n \rightarrow \infty} 4^{\beta_2 n} \left(\left\| 2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - \frac{3}{2}f\left(\frac{x}{2^n}\right) + \frac{1}{2}f\left(\frac{-x}{2^n}\right) - \frac{1}{2}f\left(\frac{y}{2^n}\right) - \frac{1}{2}f\left(\frac{-y}{2^n}\right) \right\| \right) \\ &\leq \lim_{n \rightarrow \infty} 4^{\beta_2 n} \left\| \rho\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{-y}{2^n}\right)\right) \right\| \\ &\quad + \lim_{n \rightarrow \infty} \frac{4^{\beta_2 n} \theta}{2^{\beta_1 n r}} (\|x\|^r + \|y\|^r) \\ &= \|\rho(Q(x+y) + Q(x-y) - 2Q(x) - Q(y) - Q(-y))\| \end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned} & \left\| 2Q\left(\frac{x+y}{2}\right) + 2Q\left(\frac{x-y}{2}\right) - \frac{3}{2}Q(x) + \frac{1}{2}Q(-x) - \frac{1}{2}Q(y) - \frac{1}{2}Q(-y) \right\| \\ &\leq \|\rho(Q(x+y) + Q(x-y) - 2Q(x) - Q(y) - Q(-y))\| \end{aligned}$$

for all $x, y \in X$. By Lemma 3.1, the mapping $Q : X \rightarrow Y$ is quadratic.

Now, let $T : X \rightarrow Y$ be another quadratic mapping satisfying (3.5). Then we have

$$\begin{aligned} \|Q(x) - T(x)\| &= 4^{\beta_2 n} \left\| Q\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right) \right\| \\ &\leq 4^{\beta_2 n} \left(\left\| Q\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| + \left\| T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| \right) \\ &\leq \frac{2 \cdot 4^{\beta_2 n} \cdot 2^{\beta_1 r}}{(2^{\beta_1 r} - 4^{\beta_2}) 2^{\beta_1 n r}} \theta \|x\|^r, \end{aligned}$$

which tends to zero as $n \rightarrow \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q . Thus the mapping $Q : X \rightarrow Y$ is a unique quadratic mapping satisfying (3.5). \square

Theorem 3.4. Let $r < \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be an even mapping satisfying (3.4). Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{2^{\beta_1 r} \theta}{4^{\beta_2} - 2^{\beta_1 r}} \|x\|^r \quad (3.8)$$

for all $x \in X$.

Proof. It follows from (3.6) that $\left\| f(x) - \frac{1}{4}f(2x) \right\| \leq \frac{2^{\beta_1 r} \theta}{4^{\beta_2}} \|x\|^r$ for all $x \in X$. Hence

$$\left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^m} f(2^m x) \right\| \leq \sum_{j=l}^{m-1} \left\| \frac{1}{4^j} f(2^j x) - \frac{1}{4^{j+1}} f(2^{j+1} x) \right\| \leq \frac{2^{\beta_1 r} \theta}{4^{\beta_2}} \sum_{j=l}^{m-1} \frac{2^{\beta_1 r j}}{4^{\beta_2 j}} \|x\|^r \quad (3.9)$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.9) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ converges. So one can define the mapping $Q : X \rightarrow Y$ by

$$Q(x) := \lim_{n \rightarrow \infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (3.9), we get (3.8).

The rest of the proof is similar to the proof of Theorem 3.3. \square

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Lemma 3.5. *An odd mapping $f : X \rightarrow Y$ satisfies (3.1) if and only if $f : X \rightarrow Y$ is additive.*

Proof. Assume that $f : X \rightarrow Y$ satisfies (3.1).

Letting $x = y = 0$ in (3.1), we get $\|2f(0)\| \leq |\rho|^{\beta_2} \|2f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.1), we get

$$\left\| 4f\left(\frac{x}{2}\right) - 2f(x) \right\| \leq 0 \quad (3.10)$$

for all $x \in X$. So $f\left(\frac{x}{2}\right) = \frac{1}{2}f(x)$ for all $x \in X$.

It follows from (3.1) and (3.10) that

$$\begin{aligned} & \frac{1}{2^{\beta_2}} \|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \\ &= \left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right\| \\ &\leq |\rho|^{\beta_2} \|f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)\| \end{aligned}$$

and so

$$f(x+y) + f(x-y) = 2f(x)$$

for all $x, y \in X$.

The converse is obviously true. □

Corollary 3.6. *An odd mapping $f : X \rightarrow Y$ satisfies (3.3) if and only if $f : X \rightarrow Y$ is additive.*

We prove the Hyers-Ulam stability of the additive-quadratic ρ -functional inequality (3.1) in β -homogeneous complex Banach spaces for an odd mapping case.

Theorem 3.7. *Let $r > \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be an odd mapping satisfying (3.4). Then there exists a unique additive mapping $A : X \rightarrow Y$ such that*

$$\|f(x) - A(x)\| \leq \frac{2^{\beta_1 r} \theta}{(2^{\beta_1 r} - 2^{\beta_2}) 2^{\beta_2}} \|x\|^r \quad (3.11)$$

for all $x \in X$.

Proof. Letting $x = y = 0$ in (3.4), we get $\|2f(0)\| \leq |\rho|^{\beta_2} \|2f(0)\|$. So $f(0) = 0$.

Letting $y = 0$ in (3.4), we get

$$\left\| 4f\left(\frac{x}{2}\right) - 2f(x) \right\| \leq \theta \|x\|^r \quad (3.12)$$

for all $x \in X$. So

$$\left\| 2^l f\left(\frac{x}{2^l}\right) - 2^m f\left(\frac{x}{2^m}\right) \right\| \leq \sum_{j=l}^{m-1} \left\| 2^j f\left(\frac{x}{2^j}\right) - 2^{j+1} f\left(\frac{x}{2^{j+1}}\right) \right\| \leq \sum_{j=l}^{m-1} \frac{2^{\beta_2 j}}{2^{\beta_1 r j}} \frac{\theta}{2^{\beta_2}} \|x\|^r \quad (3.13)$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.13) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $A : X \rightarrow Y$ by

$$A(x) := \lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (3.13), we get (3.11).

Since $f : X \rightarrow Y$ is odd, the mapping $A : X \rightarrow Y$ is odd.

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It follows from (3.4) that

$$\begin{aligned} & \left\| 2A\left(\frac{x+y}{2}\right) + 2A\left(\frac{x-y}{2}\right) - \frac{3}{2}A(x) + \frac{1}{2}A(-x) - \frac{1}{2}A(y) - \frac{1}{2}A(-y) \right\| \\ &= \lim_{n \rightarrow \infty} 2^{\beta_{2n}} \left(\left\| 2f\left(\frac{x+y}{2^{n+1}}\right) + 2f\left(\frac{x-y}{2^{n+1}}\right) - \frac{3}{2}f\left(\frac{x}{2^n}\right) + \frac{1}{2}f\left(\frac{-x}{2^n}\right) - \frac{1}{2}f\left(\frac{y}{2^n}\right) - \frac{1}{2}f\left(\frac{-y}{2^n}\right) \right\| \right) \\ &\leq \lim_{n \rightarrow \infty} 2^{\beta_{2n}} \left\| \rho\left(f\left(\frac{x+y}{2^n}\right) + f\left(\frac{x-y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right) - f\left(\frac{-y}{2^n}\right)\right) \right\| \\ &\quad + \lim_{n \rightarrow \infty} \frac{2^{\beta_{2n}}\theta}{2^{\beta_{1nr}}}(\|x\|^r + \|y\|^r) \\ &= \|\rho(A(x+y) + A(x-y) - 2A(x) - A(y) - A(-y))\| \end{aligned}$$

for all $x, y \in X$. So

$$\begin{aligned} & \left\| 2A\left(\frac{x+y}{2}\right) + 2A\left(\frac{x-y}{2}\right) - \frac{3}{2}A(x) + \frac{1}{2}A(-x) - \frac{1}{2}A(y) - \frac{1}{2}A(-y) \right\| \\ &\leq \|\rho(A(x+y) + A(x-y) - 2A(x) - A(y) - A(-y))\| \end{aligned}$$

for all $x, y \in X$. By Lemma 3.5, the mapping $A : X \rightarrow Y$ is additive.

Now, let $T : X \rightarrow Y$ be another additive mapping satisfying (3.11). Then we have

$$\begin{aligned} \|A(x) - T(x)\| &= 2^{\beta_{2n}} \left\| A\left(\frac{x}{2^n}\right) - T\left(\frac{x}{2^n}\right) \right\| \\ &\leq 2^{\beta_{2n}} \left(\left\| A\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| + \left\| T\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right) \right\| \right) \\ &\leq \frac{2 \cdot 2^{\beta_{2n}} \cdot 2^{\beta_{1r}}}{(2^{\beta_{1r}} - 2^{\beta_2})2^{\beta_{1nr}}} \frac{\theta}{2^{\beta_2}} \|x\|^r, \end{aligned}$$

which tends to zero as $n \rightarrow \infty$ for all $x \in X$. So we can conclude that $A(x) = T(x)$ for all $x \in X$. This proves the uniqueness of A . Thus the mapping $A : X \rightarrow Y$ is a unique additive mapping satisfying (3.11). \square

Theorem 3.8. Let $r < \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be an odd mapping satisfying (3.4). Then there exists a unique additive mapping $A : X \rightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \frac{2^{\beta_{1r}}\theta}{(2^{\beta_2} - 2^{\beta_{1r}})2^{\beta_2}} \|x\|^r \quad (3.14)$$

for all $x \in X$.

Proof. It follows from (3.12) that $\left\| f(x) - \frac{1}{2}f(2x) \right\| \leq \frac{2^{\beta_{1r}}\theta}{4^{\beta_2}} \|x\|^r$ for all $x \in X$. Hence

$$\left\| \frac{1}{2^l}f(2^l x) - \frac{1}{2^m}f(2^m x) \right\| \leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j}f(2^j x) - \frac{1}{2^{j+1}}f(2^{j+1} x) \right\| \leq \frac{2^{\beta_{1r}}\theta}{4^{\beta_2}} \sum_{j=l}^{m-1} \frac{2^{\beta_{1r}j}}{2^{\beta_{2j}}} \|x\|^r \quad (3.15)$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.15) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. So one can define the mapping $A : X \rightarrow Y$ by

$$A(x) := \lim_{n \rightarrow \infty} \frac{1}{2^n}f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \rightarrow \infty$ in (3.15), we get (3.14).

The rest of the proof is similar to the proof of Theorem 3.7. \square

By the triangle inequality, we have

$$\begin{aligned} & \left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right\| \\ & \quad - \|\rho(f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y))\| \\ & \leq \left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right. \\ & \quad \left. - \rho(f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)) \right\|. \end{aligned}$$

As corollaries of Theorems 3.3, 3.4, 3.7 and 3.8, we obtain the Hyers-Ulam stability results for the additive-quadratic ρ -functional equation (3.3) in β -homogeneous complex Banach spaces.

Corollary 3.9. Let $r > \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be an even mapping such that

$$\begin{aligned} & \left\| 2f\left(\frac{x+y}{2}\right) + 2f\left(\frac{x-y}{2}\right) - \frac{3}{2}f(x) + \frac{1}{2}f(-x) - \frac{1}{2}f(y) - \frac{1}{2}f(-y) \right. \\ & \quad \left. - \rho(f(x+y) + f(x-y) - 2f(x) - f(y) - f(-y)) \right\| \leq \theta(\|x\|^r + \|y\|^r) \end{aligned} \quad (3.16)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ satisfying (3.5).

Corollary 3.10. Let $r < \frac{2\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be an even mapping satisfying (3.16). Then there exists a unique quadratic mapping $Q : X \rightarrow Y$ satisfying (3.8).

Corollary 3.11. Let $r > \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be an odd mapping satisfying (3.16). Then there exists a unique additive mapping $A : X \rightarrow Y$ satisfying (3.11).

Corollary 3.12. Let $r < \frac{\beta_2}{\beta_1}$ and θ be nonnegative real numbers, and let $f : X \rightarrow Y$ be an odd mapping satisfying (3.16). Then there exists a unique additive mapping $A : X \rightarrow Y$ satisfying (3.14).

Remark 3.13. If ρ is a real number such that $-\frac{1}{2} < \rho < \frac{1}{2}$ and Y is a β_2 -homogeneous real Banach space, then all the assertions in this section remain valid.

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A note on stochastic functional differential equations driven by G-Brownian motion with discontinuous drift coefficients

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Abstract

In the fields of sciences and engineering, the role of discontinuous functions is of immense importance. Heaviside function, for instance, describes the switching process of voltage in an electrical circuit through mathematical process. The current paper aims at exploring the existence theory for stochastic functional differential equations driven by G-Brownian motion (G-SFDEs) whose drift coefficients may not be continuous. It is ascertain that G-SFDEs with discontinuous drift coefficients have more than one bounded and continuous solutions.

Key words: Stochastic functional differential equations, discontinuous drift coefficients, G-Brownian motion, existence.

1 Introduction

For the purpose of analysis and formulation of systems pertaining to engineering, economics and social sciences, stochastic dynamical systems play an important role. Through these equations, while considering the present status, one reconstructs the history and predicts the future of the dynamical systems. On the other hand, in several applications, analysis of the modeling system predicts that the change rate of the system's existing status depends not only on the state that is prevalent but also on the precedent record of the system. This leads to stochastic functional differential equations. The stochastic functional differential equations driven by G-Brownian motion (G-SFDEs) with Lipschitz continuous coefficients was initiated by Ren et.al. [12]. Afterwards, Faizullah used the Caratheodory approximation scheme for developing the existence and uniqueness of solution for G-SFDEs with continuous coefficients [3]. On the other hand, in this case, we study

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the existence theory for G-SFDEs with discontinuous drift coefficients, such as in the following G-SFDE

$$dX(t) = H(X_t)dt + d\langle B \rangle(t) + dB(t),$$

where $H : \mathbb{R} \rightarrow \mathbb{R}$ is the Heaviside function defined by

$$H(x) = \begin{cases} 0, & \text{if } x < 0; \\ 1, & \text{if } x \geq 0. \end{cases}$$

The above mentioned equations arise, when we take into account the effects of background noise switching systems with delays [5]. For more details on SDEs with discontinuous drift coefficients see [4, 7]. The following stochastic functional differential equation driven by G-Brownian motion (G-SFDE) with finite delay is considered

$$dX(t) = \alpha(t, X_t)dt + \beta(t, X_t)d\langle B, B \rangle(t) + \sigma(t, X_t)dB(t), \quad 0 \leq t \leq T, \quad (1.1)$$

where $X(t)$ is the value of stochastic process at time t and $X_t = \{X(t + \theta) : -\tau \leq \theta \leq 0\}$ is a $BC([-\tau, 0]; \mathbb{R})$ -valued stochastic process, which represents the family of bounded continuous \mathbb{R} -valued functions φ defined on $[-\tau, 0]$ having norm $\|\varphi\| = \sup_{-\tau \leq \theta \leq 0} |\varphi(\theta)|$. Let $\alpha : [0, T] \times BC([-\tau, 0]; \mathbb{R}) \rightarrow \mathbb{R}$, $\beta : [0, T] \times BC([-\tau, 0]; \mathbb{R}) \rightarrow \mathbb{R}$ and $\sigma : [0, T] \times BC([-\tau, 0]; \mathbb{R}) \rightarrow \mathbb{R}$ are Borel measurable. The condition $\xi(0) \in \mathbb{R}$ is given, $\{\langle B, B \rangle(t), t \geq 0\}$ is the quadratic variation process of G-Brownian motion $\{B(t), t \geq 0\}$ and $\alpha, \beta, \sigma \in M_G^2([-\tau, T]; \mathbb{R})$. Let \mathbb{L}^2 denote the space of all \mathcal{F}_t -adapted process $X(t), 0 \leq t \leq T$, such that $\|X\|_{\mathbb{L}^2} = \sup_{-\tau \leq t \leq T} |X(t)| < \infty$. We define the initial condition of equation (1.1) as follows;

$$X_{t_0} = \xi = \{\xi(\theta) : -\tau < \theta \leq 0\} \text{ is } \mathcal{F}_0 - \text{measurable, } BC([-\tau, 0]; \mathbb{R}) - \text{valued} \\ \text{random variable such that } \xi \in M_G^2([-\tau, 0]; \mathbb{R}). \quad (1.2)$$

G-SFDEs (1.1) with initial condition (1.2) can be written in the following integral form;

$$X(t) = \xi(0) + \int_0^t \alpha(s, X_s)ds + \int_0^t \beta(s, X_s)d\langle B, B \rangle(s) + \int_0^t \sigma(s, X_s)dB(s).$$

Consider the following linear growth and Lipschitz conditions respectively.

- (i) For any $t \in [0, T]$, $|\alpha(t, x)|^2 + |\beta(t, x)|^2 + |\sigma(t, x)|^2 \leq K(1 + |x|^2), K > 0$.
- (ii) For all $x, y \in (BC[-\tau, 0]; \mathbb{R})$ and $t \in [0, T]$, $|\alpha(t, x) - \alpha(t, y)|^2 + |\beta(t, x) - \beta(t, y)|^2 + |\sigma(t, x) - \sigma(t, y)|^2 \leq K(x - y)^2, K > 0$.

The above G-SFDE has a unique solution $X(t) \in M_G^2([-\tau, T]; \mathbb{R})$ if all the coefficients α, β and σ satisfy the Linear growth and Lipschitz conditions [3, 12]. However, we suppose that the drift coefficient α does not need to be continuous. The solution of equation 1.1 with initial condition 1.2 is an \mathbb{R} valued stochastic processes $X(t), t \in [-\tau, T]$ if

- (i) $X(t)$ is path-wise continuous and \mathcal{F}_t -adapted for all $t \in [0, T]$;
- (ii) $\alpha(t, X_t) \in \mathcal{L}^1([0, T]; \mathbb{R})$ and $\beta(t, X_t), \sigma(t, X_t) \in \mathcal{L}^2([0, T]; \mathbb{R})$;
- (iii) $X_0 = \xi$ and for each $t \in [0, T]$, $dX(t) = \alpha(t, X_t)dt + \beta(t, X_t)d\langle B, B \rangle(t) + \sigma(t, X_t)dB(t)$ q.s.

In the subsequent section, some preliminaries are given whereas in section 3, the comparison theorem is developed. The last section, shows that under some suitable conditions, the G-SFDE (1.1), provides more than one solutions.

2 Basic concepts and notions

In this section, we give some notions and basic definitions of the sublinear expectation [1, 2, 10, 11, 13]. Let Ω be a (non-empty) basic space and \mathcal{H} be a linear space of real valued functions defined on Ω such that any arbitrary constant $c \in \mathcal{H}$ and if $X \in \mathcal{H}$ then $|X| \in \mathcal{H}$. We consider that \mathcal{H} is the space of random variables.

Definition 2.1. A functional $\mathbb{E} : \mathcal{H} \rightarrow \mathbb{R}$ is called sub-linear expectation, if $\forall X, Y \in \mathcal{H}, c \in \mathbb{R}$ and $\lambda \geq 0$ it satisfies the following properties

- (1) (Monotonicity): If $X \geq Y$ then $\mathbb{E}[X] \geq \mathbb{E}[Y]$.
- (2) (Constant preserving): $\mathbb{E}[c] = c$.
- (3) (Sub-additivity): $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$.
- (4) (Positive homogeneity): $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$.

The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a sublinear expectation space. Consider the space of random variables \mathcal{H} such that if $X_1, X_2, \dots, X_n \in \mathcal{H}$ then $\varphi(X_1, X_2, \dots, X_n) \in \mathcal{H}$ for each $\varphi \in \mathbb{C}_{l.Lip}(\mathbb{R}^n)$, where $\mathbb{C}_{l.Lip}(\mathbb{R}^n)$ is the space of linear functions φ defined as the following

$$\mathbb{C}_{l.Lip}(\mathbb{R}^n) = \{\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \mid \exists C \in [0, \infty) : \forall x, y \in \mathbb{R}^n, \\ |\varphi(x) - \varphi(y)| \leq C(1 + |x|^C + |y|^C)|x - y|\}.$$

G-expectation and G-Brownian Motion. Let $\Omega = C_0([0, \infty))$, that is, the space of all \mathbb{R} -valued continuous paths $(w_t)_{t \in [0, \infty)}$ with $w_0 = 0$ equipped with the distance

$$\rho(w^1, w^2) = \sum_{k=1}^{\infty} \frac{1}{2^k} (\max_{t \in [0, k]} |w_t^1 - w_t^2| \wedge 1),$$

and consider the canonical process $B_t(w) = w_t$ for $t \in [0, \infty)$, $w \in \Omega$ then for each fixed $T \in [0, \infty)$ we have

$$Lip(\Omega_T) = \{\varphi(B_{t_1}, B_{t_2}, \dots, B_{t_n}) : t_1, \dots, t_n \in [0, T], \varphi \in \mathbb{C}_{l.Lip}(\mathbb{R}^n), n \in \mathbb{N}\},$$

where $L_{ip}(\Omega_t) \subseteq L_{ip}(\Omega_T)$ for $t \leq T$ and $L_{ip}(\Omega) = \cup_{m=1}^{\infty} L_{ip}(\Omega_m)$.

Consider a sequence $\{\xi_i\}_{i=1}^{\infty}$ of random variables on a sublinear expectation space $(\hat{\Omega}, \hat{\mathcal{H}}_p, \hat{\mathbb{E}})$ such that ξ_{i+1} is independent of $(\xi_1, \xi_2, \dots, \xi_i)$ for each $i = 1, 2, \dots$ and ξ_i is G-normally distributed for each $i \in \{1, 2, \dots\}$. Then a sublinear expectation $\mathbb{E}[\cdot]$ defined on $L_{ip}(\Omega)$ is introduced as follows.

For $0 = t_0 < t_1 < \dots < t_n < \infty$ ($t_0, t_1, \dots, t_n \in [t, \infty)$) [13], $\varphi \in \mathbb{C}_{L_{ip}}(\mathbb{R}^n)$ and each

$$X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}}) \in L_{ip}(\Omega),$$

$$\begin{aligned} \mathbb{E}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})] \\ = \hat{\mathbb{E}}[\varphi(\sqrt{t_1 - t_0}\xi_1, \dots, \sqrt{t_n - t_{n-1}}\xi_n)]. \end{aligned}$$

The conditional sublinear expectation of $X \in L_{ip}(\Omega_t)$ is defined by

$$\begin{aligned} \mathbb{E}[X|\Omega_t] &= \mathbb{E}[\varphi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_m} - B_{t_{m-1}})|\Omega_t] \\ &= \psi(B_{t_1}, B_{t_2} - B_{t_1}, \dots, B_{t_j} - B_{t_{j-1}}), \end{aligned}$$

where

$$\psi(x_1, \dots, x_j) = \hat{\mathbb{E}}[\varphi(x_1, \dots, x_j, \sqrt{t_{j+1} - t_j}\xi_{j+1}, \dots, \sqrt{t_n - t_{n-1}}\xi_n)].$$

Definition 2.2. The sublinear expectation $\mathbb{E} : L_{ip}(\Omega) \rightarrow \mathbb{R}$ defined above is called a G-expectation and the corresponding canonical process $\{B_t, t \geq 0\}$ is called a G-Brownian motion.

The completion of $L_{ip}(\Omega)$ under the norm $\|X\|_p = (\mathbb{E}[|X|^p])^{1/p}$ [11, 13] for $p \geq 1$ is denoted by $L_G^p(\Omega)$ and $L_G^p(\Omega_t) \subseteq L_G^p(\Omega_T) \subseteq L_G^p(\Omega)$ for $0 \leq t \leq T < \infty$. The filtration generated by the canonical process $(B_t)_{t \geq 0}$ is denoted by $\mathcal{F}_t = \sigma\{B_s, 0 \leq s \leq t\}$, $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$.

Itô's Integral of G-Brownian motion. For any $T \in [0, \infty)$, a finite ordered subset $\pi_T = \{t_0, t_1, \dots, t_N\}$ such that $0 = t_0 < t_1 < \dots < t_N = T$ is a partition of $[0, T]$ and

$$\mu(\pi_T) = \max\{|t_{i+1} - t_i| : i = 0, 1, \dots, N-1\}.$$

A sequence of partitions of $[0, T]$ is denoted by $\pi_T^N = \{t_0^N, t_1^N, \dots, t_N^N\}$ such that $\lim_{N \rightarrow \infty} \mu(\pi_T^N) = 0$.

Consider the following simple process:

Let $p \geq 1$ be fixed. For a given partition $\pi_T = \{t_0, t_1, \dots, t_N\}$ of $[0, T]$,

$$\eta_t(w) = \sum_{i=0}^{N-1} \xi_i(w) I_{[t_i, t_{i+1}]}(t), \quad (2.1)$$

where $\xi_i \in L_G^p(\Omega_{t_i})$, $i = 0, 1, \dots, N-1$. The collection containing the above type of processes, that is, containing $\eta_t(w)$ is denoted by $M_G^{p,0}(0, T)$. The completion of $M_G^{p,0}(0, T)$ under the norm $\|\eta\| = \{\int_0^T \mathbb{E}[|\eta_u|^p] du\}^{1/p}$ is denoted by $M_G^p(0, T)$ and for $1 \leq p \leq q$, $M_G^p(0, T) \supset M_G^q(0, T)$.

Definition 2.3. For each $\eta_t \in M_G^{2,0}(0, T)$, the Itô's integral of G-Brownian motion is defined as

$$I(\eta) = \int_0^T \eta_u dB_u = \sum_{i=0}^{N-1} \xi_i (B_{t_{i+1}} - B_{t_i}).$$

Definition 2.4. An increasing continuous process $\{\langle B \rangle_t : t \geq 0\}$ with $\langle B \rangle_0 = 0$, defined by

$$\langle B \rangle_t = B_t^2 - 2 \int_0^t B_u dB_u,$$

is called the quadratic variation process of G-Brownian motion.

3 Comparison theorem for G-SFDEs

The purpose of this section is to establish comparison result for problem (1.1) with initial data (1.2). Consider the following two stochastic functional differential equations

$$X(t) = \xi^1(0) + \int_{t_0}^t \alpha_1(s, X_s) ds + \int_{t_0}^t \beta(s, X_s) d\langle B, B \rangle(s) + \int_{t_0}^t \sigma(s, X_s) dB(s), \quad t \in [0, T], \quad (3.1)$$

$$X(t) = \xi^2(0) + \int_{t_0}^t \alpha_2(s, X_s) ds + \int_{t_0}^t \beta(s, X_s) d\langle B, B \rangle(s) + \int_{t_0}^t \sigma(s, X_s) dB(s), \quad t \in [0, T]. \quad (3.2)$$

Theorem 3.1. Assume that:

- (i) X^1 and X^2 are unique strong solutions of problems (3.1) and (3.2) respectively.
- (ii) $\alpha_1(s, X_s) \leq \alpha_2(s, X_s)$ componentwise for all $t \in [t_0, T]$, $x \in BC([- \tau, 0]; \mathbb{R}^d)$ and $\xi^1 \leq \xi^2$.
- (iii) α_1 or α_2 is increasing such that $f(t, x) \leq f(t, y)$ when $x \leq y$ for all $x, y \in C([- \tau, 0]; \mathbb{R})$.

Then for all $t > 0$ we have $X^1 \leq X^2$ q.s.

Proof. First, we define an operator $q(\cdot, \cdot) : C([- \tau, 0]; \mathbb{R}) \times C([- \tau, 0]; \mathbb{R}) \rightarrow C([- \tau, 0]; \mathbb{R})$ such that

$$q(x, y) = \max[x, y].$$

Obviously, $y \rightarrow q(x, y)$ satisfies the linear growth and Lipschitz conditions. Now we suppose that α_2 is increasing and consider the following equation

$$\begin{aligned} Y(t) = & \xi^2(0) + \int_{t_0}^t \alpha_2(s, q(X_s^1, Y_s)) ds + \int_{t_0}^t \beta(s, q(X_s^1, Y_s)) d\langle B, B \rangle(s) \\ & + \int_{t_0}^t \sigma(s, q(X_s^1, Y_s)) dB(s), \quad t_0 \leq t \leq T. \end{aligned} \quad (3.3)$$

Thus it is easy to see that the coefficients satisfy the linear growth and Lipschitz conditions, so 3.3 has a unique solution $Y(t)$. We shall prove that $Y(t) \geq X_s^1$ q.s. We define the following two stopping times. For more details on stopping times we refer the reader to [7, 8].

$$\begin{aligned}\tau_1 &= \inf\{t \in [t_0, T] : X_s^1 - Y(t) > 0\} \text{ where } \tau_1 < T, \\ \tau_2 &= \inf\{t \in [\tau_1, T] : X_s^1 - Y(t) < 0\}.\end{aligned}$$

Contrary suppose that there exist an interval $(\tau_1, \tau_2) \subset [t_0, T]$ such that $Y(\tau_1) = X^1(\tau_1) = \xi^*(0)$ and $Y(t) \leq X^1(t)$ for all $t \in (\tau_1, \tau_2)$. Then,

$$\begin{aligned}Y(t) - X^1(t) &= \xi^*(0) + \int_{\tau_1}^t \alpha_2(s, q(X_s^1, Y_s)) ds + \int_{\tau_1}^t \beta(s, q(X_s^1, Y_s)) d\langle B, B \rangle(s) \\ &\quad + \int_{\tau_1}^t \sigma(s, q(X_s^1, Y_s)) dB(s) - \xi^*(0) - \int_{\tau_1}^t \alpha_1(s, X_s^1) ds \\ &\quad - \int_{\tau_1}^t \beta(s, X_s^1) d\langle B, B \rangle(s) - \int_{\tau_1}^t \sigma(s, X_s^1) dB(s), \quad t \in (\tau_1, \tau_2). \\ Y(t) - X^1(t) &= \int_{\tau_1}^t [\alpha_2(s, q(X_s^1, Y_s)) - \alpha_1(s, X_s^1)] ds \\ &\quad + \int_{\tau_1}^t [\beta(s, q(X_s^1, Y_s)) - \beta(s, X_s^1)] d\langle B, B \rangle(s) \\ &\quad + \int_{\tau_1}^t [\sigma(s, q(X_s^1, Y_s)) - \sigma(s, X_s^1)] dB(s), \quad t \in (\tau_1, \tau_2).\end{aligned}$$

But our supposition $Y(t) \leq X^1(t)$ yields $q(X^1, Y) = \max[X^1, Y] = X^1$. So, we have

$$\begin{aligned}Y(t) - X^1(t) &= \int_{\tau_1}^t [\alpha_2(s, X_s^1) - \alpha_1(s, X_s^1)] ds \\ &\quad + \int_{\tau_1}^t [\beta(s, X_s^1) - \beta(s, X_s^1)] d\langle B, B \rangle(s) \\ &\quad + \int_{\tau_1}^t [\sigma(s, X_s^1) - \sigma(s, X_s^1)] dB(s) \\ Y(t) - X^1(t) &= \int_{\tau_1}^t [\alpha_2(s, X_s^1) - \alpha_1(s, X_s^1)] ds \geq 0,\end{aligned}$$

because $\alpha_2(t, x) \geq \alpha_1(t, x)$. Which gives contradiction. So, our supposition $Y(t) \leq X^1(t)$ for all $t \in (\tau_1, \tau_2)$ is wrong. Thus $Y(t) \geq X^1(t)$ q.s. and so $p(X^1, Y) = Y$. It means that $Y = X^2 \geq X^1$ because G-SFDE (3.3) has a unique solution X^2 . \square

4 G-SFDEs with discontinuous drift coefficients

We now suppose that α is left continuous, increasing and $\alpha(t, x) \geq 0$ for all $(t, x) \in [0, T] \times BC([- \tau, 0]; \mathbb{R})$ but not continuous. Consider the following sequence of problems.

$$X^n(t) = \xi(0) + \int_0^t \alpha(s, X_s^{n-1}) ds + \int_0^t \beta(s, X_s^n) d\langle B, B \rangle(s) + \int_0^t \sigma(s, X_s^n) dB(s), \quad t \in [0, T], \quad (4.1)$$

where $X^0 = L_t$, L_t is the unique solution of the following problem

$$L_t = \xi + \int_0^t \beta(s, L_s) d\langle B, B \rangle(s) + \int_0^t \sigma(s, L_s) dB(s), t \in [0, T]. \quad (4.2)$$

Thus using the comparison theorem and the fact that $\alpha(t, x) \geq 0$, we have $X^1 \geq L_t$. So, we can see that X^n is an increasing sequence. Now we shall prove that X^n is bounded in \mathbb{L}^2 norm.

Lemma 4.1. *Suppose $X^n(t)$ be a solution of problem (4.1) then there exists a positive constant C independent of n such that,*

$$E \left(\sup_{-\tau \leq s \leq T} |X^n(s)|^2 \right) \leq C.$$

Proof. For any $n \geq 1$ we define the following stopping time in a similar way as given in [9]

$$\tau_m = T \wedge \inf\{t \in [0, T] : \|X_t^n\| \geq m\}.$$

We have $\tau_m \uparrow T$ and define $X^{n,m}(t) = X^n(t \wedge \tau_m)$ for $t \in (-\tau, T)$. Then for $t \in [0, T]$,

$$X^{n,m}(t) = \xi(0) + \int_0^t \alpha(t, X_t^{n-1,m}) I_{[0, \tau_m]} dt + \int_0^t \beta(t, X_t^{n,m}) I_{[0, \tau_m]} d\langle B, B \rangle_t + \int_0^t \sigma(t, X_t^{n,m}) I_{[0, \tau_m]} dB_t.$$

$$\begin{aligned} |X^{n,m}(t)|^2 &= |\xi(0) + \int_0^t \alpha(t, X_t^{n-1,m}) I_{[0, \tau_m]} dt + \int_0^t \beta(t, X_t^{n,m}) I_{[0, \tau_m]} d\langle B, B \rangle_t \\ &\quad + \int_0^t \sigma(t, X_t^{n,m}) I_{[0, \tau_m]} dB_t|^2 \\ &\leq 4|\xi(0)|^2 + 4\left|\int_0^t \alpha(t, X_t^{n-1,m}) I_{[0, \tau_m]} dt\right|^2 + 4\left|\int_0^t \beta(t, X_t^{n,m}) I_{[0, \tau_m]} d\langle B, B \rangle_t\right|^2 \\ &\quad + 4\left|\int_0^t \sigma(t, X_t^{n,m}) I_{[0, \tau_m]} dB_t\right|^2 \end{aligned}$$

Taking G-expectation, using properties of G-integral, G-quadratic variation process [10, 11] and linear growth condition we get

$$\begin{aligned} E[|X^{n,m}(t)|^2] &\leq 4E|\xi(0)|^2 + 4C_1 \int_0^t [1 + E|X_t^{n-1,m}|^2] dt + 4C_2 \int_0^t [1 + E|X_t^{n,m}|^2] dt \\ &\quad + 4C_3 \int_0^t [1 + E|X_t^{n,m}|^2] dt \\ &\leq 4E|\xi(0)|^2 + 4C_1 \int_0^t dt + 4C_1 \int_0^t E|X_t^{n-1,m}|^2 dt + 4C_2 \int_0^t dt + 4C_2 \int_0^t E|X_t^{n,m}|^2 dt \\ &\quad + 4C_3 \int_0^t dt + 4C_3 \int_0^t E|X_t^{n,m}|^2 dt \\ &= 4E|\xi(0)|^2 + 4C_1 T + 4C_1 \int_0^t E|X_t^{n-1,m}|^2 dt + 4C_2 T + 4C_2 \int_0^t E|X_t^{n,m}|^2 dt \\ &\quad + 4C_3 T + 4C_3 \int_0^t E|X_t^{n,m}|^2 dt. \end{aligned}$$

Then for any $k \in \mathbb{N}$ we have,

$$\max_{1 \leq n \leq k} E[|X^{n,m}(t)|^2] \leq C_4 + 4C_1 \int_0^t \max_{1 \leq n \leq k} E|X_t^{n-1,m}|^2 dt + 4C_2 \int_0^t \max_{1 \leq n \leq k} E|X_t^{n,m}|^2 dt + 4C_3 \int_0^t \max_{1 \leq n \leq k} E|X_t^{n,m}|^2 dt,$$

where $C_4 = 4[E|\xi|^2 + C_1T + C_2T + C_3T]$ and thus using Doob's martingale inequality for any $n, m \in \mathbb{N}$ we have,

$$E\left[\sup_{0 \leq s \leq t} |X^{n,m}(s)|^2\right] \leq C_4 + C_5 \int_0^t E|X_s^{n,m}|^2 dt, \quad (4.3)$$

where $C_5 = 4(C_1 + C_2 + C_3)$. One can observe the fact [9],

$$\sup_{-\tau \leq s \leq t} |X^{n,m}(s)|^2 \leq \|\xi\|^2 + \sup_{0 \leq s \leq t} |X^{n,m}(s)|^2,$$

and thus 4.3 yields

$$\begin{aligned} E\left[\sup_{-\tau \leq s \leq t} |X^{n,m}(s)|^2\right] &\leq E[\|\xi\|^2] + C_4 + C_5 \int_0^t E|X_s^{n,m}|^2 dt \\ &\leq C_6 + C_5 \int_0^t E\left[\sup_{-\tau \leq r \leq s} |X^{n,m}(r)|^2\right] dt, \end{aligned}$$

where $C_6 = E[\|\xi\|^2] + C_4$. So, using the Gronwall inequality and taking $m \rightarrow \infty$ we have,

$$E\left[\sup_{-\tau \leq s \leq t} |X^n(s)|^2\right] \leq C_6 e^{C_5 t}.$$

Letting $t = T$ we get the desired result,

$$E\left[\sup_{-\tau \leq s \leq T} |X^n(s)|^2\right] \leq C^*, \quad C^* = C_6 e^{C_5 T}.$$

□

Theorem 4.2. *Suppose that:*

- (i) *The coefficient α be left continuous and increasing in the second variable x .*
- (ii) *For all $(t, x) \in [0, T] \times BC([-\tau, 0]; \mathbb{R})$, $\alpha(t, x) \geq 0$.*

Then the G-SFDE (1.1) has more than one solution $X(t) \in M_G^2([-\tau, T]; \mathbb{R})$.

Proof. By theorem 3.1 we know that $\{X^n\}$ is increasing and by Lemma 4.1 it is bounded in \mathbb{L}^2 . Then by Dominated Convergence theorem we can deduce that X^n converges in \mathbb{L}^2 . Denoting the limit of X^n by X and thus for almost all w , we get

$$\alpha(t, X^n(t)) \rightarrow \alpha(t, X(t)) \text{ as } n \rightarrow \infty,$$

and

$$|\alpha(t, X^n(t))| \leq K(1 + \sup_n |X^n(t)|) \in L^1([t_0, T]).$$

Thus, for almost all w and uniformly in t

$$\int_0^t \alpha(s, X^n(s)) ds \rightarrow \int_0^t \alpha(s, X(s)) ds, \quad n \rightarrow \infty.$$

By the properties of β, σ and by the continuity properties of G-integral and its quadratic variation process we have,

$$\sup_{0 \leq t \leq T} \left| \int_0^t \beta(s, X^n(s)) d\langle B, B \rangle_s - \int_0^t \beta(s, X(s)) d\langle B, B \rangle_s \right| \rightarrow 0 \quad (q.s), \quad n \rightarrow \infty.$$

$$\sup_{0 \leq t \leq T} \left| \int_0^t \sigma(s, X^n(s)) dB(s) - \int_0^t \sigma(s, X(s)) dB(s) \right| \rightarrow 0 \quad (q.s), \quad n \rightarrow \infty.$$

It is easy to conclude that X^n converges uniformly to X in t , hence X is continuous. Taking limit in equation (4.1), we get that X is the desired solution for stochastic functional differential equation (1.1) with initial condition (1.2). \square

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SUBCLASSES OF JANOWSKI-TYPE FUNCTIONS DEFINED BY CHO-KWON-SRIVASTAVA OPERATOR

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ABSTRACT. We introduce a new subclass of analytic functions in the unit disk U defined by using Cho-Kwon Srivastava integral operator. Inclusion results radius problem and integral preserving properties are investigated.

1. INTRODUCTION

Let \mathcal{A}_p be the class of analytic functions in U of the form

$$(1.1) \quad f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad z \in U, \quad (p \in \mathbb{N}),$$

where $\mathbb{N} := \{1, 2, \dots\}$. For $p = 1$ we denotes $\mathcal{A} := \mathcal{A}_1$. Note that the class \mathcal{A}_p is closed under *the convolution (or Hadamard) product*, that is

$$f(z) * g(z) := z^p + \sum_{n=1}^{\infty} a_{p+n} b_{p+n} z^{p+n}, \quad z \in U, \quad (p \in \mathbb{N}),$$

where f is given by (1.1) and $g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n} z^{p+n}$, $z \in U$.

The operator $L^p(d, e) : \mathcal{A}_p \rightarrow \mathcal{A}_p$ is defined by using the Hadamard (convolution) product, that is

$$(1.2) \quad L^p(d, e)f(z) := f(z) * \varphi_p(d, e; z),$$

where

$$\varphi_p(d, e; z) := z^p + \sum_{n=1}^{\infty} \frac{(d)_n}{(e)_n} z^{p+n}, \quad (d \in \mathbb{C}, e \in \mathbb{C} \setminus \mathbb{Z}_0^-),$$

and $(d)_n = d(d+1)\dots(d+n-1)$, with $(d)_0 = 1$, represents the well-known *Pochhammer symbol*.

From (1.2) it follows immediately that

$$z(L^p(d, e)f(z))' = dL^p(d+1, e)f(z) - (d-p)L^p(d, e)f(z).$$

The operator $L^p(d, e)$ was introduced by Saitoh [16] and this is an extension of the operator $L(d, e)$ which was defined by Carlson and Shaffer [2].

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Analogous to the $L^p(d, e)$ operator, Cho et. al. [4] introduced the operator $I_\mu^p(d, e) : \mathcal{A}_p \rightarrow \mathcal{A}_p$ defined by

$$(1.3) \quad I_\mu^p(d, e)f(z) := f(z) * \varphi_p^\dagger(d, e; z),$$

where

$$\varphi_p^\dagger(d, e; z) := z^p + \sum_{n=1}^{\infty} \frac{(\mu + p)_n (e)_n}{n! (d)_n} z^{p+n}, \quad (d, e \in \mathbb{C} \setminus \mathbb{Z}_0^-, \mu > -p).$$

We notice that

$$\varphi_p^\dagger(d, e; z) * \varphi_p(d, e; z) = \frac{z^p}{(1 - z)^{\mu+p}}, \quad z \in U.$$

From (1.3), the following identities can be easily obtained [4]:

$$(1.4) \quad z (I_\mu^p(d + 1, e)f(z))' = dI_\mu^p(d, e)f(z) - (d - p) I_\mu^p(d + 1, e)f(z),$$

$$(1.5) \quad z (I_\mu^p(d, e)f(z))' = (\mu + p)I_{\mu+1}^p(d, e)f(z) - \mu I_\mu^p(d, e)f(z).$$

We may easily remark the following relations

$$I_1^p(p + 1, 1)f(z) = f(z), \quad I_1^p(p, 1)f(z) = \frac{zf'(z)}{p},$$

and remark that the operator $I_\mu^1(a + 2, 1)$, with $\mu > -1$ and $a > -2$, was studied in [5].

If f and g are two analytic functions in U , we say that f is *subordinate to* g , written symbolically as $f(z) \prec g(z)$, if there exists a *Schwarz function* w , which (by definition) is analytic in U , with $w(0) = 0$, and $|w(z)| < 1$, $z \in U$, such that $f(z) = g(w(z))$, for all $z \in U$. Furthermore, if the function g is univalent in U , then we have the following equivalence, (cf., e.g., [10], see also [11, p. 4]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Definition 1.1. 1. Like in [3], for arbitrary fixed numbers A , B and β , with $-1 \leq B < A \leq 1$ and $0 \leq \beta < 1$, let $P[A, B, \beta]$ denote the family of functions p that are analytic in U , with $p(0) = 1$, and such that

$$p(z) \prec \frac{1 + [(1 - \beta)A + \beta B]z}{1 + Bz}.$$

We will use the notations $P[A, B] := P[A, B, 0]$ and $P(0) := P[1, -1, 0]$.

2. Let $P_l[A, B, \beta]$ denote the class of functions p that are analytic in U , with $p(0) = 1$, that are represented by

$$(1.6) \quad p(z) = \left(\frac{l}{4} + \frac{1}{2}\right) K_1(z) - \left(\frac{l}{4} - \frac{1}{2}\right) K_2(z),$$

where $K_1, K_2 \in P[A, B, \beta]$, $-1 \leq B < A \leq 1$, $0 \leq \beta < 1$, and $l \geq 2$.

Remarks 1.1. (i) Remark that the class $\mathcal{P}_l(\beta) := P_l[1, -1, \beta]$ was defined and studied in [12], while for $l = 2$ and $\beta = 0$ the above class was introduced by Janowski [8]. Moreover, the class $P_l := P_l[1, -1, 0]$ is the well-known class of Pinchuk [15].

Also, we see that $P_l[A, B, \beta] \subset \mathcal{P}_l(\tilde{\beta})$, where $\tilde{\beta} = \frac{1 - A_1}{1 - B}$ and $A_1 = (1 - \beta)A + \beta B$.

(ii) Notice that, if g is analytic in U with $g(0) = 1$, then there exist functions g_1 and g_2 analytic in U with $g_1(z) = g_2(z) = 1$, such that the function g could be written in the form (1.6). For example, taking

$$g_1(z) = \frac{g(z) - 1}{k} + \frac{g(z) + 1}{2} \quad \text{and} \quad g_2(z) = \frac{g(z) + 1}{2} - \frac{g(z) - 1}{k},$$

then g_1 and g_2 are analytic in U , and $g_1(z) = g_2(z) = 1$.

We will assume throughout our discussion, unless otherwise stated, that $\lambda > 0$, $d, e \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $\mu > -p$, $-1 \leq B < A \leq 1$, $\vartheta \geq 0$, and $p \in \mathbb{N}$. Moreover, all the powers are the principal ones.

Using the Cho-Kwon-Srivastava integral operator $I_\mu^p(d, e)$ defined by (1.4), we will define the following subclasses of \mathcal{A}_p .

Definition 1.2. Let $d, e \in \mathbb{R} \setminus \mathbb{Z}_0^-$, $\lambda > 0$, $\mu > -p$, $0 \leq \beta < 1$, and $\vartheta \geq 0$. For the function $f \in \mathcal{A}_p$, $p \in \mathbb{N}$, we say that $f \in \mathcal{N}_{l,p}^{\lambda, \vartheta}(d, e; \mu; \beta, A, B)$, with $l \geq 2$, if and only if

$$(1 + \vartheta) \left(\frac{z^p}{I_\mu^p(d, e)f(z)} \right)^\lambda - \vartheta \frac{I_{\mu+1}^p(d, e)f(z)}{I_\mu^p(d, e)f(z)} \left(\frac{z^p}{I_\mu^p(d, e)f(z)} \right)^\lambda \in P_l[A, B, \beta].$$

We need to notice that, since the left-hand side function from the above definition need to be analytic in U , we implicitly assumed that $I_\mu^p(d, e)f(z) \neq 0$ for all $z \in \dot{U}$.

Remarks 1.2. We remark the following special cases of the above classes:

(i) for $\beta = 0$ and $l = 2$ we obtain the subclass of non-Bazilevič functions defined by [18];

(ii) for $\mu = 0$, $l = 2$, $\vartheta = B = -1$, $A = 1$ and $\lambda > 0$, the above class reduces to the class $Q(\lambda)$ of p -valent non-Bazilevič functions (see [14]).

2. PRELIMINARIES

The following definitions and lemmas will be required in our present investigation.

Lemma 2.1. [7] Let h be a convex function in U with $h(0) = 1$. Suppose also that the function p given by

$$p(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \dots, \quad z \in U,$$

is analytic in U . Then

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z) \quad (\operatorname{Re} \gamma \geq 0, \gamma \neq 0),$$

implies

$$(2.1) \quad p(z) \prec q(z) = \frac{\gamma}{n} z^{-\frac{\gamma}{n}} \int_0^z t^{\frac{\gamma}{n}-1} h(t) dt \prec h(z),$$

and q is the best dominant of (2.1).

For real or complex numbers a, b and c , the Gauss hypergeometric function is defined by

$$(2.2) \quad {}_2F_1(a, b, c; z) = 1 + \frac{a \cdot b}{c} \frac{z}{1!} + \frac{a(a+1) \cdot b(b+1)}{c(c+1)} \frac{z^2}{2!} + \dots$$

$$= \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad a, b \in \mathbb{C}, \quad c \in \mathbb{C} \setminus \{0, -1, -2, \dots\},$$

where $(d)_k$ is the previously recalled Pochhammer symbol. The series (2.2) converges absolutely for $z \in U$, hence it represents an analytic function in U (see [19, Chapter 14]).

Each of the following identities are fairly well-known:

Lemma 2.2. [19, Chapter 14] *For all real or complex numbers a, b and c , with $c \neq 0, -1, -2, \dots$, the next equalities hold:*

$$(2.3) \quad \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} {}_2F_1(a, b, c; z)$$

where $\operatorname{Re} c > \operatorname{Re} b > 0$,

$$(2.4) \quad {}_2F_1(a, b, c; z) = (1-z)^{-a} {}_2F_1\left(a, c-b, c; \frac{z}{z-1}\right),$$

and

$$(2.5) \quad {}_2F_1(a, b, c; z) = {}_2F_1(b, a, c; z).$$

Lemma 2.3. [17] *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be analytic in U and $g(z) =$*

$\sum_{k=0}^{\infty} b_k z^k$ be analytic and convex in U . If $f(z) \prec g(z)$, then

$$|a_k| \leq |b_1|, \quad k \in \mathbb{N}.$$

3. MAIN RESULTS FOR THE CLASS $\mathcal{N}_{l,p}^{\lambda,\vartheta}(d, e; \mu; \beta, A, B)$

In this section, some properties of the class $\mathcal{N}_{l,p}^{\lambda,\vartheta}(d, e; \mu; \beta, A, B)$ such as inclusion results, integral preserving property, radius problem, coefficient bound will be discussed.

Theorem 3.1. 1. *If $f \in \mathcal{N}_{l,p}^{\lambda,\vartheta}(d, e; \mu; \beta, A, B)$, then*

$$\left(\frac{z^p}{I_{\mu}^p(d, e)f(z)} \right)^{\lambda} \in P_l[A, B, \beta].$$

2. Moreover, if $f \in \mathcal{N}_{l,p}^{\lambda,\vartheta}(d, e; \mu; \beta, \gamma)$ with $\vartheta \neq 0$, then

$$\left(\frac{z^p}{I_\mu^p(d, e)f(z)} \right)^\lambda \in \mathcal{P}_l(\beta_1),$$

where

$$\beta_1 := \beta + (1 - \beta)\vartheta_1$$

and

$$\vartheta_1 := \vartheta_1(p, \lambda, \vartheta, \mu; A, B) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1 - B)^{-1} {}_2F_1\left(1, 1, \frac{\lambda(\mu+p)}{\vartheta} + 1, \frac{B}{B-1}\right), & B \neq 0, \\ 1 - \frac{\lambda(\mu+p)}{\lambda(\mu+p)+\vartheta} A, & B = 0. \end{cases}$$

(All the powers are the principal ones).

Proof. Since the implication is obvious for $\vartheta = 0$, suppose that $\vartheta > 0$. Letting

$$(3.1) \quad K(z) = \left(\frac{z^p}{I_\mu^p(d, e)f(z)} \right)^\lambda.$$

It follows that K is analytic in U , with $K(0) = 1$, and according to the part (ii) of Remarks 1.1 the function K could be written in the form

$$(3.2) \quad K(z) = \left(\frac{k}{4} + \frac{1}{2}\right) K_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) K_2(z),$$

where K_1 and K_2 are analytic in U , with $K_1(z) = K_2(z) = 1$.

From the part 2. of Definition 1.1 we have that $K \in P_l[A, B, \beta]$, if and only if the function K has the representation given by the above relation, where $K_1, K_2 \in P[A, B, \beta]$. Consequently, supposing that K is of the form (3.2), we will prove that $K_1, K_2 \in P[A, B, \beta]$.

Differentiating the relation (3.1) and using the identity (1.5), we have

$$\frac{zK'(z)}{\lambda(\mu+p)} = K(z) - \frac{I_{\mu+1}^p(d, e)f(z)}{I_\mu^p(d, e)f(z)} \left(\frac{z^p}{I_\mu^p(d, e)f(z)} \right)^\lambda,$$

and from this relation we deduce that

$$(1 + \vartheta) \left(\frac{z^p}{I_\mu^p(d, e)f(z)} \right)^\lambda - \vartheta \frac{I_{\mu+1}^p(d, e)f(z)}{I_\mu^p(d, e)f(z)} \left(\frac{z^p}{I_\mu^p(d, e)f(z)} \right)^\lambda = K(z) + \frac{\vartheta}{\lambda(\mu+p)} zK'(z).$$

Since $f \in \mathcal{N}_{l,p}^{\lambda,\vartheta}(d, e; \mu; \beta, A, B)$, from the above relation it follows that

$$K(z) + \frac{\vartheta}{\lambda(\mu+p)} zK'(z) \in P_l[A, B, \beta],$$

and according to the second part of the Definition 1.1, this is equivalent to

$$K_i(z) + \frac{\vartheta}{\lambda(\mu + p)} z K_i'(z) \in P[A, B, \beta], \quad (i = 1, 2),$$

that is

$$\frac{1}{1 - \beta} \left[K_i(z) + \frac{\vartheta}{\lambda(\mu + p)} z K_i'(z) - \beta \right] \in P[A, B], \quad (i = 1, 2).$$

Writing

$$(3.3) \quad K_i(z) = (1 - \beta)p_i(z) + \beta, \quad (i = 1, 2),$$

from the previous relation we have

$$p_i(z) + \frac{\vartheta}{\lambda(\mu + p)} z p_i'(z) \in P[A, B], \quad (i = 1, 2).$$

By using Lemma 2.1 for $\gamma = \frac{\lambda(\mu + p)}{\vartheta}$ and $n = 1$, from the above relation we deduce that

$$p_i(z) \prec q(z) \prec \frac{1 + Az}{1 + Bz}, \quad (i = 1, 2),$$

where

$$q(z) = \frac{\lambda(\mu + p)}{\vartheta} z^{-\frac{\lambda(\mu + p)}{\vartheta}} \int_0^z t^{\frac{\lambda(\mu + p)}{\vartheta} - 1} \frac{1 + At}{1 + Bt} dt$$

is the best dominant for p_i , $i = 1, 2$.

Since $p_i(z) \prec \frac{1 + Az}{1 + Bz}$, $i = 1, 2$, from (3.3) it follows that $K_i \in P[A, B, \beta]$, $i = 1, 2$, and according to (3.1) we conclude that $K \in P_l[A, B, \beta]$, which proves the first part of the theorem.

For the second part of our result, we distinguish the following two cases:

(i) For $B = 0$, a simple computation shows that

$$p_i(z) \prec q(z) = 1 + \frac{\lambda(\mu + p)}{\lambda(\mu + p) + \vartheta} Az, \quad (i = 1, 2).$$

(ii) For $B \neq 0$, making the change of variables $s = zt$, followed by the use of the identities (2.3), (2.4) and (2.5) of Lemma 2.2, we obtain

$$p_i(z) \prec q(z) = \frac{\lambda(\mu + p)}{\vartheta} \int_0^1 s^{\frac{\lambda(\mu + p)}{\vartheta} - 1} \frac{1 + Asz}{1 + Bs z} ds = \frac{A}{B} + \left(1 - \frac{A}{B}\right) (1 + Bz)^{-1} {}_2F_1\left(1, 1, \frac{\lambda(\mu + p)}{\vartheta} + 1, \frac{Bz}{Bz + 1}\right), \quad (i = 1, 2).$$

Now, it is sufficient to show that

$$(3.4) \quad \inf \{\operatorname{Re} q(z) : z \in U\} = q(-1).$$

We may easily show that

$$\operatorname{Re} \frac{1 + Az}{1 + Bz} \geq \frac{1 - Ar}{1 - Br}, \quad \text{for } |z| \leq r < 1.$$

Denoting $G(t, z) = \frac{1 + Atz}{1 + Btz}$ and $d\mu(t) = \frac{\lambda(\mu + p)}{\vartheta} t^{\frac{\lambda(\mu+p)}{\vartheta}-1} dt$, which is a positive measure on $[0, 1]$, we have

$$q(z) = \int_0^1 G(t, z) d\mu(t),$$

hence it follows

$$\operatorname{Re} q(z) \geq \int_0^1 \frac{1 - Atr}{1 - Btr} d\mu(t) = q(-r), \quad \text{for } |z| \leq r < 1.$$

By letting $r \rightarrow 1^-$ we obtain (3.4), and from (3.3) and (3.1) we conclude that $K \in \mathcal{P}_l(\beta_1)$, which completes our proof. \square

Theorem 3.2. *If $0 \leq \vartheta_1 < \vartheta_2$, then*

$$\mathcal{N}_{l,p}^{\lambda, \vartheta_2}(d, e; \mu; \beta, A, B) \subset \mathcal{N}_{l,p}^{\lambda, \vartheta_1}(d, e; \mu; \beta, A, B)$$

Proof. The first part of Theorem 3.1 shows that the above inclusion holds whenever $\vartheta_1 = 0$.

If $0 < \vartheta_1 < \vartheta_2$, for an arbitrary $f \in \mathcal{N}_{l,p}^{\lambda, \vartheta_2}(d, e; \mu; \beta, A, B)$ let denote

$$U_1(z) = (1 + \vartheta_1) \left(\frac{z^p}{I_\mu^p(d, e)f(z)} \right)^\lambda - \vartheta_1 \frac{I_{\mu+1}^p(d, e)f(z)}{I_\mu^p(d, e)f(z)} \left(\frac{z^p}{I_\mu^p(d, e)f(z)} \right)^\lambda$$

and

$$U_0(z) = \left(\frac{z^p}{I_\mu^p(d, e)f(z)} \right)^\lambda.$$

A simple computation shows that

$$\begin{aligned} (1 + \vartheta_1) \left(\frac{z^p}{I_\mu^p(d, e)f(z)} \right)^\lambda - \vartheta_1 \frac{I_{\mu+1}^p(d, e)f(z)}{I_\mu^p(d, e)f(z)} \left(\frac{z^p}{I_\mu^p(d, e)f(z)} \right)^\lambda = \\ \left(1 - \frac{\vartheta_1}{\vartheta_2} \right) U_0(z) + \frac{\vartheta_1}{\vartheta_2} U_2(z). \end{aligned}$$

Since $P_l[A, B, \beta]$ is a convex set, from the first part of Theorem 3.1, according to the above notations it follows that

$$\left(1 - \frac{\vartheta_1}{\vartheta_2} \right) U_0(z) + \frac{\vartheta_1}{\vartheta_2} U_2(z) \in P_l[A, B, \beta],$$

that is $f \in \mathcal{N}_{l,p}^{\lambda, \vartheta_1}(d, e; \mu; \beta, A, B)$. \square

Theorem 3.3. *If $f \in \mathcal{N}_{l,p}^{\lambda, 0}(d, e; \mu; \beta, 1, -1)$, then $f(\rho z) \in \mathcal{N}_{l,p}^{\lambda, \vartheta}(d, e; \mu; \beta, 1, -1)$, where ρ is given by*

$$(3.5) \quad \rho = \frac{-\left(\beta + \frac{\vartheta}{\lambda(\mu+p)}\right) + \sqrt{\left(\beta + \frac{\vartheta}{\lambda(\mu+p)}\right)^2 + 1 - \beta^2}}{1 + \beta}.$$

Proof. For an arbitrary $f \in \mathcal{N}_{l,p}^{\lambda,0}(d, e; \mu; \beta, 1, -1)$, let denote

$$\left(\frac{z^p}{I_\mu^p(d, e)f(z)} \right)^\lambda = K(z) = \left(\frac{l}{4} + \frac{1}{2} \right) K_1(z) - \left(\frac{l}{4} - \frac{1}{2} \right) K_2(z),$$

where $K_1, K_2 \in P[1, -1, \beta]$, which is equivalent to $K_1(0) = K_2(0) = 1$ and $\operatorname{Re} K_1(z) > \beta, \operatorname{Re} K_2(z) > \beta, z \in U$.

With this notation, like in the proof of Theorem 3.1 we obtain

$$\begin{aligned} (1 + \vartheta) \left(\frac{z^p}{I_\mu^p(d, e)f(z)} \right)^\lambda - \vartheta \frac{I_{\mu+1}^p(d, e)f(z)}{I_\mu^p(d, e)f(z)} \left(\frac{z^p}{I_\mu^p(d, e)f(z)} \right)^\lambda = \\ K(z) + \frac{\vartheta}{\lambda(\mu + p)} zK'(z) = \\ \left(\frac{l}{4} + \frac{1}{2} \right) \left[K_1(z) + \frac{\vartheta}{\lambda(\mu + p)} zK_1'(z) \right] - \left(\frac{l}{4} - \frac{1}{2} \right) \left[K_2(z) + \frac{\vartheta}{\lambda(\mu + p)} zK_2'(z) \right]. \end{aligned}$$

In order to have $f(\rho z) \in \mathcal{N}_{l,p}^{\lambda,\vartheta}(d, e; \mu; \beta, 1, -1)$, according to the above formula, we need to find the (bigger) value of ρ , such that

$$\operatorname{Re} \left[K_i(z) + \frac{\vartheta}{\lambda(\mu + p)} zK_i'(z) \right] > \beta, |z| < \rho, (i = 1, 2).$$

From the well-known estimates for the class $P(0)$ (see, eq., [6]) we have

$$\begin{aligned} |K_i'(z)| &\leq \frac{2 \operatorname{Re} K_i(z)}{1 - r^2}, |z| \leq r < 1, (i = 1, 2), \\ \operatorname{Re} K_i(z) &\geq \frac{1 - r}{1 + r}, |z| \leq r < 1, (i = 1, 2), \end{aligned}$$

thus, we deduce that

$$\begin{aligned} \operatorname{Re} \left[K_i(z) + \frac{\vartheta}{\lambda(\mu + p)} zK_i'(z) \right] &\geq \operatorname{Re} K_i(z) - \frac{\vartheta}{\lambda(\mu + p)} |zK_i'(z)| \geq \\ (3.6) \quad \operatorname{Re} K_i(z) &\left[1 - \frac{\vartheta}{\lambda(\mu + p)} \frac{2r}{1 - r^2} \right], |z| \leq r < 1, (i = 1, 2). \end{aligned}$$

A simple computation shows that

$$(3.7) \quad 1 - \frac{\vartheta}{\lambda(\mu + p)} \frac{2r}{1 - r^2} \geq 0,$$

for $0 \leq r \leq r_0$, where

$$r_0 := \frac{-\vartheta + \sqrt{\vartheta^2 + \lambda^2(\mu + p)^2}}{\lambda(\mu + p)} \in (0, 1).$$

Now, from the inequality (3.6) we have

$$\operatorname{Re} \left[K_i(z) + \frac{\vartheta}{\lambda(\mu + p)} zK_i'(z) \right] \geq \frac{1 - r}{1 + r} \left[1 - \frac{\vartheta}{\lambda(\mu + p)} \frac{2r}{1 - r^2} \right], |z| \leq r_0 < 1,$$

for $i = 1, 2$. It is easy to check that

$$\frac{1-r}{1+r} \left[1 - \frac{\vartheta}{\lambda(\mu+p)} \frac{2r}{1-r^2} \right] > \beta$$

for $0 \leq r < \rho$, where ρ is given by (3.5). Moreover, since the above inequality is equivalent to

$$1 - \frac{\vartheta}{\lambda(\mu+p)} \frac{2r}{1-r^2} > \frac{1+r}{1-r} \beta$$

for $r \in [0, 1)$, it follows that (3.7) holds for $r \in [0, \rho)$, and our theorem is completely proved. \square

Next we will consider some properties of generalized p -valent Bernardi integral operator. Thus, for $f \in \mathcal{A}_p$, let $F_{\eta,p} : \mathcal{A}_p \rightarrow \mathcal{A}_p$ be defined by

$$(3.8) \quad F_{\eta,p}f(z) = \frac{\eta+p}{z^\eta} \int_0^z f(t)t^{\eta-1} dt, \quad (\eta > -p).$$

We will give a short proof that this operator is well-defined, as follows. If the function $f \in \mathcal{A}_p$ is of the form (1.1), then the definition relation (3.8) could be written as

$$F_{\eta,p}f(z) = \frac{\eta+p}{z^\eta} \int_0^z f(t)t^{\eta-1} dt = (\eta+p)I_{\eta,p}f(z),$$

where

$$I_{\eta,p}f(z) = \frac{1}{z^\eta} \int_0^z f(t)t^{\eta-1} dt.$$

We see that integral operator $I_{\eta,p}$ defined above is similar to that of Lemma 1.2c. of [11]. According to this lemma, it follows that $I_{\eta,p}$ is an analytic integral operator for any function f of the form (1.1) whenever $\operatorname{Re} \eta > -p$, and $F_{\eta,p}f \in \mathcal{A}_p$ has the form

$$F_{\eta,p}f(z) = z^p + (\eta+p) \sum_{n=1}^{\infty} \frac{a_{p+n}}{p+n+\eta} z^{p+n}, \quad z \in \mathbb{U}.$$

The operator defined in (3.8) is called *the generalized p -valent Bernardi integral operator*, and for special case $p = 1$ we get *the generalized Bernardi integral operator*. Thus, for $p = 1$ and $\eta \in \mathbb{N}$, the operator $F_\eta := F_{\eta,1}$ was introduced by Bernardi [1], and in particular, if $\eta = 1$ it reduces to the operator F_1 that was earlier introduced by Livingston [9].

Theorem 3.4. *Let $f \in \mathcal{A}_p$ and $F = F_{\eta,p}f$, where $F_{\eta,p}$ is given by (3.8). If*

$$(1+\vartheta) \left(\frac{z^p}{I_\mu^p(d,e)F(z)} \right)^\lambda - \vartheta \frac{I_\mu^p(d,e)f(z)}{I_\mu^p(d,e)F(z)} \left(\frac{z^p}{I_\mu^p(d,e)F(z)} \right)^\lambda \in P_l[A, B, \beta],$$

where $d, e \in \mathbb{R} \setminus \mathbb{Z}_0^-, \lambda > 0, \mu > -p, 0 \leq \beta < 1, \vartheta \geq 0$ and $l \geq 2$, then

$$\left(\frac{z^p}{I_\mu^p(d, e)F(z)} \right)^\lambda \in P_l[A, B, \beta].$$

(All the powers are the principal ones).

Proof. Like we mentioned after the Definition 1.2, since the left-hand side function from the relation (3.9) need to be analytic in U , we implicitly assumed that $I_\mu^p(d, e)F(z) \neq 0$ for all $z \in U$.

The implication is obvious for $\vartheta = 0$, hence suppose that $\vartheta > 0$. Letting

$$(3.10) \quad \left(\frac{z^p}{I_\mu^p(d, e)F(z)} \right)^\lambda = K(z),$$

from the assumption (3.9) it follows that K is analytic in U , with $K(0) = 1$.

It is easy to check that, if $f, g \in \mathcal{A}_p$, then

$$(3.11) \quad \frac{z}{p} (f(z) * g(z))' = \left(\frac{z}{p} f'(z) \right)' * g(z).$$

Moreover, since $F = F_{\eta, p}f$, where $F_{\eta, p}$ is given by (3.8), a simple differentiation shows that

$$(3.12) \quad z (I_\mu^p(d, e)F(z))' = (\eta + p)I_\mu^p(d, e)f(z) - \eta I_\mu^p(d, e)F(z).$$

Taking the logarithmical differentiation of (3.10), we have

$$\lambda \left[p - \frac{z (I_\mu^p(d, e)F(z))'}{I_\mu^p(d, e)F(z)} \right] = \frac{zK'(z)}{K(z)},$$

and using the relations (3.11) and (3.12), it follows that

$$\frac{I_\mu^p(d, e)f(z)}{I_\mu^p(d, e)F(z)} = 1 - \frac{1}{\lambda(\eta + p)} \frac{zK'(z)}{K(z)},$$

and thus

$$(1 + \vartheta) \left(\frac{z^p}{I_\mu^p(d, e)F(z)} \right)^\lambda - \vartheta \frac{I_\mu^p(d, e)f(z)}{I_\mu^p(d, e)F(z)} \left(\frac{z^p}{I_\mu^p(d, e)F(z)} \right)^\lambda = K(z) + \frac{\vartheta}{\lambda(p + \eta)} zK'(z).$$

From the assumption (3.9), the above relation gives that

$$K(z) + \frac{\vartheta}{\lambda(p + \eta)} zK'(z) \in P_l[A, B, \beta],$$

and using a similar proof with those of the first part of Theorem 3.1 we obtain that $K \in P_l[A, B, \beta]$, which proves our result. \square

Theorem 3.5. *If*

$$(3.13) \quad f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \in \mathcal{N}_{l,p}^{\lambda,\vartheta}(d, e; \mu; \beta, A, B),$$

where $d, e \in \mathbb{R} \setminus \mathbb{Z}_0^-, \lambda > 0, \mu > -p, 0 \leq \beta < 1, \vartheta \geq 0$ and $l \geq 2$, then

$$(3.14) \quad |a_{p+1}| \leq \left| \frac{d}{e} \right| \frac{(1-\beta)(A-B)}{|\vartheta + \lambda(\mu+p)|}.$$

The inequality (3.14) is sharp.

Proof. If we let

$$(3.15) \quad (1+\vartheta) \left(\frac{z^p}{I_{\mu}^p(d, e)f(z)} \right)^{\lambda} - \vartheta \frac{I_{\mu+1}^p(d, e)f(z)}{I_{\mu}^p(d, e)f(z)} \left(\frac{z^p}{I_{\mu}^p(d, e)f(z)} \right)^{\lambda} = p(z),$$

using the fact that

$$I_{\mu}^p(d, e)f(z) = z^p + \sum_{n=1}^{\infty} \frac{(\mu+p)_n(e)_n}{n!(d)_n} a_{p+n} z^{p+n},$$

we have

$$(3.16) \quad \begin{aligned} (1+\vartheta) \left(\frac{z^p}{I_{\mu}^p(d, e)f(z)} \right)^{\lambda} - \vartheta \frac{I_{\mu+1}^p(d, e)f(z)}{I_{\mu}^p(d, e)f(z)} \left(\frac{z^p}{I_{\mu}^p(d, e)f(z)} \right)^{\lambda} = \\ 1 - \left(1 + \frac{\vartheta}{\lambda(\mu+p)} \right) \frac{(\mu+p)_1(e)_1}{1!(d)_1} \lambda a_{p+1} z + \dots = \\ 1 - \frac{e}{d} [\vartheta + \lambda(\mu+p)] a_{p+1} z + \dots, \quad z \in U. \end{aligned}$$

Since $f \in \mathcal{N}_{l,p}^{\lambda,\vartheta}(d, e; \mu; \beta, A, B)$, it follows that the function p defined by (3.15) is of the form

$$p(z) = \left(\frac{l}{4} + \frac{1}{2} \right) p_1(z) - \left(\frac{l}{4} - \frac{1}{2} \right) p_2(z),$$

where $p_1, p_2 \in P[A, B, \beta]$. It follows that

$$p_i(z) \prec \frac{1 + [(1-\beta)A + \beta B]z}{1 + Bz} \quad (i = 1, 2),$$

and from the above relation we deduce that

$$(3.17) \quad p(z) \prec \frac{1 + [(1-\beta)A + \beta B]z}{1 + Bz}.$$

According to (3.16), from the subordination (3.17) we obtain

$$1 - \frac{e}{d} \frac{\vartheta + \lambda(\mu+p)}{1-\beta} a_{p+1} z + \dots = \frac{p(z) - \beta}{1-\beta} \prec \frac{1 + Az}{1 + Bz},$$

and from Lemma 2.3 we conclude that

$$\left| -\frac{e}{d} \frac{\vartheta + \lambda(\mu+p)}{1-\beta} \right| |a_{p+1}| \leq |A - B|,$$

which proves our result.

To prove that the inequality (3.14) is sharp we need to show that there exists a function $f \in \mathcal{N}_{l,p}^{\lambda,\vartheta}(d, e; \mu; \beta, A, B)$ of the form (3.13), such that for this function we have equality in (3.14).

Thus, we will prove that there exists $f \in \mathcal{N}_{l,p}^{\lambda,\vartheta}(d, e; \mu; \beta, A, B)$, such that the identity (3.15) holds for the special case

$$p(z) = \frac{1 + [(1 - \beta)A + \beta B]z}{1 + Bz}.$$

Setting

$$(3.18) \quad K(z) = \left(\frac{z^p}{I_\mu^p(d, e)f(z)} \right)^\lambda,$$

like in the proof of Theorem 3.1 we deduce that the relation (3.15) is equivalent to

$$(3.19) \quad K(z) + \frac{1}{\gamma} z K'(z) = p(z), \quad \text{where } \gamma := \frac{\lambda(\mu + p)}{\vartheta}.$$

(i) If $\vartheta = 0$, the above differential equation has the solution $K = p$.

(ii) If $\vartheta > 0$, then $\gamma > 0$ whenever $\lambda > 0$ and $\mu > -p$. Since the function p is convex in the unit disk U , according to Lemma 2.1 it follows that this differential equation has the solution

$$\tilde{K}(z) = \frac{\gamma}{z^\gamma} \int_0^z t^{\gamma-1} p(t) dt \prec p(z).$$

It is easy to check that $p(z) \neq 0$ for all $z \in U$, and from the above subordination we get that $\tilde{K}(z) \neq 0$, $z \in U$.

Now, if we define the function K_0 by

$$K_0(z) = \begin{cases} p(z), & \text{if } \vartheta = 0, \\ \tilde{K}(z), & \text{if } \vartheta > 0, \end{cases}$$

then K_0 is the analytic solution of the differential equation (3.19), and moreover $K_0(z) \neq 0$, $z \in U$.

Thus, for $K = K_0$ the relation (3.18) is equivalent to

$$I_\mu^p(d, e)f(z) = z^p K_0^{-1/\lambda}(z),$$

and this equation has the solution

$$(3.20) \quad f_0(z) := \psi_p(d, e; z) * \left(z^p K_0^{-1/\lambda}(z) \right),$$

where

$$\psi_p(d, e; z) := z^p + \sum_{n=1}^{\infty} \frac{n!(d)_n}{(\mu + p)_n(e)_n} z^{p+n}. \quad (d, e \in \mathbb{C} \setminus \mathbb{Z}_0^-, \mu > -p),$$

Consequently, for the function f_0 defined by (3.20) we get equality in (3.14), hence the sharpness of our result is proved. \square

As a special case, for $l = 2$ and $\beta = 0$ we obtain the corresponding result for the class $\mathcal{N}_{2,p}^{\lambda,\vartheta}(d, e; \mu; 0, A, B)$ (see [18] for $n = 1$).

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ON A PRODUCT-TYPE OPERATOR FROM MIXED-NORM SPACES TO BLOCH-ORLICZ SPACES

HAIYING LI AND ZHITAO GUO

ABSTRACT. The boundedness and compactness of a product-type operator DM_uC_ψ from mixed-norm spaces to Bloch-Orlicz spaces are characterized in this paper.

1. INTRODUCTION

Let \mathbb{D} denote the unit disk in the complex plane \mathbb{C} , $\mathcal{H}(\mathbb{D})$ the class of all analytic functions on \mathbb{D} and \mathbb{N} the set of nonnegative integers.

A positive continuous function ϕ on $[0,1)$ is called normal if there exist two positive numbers s and t with $0 < s < t$, and $\delta \in [0, 1)$ such that (see [19])

$$\begin{aligned} \frac{\phi(r)}{(1-r)^s} &\text{ is decreasing on } [\delta, 1), \quad \lim_{r \rightarrow 1} \frac{\phi(r)}{(1-r)^s} = 0; \\ \frac{\phi(r)}{(1-r)^t} &\text{ is increasing on } [\delta, 1), \quad \lim_{r \rightarrow 1} \frac{\phi(r)}{(1-r)^t} = \infty. \end{aligned}$$

For $p, q \in (0, \infty)$ and ϕ normal, the mixed-norm space $H(p, q, \phi)(\mathbb{D}) = H(p, q, \phi)$ is the space of all functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{H(p,q,\phi)} = \left(\int_0^1 M_q^p(f, r) \frac{\phi^p(r)}{1-r} dr \right)^{\frac{1}{p}} < \infty,$$

where

$$M_q(f, r) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^q d\theta \right)^{\frac{1}{q}}.$$

For $1 \leq p, q < \infty$, $H(p, q, \phi)$, equipped with the norm $\|f\|_{H(p,q,\phi)}$, is a Banach space, while for the other vales of p and q , $\|\cdot\|_{H(p,q,\phi)}$ is a quasinorm on $H(p, q, \phi)$, $H(p, q, \phi)$ is a Fréchet space but not a Banach space. Note that if $\phi(r) = (1-r)^{\frac{\alpha+1}{p}}$, then $H(p, q, \phi)$ is equivalent to the weighted Bergman space $A_\alpha^p(\mathbb{D}) = A_\alpha^p$ defined for $0 < p < \infty$ and $\alpha > -1$, as the spaces of all $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{A_\alpha^p}^p = (\alpha + 1) \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha dm(z) < \infty,$$

where $dm(z) = \frac{1}{\pi} r dr d\theta$ is the normalized Lebesgue area measure on \mathbb{D} ([8, 12, 18, 25, 27, 33, 35, 48, 51]). For more details on the mixed-norm space on various domains and operators on them, see, e.g., [1, 7, 10, 20, 22, 23, 24, 28, 29, 34, 36, 37, 38, 41, 42, 43, 44, 46, 47, 54].

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For every $0 < \alpha < \infty$, the α -Bloch space, denoted by \mathcal{B}^α , consists of all functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

\mathcal{B}^α is a Banach space under the norm

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)|.$$

For $\alpha = 1$ is obtained the Bloch space. α -Bloch space is introduced and studied by numerous authors. Recently, many authors studied different classes of Bloch-type spaces, where the typical weight function, $\omega(z) = 1 - |z|^2$ ($z \in \mathbb{D}$) is replaced by a bounded continuous positive function μ defined on \mathbb{D} . More precisely, a function $f \in \mathcal{H}(\mathbb{D})$ is called a μ -Bloch function, denoted by $f \in \mathcal{B}^\mu$, if

$$\|f\|_\mu = \sup_{z \in \mathbb{D}} \mu(z) |f'(z)| < \infty.$$

Clearly, if $\mu(z) = \omega(z)^\alpha$ with $\alpha > 0$, \mathcal{B}^μ is just the α -Bloch space \mathcal{B}^α . It is readily seen that \mathcal{B}^μ is a Banach space with the norm

$$\|f\|_{\mathcal{B}^\mu} = |f(0)| + \|f\|_\mu.$$

For some information on the Bloch, α -Bloch and Bloch-type spaces, as well as some operators on them see, e.g., [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21, 23, 25, 26, 27, 29, 30, 31, 32, 34, 37, 38, 39, 40, 41, 43, 44, 45, 46, 47, 50, 51, 52, 53, 55].

Recently, Fernández in [17] used Young's functions to define the Bloch-Orlicz space. More precisely, let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a strictly increasing convex function such that $\varphi(0) = 0$ and $\lim_{t \rightarrow \infty} \varphi(t) = \infty$. The Bloch-Orlicz space associated with the function φ , denoted by \mathcal{B}^φ , is the class of all analytic functions f in \mathbb{D} such that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) \varphi(\lambda |f'(z)|) < \infty$$

for some $\lambda > 0$ depending on f . Also, since φ is convex, it is not hard to see that the Minkowski's functional

$$\|f\|_\varphi = \inf \left\{ k > 0 : S_\varphi \left(\frac{f'}{k} \right) \leq 1 \right\}$$

define a seminorm for \mathcal{B}^φ , which, in this case, is known as Luxemburg's seminorm, where

$$S_\varphi(f) = \sup_{z \in \mathbb{D}} (1 - |z|^2) \varphi(|f(z)|)$$

We know that \mathcal{B}^φ is a Banach space with the norm $\|f\|_{\mathcal{B}^\varphi} = |f(0)| + \|f\|_\varphi$. We also have that the Bloch-Orlicz space is isometrically equal to μ -Bloch space, where

$$\mu(z) = \frac{1}{\varphi^{-1}(\frac{1}{1-|z|^2})}, \quad z \in \mathbb{D}.$$

Thus for any $f \in \mathcal{B}^\varphi$, we have

$$\|f\|_{\mathcal{B}^\varphi} = |f(0)| + \sup_{z \in \mathbb{D}} \mu(z) |f'(z)|.$$

It is well known that the differentiation operator D is defined by

$$(Df)(z) = f'(z), \quad f \in \mathcal{H}(\mathbb{D}).$$

Let $u \in \mathcal{H}(\mathbb{D})$, then the multiplication operator M_u is defined by

$$(M_u f)(z) = u(z) f(z), \quad f \in \mathcal{H}(\mathbb{D}).$$

Let ψ be an analytic self-map of \mathbb{D} . The composition operator C_ψ is defined by

$$(C_\psi f)(z) = f(\psi(z)), \quad f \in \mathcal{H}(\mathbb{D}).$$

Investigation of products of these and integral-type operators attracted a lot of attention recently (see, e.g., [2]-[49], [51]-[55]). For example, in [3] and [17], the authors investigated bounded superposition operators between Bloch-Orlicz and α -Bloch spaces and composition operators on Bloch-Orlicz type spaces. In [37] and [38], S. Stević investigated extended Cesàro operators between mixed-norm spaces and Bloch-type spaces and an integral-type operator from logarithmic Bloch-type spaces to mixed-norm spaces on the unit ball. In [36] and [41], S. Stević investigated an integral-type operator from logarithmic Bloch-type and mixed-norm spaces to Bloch-type spaces and weighted differentiation composition operators from mixed-norm spaces to weighted-type spaces. In [42] and [46], S. Stević investigated an integral-type operator from Zygmund-type spaces to mixed-norm spaces on the unit ball and weighted differentiation composition operators from the mixed-norm space to the n th weighted-type space on the unit disk. S. Stević in [34] gave the properties of products of integral-type operators and composition operators from the mixed norm space to Bloch-type spaces. In [47], S. Stević investigated weighted radial operator from the mixed-norm space to the n th weighted-type space on the unit ball. In [54], X. Zhu studied extended Cesàro operators from mixed-norm spaces to Zygmund type spaces.

Motivated, among others, by these papers, we will study here the boundedness and compactness of the following operator, which is also a product-type one,

$$(DM_u C_\psi f)(z) = u'(z)f(\psi(z)) + u(z)\psi'(z)f'(\psi(z)), \quad f \in \mathcal{H}(\mathbb{D}),$$

from $H(p, q, \phi)$ to \mathcal{B}^φ .

In what follows,

$$\mu(z) = \frac{1}{\varphi^{-1}\left(\frac{1}{1-|z|^2}\right)},$$

and we use the letter C to denote a positive constant whose value may change at each occurrence.

2. THE BOUNDEDNESS AND COMPACTNESS OF $DM_u C_\psi : H(p, q, \phi) \rightarrow \mathcal{B}^\varphi$

In this section, we will give our main results and proofs. In order to prove our main results, we need some auxiliary results. Our first lemma characterizes compactness in terms of sequential convergence. Since the proof is standard, it is omitted here (see, Proposition 3.11 in [4]).

Lemma 1. *Suppose $u \in \mathcal{H}(\mathbb{D})$, ψ is an analytic self-map of \mathbb{D} , $0 < p, q < \infty$ and ϕ is normal. Then the operator $DM_u C_\psi : H(p, q, \phi) \rightarrow \mathcal{B}^\varphi$ is compact if and only if it is bounded and for each sequence $\{f_n\}_{n \in \mathbb{N}}$ which is bounded in $H(p, q, \phi)$ and converges to zero uniformly on compact subsets of \mathbb{D} as $n \rightarrow \infty$, we have $\|DM_u C_\psi f_n\|_{\mathcal{B}^\varphi} \rightarrow 0$ as $n \rightarrow \infty$.*

The following lemma can be found in [36].

Lemma 2. *Assume $0 < p, q < \infty$, ψ is normal and $f \in H(p, q, \phi)$. Then for every $n \in \mathbb{N}$, there is a positive constant C independent of f such that*

$$|f^{(n)}(z)| \leq \frac{C\|f\|_{H(p,q,\phi)}}{\phi(|z|)(1-|z|^2)^{\frac{1}{q}+n}}.$$

Theorem 3. Let $u \in \mathcal{H}(\mathbb{D})$, ψ be an analytic self-map of \mathbb{D} , $0 < p, q < \infty$ and ϕ be normal. Then $DM_u C_\psi : H(p, q, \phi) \rightarrow \mathcal{B}^\varphi$ is bounded if and only if

$$k_1 = \sup_{z \in \mathbb{D}} \frac{\mu(z)|u''(z)|}{\phi(|\psi(z)|)(1 - |\psi(z)|^2)^{\frac{1}{q}}} < \infty, \quad (1)$$

$$k_2 = \sup_{z \in \mathbb{D}} \frac{\mu(z)|2u'(z)\psi'(z) + u(z)\psi''(z)|}{\phi(|\psi(z)|)(1 - |\psi(z)|^2)^{\frac{1}{q}+1}} < \infty, \quad (2)$$

$$k_3 = \sup_{z \in \mathbb{D}} \frac{\mu(z)|u(z)||\psi'(z)|^2}{\phi(|\psi(z)|)(1 - |\psi(z)|^2)^{\frac{1}{q}+2}} < \infty. \quad (3)$$

Proof. Assume that (1), (2) and (3) hold. By Lemma 2, then we get

$$\begin{aligned} |f(\psi(z))| &\leq \frac{C_1 \|f\|_{H(p,q,\phi)}}{\phi(|\psi(z)|)(1 - |\psi(z)|^2)^{\frac{1}{q}}}, \\ |f'(\psi(z))| &\leq \frac{C_2 \|f\|_{H(p,q,\phi)}}{\phi(|\psi(z)|)(1 - |\psi(z)|^2)^{\frac{1}{q}+1}}, \\ |f''(\psi(z))| &\leq \frac{C_3 \|f\|_{H(p,q,\phi)}}{\phi(|\psi(z)|)(1 - |\psi(z)|^2)^{\frac{1}{q}+2}}. \end{aligned}$$

Then for each $f \in H(p, q, \phi) \setminus \{0\}$, we have:

$$\begin{aligned} &S_\varphi \left(\frac{(DM_u C_\psi f)'(z)}{C \|f\|_{H(p,q,\phi)}} \right) \\ &\leq \sup_{z \in \mathbb{D}} (1 - |z|^2) \varphi \left[\left(\frac{k_1 \phi(|\psi(z)|)(1 - |\psi(z)|^2)^{\frac{1}{q}} |f(\psi(z))|}{C \mu(z) \|f\|_{H(p,q,\phi)}} \right) \right. \\ &\quad + \left(\frac{k_2 \phi(|\psi(z)|)(1 - |\psi(z)|^2)^{\frac{1}{q}+1} |f'(\psi(z))|}{C \mu(z) \|f\|_{H(p,q,\phi)}} \right) \\ &\quad \left. + \left(\frac{k_3 \phi(|\psi(z)|)(1 - |\psi(z)|^2)^{\frac{1}{q}+2} |f''(\psi(z))|}{C \mu(z) \|f\|_{H(p,q,\phi)}} \right) \right] \\ &\leq \sup_{z \in \mathbb{D}} (1 - |z|^2) \varphi \left[\frac{k_1 C_1 + k_2 C_2 + k_3 C_3}{C \mu(z)} \right] \\ &\leq \sup_{z \in \mathbb{D}} (1 - |z|^2) \varphi \left(\varphi^{-1} \left(\frac{1}{1 - |z|^2} \right) \right) = 1 \end{aligned}$$

where C is a constant such that $C \geq k_1 C_1 + k_2 C_2 + k_3 C_3$. Now, we can conclude that there exists a constant C such that $\|DM_u C_\psi f\|_{\mathcal{B}^\varphi} \leq C \|f\|_{H(p,q,\phi)}$ for all $f \in H(p, q, \phi)$, so the product-type operator $DM_u C_\psi : H(p, q, \phi) \rightarrow \mathcal{B}^\varphi$ is bounded.

Conversely, suppose that $DM_u C_\psi : H(p, q, \phi) \rightarrow \mathcal{B}^\varphi$ is bounded, i.e., there exists $C > 0$ such that $\|DM_u C_\psi f\|_{\mathcal{B}^\varphi} \leq C \|f\|_{H(p,q,\phi)}$ for all $f \in H(p, q, \phi)$. Taking the function $f(z) = 1 \in H(p, q, \phi)$, and $\|f\|_{H(p,q,\phi)} \leq C$, then

$$S_\varphi \left(\frac{(DM_u C_\psi f)'(z)}{C} \right) = S_\varphi \left(\frac{u''(z)}{C} \right) = \sup_{z \in \mathbb{D}} (1 - |z|^2) \varphi \left(\frac{|u''(z)|}{C} \right) \leq 1.$$

It follows that

$$\sup_{z \in \mathbb{D}} \mu(z)|u''(z)| < \infty. \quad (4)$$

Taking the function $f(z) = z \in H(p, q, \phi)$, and $\|f\|_{H(p, q, \phi)} \leq C$, then

$$\begin{aligned} S_\varphi \left(\frac{(DM_u C_\psi f)'(z)}{C} \right) \\ = \sup_{z \in \mathbb{D}} (1 - |z|^2) \varphi \left(\frac{|u''(z)\psi(z) + (2u'(z)\psi'(z) + u(z)\psi''(z))|}{C} \right) \leq 1. \end{aligned}$$

Hence

$$\sup_{z \in \mathbb{D}} \mu(z) |u''(z)\psi(z) + 2u'(z)\psi'(z) + u(z)\psi''(z)| < \infty.$$

By (4) and the boundedness of $\psi(z)$, we can see that

$$\sup_{z \in \mathbb{D}} \mu(z) |2u'(z)\psi'(z) + u(z)\psi''(z)| < \infty. \quad (5)$$

Taking the function $f(z) = \frac{z^2}{2} \in H(p, q, \phi)$, similarly, we can get

$$\sup_{z \in \mathbb{D}} \mu(z) |u(z)| |\psi'(z)|^2 < \infty. \quad (6)$$

For a fixed $\omega \in \mathbb{D}$, set

$$\begin{aligned} f_{\psi(\omega)}(z) &= \frac{A(1 - |\psi(\omega)|^2)^{t+1}}{\phi(|\psi(\omega)|)(1 - \overline{\psi(\omega)}z)^{\frac{1}{q}+t+1}} + \frac{B(1 - |\psi(\omega)|^2)^{t+2}}{\phi(|\psi(\omega)|)(1 - \overline{\psi(\omega)}z)^{\frac{1}{q}+t+2}} \\ &\quad + \frac{(1 - |\psi(\omega)|^2)^{t+3}}{\phi(|\psi(\omega)|)(1 - \overline{\psi(\omega)}z)^{\frac{1}{q}+t+3}}, \end{aligned} \quad (7)$$

where the constant t is from the definition of the normality of the function ϕ .

Then $\sup_{\omega \in \mathbb{D}} \|f_{\psi(\omega)}\|_{H(p, q, \phi)} < \infty$, and we have

$$\begin{aligned} f_{\psi(\omega)}(\psi(\omega)) &= \frac{A + B + 1}{\phi(|\psi(\omega)|)(1 - |\psi(\omega)|^2)^{\frac{1}{q}}}, \\ f'_{\psi(\omega)}(\psi(\omega)) &= \frac{(AM_1 + BM_2 + M_3)\overline{\psi(\omega)}}{\phi(|\psi(\omega)|)(1 - |\psi(\omega)|^2)^{\frac{1}{q}+1}}, \\ f''_{\psi(\omega)}(\psi(\omega)) &= \frac{(AM_1M_2 + BM_2M_3 + M_3M_4)\overline{\psi(\omega)}^2}{\phi(|\psi(\omega)|)(1 - |\psi(\omega)|^2)^{\frac{1}{q}+2}}. \end{aligned}$$

where $M_i = \frac{1}{q} + t + i, i = 1, 2, 3, 4$.

To prove (1), we choose the corresponding function in (7) with

$$A = \frac{M_3}{M_1}, \quad B = -\frac{2M_3}{M_2},$$

and denote it by $f_{\psi(\omega)}$, then we have

$$f_{\psi(\omega)}(\psi(\omega)) = \frac{P}{\phi(|\psi(\omega)|)(1 - |\psi(\omega)|^2)^{\frac{1}{q}}}, \quad f'_{\psi(\omega)}(\psi(\omega)) = f''_{\psi(\omega)}(\psi(\omega)) = 0, \quad (8)$$

where $P = \frac{M_3}{M_1} - \frac{2M_3}{M_2} + 1$.

By the boundedness of $DM_u C_\psi : H(p, q, \phi) \rightarrow \mathcal{B}^\varphi$, we have $\|DM_u C_\psi f_{\psi(\omega)}\|_{\mathcal{B}^\varphi} \leq C$, then

$$\begin{aligned} 1 &\geq S_\varphi \left(\frac{(DM_u C_\psi f_{\psi(\omega)})'(z)}{C} \right) \\ &\geq \sup_{w \in \mathbb{D}} (1 - |w|^2) \varphi \left(\frac{P|u''(w)|}{C\phi(|\psi(\omega)|)(1 - |\psi(\omega)|^2)^{\frac{1}{q}}} \right), \end{aligned}$$

from which we can get (1). To prove (2), we choose the corresponding function in (7) with

$$A = \frac{-2M_2 - M_1M_2 + M_3M_4}{2M_2}, \quad B = \frac{M_1M_2 - M_3M_4}{2M_2},$$

and denote it by $g_{\psi(\omega)}$, then we have

$$g'_{\psi(\omega)}(\psi(\omega)) = \frac{E\overline{\psi(\omega)}}{\phi(|\psi(\omega)|)(1 - |\psi(\omega)|^2)^{\frac{1}{q}+1}}, \quad g_{\psi(\omega)}(\psi(\omega)) = g''_{\psi(\omega)}(\psi(\omega)) = 0, \quad (9)$$

where

$$E = \frac{-2M_1M_2 - M_1^2M_2 + M_1M_3M_4}{2M_2} + \frac{M_1M_2 - M_3M_4}{2} + M_3.$$

By the boundedness of $DM_u C_\psi : H(p, q, \phi) \rightarrow \mathcal{B}^\varphi$, we have $\|DM_u C_\psi g_{\psi(\omega)}\|_{\mathcal{B}^\varphi} \leq C$, then

$$\begin{aligned} 1 &\geq S_\varphi \left(\frac{(DM_u C_\psi g_{\psi(\omega)})'(z)}{C} \right) \\ &\geq \sup_{\frac{1}{2} < |\psi(\omega)| < 1} (1 - |\omega|^2) \varphi \left(\frac{|(DM_u C_\psi g_{\psi(\omega)})'(\omega)|}{C} \right) \\ &= \sup_{\frac{1}{2} < |\psi(\omega)| < 1} (1 - |\omega|^2) \varphi \left(\frac{E|\psi(\omega)||2u'(\omega)\psi'(\omega) + u(\omega)\psi''(\omega)|}{C\phi(|\psi(\omega)|)(1 - |\psi(\omega)|^2)^{\frac{1}{q}+1}} \right). \end{aligned}$$

It follows that

$$\begin{aligned} &\sup_{\frac{1}{2} < |\psi(\omega)| < 1} \frac{\mu(\omega)|2u'(\omega)\psi'(\omega) + u(\omega)\psi''(\omega)|}{\phi(|\psi(\omega)|)(1 - |\psi(\omega)|^2)^{\frac{1}{q}+1}} \\ &\leq 2 \sup_{\frac{1}{2} < |\psi(\omega)| < 1} \frac{\mu(\omega)|\psi(\omega)||2u'(\omega)\psi'(\omega) + u(\omega)\psi''(\omega)|}{\phi(|\psi(\omega)|)(1 - |\psi(\omega)|^2)^{\frac{1}{q}+1}} < \infty. \end{aligned} \quad (10)$$

Since ϕ is normal, and using (5), we have

$$\begin{aligned} &\sup_{|\psi(\omega)| \leq \frac{1}{2}} \frac{\mu(\omega)|2u'(\omega)\psi'(\omega) + u(\omega)\psi''(\omega)|}{\phi(|\psi(\omega)|)(1 - |\psi(\omega)|^2)^{\frac{1}{q}+1}} \\ &\leq C \sup_{|\psi(\omega)| \leq \frac{1}{2}} \mu(\omega)|2u'(\omega)\psi'(\omega) + u(\omega)\psi''(\omega)| < \infty. \end{aligned} \quad (11)$$

From (10) and (11), we can get (2). To prove (3), we choose the corresponding function in (7) with

$$A = 1, \quad B = -2,$$

and denote it by $h_{\psi(\omega)}$, then we have

$$h_{\psi(\omega)}(\psi(\omega)) = h'_{\psi(\omega)}(\psi(\omega)) = 0, \quad h''_{\psi(\omega)}(\psi(\omega)) = \frac{F\overline{\psi(\omega)}^2}{\phi(|\psi(\omega)|)(1 - |\psi(\omega)|^2)^{\frac{1}{q}+2}}, \quad (12)$$

where $F = M_1M_2 - 2M_2M_3 + M_3M_4$.

By the boundedness of $DM_u C_\psi : H(p, q, \phi) \rightarrow \mathcal{B}^\varphi$, we have $\|DM_u C_\psi h_{\psi(\omega)}\|_{\mathcal{B}^\varphi} \leq C$, then

$$\begin{aligned} 1 &\geq S_\varphi \left(\frac{(DM_u C_\psi h_{\psi(\omega)})'(z)}{C} \right) \\ &\geq \sup_{\frac{1}{2} < |\psi(\omega)| < 1} (1 - |\omega|^2)^\varphi \left(\frac{|(DM_u C_\psi h_{\psi(\omega)})'(\omega)|}{C} \right) \\ &= \sup_{\frac{1}{2} < |\psi(\omega)| < 1} (1 - |\omega|^2)^\varphi \left(\frac{F|\psi(\omega)|^2 |u(\omega)| |\psi'(\omega)|^2}{C \phi(|\psi(\omega)|) (1 - |\psi(\omega)|^2)^{\frac{1}{q}+2}} \right). \end{aligned}$$

It follows that

$$\begin{aligned} &\sup_{\frac{1}{2} < |\psi(\omega)| < 1} \frac{\mu(\omega) |u(\omega)| |\psi'(\omega)|^2}{\phi(|\psi(\omega)|) (1 - |\psi(\omega)|^2)^{\frac{1}{q}+2}} \\ &\leq 4 \sup_{\frac{1}{2} < |\psi(\omega)| < 1} \frac{\mu(\omega) |\psi(\omega)|^2 |u(\omega)| |\psi'(\omega)|^2}{\phi(|\psi(\omega)|) (1 - |\psi(\omega)|^2)^{\frac{1}{q}+2}} < \infty. \end{aligned} \quad (13)$$

Since ϕ is normal, and using (6), we have

$$\sup_{|\psi(\omega)| \leq \frac{1}{2}} \frac{\mu(\omega) |u(\omega)| |\psi'(\omega)|^2}{\phi(|\psi(\omega)|) (1 - |\psi(\omega)|^2)^{\frac{1}{q}+2}} \leq C \sup_{|\psi(\omega)| \leq \frac{1}{2}} \mu(\omega) |u(\omega)| |\psi'(\omega)|^2 < \infty. \quad (14)$$

From (13) and (14), we can get (3), finishing the proof of the theorem. \square

Theorem 4. Let $u \in \mathcal{H}(\mathbb{D})$, ψ be an analytic self-map of \mathbb{D} , $0 < p, q < \infty$ and ϕ be normal. Then $DM_u C_\psi : H(p, q, \phi) \rightarrow \mathcal{B}^\varphi$ is compact if and only if $DM_u C_\psi : H(p, q, \phi) \rightarrow \mathcal{B}^\varphi$ is bounded and

$$\lim_{|\psi(z)| \rightarrow 1} \frac{\mu(z) |u''(z)|}{\phi(|\psi(z)|) (1 - |\psi(z)|^2)^{\frac{1}{q}}} = 0, \quad (15)$$

$$\lim_{|\psi(z)| \rightarrow 1} \frac{\mu(z) |2u'(z)\psi'(z) + u(z)\psi''(z)|}{\phi(|\psi(z)|) (1 - |\psi(z)|^2)^{\frac{1}{q}+1}} = 0, \quad (16)$$

$$\lim_{|\psi(z)| \rightarrow 1} \frac{\mu(z) |u(z)| |\psi'(z)|^2}{\phi(|\psi(z)|) (1 - |\psi(z)|^2)^{\frac{1}{q}+2}} = 0. \quad (17)$$

Proof. Suppose that $DM_u C_\psi : H(p, q, \phi) \rightarrow \mathcal{B}^\varphi$ is compact. It is clear that $DM_u C_\psi : H(p, q, \phi) \rightarrow \mathcal{B}^\varphi$ is bounded. Let $\{z_n\}_{n \in \mathbb{N}}$ be a sequence in \mathbb{D} such that $|\psi(z_n)| \rightarrow 1$ as $n \rightarrow \infty$. Set

$$f_n(z) = f_{\psi(z_n)}(z), \quad g_n(z) = g_{\psi(z_n)}(z), \quad h_n(z) = h_{\psi(z_n)}(z).$$

Then by the proof of Theorem 3,

$$\sup_{n \in \mathbb{N}} \|f_n\|_{H(p, q, \phi)} < \infty, \quad \sup_{n \in \mathbb{N}} \|g_n\|_{H(p, q, \phi)} < \infty, \quad \sup_{n \in \mathbb{N}} \|h_n\|_{H(p, q, \phi)} < \infty.$$

Moreover, we can see that f_n, g_n, h_n converges to 0 uniformly on compact subsets of \mathbb{D} .

Since $DM_u C_\psi : H(p, q, \phi) \rightarrow \mathcal{B}^\varphi$ is compact, by Lemma 1, we get

$$\lim_{n \rightarrow \infty} \|DM_u C_\psi f_n\|_{\mathcal{B}^\varphi} = \lim_{n \rightarrow \infty} \|DM_u C_\psi g_n\|_{\mathcal{B}^\varphi} = \lim_{n \rightarrow \infty} \|DM_u C_\psi h_n\|_{\mathcal{B}^\varphi} = 0.$$

By (8) we have

$$f_n(\psi(z_n)) = \frac{P}{\phi(|\psi(z_n)|) (1 - |\psi(z_n)|^2)^{\frac{1}{q}}}, \quad f'_n(\psi(z_n)) = f''_n(\psi(z_n)) = 0,$$

Then

$$\begin{aligned} 1 &\geq S_\varphi \left(\frac{(DM_u C_\psi f_n)'(z_n)}{\|DM_u C_\psi f_n\|_{\mathcal{B}^\varphi}} \right) \\ &\geq (1 - |z_n|^2) \varphi \left(\frac{P|u''(z_n)|}{\phi(|\psi(z_n)|)(1 - |\psi(z_n)|^2)^{\frac{1}{q}} \|DM_u C_\psi f_n\|_{\mathcal{B}^\varphi}} \right). \end{aligned}$$

It follows that

$$\frac{\mu(z_n)|u''(z_n)|}{\phi(|\psi(z_n)|)(1 - |\psi(z_n)|^2)^{\frac{1}{q}}} \leq C \|DM_u C_\psi f_n\|_{\mathcal{B}^\varphi}.$$

Therefore

$$\begin{aligned} &\lim_{|\psi(z_n)| \rightarrow 1} \frac{\mu(z_n)|u''(z_n)|}{\phi(|\psi(z_n)|)(1 - |\psi(z_n)|^2)^{\frac{1}{q}}} \\ &= \lim_{n \rightarrow \infty} \frac{\mu(z_n)|u''(z_n)|}{\phi(|\psi(z_n)|)(1 - |\psi(z_n)|^2)^{\frac{1}{q}}} = 0. \end{aligned} \quad (18)$$

So (15) follows. By (9) we have

$$g'_n(\psi(z_n)) = \frac{E \cdot \overline{\psi(z_n)}}{\phi(|\psi(z_n)|)(1 - |\psi(z_n)|^2)^{\frac{1}{q}+1}}, \quad g_n(\psi(z_n)) = g''_n(\psi(z_n)) = 0,$$

Then

$$\begin{aligned} 1 &\geq S_\varphi \left(\frac{(DM_u C_\psi g_n)'(z_n)}{\|DM_u C_\psi g_n\|_{\mathcal{B}^\varphi}} \right) \\ &\geq (1 - |z_n|^2) \varphi \left(\frac{E|\psi(z_n)||2u'(z_n)\psi'(z_n) + u(z_n)\psi''(z_n)|}{\phi(|\psi(z_n)|)(1 - |\psi(z_n)|^2)^{\frac{1}{q}+1} \|DM_u C_\psi g_n\|_{\mathcal{B}^\varphi}} \right). \end{aligned}$$

It follows that

$$\frac{\mu(z_n)|\psi(z_n)||2u'(z_n)\psi'(z_n) + u(z_n)\psi''(z_n)|}{\phi(|\psi(z_n)|)(1 - |\psi(z_n)|^2)^{\frac{1}{q}+1}} \leq C \|DM_u C_\psi g_n\|_{\mathcal{B}^\varphi}.$$

Therefore

$$\begin{aligned} &\lim_{|\psi(z_n)| \rightarrow 1} \frac{\mu(z_n)|\psi(z_n)||2u'(z_n)\psi'(z_n) + u(z_n)\psi''(z_n)|}{\phi(|\psi(z_n)|)(1 - |\psi(z_n)|^2)^{\frac{1}{q}+1}} \\ &= \lim_{n \rightarrow \infty} \frac{\mu(z_n)|\psi(z_n)||2u'(z_n)\psi'(z_n) + u(z_n)\psi''(z_n)|}{\phi(|\psi(z_n)|)(1 - |\psi(z_n)|^2)^{\frac{1}{q}+1}} = 0. \end{aligned} \quad (19)$$

So (16) follows. By (12), we have

$$h_n(\psi(z_n)) = h'_n(\psi(z_n)) = 0, \quad h''_n(\psi(z_n)) = \frac{F \cdot \overline{\psi(z_n)}^2}{\phi(|\psi(z_n)|)(1 - |\psi(z_n)|^2)^{\frac{1}{q}+2}}.$$

Then

$$\begin{aligned} 1 &\geq S_\varphi \left(\frac{(DM_u C_\psi h_n)'(z_n)}{\|DM_u C_\psi h_n\|_{\mathcal{B}^\varphi}} \right) \\ &\geq (1 - |z_n|^2) \varphi \left(\frac{F|\psi(z_n)|^2|u(z_n)||\psi'(z_n)|^2}{\phi(|\psi(z_n)|)(1 - |\psi(z_n)|^2)^{\frac{1}{q}+2} \|DM_u C_\psi h_n\|_{\mathcal{B}^\varphi}} \right). \end{aligned}$$

It follows that

$$\frac{\mu(z_n)|\psi(z_n)|^2|u(z_n)||\psi'(z_n)|^2}{\phi(|\psi(z_n)|)(1-|\psi(z_n)|^2)^{\frac{1}{q}+2}} \leq C\|DM_uC_\psi h_n\|_{\mathcal{B}^\varphi}.$$

Therefore

$$\lim_{|\psi(z_n)| \rightarrow 1} \frac{\mu(z_n)|u(z_n)||\psi'(z_n)|^2}{\phi(|\psi(z_n)|)(1-|\psi(z_n)|^2)^{\frac{1}{q}+2}} = \lim_{n \rightarrow \infty} \frac{\mu(z_n)|\psi(z_n)|^2|u(z_n)||\psi'(z_n)|^2}{\phi(|\psi(z_n)|)(1-|\psi(z_n)|^2)^{\frac{1}{q}+2}} = 0.$$

So (17) follows.

Conversely, suppose $DM_uC_\psi : H(p, q, \phi) \rightarrow \mathcal{B}^\varphi$ is bounded and (15), (16), (17) hold. Then (4), (5), (6) hold by Theorem 3 and for every $\epsilon > 0$, there is a $\delta \in (0, 1)$ such that

$$\frac{\mu(z)|u''(z)|}{\phi(|\psi(z)|)(1-|\psi(z)|^2)^{\frac{1}{q}}} < \epsilon, \quad (20)$$

$$\frac{\mu(z)|2u'(z)\psi'(z) + u(z)\psi''(z)|}{\phi(|\psi(z)|)(1-|\psi(z)|^2)^{\frac{1}{q}+1}} < \epsilon, \quad (21)$$

$$\frac{\mu(z)|u(z)||\psi'(z)|^2}{\phi(|\psi(z)|)(1-|\psi(z)|^2)^{\frac{1}{q}+2}} < \epsilon. \quad (22)$$

whenever $\delta < |\psi(z)| < 1$.

Assume that $\{t_n\}_{n \in \mathbb{N}}$ is a sequence in $H(p, q, \phi)$ such that $\sup_{n \in \mathbb{N}} \|t_n\|_{H(p, q, \phi)} \leq L$, and $\{t_n\}$ converges to 0 uniformly on compact subsets of \mathbb{D} as $n \rightarrow \infty$. Let $K = \{z \in \mathbb{D} : |\psi(z)| \leq \delta\}$. Then by Lemma 2, (4), (5), (6), (21), (22) and (23), we have

$$\begin{aligned} & \sup_{z \in \mathbb{D}} \mu(z)|(DM_uC_\psi t_n)'(z)| \\ & \leq \sup_{z \in \mathbb{D}} \mu(z)|u''(z)||t_n(\psi(z))| + \sup_{z \in \mathbb{D}} \mu(z)|2u'(z)\psi'(z) + u(z)\psi''(z)||t_n'(\psi(z))| \\ & \quad + \sup_{z \in \mathbb{D}} \mu(z)|u(z)||\psi'(z)|^2|t_n''(\psi(z))| \\ & \leq \sup_{z \in K} \mu(z)|u''(z)||t_n(\psi(z))| + C_1 \sup_{z \in \mathbb{D} \setminus K} \frac{\mu(z)|u''(z)||t_n\|_{H(p, q, \phi)}}{\phi(|\psi(z)|)(1-|\psi(z)|^2)^{\frac{1}{q}}} \\ & \quad + \sup_{z \in K} \mu(z)|2u'(z)\psi'(z) + u(z)\psi''(z)||t_n'(\psi(z))| \\ & \quad + C_2 \sup_{z \in \mathbb{D} \setminus K} \frac{\mu(z)|2u'(z)\psi'(z) + u(z)\psi''(z)||t_n\|_{H(p, q, \phi)}}{\phi(|\psi(z)|)(1-|\psi(z)|^2)^{\frac{1}{q}+1}} \\ & \quad + \sup_{z \in K} \mu(z)|u(z)||\psi'(z)|^2|t_n''(\psi(z))| + C_3 \sup_{z \in \mathbb{D} \setminus K} \frac{\mu(z)|u(z)||\psi'(z)|^2|t_n\|_{H(p, q, \phi)}}{\phi(|\psi(z)|)(1-|\psi(z)|^2)^{\frac{1}{q}+2}} \\ & \leq C \left(\sup_{|\omega| \leq \delta} |t_n(\omega)| + \sup_{|\omega| \leq \delta} |t_n'(\omega)| + \sup_{|\omega| \leq \delta} |t_n''(\omega)| \right) + 3L\epsilon. \end{aligned}$$

So we obtain

$$\begin{aligned} \|DM_uC_\psi t_n\|_{\mathcal{B}^\varphi} &= |u'(0)t_n(\psi(0)) + u(0)\psi'(0)t_n'(\psi(0))| + \sup_{z \in \mathbb{D}} \mu(z)|(DM_uC_\psi t_n)'(z)| \\ &\leq |u'(0)||t_n(\psi(0))| + |u(0)||\psi'(0)||t_n'(\psi(0))| \\ &\quad + C \left(\sup_{|\omega| \leq \delta} |t_n(\omega)| + \sup_{|\omega| \leq \delta} |t_n'(\omega)| + \sup_{|\omega| \leq \delta} |t_n''(\omega)| \right) + 3L\epsilon. \quad (23) \end{aligned}$$

Since t_n converges to 0 uniformly on compact subsets of \mathbb{D} as $n \rightarrow \infty$, Cauchy's estimation gives that t'_n, t''_n also do as $n \rightarrow \infty$. In particular, since $\{\omega : |\omega| \leq \delta\}$ and $\{\psi(0)\}$ are compact it follows that

$$\begin{aligned} \lim_{n \rightarrow \infty} |u'(0)||t_n(\psi(0))| + |u(0)||\psi'(0)||t'_n(\psi(0))| &= 0, \\ \lim_{n \rightarrow \infty} \sup_{|\omega| \leq \delta} |t_n(\omega)| &= \lim_{n \rightarrow \infty} \sup_{|\omega| \leq \delta} |t'_n(\omega)| = \lim_{n \rightarrow \infty} \sup_{|\omega| \leq \delta} |t''_n(\omega)| = 0. \end{aligned}$$

Hence, letting $n \rightarrow \infty$ in (24), we get

$$\lim_{n \rightarrow \infty} \|DM_u C_\psi t_n\|_{B^p} = 0.$$

Employing Lemma 1 the implication follows. \square

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A SHORT NOTE ON INTEGRAL INEQUALITY OF TYPE HERMITE-HADAMARD THROUGH CONVEXITY

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ABSTRACT. In this short note, a Riemann-Liouville fractional integral identity including first order derivative of a given function is established. With the help of this fractional-type integral identity, some new Hermite-Hadamard-type inequality involving Riemann-Liouville fractional integrals for (m, h_1, h_2) -convex function are considered. Our method considered here may be a stimulant for further investigations concerning Hermite-Hadamard-type inequalities involving fractional integrals.

1. Introduction and Defintions

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard inequality, due to its rich geometrical significance and applications, which is stated as in [12]

Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on the closed interval I of real numbers and $a, b \in I$ with $a < b$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}$$

Both the inequalities hold in reversed direction if f is concave. We recall some preliminary concepts about convex functions:

Definition 1. [7]. A function $f : [0, \infty) \rightarrow \mathbb{R}$ is said to be s -convex function or f belongs to the class K_s^i if for all $x, y \in [0, \infty)$ and $\mu, \nu \in [0, 1]$, the following inequality holds

$$f(\mu x + \nu y) \leq \mu^s f(x) + \nu^s f(y)$$

for some fixed $s \in (0, 1]$.

Note that, if $\mu^s + \nu^s = 1$, the above class of convex functions is called s -convex functions in first sense and represented by K_s^1 and if $\mu + \nu = 1$ the above class is called s -convex in second sense and represented by K_s^2 .

Definition 2. [11]. A function $f : [0, b] \rightarrow \mathbb{R}$ is said to be (α, m) -convex, where $(\alpha, m) \in [0, 1]^2$, if for every $x, y \in [0, b]$ and for $\lambda \in [0, 1]$, the following inequality holds

$$f(\lambda y + m(1-\lambda)x) \leq \lambda^\alpha f(y) + m(1-\lambda^\alpha)f(x)$$

where $(\alpha, m) \in [0, 1]^2$ and for some fixed $m \in (0, 1]$.

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Theorem 1. [3]. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I° (interior of I) where $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then we have

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{8} [|f'(a)| + |f'(b)|].$$

Theorem 2. [8]. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I° (interior of I) where $a, b \in I$ with $a < b$. If the mapping $|f'|^q$ is convex on $[a, b]$, for some $q \geq 1$, then the following inequality holds:

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left[\frac{|f'(a)|^q + |f'(b)|^q}{2} \right]^{\frac{1}{q}}$$

Theorem 3. [13]. Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be differentiable mapping on I° $a, b \in I^\circ$ with $a < b$, and If $|f'|^q$ is quasi-convex on $[a, b]$, $p > 1$. Then the following inequality holds:

$$\begin{aligned} \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| &\leq \frac{(b-a)}{16} \left(\frac{4}{1+p} \right)^{\frac{1}{p}} \\ &\left\{ \left[|f'(a)|^{\frac{p}{p-1}} + 3 |f'(b)|^{\frac{p}{p-1}} \right]^{1-\frac{1}{p}} \right. \\ &\left. + \left[3 |f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}} \right]^{1-\frac{p}{p-1}} \right\} \end{aligned}$$

Theorem 4. [9]. Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on I° where $a, b \in I^\circ$ with $a < b$ such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If the mapping $|f''|^q$ is s -convex on $[a, b]$ for some fixed $s \in (0, 1]$ and $q \geq 1$, then the following inequality holds:

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{2^{\frac{1}{q}}} \left[\frac{s + (\frac{1}{2})^s}{(s+1)(s+2)} \right]^{\frac{1}{q}} [|f'(a)|^q + |f'(b)|^q]^{\frac{1}{q}}$$

Theorem 5. [4]. Suppose that $f : [0, \infty) \rightarrow [0, \infty)$ is a convex function in the second sense where $\alpha \in (0, 1)$ and let $a, b \in [0, \infty)$, $a < b$. if $f \in L_1([a, b])$, then the following inequality holds

$$2^{\alpha-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{\alpha+1}.$$

Fraction calculus [2, 6, 1, 5] was introduced at the end of the nineteenth century by Riemann and Liouville the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics. We recall some definitions and preliminary facts of fractional calculus theory which will be used in this paper.

Definition 3. [6]. Let $f \in L_1[a, b]$. The Riemann-Liouville fractional integrals $J_{\alpha+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $\alpha \geq 0$ are defined by

$$J_{\alpha+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad (a < x),$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad (b > x),$$

respectively.

Here $\Gamma(\alpha) = \int_0^{\infty} e^{-u} u^{\alpha-1} du$. Here is $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

In case of $\alpha = 1$, the fractional integral reduces to the classical integral. The aim of this paper is to establish Hermite-Hadamard type inequalities based on (m, h_1, h_2) -convexity. Using these results we obtained new inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

2. Main Results

Before proceeding to our main results, we present some necessary definition and lemma which are used further in this paper.

Definition 4. Let $f : I \subseteq R_0 \rightarrow R$, $h_1, h_2 : [0, 1] \rightarrow R_0$, and $m \in (0, 1]$, then f is said to be (m, h_1, h_2) -convex. if f is non-negative and the following inequality

$$f(\lambda x + m(1-\lambda)y) \leq h_1(\lambda)f(x) + mh_2(1-\lambda)f(y),$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

If the above inequality is reversed then f is said to be (m, h_1, h_2) -concave.

$$M_{\alpha}(a, b) = \frac{1}{4} \left[f(a) + \frac{1}{2} \left(f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right) + f(b) \right] - \frac{\Gamma(\alpha+1)4^{\alpha-1}}{(b-a)^{\alpha}} \left[J_{\alpha+}^{\alpha} f\left(\frac{3a+b}{4}\right) + J_{\frac{3a+b}{4}}^{\alpha} f\left(\frac{a+3b}{4}\right) + J_{\frac{a+3b}{4}}^{\alpha} f(b) \right].$$

Specially, when $\alpha = 1$, we have

$$M_1(a, b) = \frac{1}{4} \left[f(a) + \frac{1}{2} \left(f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(t) dt.$$

Lemma 1. Suppose $f : [a, b] \rightarrow R$ is a differentiable mapping on (a, b) . If $f' \in L_1([a, b])$, then we have the following identity

$$M_{\alpha}(a, b) = \frac{b-a}{16} \times \left\{ \begin{aligned} & \int_0^1 (0-\lambda^{\alpha}) f'(\lambda a + (1-\lambda)\frac{3a+b}{4}) d\lambda \\ & + \int_0^1 (\frac{1}{2}-\lambda^{\alpha}) f'(\lambda\frac{3a+b}{4} + (1-\lambda)\frac{a+3b}{4}) d\lambda \\ & + \int_0^1 (1-\lambda^{\alpha}) f'(\lambda\frac{a+3b}{4} + (1-\lambda)b) d\lambda \end{aligned} \right\}$$

Proof. By integrating, and by making use of the substitution $u = \lambda a + (1-\lambda)\frac{3a+b}{4}$ we have

$$\begin{aligned} & \frac{b-a}{16} \left\{ \int_0^1 (-\lambda^{\alpha}) f'(\lambda a + (1-\lambda)\frac{3a+b}{4}) d\lambda \right\} \\ &= \frac{1}{4} \left[f(a) - \alpha \int_0^1 (-\lambda^{\alpha}) f'(\lambda a + (1-\lambda)\frac{3a+b}{4}) \lambda^{\alpha-1} d\lambda \right] \\ &= \frac{1}{4} f(a) - \frac{\alpha 4^{\alpha-1}}{(b-a)^{\alpha}} \int_a^{\frac{3a+b}{4}} \left(\frac{3a+b}{4} - u \right)^{\alpha-1} f(u) du \\ &= \frac{1}{4} f(a) - \frac{\Gamma(\alpha+1)4^{\alpha-1}}{(b-a)^{\alpha}} J_{\alpha+}^{\alpha} f\left(\frac{3a+b}{4}\right) \\ & \quad \frac{b-a}{16} \left\{ \int_0^1 (\frac{1}{2}-\lambda^{\alpha}) f'(\lambda\frac{3a+b}{4} + (1-\lambda)\frac{a+3b}{4}) d\lambda \right\} \\ &= \frac{1}{8} \left[\left(f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right) \right] \\ &= \frac{b-a}{16} \left\{ \int_0^1 (\frac{1}{2}-\lambda^{\alpha}) f'(\lambda\frac{3a+b}{4} + (1-\lambda)\frac{a+3b}{4}) d\lambda \right\} \\ &= \frac{1}{8} \left[\left(f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right) \right] - \frac{\Gamma(\alpha+1)4^{\alpha-1}}{(b-a)^{\alpha}} J_{\alpha+}^{\alpha} f\left(\frac{a+3b}{4}\right) \end{aligned}$$

$$\begin{aligned} & \int_0^1 (1-\lambda^\alpha) f' \left(\lambda \frac{a+3b}{4} + (1-\lambda)b \right) d\lambda \\ &= \frac{1}{4} f(b) - \frac{\Gamma(\alpha+1)4^{\alpha-1}}{(b-a)^\alpha} J_{\frac{a+3b}{4}}^\alpha f(b) \end{aligned}$$

This proves as required. \square

Theorem 6. Suppose $f : [a, b] \rightarrow R$ is a differentiable mapping on (a, b) with $a < b$. such that $f' \in L_1([a, b])$ for $0 < a < b$. If $|f'|^q$ is (m, h_1, h_2) -convex on $[a, b]$ for some fixed $q > 1$ and $h_1, h_2 \in L_1([a, b])$, then we have the following inequality

$$M_\alpha(a, b) \leq \frac{b-a}{16} \times \left[\left(\frac{q-1}{\alpha q + q-1} \right)^{1-1/q} \times \left(|f'(a)|^q \|h_1\|_1 + m \left| f' \left(\frac{3a+b}{4m} \right) \right|^q \|h_2\|_1 \right)^{1/q} + \frac{1}{2} \left(\frac{\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1}{\alpha + 1} \right)^{1-1/q} \times \left(\left| f' \left(\frac{3a+b}{4} \right) \right|^q \|h_1\|_1 + m \left| f' \left(\frac{a+3b}{4m} \right) \right|^q \|h_2\|_1 \right)^{1/q} + \left(\frac{1}{\alpha} B \left(\frac{2q-1}{q-1}, \frac{1}{\alpha} \right) \right)^{1-1/q} \times \left(\left| f' \left(\frac{a+3b}{4} \right) \right|^q \|h_1\|_1 + m \left| f' \left(\frac{b}{m} \right) \right|^q \|h_2\|_1 \right)^{1/q} \right]$$

Suppose $\|h_1\|_p = \left(\int_0^1 h_1^p(\lambda) d\lambda \right)^{1/p}$ for $p \geq 1$ with $B(x, y)$ is the classical Beta function which may be defined $B(x, y) = \int_0^1 \lambda^{x-1} (1-\lambda)^{y-1} d\lambda$, $x > 0, y > 0$. for $0 < a < b$.

Proof. Hölder integral inequality and Lemma 1 together implies with (m, h_1, h_2) -convexity of $|f'|^q$

$$M_\alpha(a, b) \leq \frac{b-a}{16} \times \left[\left(\int_0^1 \lambda^{\alpha q/q-1} d\lambda \right)^{1-1/q} \times \left(|f'(a)|^q \|h_1\|_1 + m \left| f' \left(\frac{3a+b}{4m} \right) \right|^q \|h_2\|_1 \right)^{1/q} + \left(\frac{1}{2} \right)^{1/q} \left(\int_0^1 \left| \frac{1}{2} - \lambda \right|^{q-1} d\lambda \right)^{1-1/q} \times \left(\left| f' \left(\frac{3a+b}{4} \right) \right|^q \|h_1\|_1 + m \left| f' \left(\frac{a+3b}{4m} \right) \right|^q \|h_2\|_1 \right)^{1/q} + \left(\int_0^1 (1-\lambda)^{q/q-1} d\lambda \right)^{1-1/q} \times \left(\left| f' \left(\frac{a+3b}{4} \right) \right|^q \|h_1\|_1 + m \left| f' \left(\frac{b}{m} \right) \right|^q \|h_2\|_1 \right)^{1/q} \right]$$

Therefore,

$$\begin{aligned} \int_0^1 \lambda^{\alpha q/q-1} d\lambda &= \frac{q-1}{\alpha q + q-1}, \quad \int_0^1 (1-\lambda)^{q/q-1} d\lambda = \frac{1}{\alpha} B \left(\frac{2q-1}{q-1}, \frac{1}{\alpha} \right), \\ \int_0^1 \left| \frac{1}{2} - \lambda \right|^{q-1} d\lambda &= \frac{\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1}{\alpha + 1} \end{aligned}$$

This completes the proof. \square

Corollary 1. In Theorem 6, if we choose $h_1(\lambda) = h(\lambda)$, $h_2(\lambda) = h(1-\lambda)$, then we have,

$$M_\alpha(a, b) \leq \frac{(b-a)\|h_1\|_1^{1/q}}{16} \times \left[\left(\frac{q-1}{\alpha q + q-1} \right)^{1-1/q} \times \left(|f'(a)|^q + m \left| f' \left(\frac{3a+b}{4m} \right) \right|^q \right)^{1/q} + \frac{1}{2} \left(\frac{\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1}{\alpha + 1} \right)^{1-1/q} \times \left(\left| f' \left(\frac{3a+b}{4} \right) \right|^q + m \left| f' \left(\frac{a+3b}{4m} \right) \right|^q \right)^{1/q} + \left(\frac{1}{\alpha} B \left(\frac{2q-1}{q-1}, \frac{1}{\alpha} \right) \right)^{1-1/q} \times \left(\left| f' \left(\frac{a+3b}{4} \right) \right|^q + m \left| f' \left(\frac{b}{m} \right) \right|^q \right)^{1/q} \right]$$

Furthermore if we choose $m = 1$, we have

$$M_\alpha(a, b) \leq \frac{(b-a)\|h_1\|_1^{1/q}}{16} \times \left[\left(\frac{q-1}{\alpha q + q-1} \right)^{1-1/q} \times \left(|f'(a)|^q + \left| f' \left(\frac{3a+b}{4} \right) \right|^q \right)^{1/q} + \frac{1}{2} \left(\frac{\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1}{\alpha + 1} \right)^{1-1/q} \times \left(\left| f' \left(\frac{3a+b}{4} \right) \right|^q + \left| f' \left(\frac{a+3b}{4} \right) \right|^q \right)^{1/q} + \left(\frac{1}{\alpha} B \left(\frac{2q-1}{q-1}, \frac{1}{\alpha} \right) \right)^{1-1/q} \times \left(\left| f' \left(\frac{a+3b}{4} \right) \right|^q + |f'(b)|^q \right)^{1/q} \right]$$

Corollary 2. Under the conditions of Corollary 1, if we choose $h_1(\lambda) = h(\lambda) = \lambda^s, m = 1$, we have the

$$M_\alpha(a, b) \leq \frac{(b-a)}{16} \left(\frac{1}{s+1} \right)^{1/q} \times \left[\begin{aligned} & \left(\frac{q-1}{\alpha q + q - 1} \right)^{1-1/q} \times \left(|f'(a)|^q + |f'(\frac{3a+b}{4})|^q \right)^{1/q} \\ & + \frac{1}{2} \left(\frac{\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1}{\alpha + 1} \right)^{1-1/q} \times \left(|f'(\frac{3a+b}{4})|^q + |f'(\frac{a+3b}{4})|^q \right)^{1/q} \\ & + \left(\frac{1}{\alpha} B \left(\frac{2q-1}{q-1}, \frac{1}{\alpha} \right) \right)^{1-1/q} \times \left(|f'(\frac{a+3b}{4})|^q + |f'(b)|^q \right)^{1/q} \end{aligned} \right]$$

Specially, if we choose $\alpha = s = m = 1$, we have the

$$M_\alpha(a, b) \leq \frac{(b-a)}{16} \left(\frac{1}{2} \right)^{1/q} \times \left[\begin{aligned} & \left(\frac{q-1}{2q-1} \right)^{1-1/q} \times \left(|f'(a)|^q + |f'(\frac{3a+b}{4})|^q \right)^{1/q} \\ & + \left(\frac{1}{2} \right)^{2-1/q} \times \left(|f'(\frac{3a+b}{4})|^q + |f'(\frac{a+3b}{4})|^q \right)^{1/q} \\ & + \left(\frac{q-1}{2q-1} \right)^{1-1/q} \times \left(|f'(\frac{a+3b}{4})|^q + |f'(b)|^q \right)^{1/q} \end{aligned} \right]$$

Corollary 3. Under the conditions of Theorem 6, if we choose $h_1(\lambda) = \lambda^{\alpha_1}$, $h_2(\lambda) = 1 - \lambda^{\alpha_1}$, we have the

$$M_\alpha(a, b) \leq \frac{(b-a)}{16} \left(\frac{1}{\alpha_1 + 1} \right)^{1/q} \times \left[\begin{aligned} & \left(\frac{q-1}{\alpha q + q - 1} \right)^{1-1/q} \times \left(|f'(a)|^q + m \alpha_1 |f'(\frac{3a+b}{4m})|^q \right)^{1/q} \\ & + \frac{1}{2} \left(\frac{\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1}{\alpha + 1} \right)^{1-1/q} \times \left(|f'(\frac{3a+b}{4})|^q + m |f'(\frac{a+3b}{4m})|^q \right)^{1/q} \\ & + \left(\frac{1}{\alpha} B \left(\frac{2q-1}{q-1}, \frac{1}{\alpha} \right) \right)^{1-1/q} \times \left(|f'(\frac{a+3b}{4})|^q + m \alpha_1 |f'(\frac{b}{m})|^q \right)^{1/q} \end{aligned} \right]$$

Specially, if we choose $m = 1$, we have the,

$$M_\alpha(a, b) \leq \frac{(b-a)}{16} \left(\frac{1}{\alpha_1 + 1} \right)^{1/q} \times \left[\begin{aligned} & \left(\frac{q-1}{\alpha q + q - 1} \right)^{1-1/q} \times \left(|f'(a)|^q + \alpha_1 |f'(\frac{3a+b}{4})|^q \right)^{1/q} \\ & + \frac{1}{2} \left(\frac{\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1}{\alpha + 1} \right)^{1-1/q} \times \left(|f'(\frac{3a+b}{4})|^q + |f'(\frac{a+3b}{4})|^q \right)^{1/q} \\ & + \left(\frac{1}{\alpha} B \left(\frac{2q-1}{q-1}, \frac{1}{\alpha} \right) \right)^{1-1/q} \times \left(|f'(\frac{a+3b}{4})|^q + \alpha_1 |f'(b)|^q \right)^{1/q} \end{aligned} \right]$$

Theorem 7. Suppose $f : [a, b] \rightarrow R$ is a differentiable mapping on (a, b) with $a < b$. such that $f' \in L_1([a, b])$ for $0 < a < b$. If $|f'|^q$ is and (m, h_1, h_2) -convex on $[a, b]$ for $q \geq 1$, and $h_1, h_2 \in L_1([a, b])$, then we have the following inequality

$$M_\alpha(a, b) \leq \frac{(b-a)}{16} \left(\frac{1}{\alpha + 1} \right)^{1-1/q} \times \left\{ \begin{aligned} & \left[\frac{1}{2} |f'(a)|^q \left(\frac{1}{2\alpha + 1} + \|h_1\|_2^2 \right) + \frac{m}{2} |f'(\frac{3a+b}{4m})|^q \left(\frac{1}{2\alpha + 1} + \|h_2\|_2^2 \right) \right]^{1/q} \\ & + \frac{1}{2} \left(\frac{\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1}{\alpha + 1} \right)^{1-1/q} \times \left(|f'(\frac{3a+b}{4})|^q \|h_1\|_1 + m |f'(\frac{a+3b}{4m})|^q \|h_2\|_1 \right)^{1/q} \\ & + \alpha^{1-1/q} \left[\frac{1}{2} |f'(\frac{a+3b}{4})|^q \left(\frac{2\alpha^2}{(2\alpha+1)(\alpha+1)} + \|h_1\|_2^2 \right) \right. \\ & \quad \left. + \frac{m}{2} |f'(\frac{b}{m})|^q \left(\frac{2\alpha^2}{(2\alpha+1)(\alpha+1)} + \|h_1\|_2^2 \right) \right]^{1/q} \end{aligned} \right\}$$

Where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using Holder's inequality and by Lemma 1, and (m, h_1, h_2) - convexity of $|f'|^q$, we get

$$M_{\alpha}(a, b) \leq \frac{(b-a)}{16} \times \left\{ \left(\int_0^1 \lambda^{\alpha} d\lambda \right)^{1-1/q} \left[|f'(a)|^q \int_0^1 \lambda^{\alpha} h_1(\lambda) d\lambda + \frac{m}{2} |f'(\frac{3a+b}{4m})|^q \int_0^1 \lambda^{\alpha} h_2(\lambda) d\lambda \right]^{1/q} \right. \\ \left. + \left(\frac{1}{2} \right)^{1/q} \left(\int_0^1 \left| \frac{1}{2} - \lambda \right|^{\alpha} d\lambda \right)^{1-1/q} \times \left(|f'(\frac{3a+b}{4})|^q \int_0^1 h_1(\lambda) d\lambda + m |f'(\frac{a+3b}{4m})|^q \int_0^1 h_2(\lambda) d\lambda \right)^{1/q} \right. \\ \left. + \left(\int_0^1 (1-\lambda)^{\alpha} d\lambda \right)^{1-1/q} \left[|f'(\frac{a+3b}{4})|^q \int_0^1 (1-\lambda)^{\alpha} h_1(\lambda) d\lambda \right. \right. \\ \left. \left. + m |f'(\frac{b}{m})|^q \int_0^1 (1-\lambda)^{\alpha} h_2(\lambda) d\lambda \right]^{1/q} \right\}$$

$$M_{\alpha}(a, b) \leq \frac{(b-a)}{16} \times \left\{ \left(\int_0^1 \lambda^{\alpha} d\lambda \right)^{1-1/q} \left[|f'(a)|^q \int_0^1 \frac{\lambda^{2\alpha+h_1^2(\lambda)}}{2} d\lambda + m |f'(\frac{3a+b}{4m})|^q \int_0^1 \frac{\lambda^{2\alpha+h_2^2(\lambda)}}{2} d\lambda \right]^{1/q} \right. \\ \left. + \left(\frac{1}{2} \right)^{1/q} \left(\int_0^1 \left| \frac{1}{2} - \lambda \right|^{\alpha} d\lambda \right)^{1-1/q} \times \left(|f'(\frac{3a+b}{4})|^q \int_0^1 h_1(\lambda) d\lambda + m |f'(\frac{a+3b}{4m})|^q \int_0^1 h_2(\lambda) d\lambda \right)^{1/q} \right. \\ \left. + \left(\int_0^1 (1-\lambda)^{\alpha} d\lambda \right)^{1-1/q} \left[|f'(\frac{a+3b}{4})|^q \int_0^1 \frac{(1-\lambda)^{2\alpha+h_1^2(\lambda)}}{2} d\lambda \right. \right. \\ \left. \left. + m |f'(\frac{b}{m})|^q \int_0^1 \frac{(1-\lambda)^{2\alpha+h_2^2(\lambda)}}{2} d\lambda \right]^{1/q} \right\} \\ = \frac{(b-a)}{16} \left(\frac{1}{\alpha+1} \right)^{1-1/q} \times \left\{ \left[\frac{1}{2} |f'(a)|^q \left(\frac{1}{2\alpha+1} + \|h_1\|_2^2 \right) + \frac{m}{2} |f'(\frac{3a+b}{4m})|^q \left(\frac{1}{2\alpha+1} + \|h_2\|_2^2 \right) \right]^{1/q} \right. \\ \left. + \frac{1}{2} \left(\frac{\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1}{\alpha+1} \right)^{1-1/q} \times \left(|f'(\frac{3a+b}{4})|^q \|h_1\|_1 + m |f'(\frac{a+3b}{4m})|^q \|h_2\|_1 \right)^{1/q} \right. \\ \left. + \alpha^{1-1/q} \left[\frac{1}{2} |f'(\frac{a+3b}{4})|^q \left(\frac{2\alpha^2}{(2\alpha+1)(\alpha+1)} + \|h_1\|_2^2 \right) \right. \right. \\ \left. \left. + \frac{m}{2} |f'(\frac{b}{m})|^q \left(\frac{2\alpha^2}{(2\alpha+1)(\alpha+1)} + \|h_1\|_2^2 \right) \right]^{1/q} \right\}$$

This completes the proof. \square

Corollary 4. In Theorem 7, if we choose $h_1(\lambda) = \lambda^{\alpha_1}$, $h_2(\lambda) = 1 - \lambda^{\alpha_1}$, we have

$$M_{\alpha}(a, b) \leq \frac{(b-a)}{16} \left(\frac{1}{\alpha+1} \right)^{1-1/q} \times \left\{ \left[\frac{1}{2} |f'(a)|^q \left(\frac{1}{2\alpha+1} + \frac{1}{2\alpha_1+1} \right) + \frac{m}{2} |f'(\frac{3a+b}{4m})|^q \left(\frac{1}{2\alpha+1} + \frac{2\alpha_1^2}{(2\alpha_1+1)(\alpha_1+1)} \right) \right]^{1/q} \right. \\ \left. + \frac{1}{2} \left(\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1 \right)^{1-1/q} \times \left(|f'(\frac{3a+b}{4})|^q \frac{1}{\alpha_1+1} + m |f'(\frac{a+3b}{4m})|^q \frac{\alpha_1}{\alpha_1+1} \right)^{1/q} \right. \\ \left. + \alpha^{1-1/q} \left[\frac{1}{2} |f'(\frac{a+3b}{4})|^q \left(\frac{2\alpha^2}{(2\alpha+1)(\alpha+1)} + \frac{1}{2\alpha_1+1} \right) \right. \right. \\ \left. \left. + \frac{m}{2} |f'(\frac{b}{m})|^q \left(\frac{2\alpha^2}{(2\alpha+1)(\alpha+1)} + \frac{2\alpha_1^2}{(2\alpha_1+1)(\alpha_1+1)} \right) \right]^{1/q} \right\}$$

If we choose $h_1(\lambda) = h(\lambda)$, $h_2(\lambda) = h(1-\lambda)$, $m = 1$ we have

$$M_{\alpha}(a, b) \leq \frac{(b-a)}{16} \times \left\{ \left(\frac{1}{\alpha+1} \right)^{1-1/q} \left[\frac{1}{2} \left(\frac{1}{2\alpha+1} + \|h_1\|_2^2 \right) + \left(|f'(a)|^q + |f'(\frac{3a+b}{4})|^q \right) \right]^{1/q} \right. \\ \left. + \frac{1}{2} \left(\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1 \right)^{1-1/q} \times \left(\|h_1\|_1 \left(|f'(\frac{3a+b}{4})|^q + |f'(\frac{a+3b}{4})|^q \right) \right)^{1/q} \right. \\ \left. + \alpha^{1-1/q} \left[\frac{1}{2} \left(\frac{2\alpha^2}{(2\alpha+1)(\alpha+1)} + \|h_1\|_2^2 \right) \right. \right. \\ \left. \left. \times \left(|f'(\frac{a+3b}{4})|^q + |f'(b)|^q \right) \right]^{1/q} \right\}$$

If we choose $h_1(\lambda) = h(\lambda) = \lambda^s$, $h_2(\lambda) = h(1 - \lambda)$, $m = 1$ we have the,

$$M_\alpha(a, b) \leq \frac{(b-a)}{16} \times \left\{ \begin{aligned} &\left(\frac{1}{\alpha+1} \right)^{1-1/q} \left[\frac{1}{2} \left(\frac{1}{2\alpha+1} + \frac{1}{2s+1} \right) + \left(|f'(a)|^q + |f'(\frac{3a+b}{4})|^q \right) \right]^{1/q} \\ &+ \frac{1}{2} (\alpha 2^{(\alpha-1)/\alpha} - \alpha + 1)^{1-1/q} \times \left(\frac{1}{s+1} \left(|f'(\frac{3a+b}{4})|^q + |f'(\frac{a+3b}{4})|^q \right) \right)^{1/q} \\ &+ \alpha^{1-1/q} \left[\frac{1}{2} \left(\frac{2\alpha^2}{(2\alpha+1)(\alpha+1)} + \frac{1}{2s+1} \right) \right. \\ &\quad \left. \times \left(|f'(\frac{a+3b}{4})|^q + |f'(b)|^q \right) \right]^{1/q} \end{aligned} \right\}$$

In Theorem 7, if we choose $h_1(\lambda) = h(\lambda) = \lambda^s$, $h_2(\lambda) = h(1 - \lambda)$, $\alpha = s = m = 1$, we have

$$M_\alpha(a, b) \leq \frac{(b-a)}{16} \times \left\{ \begin{aligned} &\left(\frac{1}{2} \right)^{1-1/q} \left[\frac{1}{3} \left(|f'(a)|^q + |f'(\frac{3a+b}{4})|^q \right) \right]^{1/q} \\ &+ \frac{1}{2} \left(\frac{1}{2} \left(|f'(\frac{3a+b}{4})|^q + |f'(\frac{a+3b}{4})|^q \right) \right)^{1/q} \\ &+ \left[\frac{1}{3} \left(|f'(\frac{a+3b}{4})|^q + |f'(b)|^q \right) \right]^{1/q} \end{aligned} \right\}.$$

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DEGENERATE POLY-BERNOULLI POLYNOMIALS OF THE SECOND KIND

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ABSTRACT. In this paper, we introduce the degenerate poly-Bernoulli polynomials of the second kind, which reduce in the limit to the poly-Bernoulli polynomials of the second kind. We present several explicit formulas and recurrence relations for these polynomials. Also, we establish a connection between our polynomials and several known families of polynomials.

1. INTRODUCTION

The *Korobov polynomials of the first kind* $K_n(\lambda, x)$ with $\lambda \neq 0$ introduced by Korobov (actually he defined the polynomials $\frac{1}{n!}K_n(\lambda, x)$) (see [13, 14, 18]) are given by

$$(1.1) \quad \frac{\lambda t}{(1 + \lambda t)^\lambda - 1} (1 + t)^x = \sum_{n \geq 0} K_n(\lambda, x) \frac{t^n}{n!}.$$

When $x = 0$, we define $K_n(\lambda) = K_n(\lambda, 0)$. These are what would have been called the degenerate Bernoulli polynomials of the second kind, since $\lim_{\lambda \rightarrow 0} K_n(\lambda, x) = b_n(x)$, where $b_n(x)$ is the n th *Bernoulli polynomial of the second kind* (see [15]) given by

$$\frac{t}{\log(1 + t)} (1 + t)^x = \sum_{n \geq 0} b_n(x) \frac{t^n}{n!}.$$

On the other hand, the *poly-Bernoulli polynomials of the second kind* $Pb_n^{(k)}(x)$ (of index k) are introduced in [12] (see also [5, 7, 10]) and given by

$$(1.2) \quad \frac{Li_k(1 - e^{-t})}{\log(1 + t)} (1 + t)^x = \sum_{n \geq 0} Pb_n^{(k)}(x) \frac{t^n}{n!},$$

where $Li_k(x)$ ($k \in \mathbb{Z}$) is the classical *polylogarithm function* given by $Li_k(x) = \sum_{n \geq 1} \frac{x^n}{n^k}$.

In this paper, we introduce the *degenerate poly-Bernoulli polynomials of the second kind* $Pb_n^{(k)}(\lambda, x)$ with $\lambda \neq 0$ (of index k) (see [3, 6, 8]) which are given by

$$(1.3) \quad \frac{\lambda Li_k(1 - e^{-t})}{(1 + t)^\lambda - 1} (1 + t)^x = \sum_{n \geq 0} Pb_n^{(k)}(\lambda, x) \frac{t^n}{n!}.$$

When $x = 0$, $Pb_n^{(k)}(\lambda, 0)$ are called the *degenerate poly-Bernoulli numbers of the second kind*. Clearly, $Pb_n^{(1)}(\lambda, x) = K_n(\lambda, x)$ and $\lim_{\lambda \rightarrow 0} Pb_n^{(k)}(\lambda, x) = Pb_n^{(k)}(x)$.

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Recall here that the λ -Daehee polynomials of the first kind $D_{n,\lambda}(x)$ (see [9]) are given by

$$(1.4) \quad \frac{\lambda \log(1+t)}{(1+t)^\lambda - 1} (1+t)^x = \sum_{n \geq 0} D_{n,\lambda}(x) \frac{t^n}{n!}.$$

When $x = 0$, $D_{n,\lambda} = D_{n,\lambda}(0)$ are called the λ -Daehee numbers of the first kind. Note that, as $\frac{\lambda \log(1+t)}{(1+t)^\lambda - 1} \frac{Li_k(1-e^{-t})}{\log(1+t)} (1+t)^x = \sum_{n \geq 0} Pb_n^{(k)}(\lambda, x) \frac{t^n}{n!}$, the degenerate poly-Bernoulli polynomials of the second kind are mixed-type of the λ -Daehee polynomials of the first kind and the poly-Bernoulli polynomials of the second kind.

The goal of this paper is to use umbral calculus to obtain several new and interesting identities of degenerate poly-Bernoulli polynomials of the second kind. To do that we refer the reader to umbral algebra and umbral calculus as given in [16, 17]. More precisely, we give some properties, explicit formulas, recurrence relations and identities about the degenerate poly-Bernoulli polynomials of the second kind. Also, we establish a connection between our polynomials and several known families of polynomials.

2. EXPLICIT FORMULAS

In this section we present several explicit formulas for the degenerate poly-Bernoulli polynomials of the second kind, namely $Pb_n^{(k)}(\lambda, x)$. It is immediate from (1.3) that the degenerate poly-Bernoulli polynomials of the second kind are given by the Sheffer sequence for the pair

$$(2.1) \quad Pb_n^{(k)}(\lambda, x) \sim (g_k(t), f(t)) \equiv \left(\frac{e^{\lambda t} - 1}{\lambda Li_k(1 - e^{1-e^t})}, e^t - 1 \right).$$

To do so, we recall that Stirling numbers $S_1(n, k)$ of the first kind can be defined by means of exponential generating functions as

$$(2.2) \quad \sum_{\ell \geq j} S_1(\ell, j) \frac{t^\ell}{\ell} = \frac{1}{j!} \log^j(1+t)$$

and can be defined by means of ordinary generating functions as

$$(2.3) \quad (x)_n = \sum_{m=0}^n S_1(n, m) x^m \sim (1, e^t - 1),$$

where $(x)_n = x(x-1)(x-2) \cdots (x-n+1)$ with $(x)_0 = 1$.

Theorem 2.1. For all $n \geq 0$,

$$\begin{aligned} Pb_n^{(k)}(\lambda, x) &= \sum_{j=0}^n \left(\sum_{\ell=j}^n \binom{n}{\ell} S_1(\ell, j) Pb_{n-\ell}^{(k)}(\lambda, 0) \right) x^j \\ &= \sum_{j=0}^n \left(\sum_{\ell=j}^n \sum_{m=0}^{n-\ell} \binom{n}{\ell} \binom{n-\ell}{m} S_1(\ell, j) Pb_m^{(k)} D_{n-\ell-m, \lambda} \right) x^j. \end{aligned}$$

Proof. By applying the fact that

$$(2.4) \quad s_n(x) = \sum_{j=0}^n \frac{1}{j!} \langle g(\bar{f}(t))^{-1} \bar{f}(t)^j \mid x^n \rangle x^j,$$

for any $s_n(x) \sim (g(t), f(t))$ (see [16, 17]) in the case of degenerate poly-Bernoulli polynomials of the second kind (see (2.1)), we have

$$(2.5) \quad \begin{aligned} & \frac{1}{j!} \langle g_k(\bar{f}(t))^{-1} \bar{f}(t)^j \mid x^n \rangle \\ &= \frac{1}{j!} \left\langle \frac{\lambda Li_k(1 - e^{-t})}{(1+t)^\lambda - 1} (\log(1+t))^j \mid x^n \right\rangle = \left\langle \frac{\lambda Li_k(1 - e^{-t})}{(1+t)^\lambda - 1} \mid \frac{\log^j(1+t)}{j!} x^n \right\rangle, \end{aligned}$$

which, by (2.3), we have

$$\begin{aligned} & \frac{1}{j!} \langle g_k(\bar{f}(t))^{-1} \bar{f}(t)^j \mid x^n \rangle \\ &= \left\langle \frac{\lambda Li_k(1 - e^{-t})}{(1+t)^\lambda - 1} \mid \sum_{\ell \geq j} S_1(\ell, j) \frac{t^\ell}{\ell!} x^n \right\rangle = \sum_{\ell=j}^n \binom{n}{\ell} S_1(\ell, j) \left\langle \frac{\lambda Li_k(1 - e^{-t})}{(1+t)^\lambda - 1} \mid x^{n-\ell} \right\rangle \\ &= \sum_{\ell=j}^n \binom{n}{\ell} S_1(\ell, j) Pb_{n-\ell}^{(k)}(\lambda, 0), \end{aligned}$$

which completes the proof of the first equality.

Now let us calculate $a_j = \frac{1}{j!} \langle g_k(\bar{f}(t))^{-1} \bar{f}(t)^j \mid x^n \rangle$ in another way. By (2.5), we have

$$a_j = \sum_{\ell=j}^n \binom{n}{\ell} S_1(\ell, j) \left\langle \frac{\lambda \log(1+t)}{(1+t)^\lambda - 1} \mid \frac{Li_k(1 - e^{-t})}{\log(1+t)} x^{n-\ell} \right\rangle,$$

which, by (1.3) and (1.4), implies

$$\begin{aligned} a_j &= \sum_{\ell=j}^n \sum_{m=0}^{n-\ell} \binom{n}{\ell} \binom{n-\ell}{m} S_1(\ell, j) Pb_m^{(k)} \left\langle \frac{\lambda \log(1+t)}{(1+t)^\lambda - 1} \mid x^{n-\ell-m} \right\rangle \\ &= \sum_{\ell=j}^n \sum_{m=0}^{n-\ell} \binom{n}{\ell} \binom{n-\ell}{m} S_1(\ell, j) Pb_m^{(k)} D_{n-\ell-m, \lambda}. \end{aligned}$$

Thus,

$$Pb_n^{(k)}(\lambda, x) = \sum_{j=0}^n \left(\sum_{\ell=j}^n \sum_{m=0}^{n-\ell} \binom{n}{\ell} \binom{n-\ell}{m} S_1(\ell, j) Pb_m^{(k)} D_{n-\ell-m, \lambda} \right) x^j,$$

as required. □

Theorem 2.2. For all $n \geq 0$,

$$Pb_n^{(k)}(\lambda, x) = \sum_{m=0}^n \binom{n}{m} D_{n-m, \lambda} Pb_m^{(k)}(x) = \sum_{m=0}^n \binom{n}{m} Pb_{n-m}^{(k)} D_{m, \lambda}(x).$$

Proof. By (1.3), we have

$$Pb_n^{(k)}(\lambda, y) = \left\langle \frac{\lambda \log(1+t)}{(1+t)^\lambda - 1} \mid \frac{Li_k(1-e^{-t})}{\log(1+t)} (1+t)^y x^n \right\rangle,$$

which, by (1.2), we obtain

$$Pb_n^{(k)}(\lambda, y) = \sum_{m=0}^n \binom{n}{m} Pb_m^{(k)}(y) \left\langle \frac{\lambda \log(1+t)}{(1+t)^\lambda - 1} \mid x^{n-m} \right\rangle.$$

Therefore, by (1.4), we obtain the first equality. To obtain the second equality, we reverse the order, namely we use at first (1.4) and then (1.2), to obtain

$$Pb_n^{(k)}(\lambda, y) = \sum_{m=0}^n \binom{n}{m} D_{m,\lambda}(y) \left\langle \frac{Li_k(1-e^{-t})}{\log(1+t)} \mid x^{n-m} \right\rangle = \sum_{m=0}^n \binom{n}{m} D_{m,\lambda}(y) Pb_{n-m}^{(k)},$$

which completes the proof. \square

Note that it was shown in [9] that $D_{n,\lambda}(x)$ is given by $\sum_{j=0}^n S_1(n, j) \lambda^j B_j(x/\lambda)$, where $B_m(x)$ is the m th Bernoulli polynomial. Thus, for $x = 0$, we have

$$D_{n,\lambda} = \sum_{j=0}^n S_1(n, j) \lambda^j B_j,$$

where B_m is the m th Bernoulli number. Hence, we obtain

$$\begin{aligned} Pb_n^{(k)}(\lambda, x) &= \sum_{m=0}^n \left(\sum_{\ell=0}^{n-m} \binom{n}{m} S_1(n-m, \ell) \lambda^\ell B_\ell \right) Pb_m^{(k)}(x) \\ &= \sum_{m=0}^n \left(\sum_{\ell=m}^n \binom{n}{\ell} S_1(\ell, m) \lambda^m Pb_{n-\ell}^{(k)} \right) B_m(x/\lambda). \end{aligned}$$

Note that Stirling number $S_2(n, k)$ of the second kind can be defined by the exponential generating functions as

$$(2.6) \quad \sum_{n \geq k} S_2(n, k) \frac{x^n}{n!} = \frac{(e^t - 1)^k}{k!}.$$

Theorem 2.3. For all $n \geq 1$,

$$Pb_n^{(k)}(\lambda, x) = \sum_{r=0}^n \left(\sum_{\ell=0}^{n-r} \sum_{m=0}^{n-\ell-r} \binom{n-1}{\ell} \binom{n-\ell}{r} B_\ell^{(n)} Pb_m^{(k)} S_2(n-\ell-r, m) \lambda^r \right) B_r(x/\lambda).$$

Proof. By 2.1, $x^n \sim (1, t)$, and the transfer formula (see [16, 17]), we obtain, for $n \geq 1$,

$$\begin{aligned} &\frac{e^{\lambda t} - 1}{\lambda Li_k(1 - e^{1-e^t})} Pb_n^{(k)}(\lambda, x) \\ &= x \frac{t^n}{(e^t - 1)^n} x^{-1} x^n = x \sum_{\ell \geq 0} B_\ell^{(n)} \frac{t^\ell}{\ell!} x^{n-1} = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} B_\ell^{(n)} x^{n-\ell}. \end{aligned}$$

Thus,

$$Pb_n^{(k)}(\lambda, x) = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} B_\ell^{(n)} \frac{\lambda t}{e^{\lambda t} - 1} \frac{\lambda Li_k(1 - e^{-s})}{\log(1 + s)} \Big|_{s=e^t-1} x^{n-\ell},$$

which, by (1.2) and (2.6), implies

$$\begin{aligned} Pb_n^{(k)}(\lambda, x) &= \sum_{\ell=0}^{n-1} \left(\binom{n-1}{\ell} B_\ell^{(n)} \frac{\lambda t}{e^{\lambda t} - 1} \sum_{m \geq 0} Pb_m^{(k)} \frac{(e^t - 1)^m}{m!} x^{n-\ell} \right) \\ &= \sum_{\ell=0}^{n-1} \sum_{m=0}^{n-\ell} \sum_{r=m}^{n-\ell} \binom{n-1}{\ell} B_\ell^{(n)} Pb_m^{(k)} S_2(r, m) \left(\frac{\lambda t}{e^{\lambda t} - 1} \frac{t^r}{r!} x^{n-\ell} \right) \\ &= \sum_{\ell=0}^{n-1} \sum_{m=0}^{n-\ell} \sum_{r=m}^{n-\ell} \binom{n-1}{\ell} \binom{n-\ell}{r} B_\ell^{(n)} Pb_m^{(k)} S_2(r, m) \left(\frac{\lambda t}{e^{\lambda t} - 1} x^{n-\ell-r} \right) \\ &= \sum_{\ell=0}^{n-1} \sum_{m=0}^{n-\ell} \sum_{r=m}^{n-\ell} \binom{n-1}{\ell} \binom{n-\ell}{r} B_\ell^{(n)} Pb_m^{(k)} S_2(r, m) \lambda^{n-\ell-r} B_{n-\ell-r}(x/\lambda). \end{aligned}$$

Here we used the following fact: $\frac{1}{g(\lambda t)} x^n = \lambda^n s_n(x/\lambda)$ for any $s_n(x) \sim (g(t), t)$ and $\lambda \neq 0$. Indeed, $\langle t^k | 1/g(\lambda t) x^n \rangle = \lambda^{-k} \langle (\lambda t)^k / g(\lambda t) | x^n \rangle = \lambda^{-k} \langle t^k / g(t) | \lambda^n x^n \rangle = \lambda^{n-k} \langle t^k | 1/g(t) x^n \rangle = \lambda^n \langle (t/\lambda)^k | s_n(x) \rangle = \langle t^k | \lambda^n s_n(x/\lambda) \rangle$.

By exchanging the indices of the summations, we obtain that

$$\begin{aligned} Pb_n^{(k)}(\lambda, x) &= \sum_{\ell=0}^{n-1} \sum_{m=0}^{n-\ell} \sum_{r=0}^{n-\ell-m} \binom{n-1}{\ell} \binom{n-\ell}{r} B_\ell^{(n)} Pb_m^{(k)} S_2(n-\ell-r, m) \lambda^r B_r(x/\lambda) \\ &= \sum_{r=0}^n \left(\sum_{\ell=0}^{n-r} \sum_{m=0}^{n-\ell-r} \binom{n-1}{\ell} \binom{n-\ell}{r} B_\ell^{(n)} Pb_m^{(k)} S_2(n-\ell-r, m) \lambda^r \right) B_r(x/\lambda), \end{aligned}$$

as claimed. □

Theorem 2.4. For all $n \geq 0$,

$$Pb_n^{(k)}(\lambda, x) = \sum_{r=0}^n \left(\sum_{\ell=r}^n \sum_{m=0}^{\ell-r} \binom{\ell}{r} S_1(n, \ell) S_2(\ell-r, m) \lambda^r Pb_m^{(k)} \right) B_r(x/\lambda).$$

Proof. By (2.1) we have that $\frac{e^{\lambda t}-1}{\lambda Li_k(1-e^{1-e^t})} Pb_n^{(k)}(\lambda, x) \sim (1, e^t - 1)$. Thus, by (2.3), we obtain

$$(2.7) \quad Pb_n^{(k)}(\lambda, x) = \frac{\lambda Li_k(1 - e^{1-e^t})}{e^{\lambda t} - 1} (x)_n = \sum_{\ell=0}^n S_1(n, \ell) \frac{\lambda Li_k(1 - e^{1-e^t})}{e^{\lambda t} - 1} x^\ell.$$

By replacing the function $\frac{\lambda Li_k(1-e^{1-e^t})}{e^{\lambda t}-1}$ by

$$\frac{\lambda t}{e^{\lambda t} - 1} \frac{\lambda Li_k(1 - e^{-s})}{\log(1 + s)} \Big|_{s=e^t-1},$$

and by using very similar arguments as in the proof of Theorem 2.3, one can complete the proof. □

Note that $Li_2(1 - e^{-t}) = \int_0^t \frac{y}{e^y - 1} dy = \sum_{j \geq 0} B_j \frac{1}{j!} \int_0^t y^j dy = \sum_{j \geq 0} \frac{B_j t^{j+1}}{j!(j+1)}$. For general $k \geq 2$, the function $Li_k(1 - e^{-t})$ has integral representation as

$$Li_k(1 - e^{-t}) = \int_0^t \underbrace{\frac{1}{e^y - 1} \int_0^y \frac{1}{e^y - 1} \int_0^y \cdots \frac{1}{e^y - 1} \int_0^y \frac{y}{e^y - 1} dy \cdots dy}_{(k-2) \text{ times}} dy,$$

which, by induction on k , implies

$$(2.8) \quad Li_k(1 - e^{-t}) = \sum_{j_1 \geq 0} \cdots \sum_{j_{k-1} \geq 0} t^{j_1 + \cdots + j_{k-1} + 1} \prod_{i=1}^{k-1} \frac{B_{j_i}}{j_i!(j_1 + \cdots + j_i + 1)}.$$

Theorem 2.5. For all $n \geq 0$ and $k \geq 2$,

$$Pb_n^{(k)}(\lambda, x) = \sum_{\ell=0}^n (n)_\ell K_{n-\ell}(\lambda, x) \left(\sum_{j_1 + \cdots + j_{k-1} = \ell} \prod_{i=1}^{k-1} \frac{B_{j_i}}{j_i!(j_1 + \cdots + j_i + 1)} \right).$$

Proof. By (2.1), we have

$$\begin{aligned} Pb_n^{(k)}(\lambda, y) &= \left\langle \frac{\lambda Li_k(1 - e^{-t})}{(1+t)^\lambda - 1} (1+t)^y \mid x^n \right\rangle \\ &= \left\langle \frac{Li_k(1 - e^{-t})}{t} \mid \frac{\lambda t}{(1+t)^\lambda - 1} (1+t)^y x^n \right\rangle, \end{aligned}$$

which, by (1.1), implies

$$\begin{aligned} Pb_n^{(k)}(\lambda, y) &= \left\langle \frac{Li_k(1 - e^{-t})}{t} \mid \sum_{\ell \geq 0} K_\ell(\lambda, y) \frac{t^\ell}{\ell!} x^n \right\rangle \\ &= \sum_{\ell=0}^n \binom{n}{\ell} K_\ell(\lambda, y) \left\langle \frac{Li_k(1 - e^{-t})}{t} \mid x^{n-\ell} \right\rangle. \end{aligned}$$

Thus, by (2.8), we complete the proof. \square

3. RECURRENCES

Note that, by (1.3) and the fact that $(x)_n \sim (1, e^t - 1)$, we obtain the following identity. $Pb_n^{(k)}(\lambda, x + y) = \sum_{j=0}^n \binom{n}{j} Pb_j^{(k)}(\lambda, x) (y)_{n-j}$. Moreover, in the next results, we present several recurrences for the degenerate poly-Bernoulli polynomials, namely $Pb_n^{(k)}(x)$.

Theorem 3.1. For all $n \geq 1$, $Pb_n^{(k)}(\lambda, x + 1) = Pb_n^{(k)}(\lambda, x) + n Pb_{n-1}^{(k)}(\lambda, x)$,

Proof. It is well-known that $f(t)s_n(x) = ns_{n-1}(x)$ for all $s_n(x) \sim (g(t), f(t))$ (see [16, 17]). Thus, by (2.1), we have $(e^t - 1)Pb_n^{(k)}(\lambda, x) = n Pb_{n-1}^{(k)}(\lambda, x)$, which gives $Pb_n^{(k)}(\lambda, x + 1) - Pb_n^{(k)}(\lambda, x) = n Pb_{n-1}^{(k)}(\lambda, x)$, as required. \square

Theorem 3.2. For all $n \geq 1$,

$$\begin{aligned} Pb_{n+1}^{(k)}(\lambda, x) &= xPb_n^{(k)}(\lambda, x-1) \\ &\quad - \sum_{m=0}^n \sum_{\ell=0}^{m+1} \sum_{j=\ell}^{m+1} \frac{S_1(n, m)S_2(j, \ell)}{m+1} \binom{m+1}{j} Pb_\ell^{(k)}(\lambda, 0) \lambda^{m+1-j} B_{m+1-j}\left(\frac{x+\lambda-1}{\lambda}\right) \\ &\quad + \sum_{m=0}^n \sum_{\ell=0}^{m+1} \sum_{j=\ell}^{m+1} \frac{S_1(n, m)S_2(j, \ell)}{m+1} \binom{m+1}{j} PB_\ell^{(k-1)} \lambda^{m+1-j} B_{m+1-j}\left(\frac{x}{\lambda}\right). \end{aligned}$$

Proof. It is well-known that that $s_{n+1}(x) = (x - g'(t)/g(t))\frac{1}{f'(t)}s_n(x)$ for all $s_n(x) \sim (g(t), f(t))$ (see [16, 17]). Thus, by 2.1, we have

$$(x - g'_k(t)/g_k(t))\frac{1}{f'(t)} = xe^{-t} - e^{-t}g'_k(t)/g_k(t),$$

which gives

$$(3.1) \quad Pb_{n+1}^{(k)}(\lambda, x) = xPb_n^{(k)}(\lambda, x-1) - e^{-t}g'_k(t)/g_k(t)Pb_n^{(k)}(\lambda, x),$$

where

$$\begin{aligned} e^{-t}\frac{g'_k(t)}{g_k(t)} &= e^{-t} \left(\frac{\lambda e^{\lambda t}}{e^{\lambda t} - 1} - \frac{1}{Li_k(1 - e^{1-e^t})} \frac{Li_{k-1}(1 - e^{1-e^t})}{1 - e^{1-e^t}} e^t e^{1-e^t} \right) \\ &= \frac{1}{t} \left(\frac{\lambda t e^{(\lambda-1)t}}{e^{\lambda t} - 1} \frac{\lambda Li_k(1 - e^{1-e^t})}{e^{\lambda t} - 1} - \frac{\lambda t}{e^{\lambda t} - 1} \frac{Li_{k-1}(1 - e^{1-e^t})}{e^{e^t-1} - 1} \right) \frac{e^{\lambda t} - 1}{\lambda Li_k(1 - e^{1-e^t})}. \end{aligned}$$

Note that the order $te^{-t}\frac{g'_k(t)}{g_k(t)}$ is at least one, and by (2.7) we have $\frac{e^{\lambda t}-1}{\lambda Li_k(1-e^{1-e^t})}Pb_n^{(k)}(\lambda, x) = \sum_{m=0}^n S_1(n, m)x^m$. Thus, by (1.3), we have

$$\begin{aligned} e^{-t}\frac{g'_k(t)}{g_k(t)}Pb_n^{(k)}(\lambda, x) &= \sum_{m=0}^n \frac{S_1(n, m)}{m+1} \left(\frac{\lambda t e^{(\lambda-1)t}}{e^{\lambda t} - 1} \frac{\lambda Li_k(1 - e^{1-e^t})}{e^{\lambda t} - 1} - \frac{\lambda t}{e^{\lambda t} - 1} \frac{Li_{k-1}(1 - e^{1-e^t})}{e^{e^t-1} - 1} \right) x^{m+1}. \end{aligned}$$

Therefore, by (1.2) and (1.3), we have

$$\begin{aligned} e^{-t}\frac{g'_k(t)}{g_k(t)}Pb_n^{(k)}(\lambda, x) &= \sum_{m=0}^n \frac{S_1(n, m)}{m+1} \left(\frac{\lambda t e^{(\lambda-1)t}}{e^{\lambda t} - 1} \frac{\lambda Li_k(1 - e^{-s})}{(1+s)^\lambda - 1} - \frac{\lambda t}{e^{\lambda t} - 1} \frac{Li_{k-1}(1 - e^{-s})}{e^s - 1} \Big|_{s=e^t-1} \right) x^{m+1} \\ &= \sum_{m=0}^n \frac{S_1(n, m)}{m+1} \frac{\lambda t}{e^{\lambda t} - 1} \left(e^{(\lambda-1)t} \sum_{\ell \geq 0} Pb_\ell^{(k)}(\lambda, 0) \frac{(e^t - 1)^\ell}{\ell!} - \sum_{\ell \geq 0} PB_\ell^{(k-1)} \frac{(e^t - 1)^\ell}{\ell!} \right) x^{m+1}, \end{aligned}$$

where $PB_\ell^{(k)}$ are the poly-Bernoulli numbers(of index k). So with help of (2.6), we obtain

$$\begin{aligned} & e^{-t} \frac{g'(t)}{g(t)} Pb_n^{(k)}(\lambda, x) \\ &= \sum_{m=0}^n \sum_{\ell=0}^{m+1} \sum_{j=\ell}^{m+1} \frac{S_1(n, m) S_2(j, \ell)}{m+1} \binom{m+1}{j} \left(Pb_\ell^{(k)}(\lambda, 0) \frac{\lambda t e^{(\lambda-1)t}}{e^{\lambda t} - 1} - PB_\ell^{(k-1)} \frac{\lambda t}{e^{\lambda t} - 1} \right) x^{m+1-j} \\ &= \sum_{m=0}^n \sum_{\ell=0}^{m+1} \sum_{j=\ell}^{m+1} \frac{S_1(n, m) S_2(j, \ell)}{m+1} \binom{m+1}{j} Pb_\ell^{(k)}(\lambda, 0) \lambda^{m+1-j} B_{m+1-j} \left(\frac{x + \lambda - 1}{\lambda} \right) \\ &\quad - \sum_{m=0}^n \sum_{\ell=0}^{m+1} \sum_{j=\ell}^{m+1} \frac{S_1(n, m) S_2(j, \ell)}{m+1} \binom{m+1}{j} PB_\ell^{(k-1)} \lambda^{m+1-j} B_{m+1-j} \left(\frac{x}{\lambda} \right). \end{aligned}$$

Therefore, by changing the summation on j , then substituting into (3.1), we complete the proof. \square

In next theorem, we find expression for $\frac{d}{dx} Pb_n^{(k)}(\lambda, x)$.

Theorem 3.3. For all $n \geq 0$,

$$\frac{d}{dx} Pb_n^{(k)}(\lambda, x) = n! \sum_{\ell=0}^{n-1} \frac{(-1)^{n-\ell-1}}{\ell!(n-\ell)} Pb_n^{(\ell)}(\lambda, x).$$

Proof. It is well-known that $\frac{d}{dx} s_n(x) = \sum_{\ell=0}^{n-1} \binom{n}{\ell} \langle \bar{f}(t) | x^{n-\ell} \rangle s_\ell(x)$, for all $s_n(x) \sim (g(t), f(t))$. Thus, in the case of degenerate poly-Bernoulli polynomials of the second kind (see (2.1)), we have

$$\langle \bar{f}(t) | x^{n-\ell} \rangle = \langle \log(1+t) | x^{n-\ell} \rangle = (-1)^{n-\ell-1} (n-\ell-1)!.$$

Thus

$$\frac{d}{dx} Pb_n^{(k)}(\lambda, x) = \sum_{\ell=0}^{n-1} \binom{n}{\ell} (-1)^{n-\ell-1} (n-\ell-1)! Pb_n^{(\ell)}(\lambda, x),$$

which completes the proof. \square

Theorem 3.4. For all $n \geq 1$,

$$\begin{aligned} & Pb_n^{(k)}(\lambda, x) - x Pb_{n-1}^{(k)}(\lambda, x-1) \\ &= \frac{1}{n} \sum_{m=0}^n \binom{n}{m} (Pb_m^{(k-1)}(\lambda, x) B_{n-m} - Pb_m^{(k)}(\lambda, x + \lambda - 1) K_{n-m}(\lambda)). \end{aligned}$$

Proof. By (1.3), we have, for $n \geq 1$,

$$\begin{aligned} (3.2) \quad Pb_n^{(k)}(\lambda, y) &= \left\langle \frac{\lambda Li_k(1-e^{-t})}{(1+t)^\lambda - 1} (1+t)^y | x^n \right\rangle \\ &= \left\langle \frac{\lambda Li_k(1-e^{-t})}{(1+t)^\lambda - 1} \frac{d}{dt} (1+t)^y | x^{n-1} \right\rangle \\ (3.3) \quad &+ \left\langle \frac{d}{dt} \frac{\lambda Li_k(1-e^{-t})}{(1+t)^\lambda - 1} (1+t)^y | x^{n-1} \right\rangle. \end{aligned}$$

The term in (3.2) is given by

$$(3.4) \quad y \left\langle \frac{\lambda Li_k(1-e^{-t})}{(1+t)^\lambda - 1} (1+t)^{y-1} \mid x^{n-1} \right\rangle = y Pb_{n-1}^{(k)}(\lambda, y-1).$$

For the term in (3.3), we observe that $\frac{d}{dt} \frac{\lambda Li_k(1-e^{-t})}{(1+t)^\lambda - 1} = \frac{1}{t}(A-B)$, where

$$A = \frac{t}{e^t - 1} \frac{\lambda Li_{k-1}(1-e^{-t})}{(1+t)^\lambda - 1}, \quad B = \frac{\lambda t}{(1+t)^\lambda - 1} \frac{\lambda Li_k(1-e^{-t})}{(1+t)^\lambda - 1} (1+t)^{\lambda-1}.$$

Note that the expression $A-B$ has order at least 1. Now, we ready to compute the term in (3.3). By (1.3), we have

$$\begin{aligned} & \left\langle \frac{d}{dt} \frac{\lambda Li_k(1-e^{-t})}{(1+t)^\lambda - 1} (1+t)^y \mid x^{n-1} \right\rangle = \left\langle \frac{1}{t}(A-B)(1+t)^y \mid x^{n-1} \right\rangle \\ &= \frac{1}{n} \langle A(1+t)^y \mid x^n \rangle - \frac{1}{n} \langle B(1+t)^y \mid x^n \rangle \\ &= \frac{1}{n} \left\langle \frac{t}{e^t - 1} \mid \sum_{m \geq 0} Pb_m^{(k-1)}(\lambda, y) \frac{t^m}{m!} x^n \right\rangle \\ & \quad - \frac{1}{n} \left\langle \frac{\lambda t}{(1+t)^\lambda - 1} \mid \sum_{m \geq 0} Pb_m^{(k)}(\lambda, y + \lambda - 1) \frac{t^m}{m!} x^n \right\rangle \\ &= \frac{1}{n} \sum_{m=0}^n \binom{n}{m} Pb_m^{(k-1)}(\lambda, y) \left\langle \frac{t}{e^t - 1} \mid x^{n-m} \right\rangle \\ & \quad - \frac{1}{n} \sum_{m=0}^n \binom{n}{m} Pb_m^{(k)}(\lambda, y + \lambda - 1) \left\langle \frac{\lambda t}{(1+t)^\lambda - 1} \mid x^{n-m} \right\rangle \\ (3.5) \quad &= \frac{1}{n} \sum_{m=0}^n \binom{n}{m} (Pb_m^{(k-1)}(\lambda, y) B_{n-m} - Pb_m^{(k)}(\lambda, y + \lambda - 1) K_{n-m}(\lambda)). \end{aligned}$$

Thus, if we replace (3.2) by (3.4) and (3.3) by (3.5), we obtain

$$\begin{aligned} & Pb_n^{(k)}(\lambda, x) - x Pb_{n-1}^{(k)}(\lambda, x-1) \\ &= \frac{1}{n} \sum_{m=0}^n \binom{n}{m} (Pb_m^{(k-1)}(\lambda, x) B_{n-m} - Pb_m^{(k)}(\lambda, x + \lambda - 1) K_{n-m}(\lambda)), \end{aligned}$$

as claimed. \square

4. CONNECTIONS WITH FAMILIES OF POLYNOMIALS

In this section, we present a few examples on the connections with families of polynomials. To do that we use the following fact from [16, 17]: For $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), \ell(t))$, let $s_n(x) = \sum_{k=0}^n c_{n,k} r_k(x)$. Then we have

$$(4.1) \quad c_{n,k} = \frac{1}{k!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} (\ell(\bar{f}(t)))^k \mid x^n \right\rangle.$$

We start with the connection to Korobov polynomials $K_n(\lambda, x)$ of the first kind.

Theorem 4.1. For all $n \geq 0$,

$$Pb_n^{(k)}(\lambda, x) = \sum_{m=0}^n \left(\binom{n}{m} \sum_{\ell=0}^{n-m} \frac{1}{n-m-\ell+1} \binom{n-m}{\ell} PB_\ell^{(k)} \right) K_m(\lambda, x)$$

and

$$Pb_n^{(k)}(\lambda, x) = \frac{1}{n+1} \sum_{m=0}^n \left(\sum_{\ell=0}^{n-m+1} (-1)^{n-m+1-\ell} \binom{n+1}{m} \frac{\ell!}{\ell^k} S_2(n-m+1, \ell) \right) K_m(\lambda, x).$$

Proof. By (1.1), we have that $K_n(\lambda, x) \sim \left(\frac{e^{\lambda t}-1}{\lambda(e^t-1)}, e^t - 1 \right)$. Let

$$Pb_n^{(k)}(\lambda, x) = \sum_{m=0}^n c_{n,m} K_m(\lambda, x).$$

Thus, by (2.1) and (4.1), we obtain

$$\begin{aligned} c_{n,m} &= \frac{1}{m!} \left\langle \frac{(1+t)^\lambda - 1}{\lambda t} \frac{\lambda Li_k(1-e^{-t})}{(1+t)^\lambda - 1} t^m | x^n \right\rangle = \frac{1}{m!} \left\langle \frac{Li_k(1-e^{-t})}{t} | t^m x^n \right\rangle \\ &= \binom{n}{m} \left\langle \frac{e^t - 1}{t} \left| \frac{Li_k(1-e^{-t})}{e^t - 1} x^{n-m} \right. \right\rangle = \binom{n}{m} \left\langle \frac{e^t - 1}{t} \left| \sum_{\ell \geq 0} PB_\ell^{(k)} \frac{t^\ell}{\ell!} x^{n-m} \right. \right\rangle \\ &= \binom{n}{m} \sum_{\ell=0}^{n-m} \binom{n-m}{\ell} PB_\ell^{(k)} \left\langle \frac{e^t - 1}{t} | x^{n-m-\ell} \right\rangle \\ &= \binom{n}{m} \sum_{\ell=0}^{n-m} \binom{n-m}{\ell} PB_\ell^{(k)} \int_0^1 u^{n-m-\ell} du \\ &= \binom{n}{m} \sum_{\ell=0}^{n-m} \frac{1}{n-m-\ell+1} \binom{n-m}{\ell} PB_\ell^{(k)}, \end{aligned}$$

which completes the proof of the first identity. Note that we can compute $c_{n,m}$ in another way, as follows. By using

$$c_{n,m} = \binom{n}{m} \left\langle \frac{Li_k(1-e^{-t})}{t} | x^{n-m} \right\rangle,$$

and $Li_k(1-e^{-t}) = \sum_{\ell \geq 1} \frac{(1-e^{-t})^\ell}{\ell^k}$ together with (2.6), we obtain that

$$c_{n,m} = \frac{1}{n+1} \sum_{\ell=0}^{n-m+1} (-1)^{n-m+1-\ell} \binom{n+1}{m} \frac{\ell!}{\ell^k} S_2(n-m+1, \ell),$$

which leads to the second identity. □

Theorem 4.2. For all $n \geq 0$,

$$Pb_n^{(k)}(\lambda, x) = \sum_{m=0}^n \left(\binom{n}{m} \sum_{\ell=0}^{n-m} \binom{n-m}{\ell} K_\ell(\lambda) D_{n-m-\ell} \right) Pb_m^{(k)}(x),$$

where D_n is the n th Daehee number defined by $\frac{\log(1+t)}{t} = \sum_{n \geq 0} D_n \frac{t^n}{n!}$.

Proof. By (1.2), we have that $Pb_n^{(k)}(x) \sim \left(\frac{t}{Li_k(1-e^{1-e^t})}, e^t - 1 \right)$. Let

$$Pb_n^{(k)}(\lambda, x) = \sum_{m=0}^n c_{n,m} Pb_m^{(k)}(x).$$

Thus, by (2.1) and (4.1), we obtain

$$\begin{aligned} c_{n,m} &= \frac{1}{m!} \left\langle \frac{\log(1+t)}{Li_k(1-e^{-t})} \frac{\lambda Li_k(1-e^{-t})}{(1+t)^\lambda - 1} t^m | x^n \right\rangle = \binom{n}{m} \left\langle \frac{\log(1+t)}{t} \middle| \frac{\lambda t}{(1+t)^\lambda - 1} x^{n-m} \right\rangle \\ &= \binom{n}{m} \left\langle \frac{\log(1+t)}{t} \middle| \sum_{\ell \geq 0} K_\ell(\lambda) \frac{t^\ell}{\ell!} x^{n-m} \right\rangle \\ &= \binom{n}{m} \sum_{\ell=0}^{n-m} \binom{n-m}{\ell} K_\ell(\lambda) \left\langle \frac{\log(1+t)}{t} \middle| x^{n-m-\ell} \right\rangle \\ &= \binom{n}{m} \sum_{\ell=0}^{n-m} \binom{n-m}{\ell} K_\ell(\lambda) D_{n-m-\ell} \end{aligned}$$

which completes the proof. \square

We start with the connection to *Bernoulli polynomials* $B_n^{(s)}(x)$ of order s . Recall that the *Bernoulli polynomials* $B_n^{(s)}(x)$ of order s are defined by the generating function

$$\left(\frac{t}{e^t - 1} \right)^s e^{xt} = \sum_{n \geq 0} B_n^{(s)}(x) \frac{t^n}{n!},$$

or equivalently,

$$(4.2) \quad B_n^{(s)}(x) \sim \left(\left(\frac{e^t - 1}{t} \right)^s, t \right)$$

(see [2,4]). In the next result, we express our polynomials $Pb_n^{(k)}(x)$ in terms of *Bernoulli polynomials of order s* . To do that, we recall that Bernoulli numbers of the second kind $b_n^{(s)}$ of order s are defined as

$$(4.3) \quad \frac{t^s}{\log^s(1+t)} = \sum_{n \geq 0} b_n^{(s)} \frac{t^n}{n!}.$$

Theorem 4.3. For all $n \geq 0$,

$$Pb_n^{(k)}(\lambda, x) = \sum_{m=0}^n \left(\sum_{\ell=m}^n \sum_{j=0}^{n-\ell} \binom{n}{\ell} \binom{n-\ell}{j} S_1(\ell, m) Pb_j^{(k)}(\lambda, 0) b_{n-\ell-j}^{(s)} \right) B_m^{(s)}(x).$$

Proof. Let $Pb_n^{(k)}(\lambda, x) = \sum_{m=0}^n c_{n,m} B_m^{(s)}(x)$. By (2.1), (4.1) and (4.2), we have

$$c_{n,m} = \frac{1}{m!} \left\langle \left(\frac{t}{\log(1+t)} \right)^s \frac{\lambda Li_k(1-e^{-t})}{(1+t)^\lambda - 1} (\log(1+t))^m | x^n \right\rangle,$$

which, by (2.3) and (1.3), implies

$$\begin{aligned} c_{n,m} &= \left\langle \left(\frac{t}{\log(1+t)} \right)^s \frac{\lambda Li_k(1-e^{-t})}{(1+t)^\lambda - 1} \mid \sum_{\ell \geq m} S_1(\ell, m) \frac{t^\ell}{\ell!} x^n \right\rangle \\ &= \sum_{\ell=m}^n \binom{n}{\ell} S_1(\ell, m) \left\langle \left(\frac{t}{\log(1+t)} \right)^s \mid \frac{\lambda Li_k(1-e^{-t})}{(1+t)^\lambda - 1} x^{n-\ell} \right\rangle \\ &= \sum_{\ell=m}^n \binom{n}{\ell} S_1(\ell, m) \left\langle \left(\frac{t}{\log(1+t)} \right)^s \mid \sum_{j \geq 0} Pb_j^{(k)}(\lambda, 0) \frac{t^j}{j!} x^{n-\ell} \right\rangle \\ &= \sum_{\ell=m}^n \sum_{j=0}^{n-\ell} \binom{n}{\ell} \binom{n-\ell}{j} S_1(\ell, m) Pb_j^{(k)}(\lambda, 0) \left\langle \left(\frac{t}{\log(1+t)} \right)^s \mid x^{n-\ell-j} \right\rangle. \end{aligned}$$

Thus, by (4.3), we obtain

$$c_{n,m} = \sum_{\ell=m}^n \sum_{j=0}^{n-\ell} \binom{n}{\ell} \binom{n-\ell}{j} S_1(\ell, m) Pb_j^{(k)}(\lambda, 0) b_{n-\ell-j}^{(s)},$$

which completes the proof. \square

Similar techniques as in the proof of the previous theorem, we can express our polynomials $Pb_n^{(k)}(\lambda, x)$ in terms of other families. For instance, we can express our polynomials $Pb_n^{(k)}(\lambda, x)$ in terms of Frobenius-Euler polynomials (we leave the proof to the interested reader). Note that the *Frobenius-Euler polynomials* $H_n^{(s)}(x|\mu)$ of order s are defined by the generating function $\left(\frac{1-\mu}{e^t-\mu} \right)^s e^{xt} = \sum_{n \geq 0} H_n^{(s)}(x|\mu) \frac{t^n}{n!}$, ($\mu \neq 1$), or equivalently, $H_n^{(s)}(x|\mu) \sim \left(\left(\frac{e^t-\mu}{1-\mu} \right)^s, t \right)$ (see [1, 2, 4, 11]).

Theorem 4.4. *For all $n \geq 0$,*

$$Pb_n^{(k)}(\lambda, x) = \sum_{m=0}^n \left(\sum_{\ell=m}^n \sum_{r=0}^{n-\ell} \binom{n}{\ell} \binom{n-\ell}{r} \frac{s! S_1(\ell, m) Pb_r^{(k)}(\lambda, 0)}{(1-\mu)^{n-\ell-r} (s+\ell+r-n)!} \right) H_m^{(s)}(x|\mu).$$

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Some results for meromorphic functions of several variables

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Abstract: Using the Nevanlinna theory of the value distribution of meromorphic functions, we investigate the value distribution of complex partial q -difference polynomials of meromorphic functions of zero order, and also investigate the existence of meromorphic solutions of some types of systems of complex partial q -difference equations in \mathbb{C}^n . Some existing results are improved and generalized, and some new results are obtained. Examples show that our results are precise.

Keywords: value distribution; meromorphic solution; complex partial q -difference polynomials; complex partial q -difference equations

§1 Introduction

In this paper, we assume that the reader is familiar with the standard notation and basic results of the Nevanlinna theory of meromorphic functions, see, for example [1].

The reference related to notations of this section are referred to Tu[2].

Let M be a connected complex manifold of dimension n and let

$$A(M) = \sum_{n=0}^{2m} A^n(M)$$

be the graded ring of complex valued differential forms on M . Each set $A^n(M)$ can be split into a direct sum

$$A^n(M) = \sum_{p+q=n} A^{p,q}(M),$$

where $A^{p,q}(M)$ is the forms of type p, q . The differential operators d and d^c on $A(M)$ are defined as

$$d := \partial + \bar{\partial} \quad \text{and} \quad d^c := \frac{1}{4\pi i}(\partial - \bar{\partial}).$$

where

$$\partial : A^{p,q}(M) \longrightarrow A^{p+1,q}(M),$$

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$$\bar{\partial} : A^{p,q}(M) \longrightarrow A^{p,q+1}(M).$$

Let $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, and let $r \in \mathbb{R}^+$. We define

$$\omega_n(z) := dd^c \log |z|^2 \quad \text{and} \quad \sigma_n(z) := d^c \log |z|^2 \wedge \omega_n^{n-1}(z),$$

where $z \in \mathbb{C}^n \setminus \{0\}$ and $|z|^2 := |z_1|^2 + \dots + |z_n|^2$.

Let $\mathbb{C}^n < r > = \{z \in \mathbb{C}^n : |z| = r\}$, $\mathbb{C}^n(r) = \{z \in \mathbb{C}^n : |z| < r\}$, $\mathbb{C}^n[r] = \{z \in \mathbb{C}^n : |z| \leq r\}$. Then $\sigma_n(z)$ defines a positive measure on $\mathbb{C}^n < r >$ with total measure one. In addition, by defining

$$\nu_n(z) := dd^c |z|^2 \quad \text{and} \quad \rho_n(z) := \nu_n^n(z),$$

for all $z \in \mathbb{C}^n$, it follows that $\rho_n(z)$ is the Lebesgue measure on \mathbb{C}^n normalized such that $\mathbb{C}^n(r)$ has measure r^{2n} .

Let w be a meromorphic function on \mathbb{C}^n in the sense that w can be written as a quotient of two relatively prime holomorphic functions. We will write $w = (w_0, w_1)$ where $w_0 \not\equiv 0$, thus w can be regarded as a meromorphic map $w : \mathbb{C}^n \longrightarrow \mathbb{P}^1$ such that $w^{-1}(\infty) \neq \mathbb{C}^n$.

Let \mathbb{P}^1 be the Riemann sphere. For $a, b \in \mathbb{P}^1$, the chordal distance from a to b is denoted by $\|a, b\|$, $\|a, \infty\| = \frac{1}{\sqrt{1+|a|^2}}$, $\|a, b\| = \frac{|a-b|}{\sqrt{1+|a|^2} \cdot \sqrt{1+|b|^2}}$, $a, b \in \mathbb{C}$, where $\|a, a\| = 0$ and $0 \leq \|a, b\| = \|b, a\| \leq 1$. If $a \in \mathbb{P}^1$ and $w^{-1}(a) \neq \mathbb{C}^n$, then we define the proximity function as

$$m(r, w, a) = \int_{|z|=r} \log \frac{1}{\|a, w(z)\|} \sigma_n \geq 0, \quad r > 0.$$

Let ν be a divisor on \mathbb{C}^n . We identify ν with its multiplicity function, define

$$\nu(r) = \{z \in \mathbb{C}^n : |z| < r\} \cap \text{supp} \nu, \quad r > 0.$$

The pre-counting function of ν is defined by

$$n(r, \nu) = \sum_{z \in \nu(r)} \nu(z), \quad \text{if } n = 1, \quad n(r, \nu) = r^{2-2n} \int_{\nu(r)} \nu \nu_n^{n-1}, \quad \text{if } n > 1.$$

The counting function of ν is defined by

$$N(r, \nu) = \int_s^r n(t, \nu) \frac{dt}{t}, \quad r > s.$$

Let w be a meromorphic function on \mathbb{C}^n . If $a \in \mathbb{P}^1$ and $w^{-1}(a) \neq \mathbb{C}^n$, the a -divisor $\nu(w, a) \geq 0$ is defined, and its pre-counting function and counting function will be denoted by $n(r, w, a)$ and $N(r, w, a)$, respectively.

For a divisor ν on \mathbb{C}^n , let

$$\bar{n}(r, \nu) = \sum_{z \in \nu(r)} 1, \quad \text{if } n = 1, \quad \bar{n}(r, \nu) = r^{2-2n} \int_{\nu(r)} \nu_n^{n-1}, \quad \text{if } n > 1.$$

$$\bar{N}(r, \nu) = \int_s^r \bar{n}(t, \nu) \frac{dt}{t}, \quad r > s. \quad \bar{N}(r, w, a) = \bar{N}(r, \nu(w, a)).$$

For $0 < s < r$, the characteristic of w is defined by

$$T(r, w) = \int_s^r \frac{1}{t^{2n-1}} \int_{\mathbb{C}^n[t]} w^*(\omega) \wedge \nu_n^{n-1} dt = \int_s^r \frac{1}{t} \int_{\mathbb{C}^n[t]} w^*(\omega) \wedge \omega_n^{n-1} dt.$$

where the pullback $w^*(\omega)$ satisfies $w^*(\omega) = dd^c \log(|w_0|^2 + |w_1|^2)$.

The First Main Theorem states

$$T(r, w) = N(r, w, a) + m(r, w, a) - m(s, w, a).$$

In 2012, Korhonen R has investigated the difference analogues of the lemma on the Logarithmic Derivate and of the Second Main Theorem of Nevanlinna theory for meromorphic functions of several variables, see [3]. Particularly, in 2013, Cao T B, see [4], using different method obtains difference analogues of the second main theorem for meromorphic functions in several complex variables from which difference analogues of Picard-type theorems are also obtained. His results are improvements or extensions of some results of Korhonen R.

Similarly, in 2014, Wen Z T has investigated the q -difference theory for meromorphic functions of several variables, see [5]. Some results that we will use in this paper are as follows.

Theorem A [5] *Let w be a meromorphic function in \mathbb{C}^n of zero order such that $w(0) \neq 0, \infty$, and let $q \in \mathbb{C}^n \setminus \{0\}$. Then,*

$$m(r, \frac{w(qz)}{w(z)}) = o(T(r, w)),$$

on a set of logarithmic density 1.

Theorem B [5] *Let w be a meromorphic function in \mathbb{C}^n of zero order such that $w(0) \neq 0, \infty$, and let $q \in \mathbb{C}^n \setminus \{0\}$. Then,*

$$T(r, w(qz)) = T(r, w(z)) + o(T(r, w)),$$

on a set of logarithmic density 1.

Remark: From the proof of Theorem B in [5], we have

$$N(r, w(qz)) = N(r, w(z)) + o(N(r, w)).$$

The remainder of the paper is organized as follows. In §2, we discuss Theorem A's applications to complex partial q -difference equations. We present q -shift analogues of the Clunie lemmas which can be used to study value distribution of zero-order meromorphic solutions of large classes of complex partial q -difference equations. In §3, we study the existence of meromorphic solutions of complex partial q -difference equation of several variables, and obtain four theorems, and then we give some examples, which show that the results obtained in §3 are, in a sense, the best possible. And finally, we prove these four theorems by a series of lemmas.

§2 Value distribution of complex partial q -difference polynomials

Recently, Laine I, Halburd R G, Korhonen R J, Barnett D, Morgan W, investigate complex q -difference theory, and have obtained some results, see [6,7,8,9]. Especially, in 2007, Barnett D C, Halburd R G have obtained a theorem which is analogous to the Clunie Lemma as follows

Theorem C [7] *Let $w(z)$ be a non-constant zero-order meromorphic solution of*

$$w^{n_1}(z)P_1(z, w) = Q_1(z, w),$$

where $P_1(z, w)$ and $Q_1(z, w)$ are complex q -difference polynomials in $w(z)$ of the form

$$P_1(z, w) = \sum_{\lambda_1 \in I'_1} a_{\lambda_1}(z) w(z)^{l_0^1} (w(q_1 z))^{l_1^1} \cdots (w(q_\nu z))^{l_\nu^1},$$

$$Q_1(z, w) = \sum_{\gamma_1 \in J'_1} b_{\gamma_1}(z) w(z)^{l_0^2} (w(q_1 z))^{l_1^2} \cdots (w(q_\mu z))^{l_\mu^2}.$$

If the degree of $Q_1(z, w)$ as a polynomial in $w(z)$ and its q -shifts is at most n_1 , then

$$m(r, P_1(z, w)) = S(r, w) = o\{T(r, w)\},$$

for all r on a set of logarithmic density 1.

We will investigate the problem of value distribution of complex partial q -difference polynomials (2.1), (2.2) and (2.3), where $z = (z_1, \dots, z_n) \in \mathbb{C}^n$.

$$P(z, w) = \sum_{\lambda \in I_1} a_\lambda(z) w(z)^{l_{\lambda_0}} (w(q_{\lambda_1} z))^{l_{\lambda_1}} \cdots (w(q_{\lambda_{\sigma_\lambda}} z))^{l_{\lambda_{\sigma_\lambda}}}. \quad (2.1)$$

$$Q(z, w) = \sum_{\mu \in J_1} b_\mu(z) w(z)^{m_{\mu_0}} (w(q_{\mu_1} z))^{m_{\mu_1}} \cdots (w(q_{\mu_{\tau_\mu}} z))^{m_{\mu_{\tau_\mu}}}. \quad (2.2)$$

$$U(z, w) = \sum_{\nu \in K_1} c_\nu(z) w(z)^{n_{\nu_0}} (w(q_{\nu_1} z))^{n_{\nu_1}} \cdots (w(q_{\nu_{v_\nu}} z))^{n_{\nu_{v_\nu}}}. \quad (2.3)$$

where coefficients $\{a_\lambda(z)\}, \{b_\mu(z)\}, \{c_\nu(z)\}$ are small functions of $w(z)$. I_1, J_1, K_1 are three finite sets of multi-indices, $q_j \in \mathbb{C}^n \setminus \{0\}$, $(j \in \{\lambda_1, \dots, \lambda_{\sigma_\lambda}, \mu_1, \dots, \mu_{\tau_\mu}, \nu_1, \dots, \nu_{v_\nu}\})$.

We will prove

Theorem 2.1. Let w be a meromorphic function in \mathbb{C}^n , and be a non-constant meromorphic solution of zero order of a complex partial q -difference equation of the form

$$U(z, w)P(z, w) = Q(z, w),$$

where complex partial q -difference polynomials $P(z, w), Q(z, w)$ and $U(z, w)$ are respectively as the form of (2.1), (2.2), (2.3), the total degree $\deg U(z, w) = n_1$ in $w(z)$ and its shifts, and $\deg Q(z, w) \leq n_1$. Moreover, we assume that $U(z, w)$ contains just one term of maximal total degree in $w(z)$ and its shifts. Then, we have

$$m(r, P(z, w)) = S(r, w) = o\{T(r, w)\},$$

for all r on a set of logarithmic density 1.

Corollary 2.1. Let w be a meromorphic function in \mathbb{C}^n , and be a non-constant transcendental meromorphic solution of zero order of a complex partial q -difference equation of the form

$$H(z, w)P(z, w) = Q(z, w),$$

where $H(z, w)$ is a complex partial q -difference product of total degree n_1 in $w(z)$ and its shifts, and where $P(z, w), Q(z, w)$ are complex partial q -difference polynomials such that the total degree of $Q(z, w)$ is at most n_1 . Then, we obtain

$$m(r, P(z, w)) = S(r, w) = o\{T(r, w)\},$$

for all r on a set of logarithmic density 1.

Proof of Theorem 2.1 As the proof of Theorem 1 in [10], we rearrange the expression for the complex partial q -difference polynomial $U(z, w)$ by collecting together all terms having the same total degree and then writing $U(z, w)$ as follows

$$U(z, w) = \sum_{j=0}^{n_1} d_j(z) w^j(z),$$

where $d_j(z) = \sum_{\nu=j} c_\nu(z) \left(\frac{w(q_{\nu_1} z)}{w(z)}\right)^{n_{\nu_1}} \cdots \left(\frac{w(q_{\nu_{v_\nu}} z)}{w(z)}\right)^{n_{\nu_{v_\nu}}}$, $j = 0, 1, \dots, n_1$. Since $\deg U(z, w) = n_1$ in $w(z)$ and its shifts, and $U(z, w)$ contains just one term of maximal total degree n_1 in $w(z)$

and its shifts, therefore, $d_{n_1}(z)$ contains just one product of the described form.

By Theorem A, for all r on a set of logarithmic density 1, we have

$$m(r, d_j(z)) = S(r, w) = o\{T(r, w)\}, j = 0, 1, \dots, n_1.$$

It follows from the assumption that $d_{n_1}(z)$ has just one term of maximal total degree in $U(z, w)$, thus, for all r on a set of logarithmic density 1, we get

$$m(r, \frac{1}{d_{n_1}(z)}) = S(r, w) = o\{T(r, w)\}.$$

Let

$$A(z) = \max_{1 \leq j \leq n_1} \{1, 2 \mid \frac{d_{n_1-j}}{d_{n_1}} \mid^{\frac{1}{j}}\}.$$

Then

$$m(r, A(z)) \leq \sum_{j=0}^{n_1} m(r, d_{n_1-j}) + m(r, \frac{1}{d_{n_1}}) + O(1) = S(r, w) = o\{T(r, w)\}.$$

Let

$$E_1 = \{z \in \mathbb{C}^n < r > : |w(z)| \leq A(z)\}, \quad E_2 = \mathbb{C}^n < r > \setminus E_1.$$

Thus

$$m(r, P(z, w)) = \int_{E_1} \log^+ |P(z, w)| \sigma_n(z) + \int_{E_2} \log^+ |P(z, w)| \sigma_n(z). \quad (2.4)$$

Next we estimate respectively $\int_{E_1} \log^+ |P(z, w)| \sigma_n(z)$ and $\int_{E_2} \log^+ |P(z, w)| \sigma_n(z)$ in (2.4).

When $z \in E_1$, we have

$$\begin{aligned} |P(z, w)| &= \left| \sum_{\lambda \in I_1} a_\lambda(z) w(z)^{l_{\lambda_0}} (w(q_{\lambda_1} z))^{l_{\lambda_1}} \cdots (w(q_{\lambda_{\sigma_\lambda}} z))^{l_{\lambda_{\sigma_\lambda}}} \right| \\ &\leq \sum_{\lambda \in I_1} |a_\lambda(z)| |w(z)|^{l_{\lambda_0}} |w(q_{\lambda_1} z)|^{l_{\lambda_1}} \cdots |w(q_{\lambda_{\sigma_\lambda}} z)|^{l_{\lambda_{\sigma_\lambda}}} \\ &= \sum_{\lambda \in I_1} |a_\lambda(z)| |w(z)|^{l_\lambda} \left| \frac{w(q_{\lambda_1} z)}{w(z)} \right|^{l_{\lambda_1}} \cdots \left| \frac{w(q_{\lambda_{\sigma_\lambda}} z)}{w(z)} \right|^{l_{\lambda_{\sigma_\lambda}}} \\ &\leq \sum_{\lambda \in I_1} |a_\lambda(z)| |A(z)|^{l_\lambda} \left| \frac{w(q_{\lambda_1} z)}{w(z)} \right|^{l_{\lambda_1}} \cdots \left| \frac{w(q_{\lambda_{\sigma_\lambda}} z)}{w(z)} \right|^{l_{\lambda_{\sigma_\lambda}}}, \end{aligned}$$

where $l_\lambda = l_{\lambda_0} + l_{\lambda_1} + \cdots + l_{\lambda_{\sigma_\lambda}}$. By Theorem A, for all r on a set of logarithmic density 1, we have

$$\int_{E_1} \log^+ |P(z, w)| \sigma_n(z) = S(r, w) = o\{T(r, w)\}. \quad (2.5)$$

When $z \in E_2$, we obtain

$$|w(z)| > A(z) \geq 2 \mid \frac{d_{n_1-j}}{d_{n_1}} \mid^{\frac{1}{j}}, (j = 1, 2, \dots, n_1),$$

i.e.

$$\left| \frac{w(z)}{2^j} \right|^j \geq \left| \frac{d_{n_1-j}}{d_{n_1}} \right|, (j = 1, 2, \dots, n_1).$$

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It follows from $U(z, w) = \sum_{j=0}^{n_1} d_j(z)w^j$ that

$$\begin{aligned}
 |U(z, w)| &\geq |d_{n_1}| |w|^{n_1} - (|d_{n_1-1}| |w|^{n_1-1} + |d_{n_1-2}| |w|^{n_1-2} + \dots \\
 &\quad + |d_1| |w| + |d_0|) \\
 &= |d_{n_1}| |w|^{n_1} - |d_{n_1}| |w|^{n_1} \left(\frac{|d_{n_1-1}|}{|d_{n_1}| |w|} + \frac{|d_{n_1-2}|}{|d_{n_1}| |w|^2} \right. \\
 &\quad \left. + \dots + \frac{|d_1|}{|d_{n_1}| |w|^{n_1-1}} + \frac{|d_0|}{|d_{n_1}| |w|^{n_1}} \right) \\
 &= |d_{n_1}| |w|^{n_1} - |d_{n_1}| |w|^{n_1} \left(\sum_{j=1}^{n_1} \frac{|d_{n_1-j}|}{|d_{n_1}| |w|^j} \right) \\
 &\geq |d_{n_1}| |w|^{n_1} \left(1 - \sum_{j=1}^{n_1} \frac{1}{2^j} \right) \\
 &= \frac{|d_{n_1}| |w|^{n_1}}{2^{n_1}}.
 \end{aligned}$$

Since $z \in E_2$, then

$$|w(z)| > A(z) \geq 1,$$

that is

$$\frac{1}{|w(z)|} < 1.$$

Using $U(z, w)P(z, w) = Q(z, w)$ and the total degree of $Q(z, w)$ is at most n_1 , we obtain

$$\begin{aligned}
 |P(z, w)| &= \left| \frac{Q(z, w)}{U(z, w)} \right| \\
 &\leq \frac{2^{n_1}}{|d_{n_1}| |w|^{n_1}} \sum_{\mu \in J_1} |b_\mu(z)| |w(z)|^{m_{\mu_0}} |w(q_{\mu_1} z)|^{m_{\mu_1}} \\
 &\quad \dots |w(q_{\mu_{\tau_\mu}} z)|^{m_{\mu_{\tau_\mu}}} \\
 &\leq \frac{2^{n_1}}{|d_{n_1}|} \sum_{\mu \in J_1} |b_\mu(z)| \left| \frac{w(q_{\mu_1} z)}{w(z)} \right|^{m_{\mu_1}} \dots \left| \frac{w(q_{\mu_{\tau_\mu}} z)}{w(z)} \right|^{m_{\mu_{\tau_\mu}}}.
 \end{aligned}$$

From Theorem A, for all r on a set of logarithmic density 1, we have

$$\int_{E_2} \log^+ |P(z, w)| \sigma_n(z) = S(r, w) = o\{T(r, w)\}. \quad (2.6)$$

Combining (2.4), (2.5), (2.6), yields

$$\begin{aligned}
 m(r, P(z, w)) &= \int_{E_1} \log^+ |P(z, w)| \sigma_n(z) + \int_{E_2} \log^+ |P(z, w)| \sigma_n(z) \\
 &= S(r, w) = o\{T(r, w)\}.
 \end{aligned}$$

This completes the proof of Theorem 2.1.

§3 Applications to complex partial q-difference equations

Recently, many authors, such as Chiang Y M, Halburd R G, Korhonen R J, Chen Zongxuan, Gao Lingyun have studied solutions of some types of complex difference equation, and systems of complex difference equations, and also obtained many important results, see [11, 12, 13, 14, 15, 16, 17].

Let w be a non-constant meromorphic function of zero order, if meromorphic function g satisfies $T(r, g) = o\{T(r, w)\}$, for all r outside of a set of upper logarithmic density 0, i.e. outside of a set E such that $\limsup_{r \rightarrow \infty} \frac{\int_{E \cap [1, r]} \frac{dt}{t}}{\log r} = 0$. The complement of E has lower logarithmic density 1, then g is called small function of w .

Let $q_j \in \mathbb{C}^n \setminus \{0\}$, $j = 1, \dots, n_2$, $w_i : \mathbb{C}^n \rightarrow \mathbb{P}^1, i = 1, 2$. $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, I, J, \bar{I}, \bar{J} are four finite sets of multi-indices, complex partial q -difference polynomials $\Omega_1(z, w_1, w_2)$, $\Omega_2(z, w_1, w_2)$, $\Omega_3(z, w_1, w_2)$, $\Omega_4(z, w_1, w_2)$ can be expressed as

$$\begin{aligned}\Omega_1(z, w_1, w_2) &= \sum_{(i) \in I} a_{(i)}(z) \prod_{k=1}^2 w_k^{i_{k0}} (w_k(q_1 z))^{i_{k1}} \cdots (w_k(q_{n_2} z))^{i_{kn_2}}, \\ \Omega_2(z, w_1, w_2) &= \sum_{(j) \in J} b_{(j)}(z) \prod_{k=1}^2 w_k^{j_{k0}} (w_k(q_1 z))^{j_{k1}} \cdots (w_k(q_{n_2} z))^{j_{kn_2}}, \\ \Omega_3(z, w_1, w_2) &= \sum_{(\bar{i}) \in \bar{I}} c_{(\bar{i})}(z) \prod_{k=1}^2 w_k^{\bar{i}_{k0}} (w_k(q_1 z))^{\bar{i}_{k1}} \cdots (w_k(q_{n_2} z))^{\bar{i}_{kn_2}}, \\ \Omega_4(z, w_1, w_2) &= \sum_{(\bar{j}) \in \bar{J}} d_{(\bar{j})}(z) \prod_{k=1}^2 w_k^{\bar{j}_{k0}} (w_k(q_1 z))^{\bar{j}_{k1}} \cdots (w_k(q_{n_2} z))^{\bar{j}_{kn_2}},\end{aligned}$$

where coefficients $\{a_{(i)}(z)\}, \{b_{(j)}(z)\}, \{c_{(\bar{i})}(z)\}, \{d_{(\bar{j})}(z)\}$ are small functions of w_1, w_2 .

Let $\Phi_1 = \frac{\Omega_1(z, w_1, w_2)}{\Omega_2(z, w_1, w_2)}$, $\Phi_2 = \frac{\Omega_3(z, w_1, w_2)}{\Omega_4(z, w_1, w_2)}$, for Φ_1 , we denote $\lambda_{11} = \max_{(i)} \{\sum_{l=0}^{n_2} i_{1l}\}$, $\lambda_{12} = \max_{(i)} \{\sum_{l=0}^{n_2} i_{2l}\}$, $\lambda_{21} = \max_{(j)} \{\sum_{l=0}^{n_2} j_{1l}\}$, $\lambda_{22} = \max_{(j)} \{\sum_{l=0}^{n_2} j_{2l}\}$, $\lambda_1 = \max\{\lambda_{11}, \lambda_{21}\}$, $\lambda_2 = \max\{\lambda_{12}, \lambda_{22}\}$. For Φ_2 , we denote similarly $\bar{\lambda}_1, \bar{\lambda}_2$.

We will investigate the existence of meromorphic solutions of complex partial q -difference equation of several variables (3.1) and systems of complex partial q -difference equations of several variables (3.2) and (3.3), where $z = (z_1, \dots, z_n) \in \mathbb{C}^n$.

$$\sum_{j=1}^{n_2} w(q_j z) = R(z, w(z)) = \frac{a_0(z) + a_1(z)w(z) + \cdots + a_p(z)w^p(z)}{b_0(z) + b_1(z)w(z) + \cdots + b_q(z)w^q(z)}, \quad (3.1)$$

where $q_1, \dots, q_{n_2} \in \mathbb{C}^n \setminus \{0\}$, $R(z, w(z))$ is irreducible rational function in $w(z)$, $a_0(z), \dots, a_p(z)$, $b_0(z), \dots, b_q(z)$ are rational functions.

$$\begin{cases} \Omega_1(z, w_1, w_2) = R_1(z, w_1) = \frac{a_0(z) + a_1(z)w_1(z) + \cdots + a_{p_1}(z)w_1^{p_1}(z)}{b_0(z) + b_1(z)w_1(z) + \cdots + b_{q_1}(z)w_1^{q_1}(z)}, \\ \Omega_2(z, w_1, w_2) = R_2(z, w_2) = \frac{c_0(z) + c_1(z)w_2(z) + \cdots + c_{p_2}(z)w_2^{p_2}(z)}{d_0(z) + d_1(z)w_2(z) + \cdots + d_{q_2}(z)w_2^{q_2}(z)}, \end{cases} \quad (3.2)$$

where coefficients $\{a_i(z)\}, \{b_j(z)\}$ are small functions of w_1 , $\{c_l(z)\}, \{d_m(z)\}$ are small functions of w_2 . $a_{p_1}b_{q_1} \neq 0, c_{p_2}d_{q_2} \neq 0$. The definition of $\Omega_1(z, w_1, w_2)$ and $\Omega_2(z, w_1, w_2)$ is as before.

$$\begin{cases} \Phi_1 = R_1(z, w_1, w_2), \\ \Phi_2 = R_2(z, w_1, w_2). \end{cases} \quad (3.3)$$

where $R_j (j = 1, 2)$ are irreducible rational functions with the meromorphic coefficients.

Definition 3.1. Let w_1 and w_2 be meromorphic functions in \mathbb{C}^n .

$$S(r) = \sum T(r, a_{(i)}) + \sum T(r, b_{(j)}) + \sum T(r, c'_{(\bar{i})}) + \sum T(r, d_{(\bar{j})}) + \sum T(r, d'_{(\bar{j})}),$$

where $\sum T(r, d'_{(\bar{j})})$ means the sum of characteristic functions of all coefficients in $R_j (j = 1, 2)$. $(w_1(z), w_2(z))$ be a set of meromorphic solutions of (3.2) or (3.3). If one (Let be $w_1(z)$) of meromorphic solutions $(w_1(z), w_2(z))$ of (3.2) or (3.3) satisfies $S(r) = o\{T(r, w_1)\}$, outside a possible exceptional set with finite logarithmic measure, then we say $w_1(z)$ is admissible.

We will prove

Theorem 3.1. Let w be a meromorphic function in \mathbb{C}^n . If the q -difference equation (3.1) admits a transcendental meromorphic solution of zero order, then

$$\max\{p, q\} \leq n_2.$$

Remark 3.1. If we replace the left side of (3.1) by $\prod_{j=1}^{n_2} w(q_j z)$, then the same assertion that $\max\{p, q\} \leq n_2$ holds.

Theorem 3.2. Let w_1 and w_2 be meromorphic functions in \mathbb{C}^n , and $(w_1(z), w_2(z))$ be a set of zero order meromorphic solution of (3.2). If

$$\max\{p_1, q_1\} > \lambda_{11}, \max\{p_2, q_2\} > \lambda_{22},$$

and both w_1 and w_2 are admissible, then

$$[\max\{p_1, q_1\} - \lambda_{11}][\max\{p_2, q_2\} - \lambda_{22}] \leq \lambda_{12}\lambda_{21}.$$

Example 3.1. $(w_1, w_2) = (z_1 z_2, \frac{1}{z_1 z_2})$ is a set of zero order admissible meromorphic solution of the following system of complex partial q -difference equations

$$\begin{cases} w_2^2(-2z_1, -2z_2) = \frac{1}{16w_1^2}, \\ w_1^2(\frac{1}{3}z_1, \frac{1}{3}z_2) = \frac{1}{81w_2^2}. \end{cases}$$

Easily, we obtain

$$\lambda_{11} = 0, \lambda_{22} = 0, \lambda_{12} = 2, \lambda_{21} = 2, \max\{p_1, q_1\} = 2, \max\{p_2, q_2\} = 2.$$

Thus

$$[\max\{p_1, q_1\} - \lambda_{11}][\max\{p_2, q_2\} - \lambda_{22}] = 4 = \lambda_{12}\lambda_{21}.$$

This example shows the upper bound in Theorem 3.2 can be reached.

Example 3.2. For a system of complex partial q -difference equations

$$\begin{cases} w_1^2(-2z_1, -2z_2)w_2(-\frac{1}{2}z_1, -\frac{1}{2}z_2) = \frac{w_1^4 - (\frac{7}{2}z_1 z_2 - \frac{49}{16})w_1^2 - \frac{45}{8}z_1 z_2 - \frac{65}{16}}{w_1^2 - 2w_1 - z_1^2 z_2^2 + 2}, \\ w_1(-2z_1, -2z_2)w_2^2(-\frac{1}{2}z_1, -\frac{1}{2}z_2) = \frac{\frac{1}{256}w_2^2 + \frac{1}{64z_1 z_2}w_2^3}{\frac{3}{z_1 z_2}w_2 - 3z_1 z_2 + 1}. \end{cases}$$

$(w_1, w_2) = (z_1 z_2 + 1, z_1^2 z_2^2)$ is a set of non-admissible solutions.

Clearly, we know

$$\lambda_{11} = 2, \lambda_{22} = 2, \lambda_{12} = 1, \lambda_{21} = 1, \max\{p_1, q_1\} = 4, \max\{p_2, q_2\} = 3.$$

Thus

$$[\max\{p_1, q_1\} - \lambda_{11}] [\max\{p_2, q_2\} - \lambda_{22}] = 2 > 1 = \lambda_{12}\lambda_{21}.$$

This example shows that we can not omit 'admissible' in Theorem 3.2.

Theorem 3.3. Let w_1 and w_2 be meromorphic functions in \mathbb{C}^n . Let (w_1, w_2) be a set of zero order meromorphic solution of (3.2). If one of the following conditions is satisfied

$$(i) \max\{p_1, q_1\} > \lambda_{11}, \quad (ii) \max\{p_2, q_2\} > \lambda_{22},$$

then both w_1 and w_2 are admissible or none of w_1 and w_2 is admissible.

Theorem 3.4. Let w_1 and w_2 be meromorphic functions in \mathbb{C}^n . Let (w_1, w_2) be a set of zero order meromorphic solution of (3.3). If one of the following conditions is satisfied

$$(i) p_1 > \lambda_1, \quad q_2 > \overline{\lambda_2}, \quad (ii) p_2 > \lambda_2, \quad q_1 > \overline{\lambda_1},$$

then both w_1 and w_2 are admissible or none of w_1 and w_2 is admissible, where p_1 and p_2 are the highest degree of w_1 and w_2 in $R_1(z, w_1, w_2)$, we denote similarly q_1, q_2 in $R_2(z, w_1, w_2)$.

Example 3.3. For a system of complex partial q -difference equations

$$\begin{cases} \frac{w_1^2(-\frac{1}{2}z_1, -\frac{1}{2}z_2)w_2(\frac{1}{2}z_1, \frac{1}{2}z_2)}{w_1(3z_1, 3z_2) + w_2(-\sqrt{3}z_1, -\sqrt{3}z_2)} = \\ \frac{5w_1^3w_2^2 + 3w_1^2w_2^2 - w_1^2w_2 + \frac{2}{z_1z_2}w_1w_2^2 + 4w_1w_2 - w_2 + 1}{5 - \frac{21}{z_1z_2}w_1^3w_2^3 + 37w_1w_2^2 - 23z_1z_2w_1^2w_2^2 - 4w_1^2w_2}, \\ \frac{(1-z_1)(1-z_2)w_1^3(\frac{1}{\sqrt{2}}z_1, \frac{1}{\sqrt{2}}z_2)}{w_2(\sqrt{3}z_1, \sqrt{3}z_2)} = \\ \frac{(\frac{8}{z_2} + \frac{8}{z_1})w_1^4w_2 + 8w_1^3 - 8w_1^2}{3w_1^4w_2^3 - 7w_1^4w_2^2 + 2w_1^2w_2^2 + 5w_1^2w_2 + 4w_2 + 2}, \end{cases}$$

admits a non-admissible meromorphic solution $(w_1, w_2) = (-\frac{1}{z_1z_2}, z_1^2z_2^2)$. Clearly, we obtain

$$\lambda_1 = 2, \lambda_2 = 1, p_1 = 3, p_2 = 3, \quad \overline{\lambda_1} = 3, \overline{\lambda_2} = 1, q_1 = 4, q_2 = 3.$$

In this case

$$p_1 > \lambda_1, \quad q_2 > \overline{\lambda_2}, \quad p_2 > \lambda_2, \quad q_1 > \overline{\lambda_1}.$$

This example shows that Theorem 3.4 holds.

To prove theorems, we need some lemmas as follows.

Lemma 3.1. [2] Let $R(z, w(z)) = \frac{a_0(z) + a_1(z)w(z) + \cdots + a_{p'}(z)w^{p'}(z)}{b_0(z) + b_1(z)w(z) + \cdots + b_{q'}(z)w^{q'}(z)}$ be an irreducible rational function in $w(z)$ with the meromorphic coefficients $\{a_i(z)\}$ and $\{b_j(z)\}$. If $w(z)$ is a meromorphic function in \mathbb{C}^n , then

$$T(r, R(z, w)) = \max\{p', q'\}T(r, w) + O\left\{\sum T(r, a_i) + \sum T(r, b_j)\right\}.$$

Lemma 3.2. Let w_1 and w_2 be non-constant meromorphic functions of zero order in \mathbb{C}^n ,

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$q_i \in \mathbb{C}^n \setminus \{0\}, i = 1, \dots, n_2$. If

$$\Omega_1(z, w_1, w_2) = \sum_{(i) \in I} a_{(i)}(z) \prod_{k=1}^2 w_k^{i_{k0}} (w_k(q_1 z))^{i_{k1}} \cdots (w_k(q_{n_2} z))^{i_{kn_2}},$$

$\{a_{(i)}(z)\}$ is a small function of w_1 and w_2 . $\lambda_{1k} = \max\{\sum_{l=0}^{n_2} i_{kl}\} (k = 1, 2)$, then

$$T(r, \Omega_1(z, w_1, w_2)) \leq \lambda_{11}T(r, w_1) + \lambda_{12}T(r, w_2) + S(r, w_1) + S(r, w_2) + S(r).$$

Proof It is easy to prove by Theorem B.

As the proof of Theorem 2.1 in [18], we have

Lemma 3.3. Let w_1 and w_2 be nonconstant meromorphic functions in \mathbb{C}^n . If

$$\lim_{r \rightarrow \infty} \sup_{r \notin I_1} \frac{S(r)}{T(r, w_1)} = 0, T(r, w_2) = O\{S(r)\} (r \notin I_2),$$

then

$$\lim_{r \rightarrow \infty} \sup_{r \notin I_1 \cup I_2} \frac{T(r, w_2)}{T(r, w_1)} = 0,$$

where I_1, I_2 are both exceptional sets with upper logarithmic density 0.

Lemma 3.4. Let w_1 and w_2 be non-constant meromorphic functions of zero order in \mathbb{C}^n ,

$q_i \in \mathbb{C}^n \setminus \{0\}, i = 1, \dots, n_2$. Let $\Phi_1 = \frac{\Omega_1(z, w_1, w_2)}{\Omega_2(z, w_1, w_2)}$, $\{a_{(i)}(z)\}, \{b_{(j)}(z)\}$ are both small functions of w_1 and w_2 . If

$$\lambda_1 = \max\{\lambda_{11}, \lambda_{21}\}, \lambda_2 = \max\{\lambda_{12}, \lambda_{22}\},$$

then

$$T(r, \Phi_1) \leq \lambda_1 T(r, w_1) + \lambda_2 T(r, w_2) + S(r, w_1) + S(r, w_2).$$

Proof Let $\mathbb{C}^n < r > = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 = r^2\}$.

Firstly, we estimate $m(r, \Phi_1)$. Set

$$u(z) = \max\{|\Omega_1(z, w_1, w_2)|, |\Omega_2(z, w_1, w_2)|\},$$

we have

$$\log^+ |\Phi_1| = \log u(z) - \log |\Omega_2(z, w_1, w_2)|,$$

thus

$$\int_{\mathbb{C}^n < r >} \log^+ |\Phi_1| \sigma_n = \int_{\mathbb{C}^n < r >} \log u(z) \sigma_n - \int_{\mathbb{C}^n < r >} \log |\Omega_2(z, w_1, w_2)| \sigma_n.$$

As the proof of Lemma 3.3 in [18], and using Theorem A and Theorem B, we have

$$T(r, \Phi_1) \leq \lambda_1 T(r, w_1) + \lambda_2 T(r, w_2) + S(r, w_1) + S(r, w_2).$$

This completes the proof of Lemma 3.4.

Proof of Theorem 3.1 Let w be a meromorphic function in \mathbb{C}^n , and $w(z)$ be a transcendental meromorphic solution of zero order of (3.1). It follows from Lemma 3.1 and Theorem B

that

$$\begin{aligned}\max\{p, q\}T(r, w(z)) &= T(r, R(z, w)) + S(r, w) \\ &= T(r, \sum_{j=1}^{n_2} w(q_j z)) + S(r, w) \\ &\leq n_2 T(r, w) + S(r, w).\end{aligned}$$

Thus, we have

$$\max\{p, q\} \leq n_2.$$

This completes the proof of Theorem 3.1.

Proof of Theorem 3.2 Let w_1 and w_2 be meromorphic functions in \mathbb{C}^n . Let (w_1, w_2) be a set of admissible meromorphic function of (3.2). From the first and the second equation of (3.2), and also using Lemma 3.1 and Lemma 3.2, we obtain

$$\max\{p_1, q_1\}T(r, w_1) \leq \lambda_{11}T(r, w_1) + \lambda_{12}T(r, w_2) + S(r, w_1) + S(r, w_2) + S(r). \quad (3.4)$$

$$\max\{p_2, q_2\}T(r, w_2) \leq \lambda_{21}T(r, w_1) + \lambda_{22}T(r, w_2) + S(r, w_1) + S(r, w_2) + S(r). \quad (3.5)$$

By (3.4) and (3.5), we have

$$[\max\{p_1, q_1\} - \lambda_{11} + o(1)]T(r, w_1) \leq (\lambda_{12} + o(1))T(r, w_2). \quad (3.6)$$

$$[\max\{p_2, q_2\} - \lambda_{22} + o(1)]T(r, w_2) \leq (\lambda_{21} + o(1))T(r, w_1). \quad (3.7)$$

Combining (3.6) and (3.7), we obtain

$$[\max\{p_1, q_1\} - \lambda_{11}][\max\{p_2, q_2\} - \lambda_{22}] \leq \lambda_{12}\lambda_{21}.$$

This completes the proof of Theorem 3.2.

Proof of Theorem 3.3 Let w_1 and w_2 be nonconstant meromorphic functions of zero order in \mathbb{C}^n . It follows from Lemma 3.1 and Lemma 3.2 that

$$\max\{p_1, q_1\}T(r, w_1) \leq \lambda_{11}T(r, w_1) + \lambda_{12}T(r, w_2) + S(r, w_1) + S(r, w_2) + S(r). \quad (3.8)$$

$$\max\{p_2, q_2\}T(r, w_2) \leq \lambda_{21}T(r, w_1) + \lambda_{22}T(r, w_2) + S(r, w_1) + S(r, w_2) + S(r). \quad (3.9)$$

If w_1 is admissible and w_2 is non-admissible, then the inequality (3.8) becomes

$$\max\{p_1, q_1\} \leq \lambda_{11} + (\lambda_{12} + o(1))\frac{T(r, w_2)}{T(r, w_1)} + \frac{S(r)}{T(r, w_1)},$$

using Lemma 3.3, we get

$$\max\{p_1, q_1\} \leq \lambda_{11},$$

outside of a set with upper logarithmic density 0. It is in contradiction with the condition (i).

If w_2 is admissible and w_1 is non-admissible, then the inequality (3.9) becomes

$$\max\{p_2, q_2\} \leq \lambda_{22} + (\lambda_{21} + o(1))\frac{T(r, w_1)}{T(r, w_2)} + \frac{S(r)}{T(r, w_2)},$$

using Lemma 3.3, we get

$$\max\{p_2, q_2\} \leq \lambda_{22},$$

outside of a set with upper logarithmic density 0. It is in contradiction with the condition (ii).

Thus both w_1 and w_2 are admissible or none of w_1 and w_2 is admissible.

This proves Theorem 3.3.

Proof of Theorem 3.4 Let w_1 and w_2 be meromorphic functions in \mathbb{C}^n . Let (w_1, w_2) be

a zero order meromorphic solution of (3.3). Using Lemma 3.1 and Lemma 3.4, we obtain

$$p_1 T(r, w_1) + O\{T(r, w_2)\} \leq \lambda_1 T(r, w_1) + \lambda_2 T(r, w_2) + S(r, w_1) + S(r, w_2) + S(r). \quad (3.10)$$

$$p_2 T(r, w_2) + O\{T(r, w_1)\} \leq \lambda_1 T(r, w_1) + \lambda_2 T(r, w_2) + S(r, w_1) + S(r, w_2) + S(r). \quad (3.11)$$

$$q_1 T(r, w_1) + O\{T(r, w_2)\} \leq \overline{\lambda}_1 T(r, w_1) + \overline{\lambda}_2 T(r, w_2) + S(r, w_1) + S(r, w_2) + S(r). \quad (3.12)$$

$$q_2 T(r, w_2) + O\{T(r, w_1)\} \leq \overline{\lambda}_1 T(r, w_1) + \overline{\lambda}_2 T(r, w_2) + S(r, w_1) + S(r, w_2) + S(r). \quad (3.13)$$

If w_1 is admissible and w_2 is non-admissible, then the inequality (3.10) becomes

$$p_1 + \frac{O\{T(r, w_2)\}}{T(r, w_1)} \leq \lambda_1 + (\lambda_2 + o(1)) \frac{T(r, w_2)}{T(r, w_1)} + \frac{S(r)}{T(r, w_1)}.$$

Using Lemma 3.3, we get

$$p_1 \leq \lambda_1,$$

outside of a set with upper logarithmic density 0. It is in contradiction with the first inequality of (i).

If w_2 is admissible and w_1 is non-admissible, then the inequality (3.13) becomes

$$q_2 + \frac{O\{T(r, w_1)\}}{T(r, w_2)} \leq \overline{\lambda}_2 + (\overline{\lambda}_1 + o(1)) \frac{T(r, w_1)}{T(r, w_2)} + \frac{S(r)}{T(r, w_2)}.$$

Using Lemma 3.3, we get

$$q_2 \leq \overline{\lambda}_2,$$

outside of a set with upper logarithmic density 0. It is in contradiction with the second inequality of (i).

Similarly, we can prove for conditions (ii).

Thus both w_1 and w_2 are admissible or none of w_1 and w_2 is admissible.

This completes the proof of Theorem 3.4.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors drafted the manuscript, read and approved the final manuscript.

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Nonstationary refinable functions based on generalized Bernstein polynomials

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Abstract In this paper, we introduce a new family of nonstationary refinable functions from Generalized Bernstein Polynomials, which include a class of nonstationary refinable functions generated from the family of masks for the pseudo splines of type II (see [17]). Furthermore, a proof of the convergence of nonstationary cascade algorithms associated with the new masks is completed. We then construct symmetric compacted supported nonstationary C^∞ tight wavelet frames in $L_2(\mathbb{R})$ with the spectral frame approximation order.

Keywords Nonstationary tight wavelet frames; Nonstationary refinable functions; Nonstationary cascade algorithms; Generalized Bernstein Polynomials; Spectral frame approximation order.

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1 Introduction

In frame systems, because tight wavelet frames (in the stationary case) can not satisfy compactly supported C^∞ properties, the nonstationary case was considered to obtain C^∞ tight wavelet frames with compacted support. Recently, the development of nonstationary tight wavelet frames has attracted a considerable amount of attention.

In 2008, Han and Shen [17] obtained symmetric compactly supported C^∞ tight wavelet frames in $L_2(\mathbb{R})$ with the spectral frame approximation order based on pseudo-splines of type II. In 2009, compactly supported nonstationary C^∞ tight wavelet frames in $L_2(\mathbb{R}^s)$ with the spectral frame approximation order from pseudo box splines were constructed in [22]. Li and

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Shen [22] generalized univariate pseudo-splines to the multivariate setting and got a new class of refinable functions named pseudo box splines. Next, in [18] and [23], the analysis of characterization of nonstationary tight wavelet frames in Sobolev spaces was given. Han and Shen [18] characterized Sobolev spaces of an arbitrary order of smoothness using nonstationary tight wavelet frames for $L_2(\mathbb{R}^s)$. Also, approximation order of nonstationary tight wavelet frames in Sobolev spaces was obtained in [23]. Recently, the nonstationary subdivision scheme, which nonstationary cascade algorithms is closely related to, was studied in [2–10, 12, 15, 21, 24, 26]. In particular, in [14] and [20], the properties of nonstationary subdivision scheme were performed. Daniel et al.[14] and Jeonga et al.[20] showed C^2 approximating and Hölder regularities of nonstationary subdivision scheme, respectively.

This paper is concerned with the study of symmetric C^∞ nonstationary tight wavelet frames in $L_2(\mathbb{R})$ with compacted support and the spectral frame approximation order, which generalize nonstationary tight wavelet frames from pseudo-splines of type II in [17]. We discover a new extensive function based on Generalized Bernstein polynomials [1]. Furthermore, existence of L_2 -solutions of nonstationary refinable functions from the new extensive function is implemented. At last, we prove the convergence of nonstationary cascade algorithms of the new family of nonstationary refinable functions.

The remainder of this paper is organized as follows: Section 2 collects some notations. Section 3 elaborates on existence of L_2 -solutions of nonstationary refinable functions. Section 4 implements convergence of nonstationary cascade algorithms. Section 5 constructs symmetric C^∞ nonstationary tight wavelet frames in $L_2(\mathbb{R})$ with compacted support and the spectral frame approximation order. Section 6 gives the conclusion.

2 Preliminaries

For the convenience of the readers, we review some definitions about nonstationary refinable functions in this section.

Generalized Bernstein polynomials [1] are defined as

$$S_k^{(n)}(t) = \binom{n}{k} \frac{t(t+\alpha) \cdots (t+[k-1]\alpha)(1-t)(1-t+\alpha) \cdots (1-t+[n-k-1]\alpha)}{(1+\alpha)(1+2\alpha) \cdots (1+[n-1]\alpha)}, \quad (2.1)$$

where $\alpha \geq 0$. We apply (2.1) to marks of new refinable functions by substituting $t = \sin^2(\frac{\omega}{2})$, $n = m + l$ in (2.1) and the summation of $l + 1$ terms of them as follows:

$$\tau_0^{m,l,\alpha}(\omega) := \sum_{k=0}^l \binom{m+l}{k} \left(\prod_{i=0}^{k-1} (\sin^2(\frac{\omega}{2}) + i\alpha) \prod_{i=0}^{m+l-k-1} (\cos^2(\frac{\omega}{2}) + i\alpha) \right) / \prod_{i=1}^{m+l-1} (1+i\alpha). \quad (2.2)$$

Let $\tau_{0,j}^{m,l,\alpha}(\omega) = \tau_0^{m_j,l_j,\alpha_j}(\omega)$ ($j \in \mathbb{N}$) be defined in (2.2), we obtain

$$\tau_{0,j}^{m,l,\alpha}(\omega) := \sum_{k=0}^{l_j} \binom{m_j+l_j}{k} \left(\prod_{i=0}^{k-1} (\sin^2(\frac{\omega}{2}) + i\alpha_j) \prod_{i=0}^{m_j+l_j-k-1} (\cos^2(\frac{\omega}{2}) + i\alpha_j) \right) / \prod_{i=1}^{m_j+l_j-1} (1+i\alpha_j), \quad (2.3)$$

where two positive integers l_j , m_j and α_j ($j \in \mathbb{N}$) satisfy $l_j < m_j - 5$, $\sum_{j=1}^{\infty} 2^{-j} m_j < \infty$, $\lim_{j \rightarrow \infty} m_j = \infty$ and $0 \leq \alpha_j < \frac{1}{3(m_j + l_j) - 7}$.

A class of 2π -periodic trigonometric polynomials $\hat{a}_j, j \in \mathbb{N}$, and their associated nonstationary refinable functions $\hat{\phi}_{j-1}, j \in \mathbb{N}$, defined by

$$\hat{\phi}_{j-1}(\omega) := \hat{a}_j(\omega/2) \hat{\phi}_j(\omega/2) = \prod_{n=1}^{\infty} \hat{a}_{n+j-1}(2^{-n}\omega) \quad \omega \in \mathbb{R}, j \in \mathbb{N}, \quad (2.4)$$

where the 2π -periodic trigonometric polynomials $\hat{a}_j, j \in \mathbb{N}$, are called refinement masks. Here the Fourier transform \hat{f} of a function $f \in L_1(\mathbb{R})$ is defined to be $\hat{f}(\omega) := \int_{\mathbb{R}} f(t) e^{-it\omega} dt$ and can be naturally extended to square integrable functions.

The notation \mathbb{T} was introduced in [25], which is defined by

$$\mathbb{T} := \mathbb{R}/[2\pi\mathbb{Z}].$$

Denote $\deg(\hat{a})$ the smallest nonnegative integer such that its Fourier coefficients of \hat{a} vanish outside $[-\deg(\hat{a}), \deg(\hat{a})]$. $\deg(\hat{a})$ here is the minimal integer k such that $[-k, k]$ contains the support of the Fourier coefficients of both \hat{a} and $\hat{a}(-\cdot)$, which was introduced in [17].

In the following, we will adopt some of the notations from [19]. The transition operator $T_{\hat{a}}$ for 2π -periodic functions \hat{a} and f can be defined as

$$[T_{\hat{a}} f](\omega) := |\hat{a}(\omega/2)|^2 f(\omega/2) + |\hat{a}(\omega/2 + \pi)|^2 f(\omega/2 + \pi), \quad \omega \in \mathbb{R}.$$

For $\tau \in \mathbb{R}$, a quantity is defined by

$$\rho_{\tau}(\hat{a}, \infty) := \limsup_{n \rightarrow \infty} \left\| T_{\hat{a}}^n \left(\left| \sin \left(\frac{\omega}{2} \right) \right|^{\tau} \right) \right\|_{L_{\infty}(\mathbb{T})}^{1/n}.$$

The notation $\rho(\hat{a})$ is defined by

$$\rho(\hat{a}) := \inf \{ \rho_{\tau}(\hat{a}, \infty) : |\hat{a}(\omega + \pi)|^2 |\sin(\omega/2)|^{\tau} \in L_{\infty}(\mathbb{T}) \text{ and } \tau \geq 0 \}.$$

A function $f \in W_2^{\nu}(\mathbb{R})$ if it satisfies

$$\|f\|_{W_2^{\nu}(\mathbb{R})}^2 := \int_{\mathbb{R}} (1 + |\omega|^{2\nu}) |\hat{f}(\omega)|^2 d\omega < \infty.$$

As [17], let $\{\hat{a}_j\}_{j=1}^{\infty}$ be a sequence of 2π -periodic measurable functions. Define $\{f_n\}_{n=1}^{\infty}$ by

$$\hat{f}_n(\omega) := \chi_{[-\pi, \pi]}(2^{-n}\omega) \prod_{j=1}^n \hat{a}_j(2^{-n}\omega), \quad \omega \in \mathbb{R}, n \in \mathbb{N}, \quad (2.5)$$

where $\chi_{[-\pi, \pi]}$ denotes the characteristic function of the interval $[-\pi, \pi]$. This can be understood as a representation of the nonstationary cascade algorithm associated with the masks $\{\hat{a}_j\}_{j=1}^{\infty}$ in the frequency domain. For a sequence of masks $\{\hat{a}_j\}_{j=1}^{\infty}$ and an initial function $f \in W_2^{\nu}(\mathbb{R})$, we say that the (nonstationary) cascade algorithm associated with masks $\{\hat{a}_j\}_{j=1}^{\infty}$ and an initial

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function f converges in the Sobolev space $W_2'(\mathbb{R})$, if $f_n \in W_2'(\mathbb{R})$ for all $n \in \mathbb{N}$ and the sequence $\{f_n\}_{j=1}^\infty$ is convergent in $W_2'(\mathbb{R})$.

Denote $\rho(\hat{a})$ the spectral radius of the square matrix $(c_{2j-k})_{-K \leq j, k \leq K}$ and define $\nu_2(\hat{a}) := -1/2 - \log_2 \sqrt{\rho(\hat{a})}$. It is known ([16], Theorem 4.3 and Proposition 7.2] and [[19], Theorem 2.1]) that the stationary cascade algorithm associated with a 2π -periodic trigonometric polynomial mask \hat{a} converges in $f \in W_2'(\mathbb{R})$ if and only if $\nu_2(\hat{a}) > \nu$.

For a sequence $\{\phi_n\}_{n=0}^\infty$ of functions in $L_2(\mathbb{R})$, we define the linear operators $P_n(f)$, $n \in \mathbb{N}_0$, by

$$P_n(f) := \sum_{k \in \mathbb{Z}} \langle f, \phi_{n;n,k} \rangle \phi_{n;n,k}, \quad f \in L_2(\mathbb{R}) \quad \text{with} \quad \phi_{n;n,k} := 2^{n/2} \phi_n(2^n \cdot -k). \quad (2.6)$$

Wavelet functions ψ_{j-1}^ℓ , $j \in \mathbb{N}$ and $\ell \in \{1, \dots, \mathcal{J}_j\}$, are obtained from ϕ_j by

$$\widehat{\psi_{j-1}^\ell}(\omega) := \widehat{b_j^\ell}(\omega/2) \widehat{\phi_j}(\omega/2), \quad \ell \in \{1, \dots, \mathcal{J}_j\}, \quad (2.7)$$

where \mathcal{J}_j are positive integers and each b_j^ℓ , $\ell = 1, \dots, \mathcal{J}_j$, is called a (high-pass) wavelet mask. Denote $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We say that $\{\phi_0\} \cup \{\psi_j^\ell : j \in \mathbb{N}_0, \ell = 1, \dots, \mathcal{J}_{j+1}\}$ generates a nonstationary tight wavelet frame in $L_2(\mathbb{R})$ if

$$\{\phi_0(\cdot - k) : k \in \mathbb{Z}\} \cup \{\psi_{j;j,k}^\ell := 2^{j/2} \psi_j^\ell(2^j \cdot -k) : j \in \mathbb{N}_0, k \in \mathbb{Z}, \ell = 1, \dots, \mathcal{J}_{j+1}\} \quad (2.8)$$

is a tight frame of $L_2(\mathbb{R})$.

Finally, we note that the 2π -periodic trigonometric polynomial wavelet masks $\widehat{b_j^\ell}$, $j \in \mathbb{N}$ and $\ell \in \{1, \dots, \gamma\}$, can be constructed from the masks $\widehat{a_j}$ by many ways provided that the refinement masks $\widehat{a_j}$, $j \in \mathbb{N}$, satisfy $|\widehat{a_j}(\omega)|^2 + |\widehat{a_j}(\omega + \pi)|^2 \leq 1$, a.e. $\omega \in \mathbb{R}$. Define

$$\begin{aligned} \widehat{b_j^1}(\omega) &:= e^{-i\omega} \overline{\widehat{a_j}(\omega + \pi)}, \\ \widehat{b_j^2}(\omega) &:= 2^{-1} [A_j(\omega) + e^{-i\omega} \overline{A_j(\omega)}], \\ \widehat{b_j^3}(\omega) &:= 2^{-1} [A_j(\omega) + e^{-i\omega} \overline{A_j(\omega)}], \end{aligned} \quad (2.9)$$

where A_j is a π -periodic trigonometric polynomial with real coefficients such that

$$|A_j(\omega)|^2 = 1 - |\widehat{a_j}(\omega)|^2 - |\widehat{a_j}(\omega + \pi)|^2.$$

Then, $\widehat{a_j}$, $\widehat{b_j^1}$, $\widehat{b_j^2}$ and $\widehat{b_j^3}$, $j \in \mathbb{N}$, satisfy

$$|\widehat{a_j}(\omega)|^2 + \sum_{\ell=1}^{\mathcal{J}_j} |\widehat{b_j^\ell}(\omega)|^2 = 1 \quad \text{and} \quad \widehat{a_j}(\omega) \overline{\widehat{a_j}(\omega + \pi)} + \sum_{\ell=1}^{\mathcal{J}_j} \widehat{b_j^\ell}(\omega) \overline{\widehat{b_j^\ell}(\omega + \pi)} = 0, \quad (2.10)$$

with $\mathcal{J} = 3$. Thus, the wavelet system in (2.8) is a compactly supported tight wavelet frame in $L_2(\mathbb{R})$ (see, [17], Theorem 1.1). Furthermore, the corresponding wavelets defined by (2.7) using masks in (2.9) are symmetric or antisymmetric whenever ϕ_j is symmetric.

3 Existence of L_2 -solutions of nonstationary refinable functions

In this section, demonstration of the existence of L_2 -solutions of nonstationary refinable functions is given. For notational simplicity, we will introduce the following two definitions:

$$B_{k,j}(\omega) := \left(\prod_{i=0}^{k-1} \left(\sin^2\left(\frac{\omega}{2}\right) + i\alpha_j \right) \prod_{i=1}^{m_j+l_j-k-1} \left(\cos^2\left(\frac{\omega}{2}\right) + i\alpha_j \right) \right) / \prod_{i=1}^{m_j+l_j-1} (1 + i\alpha_j), j \in \mathbb{N}.$$

$$T_{0,j}(\omega) := \sum_{k=0}^l \binom{m_j+l_j}{k} \left(\prod_{i=0}^{k-1} \left(\sin^2\left(\frac{\omega}{2}\right) + i\alpha_j \right) \prod_{i=1}^{m_j+l_j-k-1} \left(\cos^2\left(\frac{\omega}{2}\right) + i\alpha_j \right) \right) / \prod_{i=1}^{m_j+l_j-1} (1+i\alpha_j), j \in \mathbb{N}.$$

Two lemmas about the relations of the quantities $\rho_\tau(\tau_{0,j}^{m,l,\alpha}(\omega), \infty), j \in \mathbb{N}$ associated with masks (2.3) will be provided in the following.

Lemma 3.1 ([19], Theorem 4.1) *Let \hat{a} be a 2π -periodic measurable function such that $|\hat{a}|^2 \in C^\beta(\mathbb{T})$ with $|\hat{a}|^2(0) \neq 0$ and $\beta > 0$. If $|\hat{a}(\omega)|^2 = |1 + e^{-i\omega}|^{2\tau} |\hat{A}(\omega)|^2$ a.e. $\omega \in \mathbb{R}$ for some $\tau \geq 0$ such that $\hat{A}(\omega) \in L_\infty(\mathbb{T})$, then*

$$\rho_{2\tau}(\hat{a}, \infty) = \inf_{n \in \mathbb{N}} \|T_{\hat{a}}^n 1\|_{L_\infty(\mathbb{T})}^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \|T_{\hat{a}}^n 1\|_{L_\infty(\mathbb{T})}^{\frac{1}{n}} = \rho_0(\hat{A}, \infty).$$

Lemma 3.2 ([19], Theorem 4.3) *Let \hat{a} and \hat{c} be 2π -periodic measurable functions such that*

$$|\hat{a}(\omega)| \leq |\hat{c}(\omega)|$$

for almost every $\omega \in \mathbb{R}$. Then

$$\rho_\tau(\hat{a}, \infty) \leq \rho_\tau(\hat{c}, \infty), \quad \tau \in \mathbb{R}.$$

The following two lemmas are necessary for proving existence of L_2 -solutions of nonstationary refinable functions.

Lemma 3.3 ([17], Lemma 2.1) *Let $\hat{a}_j, j \in \mathbb{N}$ be a 2π -periodic trigonometric polynomials such that $\sup_{j \in \mathbb{N}} \|\hat{a}_j\|_{L_\infty(\mathbb{R})} < \infty$. If $\sum_{j=1}^\infty 2^{-j} \deg(\hat{a}_j) < \infty$ holds and $\sum_{j=1}^\infty |\hat{a}_j(0) - 1| < \infty$, then the infinite product (2.4) converges uniformly on every compact set of \mathbb{R} and all $\phi_j, j \in \mathbb{N}_0$, in (2.4) are well-defined compactly supported tempered distributions.*

Lemma 3.4 ([17], Lemma 2.2) *Let $\hat{a}_j, j \in \mathbb{N}$, be 2π -periodic measurable functions satisfying $|\hat{a}_j(\omega)|^2 + |\hat{a}_j(\omega + \pi)|^2 \leq 1$, a.e. $\omega \in \mathbb{R}$ for each $j \in \mathbb{N}$. Assume that, for every $j \in \mathbb{N}_0$, $\widehat{\phi}_j(\omega) := \lim_{N \rightarrow \infty} \prod_{n=1}^N \widehat{a_{n+j}}(2^{-n}\omega)$ is well defined for almost every $\omega \in \mathbb{R}$; that is, the infinite product in (2.4) exists for almost every point in \mathbb{R} . Then $[\widehat{\phi}_j, \widehat{\phi}_j](\omega) := \sum_{k \in \mathbb{Z}} |\widehat{\phi}_j(\omega + 2\pi k)|^2 \leq 1$, a.e. $\omega \in \mathbb{R} \forall j \in \mathbb{N}_0$ holds and consequently, $\phi_j \in L_2(\mathbb{R})$ with $\|\phi_j\|_{L_2(\mathbb{R})} \leq 1$ for every $j \in \mathbb{N}_0$.*

A useful condition of establishing existence of L_2 -solutions of nonstationary refinable functions is described in the following lemma.

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Lemma 3.5 For two positive integers l_j, m_j , $l_j < m_j - 5, j \in \mathbb{N}$, if

$$0 \leq \alpha_j < \frac{1}{3(m_j + l_j) - 7} \quad (j \in \mathbb{N}), \quad (3.1)$$

then

$$\max_{\omega \in \mathbb{T}} B_{k,j}(\omega) \leq \left(\frac{1}{2}\right)^{m_j+l_j-1}, \quad k = 1, 2, \dots, l_j, \quad (3.2)$$

Proof. For $j \in \mathbb{N}, k = 1, 2, \dots, l_j$, it is obvious that

$$B_{k,j}(\omega) = \frac{\cos^2(\frac{\omega}{2}) + (m_j + l_j - 1 - j)\alpha_j}{\sin^2(\frac{\omega}{2}) + j\alpha_j} B_{k+1,j}(\omega).$$

We claim that

$$\frac{B_{k,j}(\omega)}{B_{k+1,j}(\omega)} = \frac{\cos^2(\frac{\omega}{2}) + (m_j + l_j - 1 - j)\alpha_j}{\sin^2(\frac{\omega}{2}) + j\alpha_j} > 1. \quad (3.3)$$

Since $l_j < m_j - 5$, for $k = 1, 2, \dots, l_j$, it holds that

$$k < m_j + l_j - 1 - k. \quad (3.4)$$

There are two cases to consider:

Case I: Suppose that $\cos(\omega) \geq 0$. By (3.1) and (3.4), it is easy to see that

$$\alpha_j > 0 > \frac{-\cos(\omega)}{m_j + l_j - 1 - 2k}.$$

Then

$$\cos^2(\frac{\omega}{2}) + (m_j + l_j - 1 - k)\alpha_j > \sin^2(\frac{\omega}{2}) + k\alpha_j. \quad (3.5)$$

This implies Condition (3.3).

Case II: Suppose that $\cos(\omega) < 0$. Note that since

$$\frac{(2^{2l_j} - 2^{-1})(m_j - 1 - l_j) - l_j}{l_j(m_j - 1 - l_j)} > \frac{0.5(m_j - 1 - l_j) - l_j}{l_j(m_j - 1 - l_j)} > 0,$$

we can obtain that

$$\begin{aligned} \frac{2^{m_j+l_j-1} - 2^{-1}}{l_j} - \frac{1}{m_j - l_j - 1} &= \frac{(2^{m_j+l_j-1} - 2^{-1})(m_j - 1 - l_j) - l_j}{l_j(m_j - 1 - l_j)} \\ &> \frac{(2^{2l_j} - 2^{-1})(m_j - 1 - l_j) - l_j}{l_j(m_j - 1 - l_j)} > 0. \end{aligned}$$

By (3.1), then

$$\alpha_j \geq \frac{2^{m_j+l_j-1} - 2^{-1}}{l_j} > \frac{1}{m_j - l_j - 1} = \frac{1}{m_j + l_j - 1 - 2l_j} \geq \frac{|\cos(\omega)|}{m_j + l_j - 1 - 2k} = \frac{-\cos(\omega)}{m_j + l_j - 1 - 2k},$$

for $k = 1, 2, \dots, l_j$. Then (3.5) holds. This concludes the claim (3.3).

By using (3.1), one gets

$$(4(m_j + l_j - 2) - (m_j + l_j - 1))\alpha_j < \frac{4(m_j + l_j - 2) - (m_j + l_j - 1)}{3(m_j + l_j) - 7} = 1.$$

Then

$$\frac{(m_j + l_j - 2)\alpha_j}{(1 + \alpha_j)(1 + (m_j + l_j - 1)\alpha_j)} < \frac{(m_j + l_j - 2)\alpha_j}{1 + (m_j + l_j - 1)\alpha_j} < \frac{1}{4}. \quad (3.6)$$

Since $l_j < m_j - 5$, we have

$$(3(m_j + l_j) - 7) - (m_j + l_j - 4) = 2m_j + 2l_j - 3 > 0.$$

Then

$$\frac{1}{3(m_j + l_j) - 7} < \frac{1}{m_j + l_j - 4}.$$

Thus

$$(2(m_j + l_j - 3) - (m_j + l_j - 2))\alpha_j < \frac{2(m_j + l_j - 3) - (m_j + l_j - 2)}{m_j + l_j - 4} = 1.$$

Similarly, one has

$$\frac{(m_j + l_j - 3)\alpha_j}{1 + (m_j + l_j - 2)\alpha_j} < \frac{1}{2}. \quad (3.7)$$

For any x , Notice that

$$\left(\frac{x}{1 + (1 + x)}\right)' > 0 \quad (3.8)$$

and $B_{1,j}(\omega)$ which is a continuous function on $[-\pi, \pi]$ and is differentiable on $(-\pi, \pi)$, has the maximum value at $\omega = \pi$. The reason as follow:

The equation $[B_{1,j}(\omega)]' = 0$ has three zeros, at $\omega = 0, \pm\pi$. Since $[B_{1,j}(\omega)]'' > 0$, $B_{1,j}(0)$ is the minimum of $B_{1,j}(\omega)$ on $[-\pi, \pi]$. Thus $B_{1,j}(\pm\pi)$ is the maximum of $B_{1,j}(\omega)$ on $[-\pi, \pi]$. Therefore, applying (3.3), (3.6), (3.7), (3.8) and

$$\begin{aligned} B_{1,j}(\omega) &= \left(\sin^2\left(\frac{\omega}{2}\right) \prod_{i=1}^{m_j+l_j-2} (\cos^2\left(\frac{\omega}{2}\right) + i\alpha_j) \right) / \prod_{i=1}^{m_j+l_j-1} (1 + i\alpha_j) \\ &\leq \prod_{i=1}^{m_j+l_j-2} i\alpha_j / \prod_{i=1}^{m_j+l_j-1} (1 + i\alpha_j) \\ &\leq \prod_{i=1}^{m_j+l_j-3} \frac{i\alpha_j}{1 + (i+1)\alpha_j} \cdot \frac{(m_j + l_j - 2)\alpha_j}{(1 + \alpha_j)(1 + (m_j + l_j - 1)\alpha_j)} \\ &\leq \left(\frac{(m_j + l_j - 3)\alpha_j}{1 + (m_j + l_j - 2)\alpha_j} \right)^{m_j+l_j-3} \cdot \frac{1}{4} \\ &= \left(\frac{1}{2} \right)^{m_j+l_j-1}, \end{aligned}$$

we get the inequality (3.2). ◻

Theorem 3.1 Let $\tau_{0,j}^{m,l,\alpha}(\omega)$ be the mark (2.3), which are defined in (2.4), then the infinite product in (2.4) converges uniformly on every compact set of \mathbb{R} .

Proof. Since $\tau_{0,j}^{m,l,\alpha}(\omega) = \tau_{0,j}^{m,l,\alpha}(\omega + 2\pi)$, $j \in \mathbb{N}$, we obtain $\tau_{0,j}^{m,l,\alpha}(\omega)$ are 2π -periodic trigono-

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metric polynomials. Applying Lemma 3.5, one has

$$\begin{aligned}
 |\tau_{0,j}^{m,l,\alpha}(\omega)| &= \left| \sum_{k=0}^{l_j} \binom{m_j+l_j}{k} \left(\prod_{i=0}^{k-1} \left(\sin^2\left(\frac{\omega}{2}\right) + i\alpha_j \right) \prod_{i=0}^{m_j+l_j-k-1} \left(\cos^2\left(\frac{\omega}{2}\right) + i\alpha_j \right) \right) / \prod_{i=1}^{m_j+l_j-1} (1+i\alpha_j) \right| \\
 &\leq \left| 1 + \left(\max_{\omega \in \mathbb{T}} B_{k,j}(\omega) \right) \sum_{k=1}^{l_j} \binom{m_j+l_j}{k} \right| \\
 &= 1 + \left(\frac{1}{2} \right)^{m_j+l_j-1} \left| \sum_{k=1}^{l_j} \binom{m_j+l_j}{k} \right| \\
 &\leq 1 + \left(\frac{1}{2} \right)^{m_j+l_j-1} \cdot 2^{m_j+l_j} = 3.
 \end{aligned}$$

Thus, there exists $M = 3$, for any $j \in \mathbb{N}$, we derive that $|\tau_{0,j}^{m,l,\alpha}(\omega)| \leq 3$ holds.

For $\tau_{0,j}^{m,l,\alpha}(0) = 1$, we obtain

$$\sum_{j=1}^{\infty} |\tau_{0,j}^{m,l,\alpha}(0) - 1| = 0 < \infty.$$

By using $l_j < m_j - 5$, $\sum_{j=1}^{\infty} 2^{-j} m^j < \infty$, ones get

$$\begin{aligned}
 \sum_{j=1}^{\infty} 2^{-j} \deg(\tau_{0,j}^{m,l,\alpha}(\omega)) &= \sum_{j=1}^{\infty} 2^{-j} (2(m_j + l_j) + 1) \\
 &< \sum_{j=1}^{\infty} 2^{-j} (4m_j - 9) < \infty.
 \end{aligned}$$

Therefore, by Lemma 3.3, the infinite product in (2.4) converges uniformly on every compact set of \mathbb{R} . \P

Theorem 3.2 Let $\tau_{0,j}^{m,l,\alpha}(\omega)$ be the mark (2.3), which are defined in (2.4), then the corresponding nonstationary refinable functions $\phi_j \in L_2(\mathbb{R})$, $j \in \mathbb{N}_0$.

Proof. By Theorem 3.1, we obtain that

$$\widehat{\phi_j}(\omega) = \lim_{N \rightarrow \infty} \prod_{n=1}^N \tau_{0,n+j-1}^{m,l,\alpha}(\omega) (2^{-n} \omega)$$

is well defined for almost every $\omega \in \mathbb{R}$. In the following, we claim that

$$|\tau_{0,j}^{m,l,\alpha}(\omega)|^2 + |\tau_{0,j}^{m,l,\alpha}(\omega + \pi)|^2 \leq 1, a.e. \omega \in \mathbb{R}. \quad (3.9)$$

There are two cases to consider:

Case I: Suppose that $\omega = 0$. One has

$$|\tau_{0,j}^{m,l,\alpha}(0)|^2 + |\tau_{0,j}^{m,l,\alpha}(\pi)|^2 = 0 + 1 = 1, a.e. \omega \in \mathbb{R}.$$

Case II: Suppose that $\omega \neq 0$. Set $E_0 = \{0\}$, for $\omega \in \mathbb{R} \setminus E_0$, let $t = 2\omega$, ones get

$$\begin{aligned}
 |\tau_{0,j}^{m,l,\alpha}(\omega)|^2 &= 2^{-4} (1 + e^{-it})^4 |T_{0,j}(t)|^2 \\
 &= (1 + e^{-it})^4 |2^{-2} T_{0,j}(t)|^2.
 \end{aligned}$$

Applying

$$\begin{aligned} B_{0,j}(t) &= \prod_{i=0}^{m+l-1} (\cos^2(\frac{t}{2}) + i\alpha_j) / \prod_{i=1}^{m+l-1} (1 + i\alpha_j) \\ &\leq \prod_{i=1}^{m+l-1} (1 + i\alpha_j) / \prod_{i=1}^{m+l-1} (1 + i\alpha_j) = 1 \end{aligned}$$

and Lemma 3.5, we obtain

$$\begin{aligned} \max_{t \in \mathbb{T}_{\nu,1}} 2|2^{-2}T_0^{m,l,\alpha}(t)|^2 &= \max_{t \in \mathbb{T}} 2^{-3} \left| B_{0,j}(t) + \sum_{j=0}^l \binom{m+l}{j} B_{k,j}(t) \right|^2 \\ &< \max_{t \in \mathbb{T}} 2^{-3} \left| 1 + \left(\max_{t \in [-\pi, \pi]} B_{k,j}(t) \right) \sum_{k=1}^l \binom{m+l}{k} \right|^2 \\ &< \max_{t \in \mathbb{T}} 2^{-3} \left| \left(1 + \left(\frac{1}{2} \right)^{m+l-1} (2)^{m+l-1} \right) \right|^2 = \frac{1}{2}. \end{aligned} \quad (3.10)$$

Bringing Lemma 3.1 and Lemma 3.2 together yields

$$\rho(\tau_{0,j}^{m,l,\alpha}(t)) \leq \rho_4(\tau_{0,j}^{m,l,\alpha}(t), \infty) = \rho_0(2^{-2}T(t), \infty) < 1.$$

For

$$\begin{aligned} \rho_0(2^{-2}T_{0,j}(t), \infty) &= \limsup_{n \rightarrow \infty} \|T_{\tau_{0,j}^{m,l,\alpha}(t)}^n\|_{L_\infty(\mathbb{T})}^{1/n} \\ &> T_{\tau_{0,j}^{m,l,\alpha}(t)} \\ &= |\tau_{0,j}^{m,l,\alpha}(\omega)|^2 + |\tau_{0,j}^{m,l,\alpha}(\omega + \pi)|^2. \end{aligned}$$

Thus,

$$|\tau_{0,j}^{m,l,\alpha}(\omega)|^2 + |\tau_{0,j}^{m,l,\alpha}(\omega + \pi)|^2 < 1.$$

This concludes the claim (3.9).

By Lemma 3.4, the corresponding nonstationary refinable functions $\phi_j \in L_2(\mathbb{R})$, $j \in \mathbb{N}_0$. \blacktriangleleft

4 Convergence of nonstationary cascade algorithms

In this section, demonstration of the convergence of nonstationary cascade algorithms in the Sobolev space $W_2^\nu(\mathbb{R})$ is given. We will show a lemma about a sufficient condition on the convergence of a nonstationary cascade algorithm which is necessary for the following theorem.

Lemma 4.1 ([17], Proposition 2.6) *Let \hat{a}_j and \hat{b}_j ($j \in \mathbb{N}$) be 2π -periodic measurable functions such that for all $j \in \mathbb{N}$,*

$$|\hat{a}_j(\omega)| \leq |\hat{b}_j(\omega)|, \quad \text{a.e. } \omega \in \mathbb{R}. \quad (4.1)$$

Let $\eta \in W_2^\nu(\mathbb{R})$ such that $\lim_{j \rightarrow \infty} \hat{\eta}(2^{-j}\omega) = 1$ for almost every $\omega \in \mathbb{R}$. Define

$$\widehat{f_n}(\omega) := \hat{\eta}(2^{-n}\omega) \prod_{j=1}^n \hat{a}_j(2^{-n}\omega) \quad \text{and} \quad \widehat{g_n}(\omega) := \hat{\eta}(2^{-n}\omega) \prod_{j=1}^n \hat{b}_j(2^{-n}\omega), \omega \in \mathbb{R}.$$

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Assume that $\widehat{f_\infty}(\omega) := \lim_{n \rightarrow \infty} \prod_{j=1}^n \widehat{a}_j(2^{-n}\omega)$ and $\widehat{g_\infty}(\omega) := \lim_{n \rightarrow \infty} \prod_{j=1}^n \widehat{b}_j(2^{-n}\omega)$ are well defined for almost every $\omega \in \mathbb{R}$. Then, $\lim_{n \rightarrow \infty} \|g_n - g_\infty\|_{W_2^\nu(\mathbb{R})} = 0$ implies $\lim_{n \rightarrow \infty} \|f_n - f_\infty\|_{W_2^\nu(\mathbb{R})} = 0$. In particular, suppose that there are a positive integer J and a 2π -periodic measurable function \hat{b} such that

$$|\widehat{a}_j(\omega)| \leq |\hat{b}(\omega)|, a.e. \omega \in \mathbb{R}, \forall j > J \text{ and } \widehat{a}_j \in L_\infty(\mathbb{R}), 1 \leq j \leq J. \quad (4.2)$$

For $n \in \mathbb{N}$, define $\widehat{h_n}(\omega) := \hat{\eta}(2^{-n}\omega) \prod_{j=1}^n \widehat{b}_j(2^{-n}\omega)$. If $\{h_n\}_{n=1}^\infty$ converges in $W_2^\nu(\mathbb{R})$, then f_n converges to f_n in $W_2^\nu(\mathbb{R})$, i.e., $\lim_{n \rightarrow \infty} \|f_n - f_\infty\|_{W_2^\nu(\mathbb{R})} = 0$.

Theorem 4.1 Let $\tau_{0,j}^{m,l,\alpha}(\omega)$ be the mask (2.3), which are defined in (2.4), then, for every $n \in \mathbb{N}_0$, the nonstationary cascade algorithm (2.5) associated with $\{\tau_{0,j+n}^{m_j,l_j,\alpha}(\omega)\}_{j=1}^\infty$ converges in $W_2^\nu(\mathbb{R})$, for any $\nu \geq 0$. Consequently, the nonstationary refinable functions $\phi_j^{m_j,l_j,\alpha}$, $j \in \mathbb{N}_0$, in (2.4) must be well-defined compactly supported $C^\infty(\mathbb{R})$ functions.

Proof. Since $\deg(\tau_{0,j}^{m,l,\alpha}(\omega)(\omega)) \leq 2(m_j + l_j) + 1 < 4m_j - 9$, and $\tau_{0,j}^{m,l,\alpha}(\omega) = 1$, applying $\sum_{j=1}^\infty 2^{-j}m_j < \infty$, we get

$$\sum_{j=1}^\infty 2^{-j} \deg(\tau_{0,j}^{m,l,\alpha}(\omega)) \leq \sum_{j=1}^\infty 2^{-j} (4m_j - 9) \leq 4 \sum_{j=1}^\infty 2^{-j} m_j < \infty.$$

Moreover, by (3.9), ones obtain $|\tau_{0,j}^{m,l,\alpha}(\omega)| \leq 1$. By using Lemma (3.3), we can derive that $\phi_{0,j}^{m_j,l_j,\alpha}$, $j \in \mathbb{N}_0$ are well defined compactly supported functions.

Because $\tau_{0,j}^{m,l,\alpha}(\omega)$ in the case $\alpha = 0$ have $\nu_2(\tau_{0,j}^{m,l,\alpha}(\omega)) \rightarrow \infty$ ([11, 13]). The same proof is carried out for any α . So, there exists a positive integer J such that $\nu_2(\tau_{0,j}^{m,l,\alpha}(\omega)) \geq \nu + 2$. By $\lim_{j \rightarrow \infty} m_j = \infty$, there exists a positive integer N such that $m_j > J$ and it is obtained that

$$\nu_2(\tau_{0,j}^{m,l,\alpha}(\omega)) \geq \nu + 2. \quad (4.3)$$

Let \hat{b} be the unique 2π -periodic trigonometric polynomial such that $\tau_{0,j}^{m,l,\alpha}(\omega) = 2^{-1}(1 + e^{-i\omega})\hat{b}(\omega)$. Bringing the definition of $\nu_2(\hat{b})$ and (4.3) together yields $\nu_2(\hat{b}) = \nu_2(\tau_{0,j}^{m,l,\alpha}(\omega)) - 1 \geq \nu + 1 > \nu$, thus, the stationary cascade algorithm associated with the mask \hat{b} converges in $W_2^\nu(\mathbb{R})$ (see ([16]), Theorem 4.3). Since $|\tau_{0,j}^{m,l,\alpha}(\omega)| \leq |\hat{b}(\omega)|$, applying Lemma (4.1), we derive that the nonstationary cascade algorithm associated with masks $\tau_{0,j}^{m,l,\alpha}(\omega)$ converges in $W_2^\nu(\mathbb{R})$. The same proof is carried out for every $\phi_n^{m_j,l_j,\alpha}$ and for the nonstationary cascade algorithm associated with masks $\{\tau_{0,j}^{m,l,\alpha}(\omega)\}_{j=1}^\infty$. \blacksquare

Remark 4.1 Notice that when $\alpha_j = 0$ ($j \in \mathbb{N}$), the Theorem 4.1 in this paper is the same as corresponding Theorem 2.8 given in [17].

5 Construction of nonstationary tight wavelet frames

In this section, we shall construct the symmetric C^∞ tight wavelet frames in $L_2(\mathbb{R})$ with compact support and the spectral frame approximation order based on the masks (2.3). The

following two lemmas analyze the approximation properties of the operators P_n and relations of $\tau_{0,j}^{m,l,\alpha}(\omega)$, respectively, which are useful for construction of C^∞ tight wavelet frames.

Lemma 5.1 ([17], Theorem 3.2) *Let $\widehat{a}_j, j \in \mathbb{N}$ be 2π -periodic measurable functions such that $|\widehat{a}_j(\omega)|^2 + |\widehat{a}_j(\omega + \pi)|^2 \leq 1$, a.e. $\omega \in \mathbb{R}$ holds for all $j \in \mathbb{N}$ and for every $n \in \mathbb{N}_0$, the function $\widehat{\phi}_n(\omega) := \lim_{J \rightarrow \infty} \prod_{j=1}^J \widehat{a}_{j+n}(2^{-j}\omega)$ is well defined for almost every $\omega \in \mathbb{R}$. Let $\nu \geq 0$. If, for $n \in \mathbb{N}$,*

$$\begin{aligned} & \left| 1 - |\widehat{\phi}_n(\omega)|^2 \right|^2 \leq C_{\phi_n} |\omega|^{2\nu}, \quad \text{a.e. } \omega \in [-\pi, \pi], \\ & \sum_{k \in \mathbb{Z} \setminus \{0\}} |\widehat{\phi}_n(\omega)|^2 |\widehat{\phi}_n(\omega + 2\pi k)|^2 \leq C_{\widehat{\phi}_n} |\omega|^{2\nu}, \quad \text{a.e. } \omega \in [-\pi, \pi], \end{aligned} \quad (5.1)$$

where C_{ϕ_n} is a constant depending only on ϕ_n , then for the linear operators P_n in (2.6),

$$\|f - P_n(f)\|_{L_2(\mathbb{R})} \leq \max(2, \sqrt{C_{\phi_n}}) 2^{-\nu n} \|f\|_{W_2^\nu(\mathbb{R})} \quad \forall f \in W_2^\nu(\mathbb{R}) \quad \text{and } n \in \mathbb{N}. \quad (5.2)$$

In particular, (5.1) is satisfied if

$$1 - |\widehat{\phi}_n(\omega)|^2 \leq C_{\phi_n} |\omega|^{2\nu}. \quad (5.3)$$

Lemma 5.2 *Let $\tau_{0,j}^{m,l,\alpha}(\omega)$ be the mark (2.3), which are defined in (2.4). For any $0 < \rho \leq 1$ and $\nu \geq 0$, there exist a positive integer N and a positive constant C , (both of them depend only on ρ and ν), such that for all $N \leq \rho m < l \leq m$,*

$$0 \leq 1 - |\tau_{0,j}^{m,l,\alpha}(\omega)|^2 \leq C |\omega|^{2\nu} \quad \forall \omega \in [-\pi, \pi]. \quad (5.4)$$

Proof. Suppose that $\alpha = 0$, the case holds (see [17], Lemma 3.3). Assume that (5.4) holds for $\alpha = k - 1$. Then suppose that $\alpha = k$, since $\tau_{0,j}^{m,l,\alpha}(\omega)$ increases as α increases, we have

$$0 \leq 1 - |\tau_{0,j}^{m,l,\alpha}(\omega)|^2 \leq 1 - |\tau_{0,j}^{m,l,\alpha}(\omega)|^2 \leq C |\omega|^{2\nu}.$$

This completes the claim (5.4). \blacksquare

Theorem 5.1 *Let $\tau_{0,j}^{m,l,\alpha}(\omega)$ be the mark (2.3). For $j \in \mathbb{N}$, define ϕ_{j-1} as in (2.4) and $\psi_{j-1}^1, \psi_{j-1}^2, \psi_{j-1}^3$ as in (2.7) with the wavelet masks $\widehat{b}_j^1, \widehat{b}_j^2$ and \widehat{b}_j^3 being derived from \widehat{a}_j in (2.8). Then the following hold:*

- (1) *Each nonstationary refinable function $\phi_j, j \in \mathbb{N}_0$, is a compactly supported C^∞ real-valued function that is symmetric about the origin: $\phi_j(-\cdot) = \phi_j$.*
- (2) *Each wavelet function $\psi_j^\ell, \ell = 1, 2, 3$ and $j \in \mathbb{N}_0$, is a compactly supported C^∞ real-valued function with l_{j+1} vanishing moments and satisfies $\psi_j^\ell(1 - \cdot) = \psi_j^\ell$ for $\ell = 1, 2$ and $\psi_j^3(1 - \cdot) = -\psi_j^3$.*
- (3) *The system $\{\phi_0(\cdot - k) : k \in \mathbb{Z}\} \cup \{\psi_{j;k}^\ell := 2^{j/2} \psi_j^\ell(2^j \cdot - k) : j \in \mathbb{N}_0, k \in \mathbb{Z}, \ell = 1, 2, 3\}$ is a compactly supported symmetric C^∞ tight wavelet frame in $L_2(\mathbb{R})$.*

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(4) If in addition $\liminf_{j \rightarrow \infty} l_j/m_j > 0$, then the tight wavelet frame in item (3) has the spectral frame approximation order.

Proof. For item (1), by using Theorem 4.1, it is derived that all $\phi_j, j \in \mathbb{N}_0$ are compactly supported functions in $C^\infty(\mathbb{R})$. Combining all the masks \widehat{a}_j , which are 2π -periodic trigonometric polynomials with real coefficients and are symmetric about the origin: $\overline{\widehat{a}_j} = \widehat{a}_j$, and the definition of $\widehat{\phi}_j$ in (2.4) yields,

$$\overline{\widehat{\phi}_j(\omega)} = \widehat{\phi}_j(\omega),$$

which ϕ_j are real-valued. By the definition of $\tau_{0,j}^{m,l,\alpha}(\omega)$ ($j \in \mathbb{N}$) in (2.3), we get

$$1 - |\tau_{0,j}^{m,l,\alpha}(\omega)|^2 = O(|\omega|^{2l}), \omega \rightarrow 0. \quad (5.5)$$

$\widehat{b}_j^\ell = O(|\omega|^{l_j}), \omega \rightarrow 0$ follows from the fact that (2.10) and (5.5). Thus, ψ_j^ℓ has l_{j+1} vanishing moments.

For item (3), notice that the definition of \widehat{b}_j^ℓ in (2.9), we can straightforward obtain that (2.10). Therefore, the wavelet system in (2.8) is a compactly supported tight wavelet frame in $L_2(\mathbb{R})$ (see [17], Theorem1.1). For item (4), let ν be an arbitrary positive integer and denote $\widehat{d}_j := |\tau_{0,j}^{m,l,\alpha}(\omega)|^2$. For $\liminf_{j \rightarrow \infty} l_j/m_j > 0$, there exist a positive integer N and $0 < \rho < \liminf_{j \rightarrow \infty} l_j/m_j$ such that $2\nu < N < \rho m_j < l_j \leq m_j$ for all $j \geq N$. By using Lemma 5.2, it is to see that (5.4) holds. That is, there exists a positive constant C , independent of j , such that

$$0 \leq 1 - \widehat{d}_j(\omega) \leq C|\omega|^{2\nu}, \quad \omega \in [-\pi, \pi] \text{ and } j \geq N.$$

For $n \geq N$ and $\ell \in \mathbb{N}$, since $\widehat{d_{\ell+n}}(0) = 1$, ones get

$$|\widehat{d_{\ell+n}}(0) - \widehat{d_{\ell+n}}(2^{-\ell}\omega)| = |1 - \widehat{d_{\ell+n}}(2^{-\ell}\omega)| \leq C2^{-2\nu\ell}|\omega|^{2\nu}, \quad \forall \omega \in [-\pi, \pi].$$

Since $\widehat{d_{\ell+n}}(0) = 1$, $0 \leq \widehat{d_{\ell+n}}(\omega) \leq 1$ and (3.9), we derive that

$$0 \leq 1 - |\widehat{\phi_n}(\omega)|^2 \leq \sum_{\ell=1}^{\infty} |\widehat{d_{\ell+n}}(0) - \widehat{d_{\ell+n}}(2^{-\ell}\omega)| \quad \omega \in \mathbb{R}. \quad (5.6)$$

Applying (5.6), it is obtained that

$$1 - |\widehat{\phi_n}(\omega)|^2 \leq C|\omega|^{2\nu} \sum_{\ell=1}^{\infty} 2^{-2\nu\ell}, \quad \omega \in [-\pi, \pi].$$

Thus, (5.3) holds with

$$C_{\phi_n} := C \sum_{\ell=1}^{\infty} 2^{-2\nu\ell} = \frac{C}{1 - 2^{-2\nu}} < \infty.$$

Combining $Q_n = P_n$ and Theorem 5.1, one has

$$\|f - Q_n(f)\|_{L_2(\mathbb{R})} \leq C_1 2^{-\nu n} |f|_{W_2^\nu(\mathbb{R})}, \quad \forall f \in W_2^\nu(\mathbb{R}) \text{ and } n \in \mathbb{N},$$

where $C_1 := \max(2, \sqrt{\frac{C}{1 - 2^{-2\nu}}})$ is independent of f and n . Since ν is arbitrary, the tight wavelet frame has the desired spectral frame approximation order. \P

Remark 5.1 Under the condition $\alpha_j = 0$ ($j \in \mathbb{N}$) of Theorem 5.1, one can derive that it is consistent with the claim of Theorem 1.2 given in [17].

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The Order and Type of Meromorphic Functions and Entire Functions of Finite Iterated Order [‡]

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Abstract

In this paper, the authors investigate the p -iterated order and p -iterated type of $f_1 + f_2$, $f_1 f_2$, where $f_1(z)$, $f_2(z)$ are meromorphic functions or entire functions with the same p -iterated order and different p -iterated type, and we obtain some results which improve and extend some previous results.

Key words: meromorphic function; entire function; iterated order; iterated type

AMS Subject Classification(2010): 30D35, 30D15

1. Introduction and Notations

In this paper, we assume that readers are familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions in the complex plane (e.g. [4, 6-8, 10, 14, 15]). Throughout this paper, by a meromorphic function $f(z)$, we mean a meromorphic function in the complex plane. We use $T(r, f)$ and $M(r, f)$ to denote the characteristic function of a meromorphic function and the maximum modulus of an entire function. In the following, we will recall some notations about meromorphic functions and entire functions.

Definition 1.1. (see [4, 8, 10]) The order of a meromorphic function $f(z)$ is defined by

$$\sigma(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r}. \quad (1.1)$$

Especially, if $0 < \sigma(f) < \infty$, then the type of $f(z)$ is defined by

$$\psi(f) = \overline{\lim}_{r \rightarrow \infty} \frac{T(r, f)}{r^{\sigma(f)}}. \quad (1.2)$$

Definition 1.2. (see [4, 6 – 8, 10]) The order of an entire function $f(z)$ is defined by

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$$\sigma(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r} = \overline{\lim}_{r \rightarrow \infty} \frac{\log \log M(r, f)}{\log r}. \quad (1.3)$$

Especially, if $0 < \sigma(f) < \infty$, then the type of $f(z)$ is defined by

$$\tau(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log M(r, f)}{r^{\sigma(f)}}. \quad (1.4)$$

The order and type are two important indicators in revealing the growth of meromorphic functions in the complex plane or analytic functions in the unit disc. Many authors have investigated the growth of meromorphic functions or analytic functions in the unit disc (e.g. [3,4,7-10,12-15]) and obtain a lot of classical results in the following.

Theorem A. (see [4, 10, 14, 15]) Let $f_1(z)$ and $f_2(z)$ be meromorphic functions of finite order satisfying $\sigma(f_1) = \sigma_1$ and $\sigma(f_2) = \sigma_2$. Then

$$\sigma(f_1 + f_2) \leq \max\{\sigma_1, \sigma_2\}, \quad \sigma(f_1 f_2) \leq \max\{\sigma_1, \sigma_2\}.$$

Furthermore, if $\sigma_1 \neq \sigma_2$, then $\sigma(f_1 + f_2) = \sigma(f_1 f_2) = \max\{\sigma_1, \sigma_2\}$.

Theorem B. (see [15]) Let $f_1(z)$ and $f_2(z)$ be meromorphic functions of finite order. Then $\mu(f_1 + f_2) \leq \max\{\sigma(f_1), \mu(f_2)\}$, $\mu(f_1 f_2) \leq \max\{\sigma(f_1), \mu(f_2)\}$.

Theorem C. (see [11]) Let $f_1(z)$ and $f_2(z)$ be meromorphic functions of finite order satisfying $\sigma(f_1) < \mu(f_2)$, then $\mu(f_1 + f_2) = \mu(f_1 f_2) = \mu(f_2)$.

Theorem D. (see [7]) Let $f_1(z)$ and $f_2(z)$ be entire functions of finite order satisfying $\sigma(f_1) = \sigma(f_2) = \sigma$. Then the following two statements hold:

- (i) If $\tau(f_1) = 0$ and $0 < \tau(f_2) < \infty$, then $\sigma(f_1 f_2) = \sigma$, $\tau(f_1 f_2) = \tau(f_2)$.
- (ii) If $\tau(f_1) < \infty$ and $\tau(f_2) = \infty$, then $\sigma(f_1 f_2) = \sigma$, $\tau(f_1 f_2) = \infty$.

Theorem E. (see [4, 14, 15]) Let $f(z)$ be a meromorphic function of finite order, then $\sigma(f') = \sigma(f)$.

From Theorems A–E, a natural question is: can we get the similar results for entire functions and meromorphic functions of infinite order (i.e. finite iterated order)? In the following, we recall some notations and definitions of finite iterated order. For $r \in (0, +\infty)$, we define $\exp_1 r = e^r$ and $\exp_{i+1} r = \exp(\exp_i r)$, $i \in \mathbb{N}$; for sufficiently large $r \in (0, +\infty)$, we define $\log_1 r = \log r$ and $\log_{i+1} r = \log(\log_i r)$, $r \in \mathbb{N}$; we also denote $\exp_0 r = r = \log_0 r$ and $\exp_{-1} r = \log_1 r$. Throughout this paper, we use p to denote a positive integer. We denote the linear measure of a set $E \subset (0, +\infty)$ by $mE = \int_E dt$ and the logarithmic measure of $E \subset (0, +\infty)$ by $m_l E = \int_E \frac{dt}{t}$.

Definition 1.3. (see [1, 5, 11]) The p -iterated order and p -iterated lower-order of a meromorphic function $f(z)$ are respectively defined by

$$\sigma_p(f) = \overline{\lim}_{r \rightarrow \infty} \frac{\log_p T(r, f)}{\log r}; \quad \mu_p(f) = \lim_{r \rightarrow \infty} \frac{\log_p T(r, f)}{\log r}. \quad (1.5)$$

Definition 1.4. (see [1, 5, 11]) The p -iterated order and p -iterated lower-order of an entire

function $f(z)$ are respectively defined by

$$\sigma_p(f) = \lim_{r \rightarrow \infty} \frac{\log_{p+1} M(r, f)}{\log r} = \lim_{r \rightarrow \infty} \frac{\log_p T(r, f)}{\log r}; \quad (1.6)$$

$$\mu_p(f) = \lim_{r \rightarrow \infty} \frac{\log_{p+1} M(r, f)}{\log r} = \lim_{r \rightarrow \infty} \frac{\log_p T(r, f)}{\log r}. \quad (1.7)$$

Definition 1.5. Let $f(z)$ be a meromorphic function satisfying $0 < \sigma_p(f) = \sigma < \infty$ or $0 < \mu_p(f) = \mu < \infty$. Then the p -iterated type of order and the p -iterated lower-type of lower-order of $f(z)$ are respectively defined by

$$\psi_p(f) = \lim_{r \rightarrow \infty} \frac{\log_{p-1} T(r, f)}{r^\sigma}; \quad \underline{\psi}_p(f) = \lim_{r \rightarrow \infty} \frac{\log_{p-1} T(r, f)}{r^\mu}. \quad (1.8)$$

Definition 1.6. Let $f(z)$ be an entire function satisfying $0 < \sigma_p(f) = \sigma < \infty$ or $0 < \mu_p(f) = \mu < \infty$. Then the p -iterated type of order and the p -iterated lower-type of lower-order of $f(z)$ are respectively defined by

$$\tau_p(f) = \lim_{r \rightarrow \infty} \frac{\log_p M(r, f)}{r^\sigma}; \quad \underline{\tau}_p(f) = \lim_{r \rightarrow \infty} \frac{\log_p M(r, f)}{r^\mu}. \quad (1.9)$$

From the above definitions, we can easily obtain the following propositions:

- (i) If $f_1(z)$ and $f_2(z)$ are meromorphic functions with $\sigma_p(f_1) = \sigma_1 < \infty$ and $\sigma_p(f_2) = \sigma_2 < \infty$, then $\sigma_p(f_1 + f_2) \leq \max\{\sigma_1, \sigma_2\}$, $\sigma_p(f_1 f_2) \leq \max\{\sigma_1, \sigma_2\}$. Furthermore, if $\sigma_1 \neq \sigma_2$, then $\sigma_p(f_1 + f_2) = \sigma_p(f_1 f_2) = \max\{\sigma_1, \sigma_2\}$.
- (ii) If $f_1(z)$ and $f_2(z)$ are meromorphic functions with $\sigma_p(f_1) < \infty$ and $\mu_p(f_2) < \infty$, then $\max\{\mu_p(f_1 + f_2), \mu_p(f_1 f_2)\} \leq \max\{\sigma_p(f_1), \mu_p(f_2)\}$ or $\max\{\mu_p(f_1 + f_2), \mu_p(f_1 f_2)\} \leq \max\{\mu_p(f_1), \sigma_p(f_2)\}$.
- (iii) If $f_1(z)$ and $f_2(z)$ are meromorphic functions satisfying $\sigma_p(f_1) < \mu_p(f_2) \leq \infty$, then $\mu_p(f_1 + f_2) = \mu_p(f_1 f_2) = \mu_p(f_2)$.
- (iv) If $f(z)$ is an entire function with $0 < \sigma_p(f) < \infty$, then $\psi_p(f) = \tau_p(f)$, $\underline{\psi}_p(f) = \underline{\tau}_p(f)$ for $p \geq 2$ and $\psi(f) \leq \tau(f)$, $\underline{\psi}(f) \leq \underline{\tau}(f)$ for $p = 1$.

2. Main Results

Here our first question is : can we get the similar results as Theorem D for meromorphic function or entire function of finite iterated order? Since it is easy to see $\sigma_p(f') = \sigma_p(f)$ ($p \geq 1$) for meromorphic function $f(z)$ of finite iterated order; our second question is : can we get $\psi_p(f') = \psi_p(f)$ or $\tau_p(f') = \tau_p(f)$ for meromorphic function or entire function of finite iterated order? In fact, we obtain the following results.

Theorem 2.1. Let $f_1(z)$ and $f_2(z)$ be meromorphic functions satisfying $0 < \sigma_p(f_1) = \sigma_p(f_2) = \sigma < \infty$, $0 \leq \psi_p(f_1) < \psi_p(f_2) \leq \infty$. Set $\alpha = \psi(f_2) - \psi(f_1)$, $\beta = \psi(f_1) + \psi(f_2)$, then

- (i) $\sigma_p(f_1 + f_2) = \sigma_p(f_1 f_2) = \sigma$ ($p \geq 1$);
- (ii) If $p > 1$, we have $\psi_p(f_1 + f_2) = \psi_p(f_1 f_2) = \psi_p(f_2)$;
- (iii) If $p = 1$, we have $\alpha \leq \psi(f_1 + f_2) \leq \beta$, $\alpha \leq \psi(f_1 f_2) \leq \beta$.

Theorem 2.2. Let $f_1(z)$ and $f_2(z)$ be entire functions satisfying $0 < \sigma_p(f_1) = \sigma_p(f_2) = \sigma < \infty$, $0 \leq \tau_p(f_1) < \tau_p(f_2) \leq \infty$. Then the following statements hold:

- (i) If $p \geq 1$, then $\sigma_p(f_1 + f_2) = \sigma$, $\tau_p(f_1 + f_2) = \tau_p(f_2)$;
(ii) If $p > 1$, then $\sigma_p(f_1 f_2) = \sigma$, $\tau_p(f_1 f_2) = \tau(f_2)$.

Remark 2.1. When $p = 1$, (ii) of Theorem 2.2 does not hold. For example, set $f_1 = e^{-z}$, $f_2 = e^{2z}$ satisfy $\tau(f_1) = 1 < \tau(f_2) = 2$, but $\tau(f_1 f_2) = 1 < \tau(f_2) = 2$.

Theorem 2.3. Let $f_1(z)$ and $f_2(z)$ be meromorphic functions satisfying $\sigma_p(f_1) = \mu_p(f_2) = \mu$ ($0 < \mu < \infty$), $0 \leq \psi_p(f_1) < \underline{\psi}_p(f_2) \leq \infty$. Then the following statements hold:

- (i) $\mu_p(f_1 + f_2) = \mu_p(f_1 f_2) = \mu$ ($p \geq 1$); (ii) If $p > 1$, then $\underline{\psi}_p(f_1 + f_2) = \underline{\psi}_p(f_1 f_2) = \underline{\psi}_p(f_2)$;
(iii) If $p = 1$, then $\underline{\psi}(f_2) - \psi(f_1) \leq \max\{\underline{\psi}(f_1 + f_2), \underline{\psi}(f_1 f_2)\} \leq \psi(f_1) + \underline{\psi}(f_2)$.

Theorem 2.4. Let $f_1(z)$ and $f_2(z)$ be entire functions satisfying $\sigma_p(f_1) = \mu_p(f_2) = \mu$ ($0 < \mu < \infty$), $0 \leq \tau_p(f_1) < \underline{\tau}_p(f_2) \leq \infty$. Then the following statements hold:

- (i) $\mu_p(f_1 + f_2) = \mu_p(f_1 f_2) = \mu$ ($p \geq 1$); (ii) If $p \geq 1$, then $\underline{\tau}_p(f_1 + f_2) = \underline{\tau}_p(f_2)$;
(iii) If $p > 1$, then $\underline{\tau}_p(f_1 f_2) = \underline{\tau}_p(f_2)$; if $p = 1$, then $\underline{\tau}(f_2) - \tau(f_1) \leq \underline{\tau}(f_1 f_2) \leq \tau(f_1) + \underline{\tau}(f_2)$.

Theorem 2.5. Let $p > 1$, $f(z)$ be a meromorphic function satisfying $0 < \sigma_p(f) < \infty$. Then $\psi_p(f') = \psi_p(f)$.

Theorem 2.6. Let $p \geq 1$, $f(z)$ be an entire function satisfying $0 < \sigma_p(f) < \infty$. Then $\tau_p(f') = \tau_p(f)$.

3. Preliminary Lemmas

Lemma 3.1. (see [11]) Let $f(z)$ be an entire function of p -iterated order satisfying $0 < \sigma_p(f) = \sigma < \infty$, $0 < \tau_p(f) = \tau \leq \infty$. Then for any given $\beta < \tau$, there exists a set $E \subset [1, +\infty)$ having infinite logarithmic measure such that for all $r \in E$, we have

$$\log_p M(r, f) > \beta r^\sigma.$$

Lemma 3.2. (see [7]) Let $f(z)$ be an analytic function in the circle $|z| \leq R$ and has no zeros in this circle, and if $f(0) = 1$, then its modulus in the circle $|z| \leq r < R$ satisfies the inequality

$$\ln |f(z)| \geq -\frac{2r}{R-r} \ln M(R, f).$$

Lemma 3.3. (see [2]) Given any number $H > 0$ and complex numbers a_1, a_2, \dots, a_n , there is a system of circles in the complex plane, with the sum of the radii equal to $2H$, such that for each point z lying outside these circles one has the inequality

$$\prod_{k=1}^n |z - a_k| \geq \left(\frac{H}{e}\right)^n.$$

Lemma 3.4. Let $f(z)$ be an analytic function in the circle $|z| \leq \beta R$ ($\beta > 1$) with $f(0) = 1$, and let ε be an arbitrary positive number not exceeding 2. Then inside the circle $|z| \leq R$, but outside of a family of excluded circles the sum of whose radii is not greater than $2\varepsilon\beta R$, we have

$$\ln |f(z)| > - \left(\frac{2}{\beta - 1} + \frac{\ln 2 - \ln \varepsilon}{\ln \beta} \right) \ln M(\beta^2 R, f).$$

Proof. By the similar proof in [7, p.21], constructing the function

$$h(z) = \frac{(-\beta R)^n}{a_1 a_2 \cdots a_n} \prod_{k=1}^n \frac{\beta R(z - a_k)}{(\beta R)^2 - \bar{a}_k z},$$

where a_1, a_2, \dots, a_n are the zeros of $f(z)$ in the circle $|z| < \beta R$, then we have $h(0) = 1$ and $|h(\beta R e^{i\theta})| = \frac{(\beta R)^n}{|a_1 a_2 \cdots a_n|} > 1$, then function $g(z) = \frac{f(z)}{h(z)}$ has no zeros in the circle $|z| < \beta R$. Therefore, by Lemma 3.2, for any z satisfying $|z| \leq R < \beta R$, we have

$$\begin{aligned} \ln |g(z)| &\geq - \frac{2R}{\beta R - R} \ln M(\beta R, g) \\ &= - \frac{2}{\beta - 1} (\ln M(\beta R, f) - \ln |h(\beta R e^{i\theta})|) \\ &\geq - \frac{2}{\beta - 1} \ln M(\beta R, f). \end{aligned} \quad (3.1)$$

For $|z| \leq \beta R$, we get $\prod_{k=1}^n |(\beta R)^2 - \bar{a}_k z| \leq [2(\beta R)^2]^n = 2^n (\beta R)^{2n}$. By Lemma 3.3, we get outside of a family of excluded circles the sum of whose radii are not greater than $2\varepsilon\beta R$, we have

$$\prod_{k=1}^n |\beta R(z - a_k)| > (\beta R)^n (\beta \varepsilon R)^n = \varepsilon^n (\beta R)^{2n},$$

where $n = n(\beta R)$ denotes the number of zeros of $f(z)$ in $|z| < \beta R$. So we have

$$|h(z)| \geq \frac{(\beta R)^n}{|a_1 a_2 \cdots a_n|} \frac{\varepsilon^n (\beta R)^{2n}}{2^n (\beta R)^{2n}} \geq \left(\frac{\varepsilon}{2} \right)^n. \quad (3.2)$$

Since $0 < \varepsilon < 2$, by (3.2), we have

$$\ln |\psi(z)| \geq -n \ln \frac{2}{\varepsilon}. \quad (3.3)$$

On the other hand, by Jensen's formula, we have

$$M(\beta^2 R, f) \geq \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log |f(\beta^2 R e^{i\theta})| d\theta \right\} = \prod_{k=1}^{n(\beta^2 R)} \frac{\beta^2 R}{|a_k|} \geq \prod_{k=1}^{n(\beta R)} \frac{\beta^2 R}{|a_k|} \geq \beta^n.$$

Therefore,

$$n \leq \frac{\ln M(\beta^2 R, f)}{\ln \beta}. \quad (3.4)$$

By (3.1), (3.3) and (3.4), we have

$$\begin{aligned} \ln |f(z)| &\geq -\frac{2}{\beta-1} \ln M(\beta R, f) - \frac{\ln 2 - \ln \varepsilon}{\ln \beta} \ln M(\beta^2 R, f) \\ &\geq -\left(\frac{2}{\beta-1} + \frac{\ln 2 - \ln \varepsilon}{\ln \beta}\right) \ln M(\beta^2 R, f). \end{aligned} \quad (3.5)$$

Lemma 3.5. Let $p > 1$, $f(z)$ be an entire function satisfying $0 < \sigma_p(f) = \sigma < \infty$ and $\tau_p(f) < \infty$. Then for any given $\varepsilon > 0$, there exists a set $E \subset (0, +\infty)$ having finite logarithmic measure, such that for all $|z| = r \notin E$, we have

$$\exp\{-\exp_{p-1}\{(\tau_p(f) + \varepsilon)r^\sigma\}\} < |f(z)| < \exp_p\{(\tau_p(f) + \varepsilon)r^\sigma\}.$$

Proof. By (1.9), for any given $\varepsilon > 0$ and for sufficiently large r , we have

$$|f(z)| < \exp_p\{(\tau_p(f) + \varepsilon)r^\sigma\}, \quad M(\beta^2 r, f) < \exp_p\{(\tau_p(f) + \frac{\varepsilon}{2})\beta^{2\sigma}r^\sigma\} \quad (\beta > 1). \quad (3.6)$$

For any given $\varepsilon (0 < \varepsilon < 2)$ and any $\beta > 1$, we choose ε, β satisfying $(\tau_p(f) + \frac{\varepsilon}{2})\beta^{2\sigma} < \tau_p(f) + \varepsilon$, by Lemma 3.4 and (3.5), there exists a set $E \subset (0, +\infty)$ having finite logarithmic measure, such that for all $|z| = r \notin E$, we have

$$|f(z)| > \exp\{-\exp_{p-1}\{(\tau_p(f) + \varepsilon)r^\sigma\}\} \quad (p > 1). \quad (3.7)$$

By (3.6), (3.7), we obtain that Lemma 3.5 holds.

Lemma 3.6. (see [4, 14]) Let $f(z)$ be a meromorphic function satisfying $f(0) \neq \infty$. Then for any $\tau > 1$ and $r > 0$, we have

$$T(r, f) < C_\tau T(\tau r, f') + \log^+(\tau r) + 4 + \log^+ |f(0)|,$$

where $C_\tau > 0$ is a constant related to τ .

Lemma 3.7. (see [6]) Let $g : (0, \infty) \rightarrow R$ and $h : (0, \infty) \rightarrow R$ be monotone non-decreasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E of finite linear measure. Then for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.

4. Proofs of Theorems 2.1 - 2.6

Proof of Theorem 2.1. (i) Without loss of generality, set $0 \leq \psi_p(f_1) < \psi_p(f_2) < \infty$, by (1.8), for any $\varepsilon > 0$ and for sufficiently large r , we have

$$\begin{aligned} T(r, f_1 + f_2) &\leq T(r, f_1) + T(r, f_2) + \ln 2 \\ &\leq \exp_{p-1}\{(\psi_p(f_1) + \varepsilon)r^\sigma\} + \exp_{p-1}\{(\psi_p(f_2) + \varepsilon)r^\sigma\} \\ &\leq 2 \exp_{p-1}\{(\psi_p(f_2) + \varepsilon)r^\sigma\} \end{aligned} \quad (4.1)$$

By (4.1), we get $\sigma_p(f_1 + f_2) \leq \sigma$. On the other hand, by (1.8), for any $\varepsilon (0 < 2\varepsilon < \psi_p(f_2) - \psi_p(f_1))$, there exists a sequence $\{r_n\}_{n=1}^\infty$ tending to ∞ such that

$$\begin{aligned} T(r_n, f_1 + f_2) &\geq T(r_n, f_2) - T(r_n, f_1) - \ln 2 \\ &\geq \exp_{p-1}\{(\psi_p(f_2) - \varepsilon)r_n^\sigma\} - \exp_{p-1}\{(\psi_p(f_1) + \varepsilon)r_n^\sigma\} \end{aligned} \quad (4.2)$$

holds for sufficiently large r_n . By (4.2), we get $\sigma_p(f_1 + f_2) \geq \sigma$; therefore $\sigma_p(f_1 + f_2) = \sigma$ ($p \geq 1$).

(ii)-(iii) By (4.1) and (4.2), we have $\psi_p(f_1 + f_2) = \psi_p(f_2)$ for $p > 1$ and $\alpha \leq \psi(f_1 + f_2) \leq \beta$ for $p = 1$. Since

$$T(r, f_1 f_2) \leq T(r, f_1) + T(r, f_2), \quad T(r, f_1 f_2) \geq T(r, f_2) - T(r, f_1) - o(1)$$

and by the similar proof in (4.1) and (4.2), we can easily obtain that the conclusions in cases (ii)-(iii) holds.

By the above proof, we can easily obtain that Theorem 2.1 also holds for $0 \leq \psi_p(f_1) < \psi_p(f_2) = \infty$.

Proof of Theorem 2.2. (i) Set $0 \leq \tau_p(f_1) < \tau_p(f_2) < \infty$. By (1.9), for any $\varepsilon > 0$ and for sufficiently large r , we have

$$\begin{aligned} M(r, f_1 + f_2) &\leq M(r, f_1) + M(r, f_2) \\ &\leq \exp_p\{(\tau_p(f_1) + \varepsilon)r^\sigma\} + \exp_p\{(\tau_p(f_2) + \varepsilon)r^\sigma\} \\ &\leq 2 \exp_p\{(\tau_p(f_2) + \varepsilon)r^\sigma\}, \end{aligned} \quad (4.3)$$

by (4.3), we get $\sigma_p(f_1 + f_2) \leq \sigma$, $\tau_p(f_1 + f_2) \leq \tau_p(f_2)$. On the other hand, by (1.9), for any $\varepsilon (0 < 2\varepsilon < \tau_p(f_2) - \tau_p(f_1))$ there exists a sequence $\{r_n\}_{n=1}^\infty$ tending to ∞ such that

$$M(r_n, f_1) < \exp_p\{(\tau_p(f_1) + \varepsilon)r_n^\sigma\}, \quad M(r_n, f_2) > \exp_p\{(\tau_p(f_2) - \varepsilon)r_n^\sigma\} \quad (4.4)$$

holds for sufficiently large r_n . In each circle $|z| = r_n (n = 1, 2, \dots)$, we choose a sequence $\{z_n\}_{n=1}^\infty$ satisfying $|f_2(z_n)| = M(r_n, f_2)$ such that

$$\begin{aligned} M(r_n, f_1 + f_2) &\geq |f_1(z_n) + f_2(z_n)| \geq |f_2(z_n)| - |f_1(z_n)| \\ &\geq M(r_n, f_2) - M(r_n, f_1) \\ &\geq \exp_p\{(\tau_p(f_2) - \varepsilon)r_n^\sigma\} - \exp_p\{(\tau_p(f_1) + \varepsilon)r_n^\sigma\} \\ &\geq \frac{1}{2} \exp_p\{(\tau_p(f_2) - \varepsilon)r_n^\sigma\} \quad (r_n \rightarrow \infty), \end{aligned} \quad (4.5)$$

by (4.5), we get $\sigma_p(f_1 + f_2) \geq \sigma$, $\tau_p(f_1 + f_2) \geq \tau_p(f_2)$; therefore $\sigma_p(f_1 + f_2) = \sigma$, $\tau_p(f_1 + f_2) = \tau_p(f_2)$.

(ii) By (1.9), for any $\varepsilon > 0$ and for sufficiently large r , we have

$$\begin{aligned} M(r, f_1 f_2) &\leq M(r, f_1) M(r, f_2) \\ &\leq \exp_p\{(\tau_p(f_1) + \varepsilon)r^\sigma\} \exp_p\{(\tau_p(f_2) + \varepsilon)r^\sigma\} \\ &\leq \exp_p\{(\tau_p(f_2) + 2\varepsilon)r^\sigma\} \quad (p > 1), \end{aligned} \quad (4.6)$$

by (4.6), we get $\sigma_p(f_1 f_2) \leq \sigma$, $\tau_p(f_1 f_2) \leq \tau_p(f_2)$ ($p > 1$). On the other hand, by Lemma 3.1, for any $\varepsilon > 0$ there exists a set E_1 having infinite logarithmic measure such that for all $r \in E_1$, we have

$$M(r, f_2) > \exp_p\{(\tau_p(f_2) - \varepsilon)r^\sigma\}. \quad (4.7)$$

By Lemma 3.5, for any $\varepsilon > 0$, there exists a set E_2 having finite logarithmic measure such that $|z| = r \notin E_2$, we have

$$|f_1(z)| > \exp\{-\exp_{p-1}\{(\tau_p(f_1) + \varepsilon)r^\sigma\}\} \quad (p > 1). \quad (4.8)$$

Therefore, by (4.7) and (4.8), for any $\varepsilon > 0$ and for all $|z| = r \in E_1 \setminus E_2$, we have

$$\begin{aligned} M(r, f_1 f_2) &\geq M(r, f_2) \exp\{-\exp_{p-1}\{(\tau_p(f_1) + \varepsilon)r^\sigma\}\} \\ &\geq \exp_p\{(\tau_p(f_2) - \varepsilon)r^\sigma\} \exp\{-\exp_{p-1}\{(\tau_p(f_1) + \varepsilon)r^\sigma\}\}. \end{aligned} \quad (4.9)$$

Since $\tau_p(f_1) < \tau_p(f_2)$, we choose ε satisfying $\tau_p(f_2) - \varepsilon > \tau_p(f_1) + \varepsilon$. By (4.9), for any ε ($0 < 2\varepsilon < \tau_p(f_2) - \tau_p(f_1)$) and for all $r \in E_1 \setminus E_2$, we have

$$M(r, f_1 f_2) > \exp_p\{(\tau_p(f_2) - 2\varepsilon)r^\sigma\} \quad (p > 1). \quad (4.10)$$

By (4.10), we have $\sigma_p(f_1 f_2) \geq \sigma$, $\tau_p(f_1 f_2) \geq \tau_p(f_2)$ for $p > 1$; Therefore $\sigma_p(f_1 f_2) = \sigma$, $\tau_p(f_1 f_2) = \tau_p(f_2)$ for $p > 1$.

By the similar proof in cases (i) and (ii), we can easily obtain that Theorem 2.2 holds for $0 \leq \tau_p(f_1) < \tau_p(f_2) = \infty$.

Proof of Theorem 2.3. Set $0 \leq \psi_p(f_1) < \underline{\psi}_p(f_2) < \infty$. By (1.8), for any $\varepsilon > 0$ and for sufficiently large r , we have

$$T(r, f_1) < \exp_{p-1}\{(\psi_p(f_1) + \varepsilon)r^\mu\}, \quad T(r, f_2) > \exp_{p-1}\{(\underline{\psi}_p(f_2) - \varepsilon)r^\mu\}. \quad (4.11)$$

By (4.11), for any ε ($0 < 2\varepsilon < \underline{\psi}_p(f_2) - \psi_p(f_1)$) and for sufficiently large r , we have

$$\begin{aligned} T(r, f_1 + f_2) &\geq T(r, f_2) - T(r, f_1) - \ln 2 \\ &\geq \exp_{p-1}\{(\underline{\psi}_p(f_2) - \varepsilon)r^\mu\} - \exp_{p-1}\{(\psi_p(f_1) + \varepsilon)r^\mu\}. \end{aligned} \quad (4.12)$$

By (4.12), we have

$$\mu_p(f_1 + f_2) \geq \mu(p \geq 1), \quad \underline{\psi}_p(f_1 + f_2) \geq \underline{\psi}_p(f_2) (p > 1), \quad \underline{\psi}(f_1 + f_2) \geq \underline{\psi}(f_2) - \underline{\psi}(f_1). \quad (4.13)$$

On the other hand, by (1.8), for any $\varepsilon > 0$ there exists a sequence $\{r_n\}_{n=1}^\infty$ tending to ∞ such that

$$T(r_n, f_2) < \exp_{p-1}\{(\underline{\psi}_p(f_2) + \varepsilon)r_n^\mu\} \quad (4.14)$$

for sufficiently large r_n . By (4.11) and (4.14), for any $\varepsilon > 0$ and for sufficiently large r_n , we have

$$\begin{aligned} T(r, f_1 + f_2) &\leq T(r_n, f_1) + T(r_n, f_2) + \ln 2 \\ &\leq \exp_{p-1}\{(\psi_p(f_1) + \varepsilon)r_n^\mu\} + \exp_{p-1}\{(\underline{\psi}_p(f_2) + \varepsilon)r_n^\mu\}. \end{aligned} \quad (4.15)$$

By (4.15), we have

$$\mu_p(f_1 + f_2) \leq \mu(p \geq 1), \quad \underline{\psi}_p(f_1 + f_2) \leq \underline{\psi}_p(f_2)(p > 1), \quad \underline{\psi}(f_1 + f_2) \leq \underline{\psi}(f_2) + \underline{\psi}(f_1). \quad (4.16)$$

Therefore, by (4.13) and (4.16), the conclusions of Theorem 2.3 hold for $f_1 + f_2$. Since $T(r, f_1 f_2) \leq T(r, f_1) + T(r, f_2)$, $T(r, f_1 f_2) \geq T(r, f_2) - T(r, f_1) - o(1)$, we can easily obtain that the conclusions of Theorem 2.3 hold for $f_1 f_2$.

Theorem 2.3 also holds for $0 \leq \psi_p(f_1) < \underline{\psi}_p(f_2) = \infty$ by the above proof.

Proof of Theorem 2.4. We can obtain the conclusions of Theorem 2.4 by the similar proof in Theorem 2.2 and Theorem 2.3.

Proof of Theorem 2.5. By the Lemma of logarithmic derivative, we have that

$$T(r, f') \leq 3T(r, f) + O\{\log r\}$$

holds outside of an exceptional set E of finite linear measure. By Lemma 3.7, there exists $\alpha > 1$ such that $T(r, f') \leq 3T(\alpha r, f) + O\{\log(\alpha r)\}$ holds for sufficiently large r , so we have $\tau_p(f') \leq \tau_p(f)$ ($p > 1$). On the other hand, by Lemma 3.6, for any $\tau > 1$, there exists a constant C_τ such that

$$T(r, f) < C_\tau T(\tau r, f') + \log^+(\tau r) + 4 + \log^+ |f(0)|.$$

Set $\tau \rightarrow 1^+$, by the above inequality, we have $\tau_p(f') \geq \tau_p(f)$ ($p > 1$). Therefore $\tau_p(f') = \tau_p(f)$ for $p > 1$.

Proof of Theorem 2.6. For an entire function $f(z)$, we have

$$f(z) = f(0) + \int_0^z f'(\zeta) d\zeta, \quad (4.17)$$

where the integral route is a line from 0 to z . By (4.17), we have

$$M(r, f) \leq |f(0)| + \left| \int_0^z f'(\zeta) d\zeta \right| \leq |f(0)| + rM(r, f'), \quad (4.18)$$

By (4.18), we have

$$M(r, f') \geq \frac{M(r, f) - |f(0)|}{r}. \quad (4.19)$$

By (1.9) and (4.19), we have $\tau_p(f') \geq \tau_p(f)$. On the other hand, by Cauchy's integral formula, we have

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta, \quad (4.20)$$

where integral curve is the circle $|\zeta| = R$. By (4.20), for any $|z| = r < R$,

$$M(r, f') \leq \frac{1}{2\pi} \int_C \left\{ \max \left| \frac{f(\zeta)}{(\zeta - z)^2} \right| \right\} |d\zeta| \leq \frac{1}{2\pi} \frac{M(R, f)}{(R - r)^2} \cdot 2\pi R. \quad (4.21)$$

Set $R = \beta r$ ($\beta > 1$), then by (4.21), we have

$$M(r, f') \leq \frac{\beta}{(\beta - 1)^2 r} M(\beta r, f) \leq \frac{\beta}{(\beta - 1)^2 r} \exp_p\{(\tau_p(f) + \varepsilon)(\beta r)^{\sigma_p(f)}\}, \quad r \rightarrow \infty. \quad (4.22)$$

Since $\sigma_p(f') = \sigma_p(f)$, set $\beta \rightarrow 1$, we have $\tau_p(f') \leq \tau_p(f)$; therefore $\tau_p(f') = \tau_p(f)$.

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Journal Article

1. H.H.Gonska, Degree of simultaneous approximation of bivariate functions by Gordon operators, (journal name in italics) *J. Approx. Theory*, 62,170-191(1990).

Book

2. G.G.Lorentz, (title of book in italics) *Bernstein Polynomials* (2nd ed.), Chelsea, New York, 1986.

Contribution to a Book

3. M.K.Khan, Approximation properties of beta operators, in (title of book in italics) *Progress in Approximation Theory* (P.Nevai and A.Pinkus, eds.), Academic Press, New York, 1991, pp.483-495.

11. All acknowledgements (including those for a grant and financial support) should occur in one paragraph that directly precedes the References section.

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Some Results of a New Integral Operator

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Abstract

The main objective of the present paper is to obtain sufficient conditions for the univalence, starlikeness and convexity of a new integral operator defined on the space of normalized analytic functions in the open unit disk. Results presented in this paper may motivate further research in this fascinating field.

Keywords: analytic, univalent, starlike and convex functions, integral operator.

2010 Mathematics Subject Classifications: 30C45.

1 Introduction

Let $U = \{z : |z| < 1\}$ be the open unit disk and \mathcal{A} the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \text{ for all } z \in U, \quad (1)$$

which are analytic in U . Consider S the class of all functions in \mathcal{A} which are univalent in U .

A domain $D \subset \mathbb{C}$ is starlike with respect to a point $w_0 \in D$ if the line segment joining any point of D to w_0 lies inside D , while a domain is convex if the line segment joining any two points in D lies entirely in D . We say that the function $f \in \mathcal{A}$ is starlike if $f(U)$ is a starlike domain with respect to origin, and convex if $f(U)$ is convex. Analytically, $f \in \mathcal{A}$ is starlike if and only if

$$\operatorname{Re} \left[\frac{zf'(z)}{f(z)} \right] > 0, \text{ for all } z \in U,$$

and $f \in \mathcal{A}$ is convex if and only if

$$\operatorname{Re} \left[1 + \frac{zf''(z)}{f'(z)} \right] > 0, \text{ for all } z \in U.$$

The classes consisting of starlike and convex functions are denoted by S^* and K , respectively. Further, we denote by $S^*(\delta)$ and $K(\delta)$ the class of starlike functions of order δ and the class of convex functions of order δ ($0 \leq \delta < 1$), respectively, where

$$\operatorname{Re} \left[\frac{zf'(z)}{f(z)} \right] > \delta \text{ and } \operatorname{Re} \left[1 + \frac{zf''(z)}{f'(z)} \right] > \delta.$$

Recently, Frasin and Jahangiri [4] defined the family $B(\mu, \lambda)$, $\mu \geq 0$, $0 \leq \lambda < 1$ consisting of functions $f \in \mathcal{A}$ satisfying the condition

$$\left| f'(z) \left[\frac{z}{f(z)} \right]^\mu - 1 \right| < 1 - \lambda, \text{ for all } z \in U.$$

We note that $B(1, \lambda) = S^*(\lambda)$, $B(2, \lambda) = B(\lambda)$ (see [3]) and $B(2, 0) = S$.

For the functions $f, g \in \mathcal{A}$ and $\alpha, \zeta \in \mathbb{C}$ we define the integral operator $I_\alpha^\zeta(f, g)$ given by

$$I_\alpha^\zeta(f, g)(z) = \left[\zeta \int_0^z t^{\alpha+\zeta-1} \left(\frac{f'(t)}{g(t)} \right)^\alpha dt \right]^{\frac{1}{\zeta}}. \quad (2)$$

Note that the integral operator $I_\alpha^\zeta(f, g)(z)$ generalizes the integral operator $I_\alpha(f, g)(z)$ introduced in [2].

In this paper our purpose is to derive univalence conditions, starlikeness properties and the order of convexity for the integral operator introduced in (2). Recently, many authors studied the problem of integral operators which preserve the class S (see [5], [9]).

In order to prove our results, we have to recall here the following:

Lemma 1.1 (Mocanu and Şerb [7]) *Let $M_0 = 1, 5936\dots$, the positive solution of equation*

$$(2 - M)e^M = 2. \quad (3)$$

If $f \in \mathcal{A}$ and

$$\left| \frac{f''(z)}{f'(z)} \right| \leq M_0, \text{ for all } z \in U,$$

then

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1, \text{ for all } z \in U.$$

The edge M_0 is sharp.

Lemma 1.2 (Pascu [8]) *Let γ be a complex number, $\operatorname{Re} \gamma > 0$ and let the function $f \in \mathcal{A}$. If*

$$\frac{1 - |z|^{2\operatorname{Re} \gamma}}{\operatorname{Re} \gamma} \cdot \left| \frac{zf''(z)}{f'(z)} \right| \leq 1,$$

for all $z \in U$, then for any complex number ζ , $\operatorname{Re} \zeta \geq \operatorname{Re} \gamma$, the function

$$F_\zeta(z) = \left[\zeta \int_0^z t^{\zeta-1} f'(t) dt \right]^{\frac{1}{\zeta}}$$

is regular and univalent in U .

Lemma 1.3 (General Schwarz Lemma [6]) *Let f be regular function in the disk $U_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If f has in $z = 0$ one zero with multiply bigger than m , then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m, \quad z \in U_R.$$

The equality case hold only if $f(z) = e^{i\theta} \cdot \frac{M}{R^m} \cdot z^m$, where θ is constant.

Lemma 1.4 (Ready and Padmanabhan [10]) *Let the functions p, q be analytic in U with*

$$p(0) = q(0) = 0,$$

and let δ be a real number. If the function q maps the unit disk U onto a region which is starlike with respect to the origin, the inequality

$$\operatorname{Re} \left[\frac{p'(z)}{q'(z)} \right] > \delta, \text{ for all } z \in U$$

implies that

$$\operatorname{Re} \left[\frac{p(z)}{q(z)} \right] > \delta, \text{ for all } z \in U.$$

Lemma 1.5 (Wilken and Feng [11]) If $0 \leq \delta < 1$ and $f \in K(\delta)$, then $f \in S^*(\nu(\delta))$, where

$$\nu(\delta) = \begin{cases} \frac{1-2\delta}{2^{2(1-\delta)}-2}, & \text{if } \delta \neq \frac{1}{2}, \\ \frac{1}{2 \log 2}, & \text{if } \delta = \frac{1}{2}. \end{cases} \quad (4)$$

2 Main results

The univalence condition for the operator $I_\alpha^\zeta(f, g)$ defined in (2) is proved in the next theorem, by using Pascu univalence criterion.

Theorem 2.1 Let α, γ be complex numbers, $\operatorname{Re} \gamma > 0$, M_0 the positive solution of the equation (3), $M_0 = 1, 5936\dots$, and $f, g \in \mathcal{A}$. If

$$\left| \frac{f''(z)}{f'(z)} \right| \leq M_0, \quad \left| \frac{g''(z)}{g'(z)} \right| \leq M_0, \quad z \in U \quad (5)$$

and

$$2M_0 \operatorname{Re} \gamma + (2\operatorname{Re} \gamma + 1) \frac{2\operatorname{Re} \gamma + 1}{2\operatorname{Re} \gamma} \leq \frac{\operatorname{Re} \gamma \cdot (2\operatorname{Re} \gamma + 1) \frac{2\operatorname{Re} \gamma + 1}{2\operatorname{Re} \gamma}}{|\alpha|}, \quad (6)$$

then for any complex number ζ , $\operatorname{Re} \zeta \geq \operatorname{Re} \gamma$, the integral operator

$$I_\alpha^\zeta(f, g)(z) = \left[\zeta \int_0^z t^{\alpha+\zeta-1} \left(\frac{f'(t)}{g'(t)} \right)^\alpha dt \right]^{\frac{1}{\zeta}}$$

is in the class S .

Proof. Let the function

$$h(z) = \int_0^z \left[\frac{tf'(t)}{g(t)} \right]^\alpha dt. \quad (7)$$

The function h is regular in U and $h(0) = h'(0) - 1 = 0$.

From (7) we have

$$h'(z) = \left[\frac{zf'(z)}{g(z)} \right]^\alpha$$

and

$$h''(z) = \alpha \left(\frac{zf'(z)}{g(z)} \right)^{\alpha-1} \cdot \left[\frac{f'(z)}{g(z)} + \frac{zf''(z)}{g(z)} - z \cdot \frac{f'(z)}{g(z)} \cdot \frac{g'(z)}{g(z)} \right].$$

We get

$$\frac{zh''(z)}{h'(z)} = \alpha \left[1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} \right] = \alpha \left[\frac{zf''(z)}{f'(z)} - \left(\frac{zg'(z)}{g(z)} - 1 \right) \right]. \quad (8)$$

From (8) we obtain

$$\frac{1 - |z|^{2\operatorname{Re} \gamma}}{\operatorname{Re} \gamma} \cdot \left| \frac{zh''(z)}{h'(z)} \right| \leq \frac{1 - |z|^{2\operatorname{Re} \gamma}}{\operatorname{Re} \gamma} \cdot |z| \cdot |\alpha| \cdot \left| \frac{f''(z)}{f'(z)} \right| + \frac{1 - |z|^{2\operatorname{Re} \gamma}}{\operatorname{Re} \gamma} \cdot |\alpha| \cdot \left| \frac{zg'(z)}{g(z)} - 1 \right|.$$

From (5) and applying Lemma 1.1 we obtain

$$\left| \frac{zg'(z)}{g(z)} - 1 \right| < 1, \text{ for all } z \in U,$$

which implies that

$$\frac{1 - |z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma} \cdot \left| \frac{zh''(z)}{h'(z)} \right| \leq \frac{1 - |z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma} \cdot |z| \cdot |\alpha| \cdot M_0 + \frac{1 - |z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma} \cdot |\alpha|.$$

Since

$$\max_{|z| \leq 1} \frac{1 - |z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma} \cdot |z| = \frac{2}{(2\operatorname{Re}\gamma + 1)^{\frac{2\operatorname{Re}\gamma + 1}{2\operatorname{Re}\gamma}}},$$

we have

$$\frac{1 - |z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma} \cdot \left| \frac{zh''(z)}{h'(z)} \right| \leq \frac{2}{(2\operatorname{Re}\gamma + 1)^{\frac{2\operatorname{Re}\gamma + 1}{2\operatorname{Re}\gamma}}} \cdot |\alpha| \cdot M_0 + \frac{|\alpha|}{\operatorname{Re}\gamma}. \quad (9)$$

Using (6) in (9) we obtain

$$\frac{1 - |z|^{2\operatorname{Re}\gamma}}{\operatorname{Re}\gamma} \cdot \left| \frac{zh''(z)}{h'(z)} \right| \leq 1, \quad z \in U, \quad (10)$$

and by applying Lemma 1.2, we obtain that the function $I_\alpha^\zeta(f, g)(z)$ is in the class S. ■

If we put $\zeta = 1$ in Theorem 2.1, we obtain

Corollary 2.2 *Let α, γ be complex numbers, $0 < \operatorname{Re}\gamma \leq 1$, M_0 the positive solution of the equation (3), $M_0 = 1,5936\dots$, and $f, g \in \mathcal{A}$. If*

$$\left| \frac{f''(z)}{f'(z)} \right| \leq M_0, \quad \left| \frac{g''(z)}{g'(z)} \right| \leq M_0, \quad z \in U,$$

and

$$2M_0\operatorname{Re}\gamma + (2\operatorname{Re}\gamma + 1)^{\frac{2\operatorname{Re}\gamma + 1}{2\operatorname{Re}\gamma}} \leq \frac{\operatorname{Re}\gamma \cdot (2\operatorname{Re}\gamma + 1)^{\frac{2\operatorname{Re}\gamma + 1}{2\operatorname{Re}\gamma}}}{|\alpha|},$$

then the integral operator

$$I_\alpha(f, g)(z) = \int_0^z \left[\frac{tf'(t)}{g(t)} \right]^\alpha dt,$$

is in the class S.

Putting $\operatorname{Re}\gamma = 1$ in Corollary 2.2, we obtain

Corollary 2.3 *Let α, γ be complex numbers, $0 < \operatorname{Re}\gamma \leq 1$, M_0 the positive solution of the equation (3), $M_0 = 1,5936\dots$, and $f, g \in \mathcal{A}$. If*

$$\left| \frac{f''(z)}{f'(z)} \right| \leq M_0, \quad \left| \frac{g''(z)}{g'(z)} \right| \leq M_0, \quad z \in U,$$

and

$$|\alpha| \leq \frac{3\sqrt{3}}{2M_0 + 3\sqrt{3}}$$

then the integral operator

$$I_\alpha(f, g)(z) = \int_0^z \left[\frac{tf'(t)}{g(t)} \right]^\alpha dt,$$

is in the class S.

This result was also obtained in [2].

In the following theorem we give sufficient conditions such that the integral operator $I_\alpha^\zeta(f, g)(z) \in S^*$.

Theorem 2.4 Let α, ζ be complex numbers, $M \geq 1$, $f \in \mathcal{A}$ and $g \in B(\mu, \lambda)$ such that

$$\left| \frac{zf''(z)}{f'(z)} \right| < 1 \quad \text{and} \quad |g(z)| < M, \quad z \in U.$$

If

$$|\alpha| \leq \frac{|\zeta|}{2 + (2 - \lambda)M^{\mu-1}},$$

then the integral operator $I_{\alpha}^{\zeta}(f, g)(z)$ is in the class S^* .

Proof. Let's consider the function φ given by

$$\varphi(z) = I_{\alpha}^{\zeta}(f, g)(z), \quad z \in U. \quad (11)$$

Then, by differentiating φ with respect to z , we obtain

$$\frac{z\varphi'(z)}{\varphi(z)} = \frac{z^{\alpha+\zeta} \left[\frac{f'(z)}{g(z)} \right]^{\alpha}}{\zeta \int_0^z t^{\alpha+\zeta-1} \left(\frac{f'(t)}{g(t)} \right)^{\alpha} dt}.$$

Letting

$$p(z) = z\varphi'(z) \text{ and } q(z) = \varphi(z),$$

we find that

$$\frac{p'(z)}{q'(z)} = 1 + \frac{\alpha}{\zeta} \left[1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} \right].$$

Thus,

$$\begin{aligned} \left| \frac{p'(z)}{q'(z)} - 1 \right| &\leq \frac{|\alpha|}{|\zeta|} \left[1 + \left| \frac{zf''(z)}{f'(z)} \right| + \left| \frac{zg'(z)}{g(z)} \right| \right] \\ &\leq \frac{|\alpha|}{|\zeta|} \left[1 + \left| \frac{zf''(z)}{f'(z)} \right| + \left(\left| g'(z) \cdot \left(\frac{z}{g(z)} \right)^{\mu} - 1 \right| + 1 \right) \left| \frac{g(z)}{z} \right|^{\mu-1} \right]. \end{aligned} \quad (12)$$

Since $|g(z)| < M$, $z \in U$, by applying the Schwarz Lemma, we have

$$\left| \frac{g(z)}{z} \right| \leq M, \text{ for all } z \in U. \quad (13)$$

By using the hypothesis and (13) we obtain

$$\left| \frac{p'(z)}{q'(z)} - 1 \right| \leq \frac{|\alpha|}{|\zeta|} [2 + (2 - \lambda) \cdot M^{\mu-1}] \leq 1,$$

that is

$$\operatorname{Re} \left[\frac{p'(z)}{q'(z)} \right] > 0, \quad z \in U.$$

Therefore, applying Lemma 1.4, we find that

$$\operatorname{Re} \left[\frac{p(z)}{q(z)} \right] > 0, \quad z \in U.$$

This completes the proof. of the theorem. ■

Taking $\mu = 1$ in Theorem 2.4, we have

Corollary 2.5 Let α, ζ be complex numbers, $M \geq 1$, $f \in \mathcal{A}$ and $g \in S^*(\lambda)$ such that

$$\left| \frac{zf''(z)}{f'(z)} \right| < 1 \quad \text{and} \quad |g(z)| < M, \quad z \in U.$$

If

$$|\alpha| \leq \frac{|\zeta|}{4 - \lambda},$$

then the integral operator $I_\alpha(f, g)$ is in the class S^* .

Letting $\lambda = 0$ in Corollary 2.5, we obtain

Corollary 2.6 Let α, ζ be complex numbers with $|\alpha| = \frac{|\zeta|}{4}$ and $M \geq 1$. If $f \in \mathcal{A}$ and $g \in S^*$ satisfies

$$\left| \frac{zf''(z)}{f'(z)} \right| < 1 \quad \text{and} \quad |g(z)| < M, \quad z \in U,$$

then the integral operator $I_\alpha^\zeta(f, g)$ is in the class S^* .

Next, we find sufficient conditions such that $I_\alpha^\zeta(f, g)(z) \in K(\delta)$.

Theorem 2.7 Let α, ζ be complex numbers, $M, N \geq 1$, $f \in \mathcal{A}$ and $g \in B(\mu, \lambda)$. If

$$|g(z)| < M \quad \text{and} \quad \left| \frac{f''(z)}{f'(z)} \right| < N,$$

for all $z \in U$ then, the integral operator $I_\alpha^\zeta(f, g)$ is in the class $K(\delta)$, where

$$\delta = 1 - \left| \frac{\alpha}{\zeta} \right| [1 + N + (2 - \lambda)M^{\mu-1}] \quad \text{and} \quad 0 < \left| \frac{\alpha}{\zeta} \right| [1 + N + (2 - \lambda)M^{\mu-1}] \leq 1.$$

Proof. Letting the function φ be given by (11), we have

$$\frac{z\varphi''(z)}{\varphi'(z)} = \frac{\alpha}{\zeta} \left[1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)} \right].$$

Therefore, using the hypothesis of the theorem and applying the Schwarz Lemma, we obtain

$$\begin{aligned} \left| \frac{z\varphi''(z)}{\varphi'(z)} \right| &\leq \left| \frac{\alpha}{\zeta} \right| \left[1 + \left| \frac{zf''(z)}{f'(z)} \right| + \left| \frac{zg'(z)}{g(z)} \right| \right] \leq \left| \frac{\alpha}{\zeta} \right| \left[1 + N + \left| \frac{zg'(z)}{g(z)} \cdot \left(\frac{z}{g(z)} \right)^\mu \right| \cdot \left| \left(\frac{g(z)}{z} \right)^{\mu-1} \right| \right] \\ &\leq \left| \frac{\alpha}{\zeta} \right| \left[1 + N + \left[\left| g'(z) \left(\frac{z}{g(z)} \right)^\mu - 1 \right| + 1 \right] \cdot M^{\mu-1} \right] \leq \left| \frac{\alpha}{\zeta} \right| [1 + N + (2 - \lambda) \cdot M^{\mu-1}] = 1 - \delta. \end{aligned}$$

This evidently completes the proof. ■

Letting $\mu = 1$ in Theorem 2.7, we have

Corollary 2.8 Let α, ζ be complex numbers, $M, N \geq 1$, $f \in \mathcal{A}$ and $g \in S^*(\lambda)$. If

$$|g(z)| < M \quad \text{and} \quad \left| \frac{f''(z)}{f'(z)} \right| < N,$$

for all $z \in U$ then, the integral operator $I_\alpha^\zeta(f, g)$ is in the class $K(\delta)$, where

$$\delta = 1 - \left| \frac{\alpha}{\zeta} \right| (3 + N - \lambda) \quad \text{and} \quad 0 < \left| \frac{\alpha}{\zeta} \right| (3 + N - \lambda) \leq 1.$$

Letting $\delta = \lambda = 0$ in Corollary 2.8, we obtain

Corollary 2.9 Let α, ζ be complex numbers, $M, N \geq 1$, $f \in \mathcal{A}$ and $g \in S^*$. If

$$|g(z)| < M \quad \text{and} \quad \left| \frac{f''(z)}{f'(z)} \right| < N$$

for all $z \in U$ then, the integral operator $I_\alpha^\zeta(f, g)$ is convex in U , where

$$|\alpha| = \frac{|\zeta|}{3 + N}.$$

Theorem 2.10 If α, ζ are complex numbers and $f, g \in K(\delta)$ then $I_\zeta^\alpha(f, g)$ belongs to the class $K(b)$, where $b = 1 - \left| \frac{\alpha}{\zeta} \right| (2 - \delta - \nu(\delta))$, $0 \leq b < 1$ and $\nu(\delta)$ is given by Lemma 1.5.

Proof. Letting the function φ be given by (11), we have

$$\left| \frac{z\varphi''(z)}{\varphi'(z)} \right| \leq \left| \frac{\alpha}{\zeta} \right| \left[\left| \frac{zf''(z)}{f'(z)} \right| + \left| \frac{zg'(z)}{g(z)} - 1 \right| \right].$$

Since $g \in K(\delta)$, by applying Lemma 1.5, we yield that $g \in S^*(\nu(\delta))$. So,

$$\left| \frac{z\varphi''(z)}{\varphi'(z)} \right| \leq \left| \frac{\alpha}{\zeta} \right| (2 - \delta - \nu(\delta)) = 1 - b, \quad (14)$$

which evidently proves Theorem 2.10. ■

Corollary 2.11 Let α, ζ be complex numbers with $|\alpha/\zeta| \leq 2/3$. If $f, g \in K$ then $I_\zeta^\alpha(f, g)$ belongs to the class $K(1 - \frac{3}{2} \left| \frac{\alpha}{\zeta} \right|)$.

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A new generalization of the Ostrowski inequality and Ostrowski type inequality for double integrals on time scales

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Abstract: A new generalization of the Ostrowski inequality and Ostrowski type inequality for double integrals on time scales are established in this paper. Several interesting inequalities representing special cases of our general results are supplied.

Keywords: Ostrowski type inequalities; Double integrals; Time scales.

1 Introduction

In 1938, Ostrowski [21] proved the following interesting integral inequality.

Theorem 1.1 *Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b) and its derivative $f' : (a, b) \rightarrow \mathbb{R}$ is bounded in (a, b) . Then for any $x \in [a, b]$, we have*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left(\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right) (b-a) \|f'\|_{\infty}.$$

where $\|f'\|_{\infty} := \sup_{t \in (a,b)} |f'(x)| < \infty$. The inequality is sharp in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.

Mohammad Masjed-Jamei and Sever S. Dragomir[11] established the generalization of the Ostrowski inequality for functions in L^p -spaces and applied it to find appropriate error bounds for numerical quadrature rules of equal coefficients type using kernel (3.2) on $[a, b]$.

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The Ostrowski inequality and the Montgomery identity were generalized by Bohner et. al.[7] to an arbitrary time scale, unifying the discrete, the continuous, and the quantum cases:

Theorem 1.2 $a, b, s, t \in \mathbb{T}$, $a < b$ and $f : [a, b] \rightarrow \mathbb{R}$ be differentiable. Then

$$\left| f(t) - \frac{1}{b-a} \int_a^b f(\sigma(s)) \Delta s \right| \leq \frac{M}{b-a} (h_2(t, a) + h_2(t, b)), \quad (1.1)$$

where $h_2(.,.)$ is defined by Definition 8 and $M = \sup_{a < t < b} |f^\Delta(t)| < \infty$. This inequality is sharp in the sense that the right-hand side of (1.1) cannot be replaced by a smaller one.

During the past few years, many researchers have given considerable attention to the Ostrowski inequality on time scales. In [16, 17, 18], variants generalizations, extensions of Ostrowski inequality on time scales have established.

In 1988, S. Hilger [10] introduced the time scales theory to unify continuous and discrete analysis. For other results of Ostrowski type inequalities involving functions of two independent variables for multiple points, the Ostrowski type inequalities involving functions of two independent variables for k^2 points, generalized double integral Ostrowski type inequalities, Ostrowski type inequalities for double integrals, Ostrowski type inequality for double integrals on time scales via $\Delta\Delta$ -integral, Ostrowski and Grüss type inequalities for triple integrals, weighted Grüss type inequalities for double integrals, Grüss type inequalities, the Ostrowski type inequality for double integrals, generalized n dimensional Ostrowski type and Grüss type integral inequalities, generalized 2D Ostrowski-Grüss type integral inequalities on time scales see the papers [8, 9, 12, 14, 15, 19, 20, 22, 23, 24, 25], respectively.

This paper is organized as follows. In Section 2, we briefly present the general definitions and theorems related to the time scales calculus. A new generalization of the Ostrowski inequality and Ostrowski type inequality for double integrals are derived in Section 3. We also apply our results to the continuous and discrete calculus cases.

2 General Definitions

In this section we briefly introduce the time scales theory. For further details and proofs we refer the reader to Hilger's Ph.D. thesis [10], the books [2, 3, 13], and the survey [1].

Definition 2.1 A time scale is an arbitrary nonempty closed subset of the real numbers \mathbb{R} .

Throughout this work we assume \mathbb{T} is a time scale and \mathbb{T} has the topology that is inherited from the standard topology on \mathbb{R} . It is also assumed throughout that in \mathbb{T} the interval $[a, b]$ means the set $\{t \in \mathbb{T} : a \leq t \leq b\}$ for the points $a < b$ in \mathbb{T} . Since a time scale may not be connected, we need the following concept of jump operators.

Definition 2.2 The forward and backward jump operators $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{T}$ are defined by

$$\sigma(t) = \inf \{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup \{s \in \mathbb{T} : s < t\},$$

respectively.

The jump operators σ and ρ allow the classification of points in \mathbb{T} as follows.

Definition 2.3 If $\sigma(t) > t$, then we say that t is right-scattered, while if $\rho(t) < t$ then we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. If $\sigma(t) = t$, then t is called right-dense, and if $\rho(t) = t$ then t is called left-dense. Points that are both right-dense and left-dense are called dense.

Definition 2.4 The graininess function $\mu : \mathbb{T} \rightarrow [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$ for $t \in \mathbb{T}$. The set \mathbb{T}^k is defined as follows: if \mathbb{T} has a left-scattered maximum m , then $\mathbb{T}^k = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^k = \mathbb{T}$.

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$, and when $\mathbb{T} = \mathbb{Z}$, we have $\mu(t) = 1$.

Definition 2.5 Assume $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function and fix $t \in \mathbb{T}$. Then the (delta) derivative $f^\Delta(t) \in \mathbb{R}$ at $t \in \mathbb{T}^k$ is defined to be number (provided it exists) with property that given for any $\epsilon > 0$ there exists a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t) [\sigma(t) - s]| \leq \epsilon |\sigma(t) - s|, \quad \forall s \in U.$$

If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = \frac{df(t)}{dt}$, and if $\mathbb{T} = \mathbb{Z}$, then $\Delta f(t) = f(t+1) - f(t)$.

Theorem 2.6 Assume $f, g : \mathbb{T} \rightarrow \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$. Then the product $fg : \mathbb{T} \rightarrow \mathbb{R}$ is differentiable at t with

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t).$$

Definition 2.7 The function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be rd-continuous (denote $f \in C_{rd}(\mathbb{T}, \mathbb{R})$), if it is continuous at all right-dense points $t \in \mathbb{T}$ and its left-sided limits exist at all left-dense points $t \in \mathbb{T}$.

It follows from [2, Theorem 1.74] that every rd-continuous function has an anti-derivative.

Definition 2.8 A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \rightarrow \mathbb{R}$ provided $F^\Delta(t) = f(t)$ for any $t \in \mathbb{T}^k$. In this case, we define the Δ -integral of f as

$$\int_a^b f(s) \Delta s := F(t) - F(a), \quad t \in \mathbb{T}.$$

Theorem 2.9 Let f, g be rd-continuous, $a, b, c \in \mathbb{T}$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\begin{aligned} (1) \quad & \int_a^b [\alpha f(t) + \beta g(t)] \Delta t = \alpha \int_a^b f(t) \Delta t + \beta \int_a^b g(t) \Delta t, \\ (2) \quad & \int_a^b f(t) \Delta t = - \int_b^a f(t) \Delta t, \\ (3) \quad & \int_a^b f(t) \Delta t = \int_a^c f(t) \Delta t + \int_c^b f(t) \Delta t \\ (4) \quad & \int_a^b f(t) g^\Delta(t) \Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t) g^\sigma(t) \Delta t, \end{aligned}$$

Theorem 2.10 *If f is Δ -integrable on $[a, b]$, then so is $|f|$, and*

$$\left| \int_a^b f(t) \Delta t \right| \leq \int_a^b |f(t)| \Delta t.$$

Definition 2.11 *Let $h_k : \mathbb{T}^2 \rightarrow \mathbb{R}$, $k \in \mathbb{N}_0$ be defined by $h_0(t, s) = 1$, for all $s, t \in \mathbb{T}$ and then recursively by $h_{k+1}(t, s) = \int_s^t h_k(\tau, s) \Delta \tau$, for all $s, t \in \mathbb{T}$.*

The two-variable time scales calculus and multiple integration on time scales were introduced in [4, 5] (see also [6]). Let \mathbb{T}_1 and \mathbb{T}_2 be two time scales and put

$$\mathbb{T}_1 \times \mathbb{T}_2 = \{(t, s) : t \in \mathbb{T}_1, s \in \mathbb{T}_2\},$$

which is a complete metric space with the metric d defined by

$$d((t, s), (t', s')) = \sqrt{(t - t')^2 + (s - s')^2}, \quad \forall (t, s), (t', s') \in \mathbb{T}_1 \times \mathbb{T}_2.$$

For a given $\delta > 0$, the δ -neighborhood $U_\delta(t_0, s_0)$ of a given point $(t_0, s_0) \in \mathbb{T}_1 \times \mathbb{T}_2$ is the set of all points $(t, s) \in \mathbb{T}_1 \times \mathbb{T}_2$ such that $d((t, s), (t', s')) < \delta$. Let σ_1, ρ_1 and σ_2, ρ_2 be the forward jump and backward jump operators in \mathbb{T}_1 and \mathbb{T}_2 , respectively.

Let \mathbb{T}_1 and \mathbb{T}_2 be two time scales. For $i = 1, 2$, let σ_i, ρ_i and Δ_i denote the forward jump operator, the backward jump operator, and the delta differentiation operator, respectively, on \mathbb{T}_i . Suppose $a < b$ are points in \mathbb{T}_1 , $c < d$ are points in \mathbb{T}_2 , $[a, b]$ is the half-closed bounded interval in \mathbb{T}_1 , $[c, d]$ is the half-closed bounded interval in \mathbb{T}_2 . Let us introduce a "rectangle" in $\mathbb{T}_1 \times \mathbb{T}_2$ by

$$R = [a, b] \times [c, d] = \{(t_1, t_2) : t_1 \in [a, b], t_2 \in [c, d]\}.$$

3 Main Results

To derive main results in this section, we need the following Lemma.

Lemma 3.1 *Let $a, b, t \in \mathbb{T}_1$ and $c, d, s \in \mathbb{T}_2$ and $f \in CC_{rd}^1([a, b] \times [c, d], \mathbb{R})$. Then we have*

$$\begin{aligned} & w_1 w_2 f(x, y) \\ = & \int_a^b \int_c^d K_{w_1}(x, t) K_{w_2}(y, s) \frac{\partial^2 f(t, s)}{\Delta_2 s \Delta_1 t} \Delta_2 s \Delta_1 t - F_1(x, y) - F_2 + w_2 \int_a^b f(\sigma(t), y) \Delta_1 t \\ & + w_1 \int_c^d f(x, \sigma(s)) \Delta_2 s - \int_a^b [(c - \beta_1) f(\sigma(t), c) - (d - \beta_2) f(\sigma(t), d)] \Delta_1 t \\ & - \int_c^d [(a - \theta_1) f(a, \sigma(s)) - (b - \theta_2) f(b, \sigma(s))] \Delta_2 s - \int_a^b \int_c^d f(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \quad (3.1) \end{aligned}$$

for all $(x, y) \in [a, b] \times [c, d]$, where

$$K_{w_1}(x, t) = \begin{cases} t - \frac{(b-w_1)f(b)-af(a)}{f(b)-f(a)} = t - \theta_1, & a \leq t \leq x \\ t - \frac{bf(b)-(a+w_1)f(a)}{f(b)-f(a)} = t - \theta_2, & x < t \leq b \end{cases} \quad (3.2)$$

$$K_{w_2}(y, s) = \begin{cases} s - \frac{(d-w_2)f(d)-cf(c)}{f(d)-f(c)} = s - \beta_1, & c \leq s \leq y \\ s - \frac{df(d)-(c+w_2)f(c)}{f(d)-f(c)} = s - \beta_2, & y < s \leq d \end{cases} \quad (3.3)$$

in which $w_1, w_2 \in \mathbb{R}$, $f(b) \neq f(a)$, $f(d) \neq f(c)$, $\theta_2 - \theta_1 = w_1$, $\beta_2 - \beta_1 = w_2$,

$$F_1(x, y) = w_1[(d - \beta_2)f(x, d) - (c - \beta_1)f(x, c)] + w_2[(b - \theta_2)f(b, y) - (a - \theta_1)f(a, y)],$$

and

$$F_2 = (a - \theta_1)[(c - \beta_1)f(a, c) - (d - \beta_2)f(a, d)] + (b - \theta_2)[(d - \beta_2)f(b, d) - (c - \beta_1)f(b, c)].$$

Proof. Integrating by parts and considering (3.2) and (3.3), we get

$$\begin{aligned} & \int_a^b \int_c^d K_{w_1}(x, t) K_{w_2}(y, s) \frac{\partial^2 f(t, s)}{\Delta_2 s \Delta_1 t} \Delta_2 s \Delta_1 t \\ &= \int_a^x \int_c^y (t - \theta_1)(s - \beta_1) \frac{\partial^2 f(t, s)}{\Delta_2 s \Delta_1 t} \Delta_2 s \Delta_1 t + \int_a^x \int_y^d (t - \theta_1)(s - \beta_2) \frac{\partial^2 f(t, s)}{\Delta_2 s \Delta_1 t} \Delta_2 s \Delta_1 t \\ &+ \int_x^b \int_c^y (t - \theta_2)(s - \beta_1) \frac{\partial^2 f(t, s)}{\Delta_2 s \Delta_1 t} \Delta_2 s \Delta_1 t + \int_x^b \int_y^d (t - \theta_2)(s - \beta_2) \frac{\partial^2 f(t, s)}{\Delta_2 s \Delta_1 t} \Delta_2 s \Delta_1 t \quad (3.4) \end{aligned}$$

We have

$$\begin{aligned} & \int_a^x \int_c^y (t - \theta_1)(s - \beta_1) \frac{\partial^2 f(t, s)}{\Delta_2 s \Delta_1 t} \Delta_2 s \Delta_1 t \\ &= \int_a^x (t - \theta_1) \left[(y - \beta_1) \frac{\partial f(t, y)}{\Delta_1 t} - (c - \beta_1) \frac{\partial f(t, c)}{\Delta_1 t} - \int_c^y \frac{\partial f(t, \sigma(s))}{\Delta_1 t} \Delta_2 s \right] \Delta_1 t \\ &= (y - \beta_1) \int_a^x (t - \theta_1) \frac{\partial f(t, y)}{\Delta_1 t} \Delta_1 t - (c - \beta_1) \int_a^x (t - \theta_1) \frac{\partial f(t, c)}{\Delta_1 t} \Delta_1 t \\ &\quad - \int_c^y \left(\int_a^x (t - \theta_1) \frac{\partial f(t, \sigma(s))}{\Delta_1 t} \Delta_1 t \right) \Delta_2 s \end{aligned}$$

$$\begin{aligned}
&= (y - \beta_1) \left[(x - \theta_1) f(x, y) - (a - \theta_1) f(a, y) - \int_a^x f(\sigma(t), y) \Delta_1 t \right] \\
&\quad - (c - \beta_1) \left[(x - \theta_1) f(x, c) - (a - \theta_1) f(a, c) - \int_a^x f(\sigma(t), c) \Delta_1 t \right] \\
&\quad - \int_c^y \left[(x - \theta_1) f(x, \sigma(s)) - (a - \theta_1) f(a, \sigma(s)) - \int_a^x f(\sigma(t), \sigma(s)) \Delta_1 t \right] \Delta_2 s \\
&= (x - \theta_1)(y - \beta_1) f(x, y) - (a - \theta_1)(y - \beta_1) f(a, y) - (y - \beta_1) \int_a^x f(\sigma(t), y) \Delta_1 t \\
&\quad - (c - \beta_1)(x - \theta_1) f(x, c) + (a - \theta_1)(c - \beta_1) f(a, c) + (c - \beta_1) \int_a^x f(\sigma(t), c) \Delta_1 t \\
&\quad - (x - \theta_1) \int_c^y f(x, \sigma(s)) \Delta_2 s + (a - \theta_1) \int_c^y f(a, \sigma(s)) \Delta_2 s \\
&\quad + \int_a^x \int_c^y f(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t, \tag{3.5}
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
&\int_a^x \int_y^d (t - \theta_1)(s - \beta_2) \frac{\partial^2 f(t, s)}{\Delta_2 s \Delta_1 t} \Delta_2 s \Delta_1 t \\
&= (d - \beta_2)(x - \theta_1) f(x, d) - (a - \theta_1)(d - \beta_2) f(a, d) - (d - \beta_2) \int_a^x f(\sigma(t), d) \Delta_1 t \\
&\quad - (x - \theta_1)(y - \beta_2) f(x, y) + (a - \theta_1)(y - \beta_2) f(a, y) + (y - \beta_2) \int_a^x f(\sigma(t), y) \Delta_1 t \\
&\quad - (x - \theta_1) \int_y^d f(x, \sigma(s)) \Delta_2 s + (a - \theta_1) \int_y^d f(a, \sigma(s)) \Delta_2 s \\
&\quad + \int_a^x \int_y^d f(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t, \tag{3.6}
\end{aligned}$$

and

$$\begin{aligned}
& \int_x^b \int_c^y (t - \theta_2) (s - \beta_1) \frac{\partial^2 f(t, s)}{\Delta_2 s \Delta_1 t} \Delta_2 s \Delta_1 t \\
&= (b - \theta_2) (y - \beta_1) f(b, y) - (x - \theta_2) (y - \beta_1) f(x, y) - (y - \beta_1) \int_x^b f(\sigma(t), y) \Delta_1 t \\
&\quad - (b - \theta_2) (c - \beta_1) f(b, c) + (c - \beta_1) (x - \theta_2) f(x, c) + (c - \beta_1) \int_x^b f(\sigma(t), c) \Delta_1 t \\
&\quad - (b - \theta_2) \int_c^y f(b, \sigma(s)) \Delta_2 s + (x - \theta_2) \int_c^y f(x, \sigma(s)) \Delta_2 s + \int_x^b \int_c^y f(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t, \quad (3.7)
\end{aligned}$$

and finally

$$\begin{aligned}
& \int_x^b \int_y^d (t - \theta_2) (s - \beta_2) \frac{\partial^2 f(t, s)}{\Delta_2 s \Delta_1 t} \Delta_2 s \Delta_1 t \\
&= (b - \theta_2) (d - \beta_2) f(b, d) - (d - \beta_2) (x - \theta_2) f(x, d) - (d - \beta_2) \int_x^b f(\sigma(t), d) \Delta_1 t \\
&\quad - (b - \theta_2) (y - \beta_2) f(b, y) + (x - \theta_2) (y - \beta_2) f(x, y) + (y - \beta_2) \int_x^b f(\sigma(t), y) \Delta_1 t \\
&\quad - (b - \theta_2) \int_y^d f(b, \sigma(s)) \Delta_2 s + (x - \theta_2) \int_y^d f(x, \sigma(s)) \Delta_2 s \\
&\quad + \int_x^b \int_y^d f(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t, \quad (3.8)
\end{aligned}$$

Substituting (3.5)-(3.8) into (3.4), we obtain the result (3.1). ■

Corollary 3.2 *In the Lemma 3.1, we choose $w_1 = b - a$, $w_2 = d - c$ and hence $\theta_1 = a$, $\theta_2 = b$, $\beta_1 = c$, $\beta_2 = d$. Then by simple computation, we get*

$$\begin{aligned}
\int_a^b \int_c^d K_{w_1}(x, t) K_{w_2}(y, s) \frac{\partial^2 f(t, s)}{\Delta_2 s \Delta_1 t} \Delta_2 s \Delta_1 t &= (b - a) (d - c) f(x, y) - (d - c) \int_a^b f(\sigma(t), y) \Delta_1 t \\
&\quad - (b - a) \int_c^d f(x, \sigma(s)) \Delta_2 s + \int_a^b \int_c^d f(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t
\end{aligned}$$

This is the result given in [22, Lemma 2.3].

The following Theorem is a new generalization of the Ostrowski inequality for double integrals on time scales.

Theorem 3.3 *Let the assumptions of Lemma 3.1 hold. Assume that $\sup_{a < t < b; c < s < d} \left| \frac{\partial^2 f(t,s)}{\Delta_2 s \Delta_1 t} \right| < \infty$. Then we have the inequality*

$$\begin{aligned} & \left| w_1 w_2 f(x, y) + F_1(x, y) + F_2 - w_2 \int_a^b f(\sigma(t), y) \Delta_1 t \right. \\ & \quad - w_1 \int_c^d f(x, \sigma(s)) \Delta_2 s + \int_a^b [(c - \beta_1) f(\sigma(t), c) - (d - \beta_2) f(\sigma(t), d)] \Delta_1 t \\ & \quad \left. + \int_c^d [(a - \theta_1) f(a, \sigma(s)) - (b - \theta_2) f(b, \sigma(s))] \Delta_2 s + \int_a^b \int_c^d f(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \right| \\ & \leq K \left[\int_a^x |t - \theta_1| \Delta_1 t + \int_x^b |t - \theta_2| \Delta_1 t \right] \left[\int_c^y |s - \beta_1| \Delta_2 s + \int_y^d |s - \beta_2| \Delta_2 s \right] \end{aligned} \quad (3.9)$$

$$\text{where } K = \sup_{a < t < b; c < s < d} \left| \frac{\partial^2 f(t,s)}{\Delta_2 s \Delta_1 t} \right|,$$

$$F_1(x, y) = w_1 [(d - \beta_2) f(x, d) - (c - \beta_1) f(x, c)] + w_2 [(b - \theta_2) f(b, y) - (a - \theta_1) f(a, y)],$$

and

$$F_2 = (a - \theta_1) [(c - \beta_1) f(a, c) - (d - \beta_2) f(a, d)] + (b - \theta_2) [(d - \beta_2) f(b, d) - (c - \beta_1) f(b, c)].$$

Proof. By applying Lemma 3.1 and using the properties of modulus, we can state that

$$\begin{aligned} & \left| w_1 w_2 f(x, y) + F_1(x, y) + F_2 - w_2 \int_a^b f(\sigma(t), y) \Delta_1 t \right. \\ & \quad - w_1 \int_c^d f(x, \sigma(s)) \Delta_2 s + \int_a^b [(c - \beta_1) f(\sigma(t), c) - (d - \beta_2) f(\sigma(t), d)] \Delta_1 t \\ & \quad \left. + \int_c^d [(a - \theta_1) f(a, \sigma(s)) - (b - \theta_2) f(b, \sigma(s))] \Delta_2 s + \int_a^b \int_c^d f(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \right| \\ & \leq \int_a^b \int_c^d |K_{w_1}(x, t)| |K_{w_2}(y, s)| \left| \frac{\partial^2 f(t,s)}{\Delta_2 s \Delta_1 t} \right| \Delta_2 s \Delta_1 t \end{aligned}$$

where $K_{w_1}(x, t)$ and $K_{w_2}(y, s)$ are given by (3.2) and (3.3). The proof is complete. ■

Theorem 3.4 Let $a, b, t \in \mathbb{T}_1$ and $c, d, s \in \mathbb{T}_2$ and $f \in CC_{rd}^1([a, b] \times [c, d], \mathbb{R})$. Assume that $\sup_{a < t < b; c < s < d} \left| \frac{\partial^2 f(t, s)}{\Delta_2 s \Delta_1 t} \right| < \infty$ and $\sup_{a < t < b; c < s < d} \left| \frac{\partial^2 g(t, s)}{\Delta_2 s \Delta_1 t} \right| < \infty$. Then for all $(x, y) \in [a, b] \times [c, d]$, we have the inequality

$$\begin{aligned}
& \left| \int_a^b \int_c^d f(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \int_a^b \int_c^d g(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \right. \\
& + \frac{w_1 w_2}{2} \left[f(x, y) \int_a^b \int_c^d g(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t + g(x, y) \int_a^b \int_c^d f(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \right] \\
& + \frac{1}{2} \left[F_1(x, y) \int_a^b \int_c^d g(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t + G_1(x, y) \int_a^b \int_c^d f(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \right] \\
& + \frac{1}{2} \left[F_2 \int_a^b \int_c^d g(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t + G_2 \int_a^b \int_c^d f(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \right] \\
& - \frac{w_2}{2} \left[\int_a^b f(\sigma(t), y) \Delta_1 t \int_a^b \int_c^d g(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \right. \\
& \left. + \int_a^b g(\sigma(t), y) \Delta_1 t \int_a^b \int_c^d f(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \right] \\
& - \frac{w_1}{2} \left[\int_c^d f(x, \sigma(s)) \Delta_2 s \int_a^b \int_c^d g(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \right. \\
& \left. + \int_c^d g(x, \sigma(s)) \Delta_2 s \int_a^b \int_c^d f(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \right] \\
& + \frac{1}{2} \left[\int_a^b [(c - \beta_1) f(\sigma(t), c) - (d - \beta_2) f(\sigma(t), d)] \Delta_1 t \int_a^b \int_c^d g(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \right. \\
& \left. + \int_a^b [(c - \beta_1) g(\sigma(t), c) - (d - \beta_2) g(\sigma(t), d)] \Delta_1 t \int_a^b \int_c^d f(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \right] \\
& + \frac{1}{2} \left[\int_c^d [(a - \theta_1) f(a, \sigma(s)) - (b - \theta_2) f(b, \sigma(s))] \Delta_2 s \int_a^b \int_c^d g(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \right. \\
& \left. + \int_c^d [(a - \theta_1) g(a, \sigma(s)) - (b - \theta_2) g(b, \sigma(s))] \Delta_2 s \int_a^b \int_c^d f(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \right] \Big|
\end{aligned}$$

$$\leq \frac{1}{2} \left[K \int_a^b \int_c^d |g(\sigma(t), \sigma(s))| \Delta_2 s \Delta_1 t + L \int_a^b \int_c^d |f(\sigma(t), \sigma(s))| \Delta_2 s \Delta_1 t \right] \\ \times \left[\int_a^x |t - \theta_1| \Delta_1 t + \int_x^b |t - \theta_2| \Delta_1 t \right] \left[\int_c^y |s - \beta_1| \Delta_2 s + \int_y^d |s - \beta_2| \Delta_2 s \right] \quad (3.10)$$

$$\text{where } K = \sup_{a < t < b; c < s < d} \left| \frac{\partial^2 f(t, s)}{\Delta_2 s \Delta_1 t} \right|, \quad L = \sup_{a < t < b; c < s < d} \left| \frac{\partial^2 g(t, s)}{\Delta_2 s \Delta_1 t} \right|.$$

Proof. From (3.1), we have following identities

$$\int_a^b \int_c^d f(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t = -w_1 w_2 f(x, y) - F_1(x, y) - F_2 \\ + w_2 \int_a^b f(\sigma(t), y) \Delta_1 t + w_1 \int_c^d f(x, \sigma(s)) \Delta_2 s \\ - \int_a^b [(c - \beta_1) f(\sigma(t), c) - (d - \beta_2) f(\sigma(t), d)] \Delta_1 t \\ - \int_c^d [(a - \theta_1) f(a, \sigma(s)) - (b - \theta_2) f(b, \sigma(s))] \Delta_2 s \\ + \int_a^b \int_c^d K_{w_1}(x, t) K_{w_2}(y, s) \frac{\partial^2 f(t, s)}{\Delta_2 s \Delta_1 t} \Delta_2 s \Delta_1 t \quad (3.11)$$

in which

$$F_1(x, y) = w_1 [(d - \beta_2) f(x, d) - (c - \beta_1) f(x, c)] + w_2 [(b - \theta_2) f(b, y) - (a - \theta_1) f(a, y)],$$

and

$$F_2 = (a - \theta_1) [(c - \beta_1) f(a, c) - (d - \beta_2) f(a, d)] + (b - \theta_2) [(d - \beta_2) f(b, d) - (c - \beta_1) f(b, c)]$$

and similarly

$$\int_a^b \int_c^d g(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t = -w_1 w_2 g(x, y) - G_1(x, y) - G_2 \\ + w_2 \int_a^b g(\sigma(t), y) \Delta_1 t + w_1 \int_c^d g(x, \sigma(s)) \Delta_2 s \\ - \int_a^b [(c - \beta_1) g(\sigma(t), c) - (d - \beta_2) g(\sigma(t), d)] \Delta_1 t$$

$$\begin{aligned}
& - \int_c^d [(a - \theta_1) g(a, \sigma(s)) - (b - \theta_2) g(b, \sigma(s))] \Delta_2 s \\
& + \int_a^b \int_c^d K_{w_1}(x, t) K_{w_2}(y, s) \frac{\partial^2 g(t, s)}{\Delta_2 s \Delta_1 t} \Delta_2 s \Delta_1 t
\end{aligned} \tag{3.12}$$

$$G_1(x, y) = w_1 [(d - \beta_2) g(x, d) - (c - \beta_1) g(x, c)] + w_2 [(b - \theta_2) g(b, y) - (a - \theta_1) g(a, y)],$$

and

$$G_2 = (a - \theta_1) [(c - \beta_1) g(a, c) - (d - \beta_2) g(a, d)] + (b - \theta_2) [(d - \beta_2) g(b, d) - (c - \beta_1) g(b, c)].$$

Now, multiplying both sides (3.11) and (3.12) by $\int_a^b \int_c^d g(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t$ and

$\int_a^b \int_c^d f(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t$, adding the resulting identities and taking absolute values, we get

$$\begin{aligned}
& \left| \int_a^b \int_c^d f(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \int_a^b \int_c^d g(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \right. \\
& + \frac{w_1 w_2}{2} \left[f(x, y) \int_a^b \int_c^d g(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t + g(x, y) \int_a^b \int_c^d f(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \right] \\
& + \frac{1}{2} \left[F_1(x, y) \int_a^b \int_c^d g(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t + G_1(x, y) \int_a^b \int_c^d f(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \right] \\
& + \frac{1}{2} \left[F_2 \int_a^b \int_c^d g(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t + G_2 \int_a^b \int_c^d f(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \right] \\
& - \frac{w_2}{2} \left[\int_a^b f(\sigma(t), y) \Delta_1 t \int_a^b \int_c^d g(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t + \int_a^b g(\sigma(t), y) \Delta_1 t \int_a^b \int_c^d f(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \right] \\
& - \frac{w_1}{2} \left[\int_c^d f(x, \sigma(s)) \Delta_2 s \int_a^b \int_c^d g(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t + \int_c^d g(x, \sigma(s)) \Delta_2 s \int_a^b \int_c^d f(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \right] \\
& + \frac{1}{2} \left[\int_a^b [(c - \beta_1) f(\sigma(t), c) - (d - \beta_2) f(\sigma(t), d)] \Delta_1 t \int_a^b \int_c^d g(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \right. \\
& \left. + \int_a^b [(c - \beta_1) g(\sigma(t), c) - (d - \beta_2) g(\sigma(t), d)] \Delta_1 t \int_a^b \int_c^d f(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \left[\int_c^d [(a - \theta_1) f(a, \sigma(s)) - (b - \theta_2) f(b, \sigma(s))] \Delta_2 s \int_a^b \int_c^d g(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \right. \\
& \left. + \int_c^d [(a - \theta_1) g(a, \sigma(s)) - (b - \theta_2) g(b, \sigma(s))] \Delta_2 s \int_a^b \int_c^d f(\sigma(t), \sigma(s)) \Delta_2 s \Delta_1 t \right] \Bigg| \\
& \leq \frac{1}{2} \left[K \int_a^b \int_c^d |g(\sigma(t), \sigma(s))| \Delta_2 s \Delta_1 t + L \int_a^b \int_c^d |f(\sigma(t), \sigma(s))| \Delta_2 s \Delta_1 t \right] \\
& \quad \times \int_a^b \int_c^d |K_w(x, t)| |K_w(y, s)| \Delta_2 s \Delta_1 t
\end{aligned}$$

Hence, we get the inequality (3.10). The proof is complete. ■

If we apply the Theorem 3.2 to different time scales, we will get some new results.

Corollary 3.5 *If we let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$ in Theorem 3.4, then we obtain the inequality*

$$\begin{aligned}
& \left| \int_a^b \int_c^d f(t, s) ds dt \int_a^b \int_c^d g(t, s) ds dt \right. \\
& + \frac{w_1 w_2}{2} \left[f(x, y) \int_a^b \int_c^d g(t, s) ds dt + g(x, y) \int_a^b \int_c^d f(t, s) ds dt \right] \\
& + \frac{1}{2} \left[F_1(x, y) \int_a^b \int_c^d g(t, s) ds dt + G_1(x, y) \int_a^b \int_c^d f(t, s) ds dt \right] \\
& + \frac{1}{2} \left[F_2 \int_a^b \int_c^d g(t, s) ds dt + G_2 \int_a^b \int_c^d f(t, s) ds dt \right] \\
& - \frac{w_2}{2} \left[\int_a^b f(t, y) \Delta_1 t \int_a^b \int_c^d g(t, s) ds dt + \int_a^b g(t, y) \Delta_1 t \int_a^b \int_c^d f(t, s) ds dt \right] \\
& - \frac{w_1}{2} \left[\int_c^d f(x, s) ds \int_a^b \int_c^d g(t, s) ds dt + \int_c^d g(x, s) ds \int_a^b \int_c^d f(t, s) ds dt \right] \\
& + \frac{1}{2} \left[\int_a^b [(c - \beta_1) f(t, c) - (d - \beta_2) f(t, d)] dt \int_a^b \int_c^d g(t, s) ds dt \right. \\
& \left. + \int_a^b [(c - \beta_1) g(t, c) - (d - \beta_2) g(t, d)] dt \int_a^b \int_c^d f(t, s) ds dt \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \left[\int_c^d [(a - \theta_1) f(a, s) - (b - \theta_2) f(b, s)] ds \int_a^b \int_c^d g(\sigma(t), s) ds dt \right. \\
& \left. + \int_c^d [(a - \theta_1) g(a, s) - (b - \theta_2) g(b, s)] ds \int_a^b \int_c^d f(t, s) ds dt \right] \\
& \leq \frac{1}{2} \left[K \int_a^b \int_c^d |g(t, s)| ds dt + L \int_a^b \int_c^d |f(t, s)| ds dt \right] \\
& \quad \times \left[\int_a^x |t - \theta_1| dt + \int_x^b |t - \theta_2| dt \right] \left[\int_c^y |s - \beta_1| ds + \int_y^d |s - \beta_2| ds \right]
\end{aligned}$$

for all $(x, y) \in [a, b] \times [c, d]$, where $K = \sup_{a < t < b; c < s < d} \left| \frac{\partial^2 f(t, s)}{\partial s \partial t} \right|$, $L = \sup_{a < t < b; c < s < d} \left| \frac{\partial^2 g(t, s)}{\partial s \partial t} \right|$.

This inequality is a new Ostrowski type inequality for double integrals in continuous case.

Corollary 3.6 If we let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$ in Theorem 3.4, then we obtain the inequality

$$\begin{aligned}
& \left| \sum_{t=a}^{b-1} \sum_{s=c}^{d-1} f(t+1, s+1) \sum_{t=a}^{b-1} \sum_{s=c}^{d-1} g(t+1, s+1) \right. \\
& + \frac{w_1 w_2}{2} \left[f(x, y) \sum_{t=a}^{b-1} \sum_{s=c}^{d-1} g(t+1, s+1) + g(x, y) \sum_{t=a}^{b-1} \sum_{s=c}^{d-1} f(t+1, s+1) \right] \\
& + \frac{1}{2} \left[F_1(x, y) \sum_{t=a}^{b-1} \sum_{s=c}^{d-1} g(t+1, s+1) + G_1(x, y) \sum_{t=a}^{b-1} \sum_{s=c}^{d-1} f(t+1, s+1) \right] \\
& + \frac{1}{2} \left[F_2 \sum_{t=a}^{b-1} \sum_{s=c}^{d-1} g(t+1, s+1) + \sum_{t=a}^{b-1} \sum_{s=c}^{d-1} f(t+1, s+1) \right] \\
& - \frac{w_2}{2} \left[\sum_{t=a}^{b-1} f(t+1, y) \sum_{t=a}^{b-1} \sum_{s=c}^{d-1} g(t+1, s+1) + \sum_{t=a}^{b-1} g(t+1, y) \sum_{t=a}^{b-1} \sum_{s=c}^{d-1} f(t+1, s+1) \right] \\
& - \frac{w_1}{2} \left[\sum_{s=c}^{d-1} f(x, s+1) \sum_{t=a}^{b-1} \sum_{s=c}^{d-1} g(t+1, s+1) + \int_c^d g(x, s+1) \sum_{t=a}^{b-1} \sum_{s=c}^{d-1} f(t+1, s+1) \right] \\
& + \frac{1}{2} \left[\sum_{t=a}^{b-1} [(c - \beta_1) f(t+1, c) - (d - \beta_2) f(t+1, d)] \sum_{t=a}^{b-1} \sum_{s=c}^{d-1} g(t+1, s+1) \right. \\
& + \sum_{t=a}^{b-1} [(c - \beta_1) g(t+1, c) - (d - \beta_2) g(t+1, d)] \sum_{t=a}^{b-1} \sum_{s=c}^{d-1} f(t+1, s+1) \left. \right] \\
& + \frac{1}{2} \left[\sum_{s=c}^{d-1} [(a - \theta_1) f(a, s+1) - (b - \theta_2) f(b, s+1)] \sum_{t=a}^{b-1} \sum_{s=c}^{d-1} g(t+1, s+1) \right.
\end{aligned}$$

$$\begin{aligned}
& + \sum_{s=c}^{d-1} [(a - \theta_1) g(a, s + 1) - (b - \theta_2) g(b, s + 1)] \sum_{t=a}^{b-1} \sum_{s=c}^{d-1} f(t + 1, s + 1) \Bigg| \\
\leq & \frac{1}{2} \left[K \sum_{t=a}^{b-1} \sum_{s=c}^{d-1} |g(t + 1, s + 1)| + L \sum_{t=a}^{b-1} \sum_{s=c}^{d-1} |f(t + 1, s + 1)| \right] \\
& \times \left[\sum_{t=a}^{x-1} |t - \theta_1| + \sum_{t=x}^{b-1} |t - \theta_2| \right] \left[\sum_{s=c}^{y-1} |s - \beta_1| + \sum_{s=y}^{d-1} |s - \beta_2| \right]
\end{aligned}$$

for all $(x, y) \in [a, b - 1] \times [c, d - 1]$, where K denotes the maximum value of the absolute value of the difference $\Delta_2 \Delta_1 f$ over $[a, b - 1]_{\mathbb{Z}} \times [c, d - 1]_{\mathbb{Z}}$ and L denotes the maximum value of the absolute value of the difference $\Delta_2 \Delta_1 g$ over $[a, b - 1]_{\mathbb{Z}} \times [c, d - 1]_{\mathbb{Z}}$.

This inequality is a new Ostrowski type inequality for double integrals in discrete case.

Note that to compute the integrals of the right hand side of inequalities (3.9) and (3.10), we need the following general identities:

$$\int_a^b |t - \theta| \Delta_1 t = \begin{cases} [h_2(a, \theta) + h_2(b, \theta)], & a < \theta < b \\ [h_2(b, \theta) - h_2(a, \theta)], & \theta < a < b \\ [h_2(a, \theta) - h_2(b, \theta)], & a < b < \theta, \end{cases}$$

and

$$\int_c^d |s - \beta| \Delta_2 s = \begin{cases} [h_2(c, \beta) + h_2(d, \beta)], & c < \beta < d \\ [h_2(d, \beta) - h_2(c, \beta)], & \beta < c < d \\ [h_2(c, \beta) - h_2(d, \beta)], & c < d < \beta. \end{cases}$$

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Approximate ternary Jordan bi-homomorphisms in Banach Lie triple systems

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Abstract. We prove the Hyers-Ulam stability of ternary Jordan bi-homomorphism in Banach Lie triple systems associated to the Cauchy functional equation.

1. INTRODUCTION AND PRELIMINARIES

We say that a functional equation (Q) is stable if any function g satisfying the equation (Q) approximately is near to true solution of (Q).

Ternary algebraic operations were considered in the 19th century by several mathematicians and physicists. Cayley [8] introduced the notion of cubic matrix which in turn was generalized by Kapranov, Gelfand and Zelevinskii [6]. As an application in physics, the quark model inspired a particular brand of ternary algebraic systems. The so-called Nambu mechanics which has been proposed by Nambu [11], is based on such structures. There are also some applications, although still hypothetical, in the fractional quantum Hall effect, the non-standard statistics (the anyons), supersymmetric theories, Yang-Baxter equation, etc, (cf. [15, 27]).

The comments on physical applications of ternary structures can be found in [1, 5, 10, 14, 17, 23, 24, 29].

A C^* -ternary algebra is a complex Banach space, equipped with a ternary product $(x, y, z) \rightarrow [x, y, z]$ of A^3 into A , which is \mathbb{C} -linear in the outer variables, conjugate \mathbb{C} -linear in the middle variable, and associative in the sense that $[x, y, [z, u, v]] = [x, [y, z, u]v] = [[x, y, z], u, v]$, and satisfies

$$\|[x, y, z]\| \leq \|x\| \cdot \|y\| \cdot \|z\|, \quad \|[x, x, x]\| = \|x\|^3$$

A normed (Banach) Lie triple system is a normed (Banach) space $(A, \|\cdot\|)$ with a trilinear mapping $(x, y, z) \mapsto [x, y, z]$ from $A \times A \times A$ to A satisfying the following axioms:

$$\begin{aligned} [x, y, z] &= -[y, x, z], \\ [x, y, z] &= -[y, z, x] - [z, x, y], \\ [u, v, [x, y, z]] &= [[u, v, x], y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]], \\ \|[x, y, z]\| &\leq \|x\| \|y\| \|z\| \end{aligned}$$

for all $u, v, x, y, z \in A$ (see [12, 16]).

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Approximate ternary Jordan bi-homomorphisms

Definition 1.1. Let A and B be normed Lie triple systems. A \mathbb{C} -bilinear mapping $H : A \times A \rightarrow B$ is called a ternary Jordan bi-homomorphism if it satisfies

$$H([x, x, x], [w, w, w]) = [H(x, w), H(x, w), H(x, w)]$$

for all $x, w \in A$.

The stability problem of functional equations originated from a question of Ulam [28] concerning the stability of group homomorphisms. Hyers [13] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Th.M. Rassias [21] for linear mappings by considering an unbounded Cauchy difference. J.M. Rassias [20] followed the innovative approach of the Th.M. Rassias theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p\|y\|^p$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$. The stability problems of various functional equations have been extensively investigated by a number of authors (see [2, 7, 9, 10, 18, 19, 22, 23, 24, 25, 26, 30, 31]).

2. HYERS-ULAM STABILITY OF TERNARY JORDAN BI-HOMOMORPHISMS IN BANACH LIE TRIPLE SYSTEMS

Throughout this section, assume that A is a normed Lie triple system and B is a Banach Lie triple systems.

For a given mapping $f : A \times A \rightarrow B$, we define

$$\begin{aligned} D_{\lambda, \mu} f(x, y, z, w) &= f(\lambda x + \lambda y, \mu z + \mu w) + f(\lambda x + \lambda y, \mu z - \mu w) \\ &\quad + f(\lambda x - \lambda y, \mu z + \mu w) + f(\lambda x - \lambda y, \mu z - \mu w) - 4\lambda\mu f(x, z) \end{aligned}$$

for all $x, y, z, w \in A$ and all $\lambda, \mu \in \mathbb{T}^1 := \{\nu \in \mathbb{C} : |\nu| = 1\}$.

From now on, assume that $f(0, z) = f(x, 0) = 0$ for all $x, z \in A$.

We need the following lemma to obtain the main results.

Lemma 2.1. ([4]) *Let $f : A \times A \rightarrow B$ be a mapping satisfying $D_{\lambda, \mu} f(x, y, z, w) = 0$ for all $x, y, z, w \in A$ and all $\lambda, \mu \in \mathbb{T}^1$. Then the mapping $f : A \times A \rightarrow B$ is \mathbb{C} -bilinear.*

Lemma 2.2. *Let $f : A \times A \rightarrow B$ be a bi-additive mapping. Then the following assertions are equivalent:*

$$f([a, a, a], [w, w, w]) = [f(a, w), f(a, w), f(a, w)] \quad (2.1)$$

for all $a, w \in A$, and

$$\begin{aligned} &f([a, b, c] + [b, c, a] + [c, a, b], [w, w, w]) \\ &= [f(a, w), f(b, w), f(c, w)] + [f(b, w), f(c, w), f(a, w)] + [f(c, w), f(a, w), f(b, w)], \end{aligned} \quad (2.2)$$

$$\begin{aligned} &f([a, a, a], [b, c, w] + [c, w, b] + [w, b, c]) \\ &= [f(a, b), f(a, c), f(a, w)] + [f(a, c), f(a, w), f(a, b)] + [f(a, w), f(a, b), f(a, c)], \end{aligned}$$

for all $a, b, c, w \in A$.

Proof. Replacing a by $a + b + c$ in (2.1), we get

$$f([(a + b + c), (a + b + c), (a + b + c)], [w, w, w]) = [f(a + b + c, w), f(a + b + c, w), f(a + b + c, w)].$$

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The we have

$$\begin{aligned}
& f([(a+b+c), (a+b+c), (a+b+c)], [w, w, w]) \\
&= f([a, a, a] + [a, b, a] + [a, c, a] + [b, a, a] + [b, b, a] + [b, c, a] + [c, a, a] + [c, b, a] + [c, c, a] \\
&+ [a, a, b] + [a, b, b] + [a, c, b] + [b, a, b] + [b, b, b] + [b, c, b] + [c, a, b] + [c, b, b] + [c, c, b] \\
&+ [a, a, c] + [a, b, c] + [a, c, c] + [b, a, c] + [b, b, c] + [b, c, c] + [c, a, c] + [c, b, c] + [c, c, c], [w, w, w]) \\
&= [f(a, w), f(a, w), f(a, w)] + [f(a, w), f(b, w), f(a, w)] + [f(a, w), f(c, w), f(a, w)] + [f(b, w), f(a, w), f(a, w)] \\
&+ [f(b, w), f(b, w), f(a, w)] + [f(b, w), f(c, w), f(a, w)] + [f(c, w), f(a, w), f(a, w)] + [f(c, w), f(b, w), f(a, w)] \\
&+ [f(c, w), f(c, w), f(a, w)] + [f(a, w), f(a, w), f(b, w)] + [f(a, w), f(b, w), f(b, w)] + [f(a, w), f(c, w), f(b, w)] \\
&+ [f(b, w), f(a, w), f(b, w)] + [f(b, w), f(b, w), f(b, w)] + [f(b, w), f(c, w), f(b, w)] + [f(c, w), f(a, w), f(b, w)] \\
&+ [f(c, w), f(b, w), f(b, w)] + [f(c, w), f(c, w), f(b, w)] + [f(a, w), f(a, w), f(c, w)] + [f(a, w), f(b, w), f(c, w)] \\
&+ [f(a, w), f(c, w), f(c, w)] + [f(b, w), f(a, w), f(c, w)] + [f(b, w), f(b, w), f(c, w)] + [f(b, w), f(c, w), f(c, w)] \\
&+ [f(c, w), f(a, w), f(c, w)] + [f(c, w), f(b, w), f(c, w)] + [f(c, w), f(c, w), f(c, w)]
\end{aligned}$$

for all $a, b, c, w \in A$.

On the other hand, for the right side of equation, we have

$$\begin{aligned}
& [f(a+b+c, w), f(a+b+c, w), f(a+b+c, w)] \\
&= [f(a, w), f(a, w), f(a, w)] + [f(a, w), f(a, w), f(b, w)] + [f(a, w), f(a, w), f(c, w)] + [f(a, w), f(b, w), f(a, w)] \\
&+ [f(a, w), f(b, w), f(b, w)] + [f(a, w), f(b, w), f(c, w)] + [f(a, w), f(c, w), f(a, w)] + [f(a, w), f(c, w), f(b, w)] \\
&+ [f(a, w), f(c, w), f(c, w)] + [f(b, w), f(a, w), f(a, w)] + [f(b, w), f(a, w), f(b, w)] + [f(b, w), f(a, w), f(c, w)] \\
&+ [f(b, w), f(b, w), f(a, w)] + [f(b, w), f(b, w), f(b, w)] + [f(b, w), f(b, w), f(c, w)] + [f(b, w), f(c, w), f(a, w)] \\
&+ [f(b, w), f(c, w), f(b, w)] + [f(b, w), f(c, w), f(c, w)] + [f(c, w), f(a, w), f(a, w)] + [f(c, w), f(a, w), f(b, w)] \\
&+ [f(c, w), f(a, w), f(c, w)] + [f(c, w), f(b, w), f(a, w)] + [f(c, w), f(b, w), f(b, w)] + [f(c, w), f(b, w), f(c, w)] \\
&+ [f(c, w), f(c, w), f(a, w)] + [f(c, w), f(c, w), f(b, w)] + [f(c, w), f(c, w), f(c, w)]
\end{aligned}$$

for all $a, b, c, w \in A$.

It follows that

$$f([a, b, c] + [b, c, a] + [c, a, b], [w, w, w]) = [f(a, w), f(b, w), f(c, w)] + [f(b, w), f(c, w), f(a, w)] + [f(c, w), f(a, w), f(b, w)]$$

for all $a, b, c, w \in A$. Hence (2.2) holds.

Similarly, we can show that

$$f([a, a, a], [b, c, w] + [c, w, b] + [w, b, c]) = [f(a, b), f(a, c), f(a, w)] + [f(a, c), f(a, w), f(a, b)] + [f(a, w), f(a, b), f(a, c)]$$

for all $a, b, c, w \in A$.

For the converse, replacing b and c by a in (2.2), we have

$$f([a, a, a] + [a, a, a] + [a, a, a], [w, w, w]) = [f(a, w), f(a, w), f(a, w)] + [f(a, w), f(a, w), f(a, w)] + [f(a, w), f(a, w), f(a, w)]$$

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and so $f(3[a, a, a], [w, w, w]) = 3([f(a, w), f(a, w), f(a, w)])$. Thus

$$f([a, a, a], [w, w, w]) = [f(a, w), f(a, w), f(a, w)]$$

for all $a, w \in A$. This completes the proof. \square

Now we prove the Hyers-Ulam stability of ternary Jordan bi-homomorphisms in Banach Lie triple systems.

Theorem 2.3. *Let p and θ be positive real numbers with $p < 2$, and let $f : A \times A \rightarrow B$ be a mapping such that*

$$\|D_{\lambda, \mu} f(x, y, z, w)\|_B \leq \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p + \|w\|_A^p), \quad (2.3)$$

$$\begin{aligned} & \|f\left([x, y, z] + [y, z, x] + [z, x, y], [w, w, w]\right) - [f(x, w), f(y, w), f(z, w)] - [f(y, w), f(z, w), f(x, w)] \\ & - [f(z, w), f(x, w), f(y, w)]\|_B + \|f\left([x, x, x], ([y, z, w] + [z, w, y] + [w, y, z])\right) - [f(x, y), f(x, z), f(x, w)] \\ & - [f(x, z), f(x, w), f(x, y)] - [f(x, w), f(x, y), f(x, z)]\|_B \leq \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p + \|w\|_A^p) \end{aligned} \quad (2.4)$$

for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y, z, w \in A$. Then there exists a unique ternary Jordan bi-homomorphism $H : A \times A \rightarrow B$ such that

$$\|f(x, y) - H(x, y)\|_B \leq \frac{2\theta}{4 - 2^p} (\|x\|_A^p + \|y\|_A^p) \quad (2.5)$$

for all $x, y \in A$.

Proof. By the same reasoning as in the proof of [4, Theorem 2.3], there exists a unique \mathbb{C} -bilinear mapping $H : A \times A \rightarrow B$ satisfying (2.5). The \mathbb{C} -bilinear mapping $H : A \times A \rightarrow B$ is given by

$$H(x, y) := \lim_{n \rightarrow \infty} \frac{1}{4^n} f(2^n x, 2^n y),$$

for all $x, y \in A$.

It follows from (2.4) that

$$\begin{aligned} & \|H\left([x, y, z] + [y, z, x] + [z, x, y], [w, w, w]\right) - [H(x, w), H(y, w), H(z, w)] - [H(y, w), H(z, w), H(x, w)] \\ & - [H(z, w), H(x, w), H(y, w)]\|_B + \|H\left([x, x, x], ([y, z, w] + [z, w, y] + [w, y, z])\right) - [H(x, y), H(x, z), H(x, w)] \\ & - [H(x, z), H(x, w), H(x, y)] - [H(x, w), H(x, y), H(x, z)]\|_B \\ & = \lim_{n \rightarrow \infty} \frac{1}{64^n} \left(\|f\left([2^n x, 2^n y, 2^n z] + [2^n y, 2^n z, 2^n x] + [2^n z, 2^n x, 2^n y], [2^n w, 2^n w, 2^n w]\right) \right. \\ & - [f(2^n x, 2^n w), f(2^n y, 2^n w), f(2^n z, 2^n w)] - [f(2^n y, 2^n w), f(2^n z, 2^n w), f(2^n x, 2^n w)] \\ & - [f(2^n z, 2^n w), f(2^n x, 2^n w), f(2^n y, 2^n w)]\|_B + \|f\left([2^n x, 2^n x, 2^n x], ([2^n y, 2^n z, 2^n w] + [2^n z, 2^n w, 2^n y] + [2^n w, 2^n y, 2^n z])\right) \\ & - [f(2^n x, 2^n y), f(2^n x, 2^n z), f(2^n x, 2^n w)] - [f(2^n x, 2^n z), f(2^n x, 2^n w), f(2^n x, 2^n y)] \\ & \left. - [f(2^n x, 2^n w), f(2^n x, 2^n y), f(2^n x, 2^n z)]\|_B \right) \leq \lim_{n \rightarrow \infty} \frac{2^{np}}{64^n} \theta(\|x\|_A^p + \|y\|_A^p + \|z\|_A^p + \|w\|_A^p) = 0 \end{aligned} \quad (2.6)$$

for all $x, y, z, w \in A$. So

$$H\left([x, y, z] + [y, z, x] + [z, x, y], [w, w, w]\right) = [H(x, w), H(y, w), H(z, w)] + [H(y, w), H(z, w), H(x, w)] + [H(z, w), H(x, w), H(y, w)]$$

and

$$H\left([x, x, x], ([y, z, w] + [z, w, y] + [w, y, z])\right) = [H(x, y), H(x, z), H(x, w)] + [H(x, z), H(x, w), H(x, y)] + [H(x, w), H(x, y), H(x, z)]$$

for all $x, y, z, w \in A$. By Lemma 2.2, the bi-additive mapping H is a unique ternary Jordan bi-homomorphism satisfying (2.5). \square

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Theorem 2.4. Let p and θ be positive real numbers with $p > 6$, and let $f : A \times A \rightarrow B$ be a mapping satisfying (2.3) and (2.4). Then there exists a unique ternary Jordan bi-homomorphism $H : A \times A \rightarrow B$ such that

$$\|f(x, y) - H(x, y)\|_B \leq \frac{2\theta}{2^p - 4} (\|x\|_A^p + \|y\|_A^p)$$

for all $x, y \in A$.

Proof. The proof is similar to the proof of Theorem 2.3. □

Theorem 2.5. Let p and θ be positive real numbers with $p < \frac{1}{2}$, and let $f : A \times A \rightarrow B$ be a mapping such that

$$\|D_{\lambda, \mu} f(x, y, z, w)\|_B \leq \theta (\|x\|_A^p \cdot \|y\|_A^p \cdot \|z\|_A^p \cdot \|w\|_A^p),$$

$$\begin{aligned} & \|f\left([x, y, z] + [y, z, x] + [z, x, y], [w, w, w]\right) - [f(x, w), f(y, w), f(z, w)] - [f(y, w), f(z, w), f(x, w)] \\ & - [f(z, w), f(x, w), f(y, w)]\|_B \\ & + \|f\left([x, x, x], ([y, z, w] + [z, w, y] + [w, y, z])\right) - [f(x, y), f(x, z), f(x, w)] - [f(x, z), f(x, w), f(x, y)] \\ & - [f(x, w), f(x, y), f(x, z)]\|_B \leq \theta (\|x\|_A^p \cdot \|y\|_A^p \cdot \|z\|_A^p \cdot \|w\|_A^p) \end{aligned}$$

for all $\lambda, \mu \in \mathbb{T}^1$ and all $x, y, z, w \in A$. Then there exists a unique ternary Jordan bi-homomorphism $H : A \times A \rightarrow B$ such that

$$\|f(x, y) - H(x, y)\|_B \leq \frac{\theta}{4 - 2^{4p}} \|x\|_A^{2p} \|y\|_A^{2p} \quad (2.7)$$

for all $x, y \in A$.

Proof. By the same reasoning as in the proof of [4, Theorem 2.6], there exists a unique \mathbb{C} -bilinear mapping $H : A \times A \rightarrow A$ satisfying (2.7). The \mathbb{C} -bilinear mapping $H : A \times A \rightarrow A$ is given by

$$H(x, y) := \lim_{n \rightarrow \infty} \frac{1}{4^n} f(2^n x, 2^n y),$$

for all $x, y \in A$.

The rest of the proof is similar to the proof of Theorem 2.3. □

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BOREL DIRECTIONS AND UNIQUENESS OF MEROMORPHIC FUNCTIONS SHARING FIVE VALUES

JIANREN LONG AND CHUNHUI QIU

ABSTRACT. We study a problem uniqueness of meromorphic functions in an angular domain concerning a Borel direction, and obtain some uniqueness results by using Nevanlinna theory of angular domain and angular distributions, that is, if the zeros of $f - a_j$ ($j = 1, 2, \dots, 5$) is also zeros of $g - a_j$ in the angular domain, then $f = g$.

1. INTRODUCTION AND MAIN RESULTS

As usual, the abbreviations IM and CM refer to sharing values ignoring multiplicities and counting multiplicities in domain $D \subseteq \mathbb{C}$, respectively, where \mathbb{C} denotes the complex plane. In addition, $\rho(f)$ denotes the order of growth of a meromorphic function f in \mathbb{C} . The standard notation and basic results in Nevanlinna theory of meromorphic functions can be found in [7] or [20].

In [12], Nevanlinna proved the remarkable five-value theorem and four-value theorem by using his value distribution theory, here the five-value theorem is stated as follows.

Theorem A. *Let f and g be two non-constant meromorphic functions in \mathbb{C} and let $a_i \in \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ ($i = 1, 2, 3, 4, 5$) be five distinct values. If f and g share the values a_i ($i = 1, 2, 3, 4, 5$) IM in $D(= \mathbb{C})$, then $f = g$.*

After his work, lots of uniqueness results of meromorphic functions in the complex plane have been obtained, which are introduced systematically in [18]. In [24, 25], Zheng first took into the uniqueness question of meromorphic functions related shared values in an angular domain, and obtained some five-value theorem and four-value theorem in some angular domain, while he posed the question: Under what conditions, must two meromorphic functions on $D(\neq \mathbb{C})$ be identical? After his work, a lot of uniqueness results of meromorphic functions in an angular domain concerning this problem were obtained. In [1, 17, 23], Nevanlinna's five value theorem and four value theorem were extended to some angular domain by using sectorial

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Nevanlinna characteristic, respectively. It is an interesting topic how to extend some interesting uniqueness results in the whole complex plane to an angular domain, more uniqueness results concerning this problem can be found in [9, 10]. Recently, this problem was studied [11] by using new idea that angular distributions of meromorphic functions is considered. In order to make our statements understand easily, we first recall the following definition and Theorem B.

Theorem B. *Let $B(r)$ be a positive and continuous function in $[0, \infty)$ which satisfies $\limsup_{r \rightarrow \infty} \frac{\log B(r)}{\log r} = \infty$. Then there exists a continuously differentiable function $\rho(r)$, which satisfies the following conditions.*

(i) $\rho(r)$ is continuous and nondecreasing for $r \geq r_0$ ($r_0 > 0$) and tends to ∞ as $r \rightarrow \infty$;

(ii) The function $U(r) = r^{\rho(r)}$ ($r \geq r_0$) satisfies the condition

$$\lim_{r \rightarrow \infty} \frac{\log U(R)}{\log U(r)} = 1, \quad R = r + \frac{r}{\log U(r)};$$

(iii)

$$\limsup_{r \rightarrow \infty} \frac{\log B(r)}{\log U(r)} = 1.$$

Theorem B is due to K.L.Hiong [8]. A simple proof of the existence of $\rho(r)$ was given by Chuang [2].

Definition 1. We define $\rho(r)$ and $U(r)$ in Theorem B by the proximate order and type function of $B(r)$, respectively. For a meromorphic function $f(z)$ of infinite order, we define its proximate order and type function as the proximate order and type function of $T(r, f)$. Let $\rho(r)$ be a proximate order of meromorphic function f of infinite order in \mathbb{C} , and let $M(\rho(r))$ be the set of all meromorphic functions g in \mathbb{C} such that

$$\limsup_{r \rightarrow \infty} \frac{\log T(r, g)}{\rho(r) \log r} \leq 1.$$

Let $\alpha < \beta$ such that $\beta - \alpha < 2\pi$ and $r > 0$, we denote

$$\Omega(\alpha, \beta) = \{z : \alpha \leq \arg z \leq \beta\},$$

$$\Omega(\alpha, \beta; r) = \{z : \alpha \leq \arg z \leq \beta\} \cap \{z : 0 < |z| \leq r\}.$$

The following definition, originally due to Hiong [8], which also be found in [3] or [4, p. 140].

Definition 2. Suppose that $\rho(r)$ is a proximate order of meromorphic function f of infinite order in \mathbb{C} . A ray $\arg z = \theta \in [0, 2\pi)$ from the origin is

called a Borel direction order $\rho(r)$ of f , if for any $\varepsilon > 0$ and any complex value $a \in \bar{\mathbb{C}}$, possibly with two exceptions, the following equality

$$(1.1) \quad \limsup_{r \rightarrow \infty} \frac{\log n(\Omega(\theta - \varepsilon, \theta + \varepsilon; r), \frac{1}{f-a})}{\rho(r) \log r} = 1$$

holds, where $n(\Omega(\theta - \varepsilon, \theta + \varepsilon; r), \frac{1}{f-a})$ is the number of zeros, counting multiplicities, of $f - a$ in the region $\Omega(\theta - \varepsilon, \theta + \varepsilon; r)$.

It is well known that every meromorphic function of infinite order must have at least one Borel direction of order $\rho(r)$. The proof can be found in [4, pp. 140-145]. In Nevanlinna theory of meromorphic functions, the angular distributions is one of main topics. Borel direction plays a basic role in the theory of angular distributions of meromorphic functions, lots of results can be found in [5, 13, 15, 16, 19, 21, 22]. In [11], the authors investigated the uniqueness of meromorphic functions in an angular domain by using theory of angular distributions, and proved the following version of five value theorem.

Theorem C. *Let $\rho(r)$ be a proximate order of meromorphic function f of infinite order in \mathbb{C} and let $g \in M(\rho(r))$. Suppose that $\arg z = \theta \in [0, 2\pi)$ is a Borel direction of order $\rho(r)$ of f . For any $\varepsilon > 0$, if f and g share five distinct values $a_i \in \bar{\mathbb{C}}$ ($i = 1, 2, 3, 4, 5$) IM in $\Omega(\theta - \varepsilon, \theta + \varepsilon)$, then $f = g$.*

In order to state the next result, we also need the following notation. Let f be a non-constant meromorphic function in \mathbb{C} , and let a be an arbitrary complex number. We use $\bar{E}(a, D, f)$ to denote the zeros set of $f - a$ in $D \subseteq \mathbb{C}$, in which each zero is counted only once. Clearly, we say that f and g share a IM in D , if $\bar{E}(a, D, f) = \bar{E}(a, D, g)$. We use $\bar{E}(a, f)$ to denote the zeros set of $f - a$ in $D = \mathbb{C}$. In [18, Theorem 3.2], C.C.Yang improved Theorem A by proving

Theorem D. *Let f and g be two non-constant meromorphic functions in \mathbb{C} and $a_i \in \bar{\mathbb{C}}$ ($i = 1, 2, 3, 4, 5$) be five distinct values. If*

$$(1.2) \quad \bar{E}(a_i, f) \subseteq \bar{E}(a_i, g), \quad i = 1, 2, 3, 4, 5,$$

and

$$(1.3) \quad \liminf_{r \rightarrow \infty} \sum_{i=1}^5 \bar{N}(r, \frac{1}{f-a_i}) / \sum_{i=1}^5 \bar{N}(r, \frac{1}{g-a_i}) > \frac{1}{2},$$

then $f = g$.

Now, it is natural to ask the following question.

Question 1. *Do f and g coincide if they satisfy the conditions of Theorem D in an angular domain?*

In the present paper, we answer to Question 1 is affirmative for some class of meromorphic functions by using Nevanlinna theory in an angular domain which is recalled in Lemma 2.1 below. The first result is stated as follows.

Theorem 1.1. *Let $\rho(r)$ be a proximate order of meromorphic function f of infinite order in \mathbb{C} and let $g \in M(\rho(r))$. Let $a_i \in \bar{\mathbb{C}}$ ($i = 1, 2, 3, 4, 5$) be five distinct values. Suppose that $\arg z = \theta \in [0, 2\pi)$ is a Brode direction of order $\rho(r)$ of f . For any given $\varepsilon > 0$, if*

$$(1.4) \quad \bar{E}(a_i, \Omega(\theta - \varepsilon, \theta + \varepsilon), f) \subseteq \bar{E}(a_i, \Omega(\theta - \varepsilon, \theta + \varepsilon), g), \quad i = 1, 2, 3, 4, 5,$$

and

$$(1.5) \quad \liminf_{r \rightarrow \infty} \sum_{i=1}^5 \bar{C}_{\theta-\varepsilon, \theta+\varepsilon}(r, \frac{1}{f-a_i}) / \sum_{i=1}^5 \bar{C}_{\theta-\varepsilon, \theta+\varepsilon}(r, \frac{1}{g-a_i}) > \frac{1}{2},$$

then $f = g$.

Before stating the following result, we need some notation concerning Ahlfors theory in an angular domain $\Omega(\alpha, \beta)$ which can be found [14, pp. 258-259], or for reference [26, pp. 66-76].

$$S_A(r, \Omega(\alpha, \beta), f) = \frac{1}{\pi} \int_0^r \int_\alpha^\beta \left(\frac{|f'(te^{i\varphi})|}{1 + |f(te^{i\varphi})|^2} \right)^2 t d\varphi dt,$$

$$T(r, \Omega(\alpha, \beta), f) = \int_0^r \frac{S_A(t, \Omega(\alpha, \beta), f)}{t} dt.$$

Especially the corresponding notation in the whole complex plane are denoted by

$$S_A(r, f) = \frac{1}{\pi} \int_0^r \int_0^{2\pi} \left(\frac{|f'(te^{i\varphi})|}{1 + |f(te^{i\varphi})|^2} \right)^2 t d\varphi dt,$$

$$T(r, f) = \int_0^r \frac{S_A(t, f)}{t} dt.$$

By using the relationship between Ahlfors characteristic function in an angular domain and sectorial Nevanlinna characteristic function which is introduced in Lemma 2.7 of Section 2, we can prove the following result.

Theorem 1.2. *Let f and g be two non-constant meromorphic functions of finite order in \mathbb{C} and $a_i \in \bar{\mathbb{C}}$ ($i = 1, 2, 3, 4, 5$) be five distinct values. Suppose*

that $\Omega(\alpha, \beta)$ is an angular domain such that f satisfies

$$(1.6) \quad \limsup_{\varepsilon \rightarrow 0^+} \limsup_{r \rightarrow \infty} \frac{\log T(r, \Omega(\alpha + \varepsilon, \beta - \varepsilon), f)}{\log r} > \omega,$$

where $\omega = \frac{\pi}{\beta - \alpha}$. If

$$(1.7) \quad \bar{E}(a_i, \Omega(\alpha, \beta), f) \subseteq \bar{E}(a_i, \Omega(\alpha, \beta), g), \quad i = 1, 2, 3, 4, 5,$$

and

$$(1.8) \quad \liminf_{r \rightarrow \infty} \sum_{i=1}^5 \bar{C}_{\alpha, \beta}(r, \frac{1}{f - a_i}) / \sum_{i=1}^5 \bar{C}_{\alpha, \beta}(r, \frac{1}{g - a_i}) > \frac{1}{2},$$

then $f = g$.

Theorem 1.3. Let f and g be two non-constant meromorphic functions of finite order in \mathbb{C} and $a_i \in \bar{\mathbb{C}}$ ($i = 1, 2, 3, 4, 5$) be five distinct values. Suppose that $\Omega(\alpha, \beta)$ is an angular domain such that for any $\varepsilon > 0$ and for some $a \in \bar{\mathbb{C}}$

$$(1.9) \quad \limsup_{r \rightarrow \infty} \frac{\log n(r, \Omega(\alpha + \varepsilon, \beta - \varepsilon), \frac{1}{f - a})}{\log r} > \omega,$$

where $\omega = \frac{\pi}{\beta - \alpha}$. If f and g satisfy (1.7) and (1.8), then $f = g$.

Remark 1.4. It is well known that every meromorphic function of order $\rho \in (0, \infty)$ must have at least one direction $\arg z = \theta \in [0, 2\pi)$ such that for sufficiently small $\varepsilon > 0$,

$$\limsup_{r \rightarrow \infty} \frac{\log n(r, \Omega(\alpha + \varepsilon, \beta - \varepsilon), \frac{1}{f - a})}{\log r} = \rho$$

holds for all $a \in \bar{\mathbb{C}}$ with at most two exceptional values, which can be found in [20, Chapter 3]. So the angular domain satisfying (1.9) must exist when f is of order $\rho \in (\frac{1}{2}, \infty)$.

This paper is organized as follows. In Section 2, we recall the properties of sectorial Nevanlinna characteristic and state some Lemmas which are needed in proving our results. The proof of Theorem 1.1 is given in Section 3. Finally, we prove Theorem 1.2 and 1.3 in Section 4.

2. AUXILIARY RESULTS

Let f be a meromorphic function in the angular domain $\Omega(\alpha, \beta) = \{z : \alpha \leq \arg z \leq \beta\}$, where $\alpha < \beta$ and $\beta - \alpha < 2\pi$. We recall the following

definitions that were found in [6, Chapter 1].

$$\begin{aligned} A_{\alpha,\beta}(r, f) &= \frac{\omega}{\pi} \int_1^r \left(\frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}} \right) \{ \log^+ |f(te^{i\alpha})| + \log^+ |f(te^{i\beta})| \} \frac{dt}{t}, \\ B_{\alpha,\beta}(r, f) &= \frac{2\omega}{\pi r^\omega} \int_\alpha^\beta \log^+ |f(te^{i\theta})| \sin \omega(\theta - \alpha) d\theta, \\ C_{\alpha,\beta}(r, f) &= 2 \sum_{1 < |b_n| < r} \left(\frac{1}{|b_n|^\omega} - \frac{|b_n|^\omega}{r^{2\omega}} \right) \sin \omega(\theta_n - \alpha), \end{aligned}$$

where $\omega = \frac{\pi}{\beta - \alpha}$ and $b_n = |b_n|e^{i\theta_n}$ are the poles of f in $\Omega(\alpha, \beta)$ counting multiplicities. The function $C_{\alpha,\beta}(r, f)$ is called the sectorial counting function of the poles of f in $\Omega(\alpha, \beta)$. In the corresponding counting function $\bar{C}_{\alpha,\beta}(r, f)$ these multiplicities are ignored. For $a \in \mathbb{C}$, the definitions of $A_{\alpha,\beta}(r, \frac{1}{f-a})$, $B_{\alpha,\beta}(r, \frac{1}{f-a})$, and $C_{\alpha,\beta}(r, \frac{1}{f-a})$ are immediate. Finally, the sectorial Nevanlinna characteristic function is given by

$$S_{\alpha,\beta}(r, f) = A_{\alpha,\beta}(r, f) + B_{\alpha,\beta}(r, f) + C_{\alpha,\beta}(r, f).$$

We state sectorial analogues of Nevanlinna's first and second main theorems as follows.

Lemma 2.1 ([6]). *Let f be a meromorphic function in \mathbb{C} and let $\Omega(\alpha, \beta)$ be an angular domain. Then, for any $a \in \mathbb{C}$,*

$$S_{\alpha,\beta}\left(r, \frac{1}{f-a}\right) = S_{\alpha,\beta}(r, f) + O(1).$$

Moreover, for any $q \geq 3$ distinct values, $a_j \in \bar{\mathbb{C}}$ ($j = 1, 2, \dots, q$),

$$(q-2)S_{\alpha,\beta}(r, f) \leq \sum_{j=1}^q \bar{C}_{\alpha,\beta}\left(r, \frac{1}{f-a_j}\right) + R_{\alpha,\beta}(r, f),$$

where

$$\begin{aligned} R_{\alpha,\beta}(r, f) &= A_{\alpha,\beta}\left(r, \frac{f'}{f}\right) + B_{\alpha,\beta}\left(r, \frac{f'}{f}\right) \\ (2.1) \quad &+ \sum_{j=1}^q \left\{ A_{\alpha,\beta}\left(r, \frac{f'}{f-a_j}\right) + B_{\alpha,\beta}\left(r, \frac{f'}{f-a_j}\right) \right\} + O(1). \end{aligned}$$

Lemma 2.2 ([6]). *Let f be a meromorphic function in \mathbb{C} and let $\Omega(\alpha, \beta)$ be an angular domain. Then*

$$\begin{aligned} A_{\alpha,\beta}\left(r, \frac{f'}{f}\right) &\leq K \left\{ \left(\frac{R}{r}\right)^\omega \int_r^R \frac{\log^+ T(t, f)}{t^{\omega+1}} dt + \log^+ \frac{r}{R-r} + \log \frac{R}{r} + 1 \right\}, \\ B_{\alpha,\beta}\left(r, \frac{f'}{f}\right) &\leq \frac{4\omega}{r^\omega} m\left(r, \frac{f'}{f}\right), \end{aligned}$$

where $\omega = \frac{\pi}{\beta - \alpha}$, $1 < r < R < \infty$, K is a nonzero constant.

The next result follows from Lemma 2.2 and Lemma on the logarithmic derivative.

Lemma 2.3. *Let f be a meromorphic function in \mathbb{C} and let $\Omega(\alpha, \beta)$ be an angular domain. Then*

$$R_{\alpha, \beta}(r, f) = \begin{cases} O(1), & f \text{ is of finite order;} \\ O(\log U(r)), & f \text{ is of infinite order;} \end{cases}$$

where $R_{\alpha, \beta}(r, f)$ is defined as in (2.1), $U(r) = r^{\rho(r)}$ and $\rho(r)$ is a proximate order of the meromorphic function f of infinite order.

Lemma 2.4 ([3]). *Suppose that $\rho(r)$ is a proximate order of meromorphic function f of infinite order in \mathbb{C} . Then, a ray $\arg z = \theta \in [0, 2\pi)$ from the origin is a Borel direction of order $\rho(r)$ of f if and only if for any $\varepsilon \in (0, \frac{\pi}{2})$, we have*

$$\limsup_{r \rightarrow \infty} \frac{\log S_{\theta-\varepsilon, \theta+\varepsilon}(r, f)}{\rho(r) \log r} = 1.$$

Lemma 2.5 ([23]). *Let f be a meromorphic function in \mathbb{C} , $\Omega(\alpha, \beta)$ be an angular domain. If the order of f is finite order and satisfy*

$$\limsup_{\varepsilon \rightarrow 0^+} \limsup_{r \rightarrow \infty} \frac{\log T(r, \Omega(\alpha + \varepsilon, \beta - \varepsilon), f)}{\log r} = \lambda > \omega,$$

where $\omega = \frac{\pi}{\beta - \alpha}$. Then

$$\limsup_{\varepsilon \rightarrow 0^+} \limsup_{r \rightarrow \infty} \frac{\log S_{\alpha+\varepsilon, \beta-\varepsilon}(r, f)}{\log r} = \lambda - \omega.$$

In order to describe the relationship between Ahlfors characteristic function in an angular domain and sectorial Nevanlinna characteristic function, we also need some notation and definition. Since $S_{\alpha, \beta}(r, f)$ is not increasing with respect to r , hence Nevanlinna defined the following function $\dot{S}_{\alpha, \beta}(r, f)$ that is increasing with respect to r ,

$$\dot{S}_{\alpha, \beta}(r, f) = \frac{1}{\pi} \int_1^r \int_{\alpha}^{\beta} \left(\frac{1}{t^{\omega}} - \frac{t^{\omega}}{r^{2\omega}} \right) \left(\frac{|f'(te^{i\theta})|}{1 + |f(te^{i\theta})|^2} \right)^2 \sin \omega(\theta - \alpha) t dt d\theta,$$

where $\omega = \frac{\pi}{\beta - \alpha}$. $\dot{S}_{\alpha, \beta}(r, f)$ and $S_{\alpha, \beta}(r, f)$ have following relationship.

Lemma 2.6. [26, Lemma 2.2.1] *Let f be a meromorphic function in $\Omega(\alpha, \beta)$. Then*

$$\dot{S}_{\alpha, \beta}(r, f) = S_{\alpha, \beta}(r, f) + O(1).$$

In [26], we can also find the relationship between $\dot{S}_{\alpha, \beta}(r, f)$ and $T(r, \Omega(\alpha, \beta), f)$ as follows.

Lemma 2.7. [26, Theorem 2.4.7] *Let f be a meromorphic function in $\Omega(\alpha, \beta)$. Then*

$$\dot{S}_{\alpha, \beta}(r, f) \leq 2\omega \frac{T(r, \Omega(\alpha, \beta), f)}{r^\omega} + \omega^2 \int_1^r \frac{T(t, \Omega(\alpha, \beta), f)}{t^{\omega+1}} dt,$$

where $\omega = \frac{\pi}{\beta-\alpha}$.

Lemma 2.8. *Let f be a meromorphic function in \mathbb{C} , and $\Omega(\alpha, \beta)$ be an angular domain. For any $\varepsilon > 0$ and for some $a \in \bar{\mathbb{C}}$, if f satisfies*

$$(2.2) \quad \limsup_{r \rightarrow \infty} \frac{\log n(r, \Omega(\alpha + \varepsilon, \beta - \varepsilon), \frac{1}{f-a})}{\log r} > \omega,$$

where $\omega = \frac{\pi}{\beta-\alpha}$, then

$$\limsup_{\varepsilon \rightarrow 0^+} \limsup_{r \rightarrow \infty} \frac{\log T(r, \Omega(\alpha + \varepsilon, \beta - \varepsilon), f)}{\log r} > \omega.$$

Proof. For any given $\varepsilon > 0$, from (2.2), there exists a sequence $\{r_n\}$, $r_n \rightarrow \infty$ as $n \rightarrow \infty$, such that

$$\lim_{n \rightarrow \infty} \frac{\log n(r_n, \Omega(\alpha + \varepsilon, \beta - \varepsilon), \frac{1}{f-a})}{\log r_n} = \lambda > \omega.$$

Let σ be a real number such that $\omega < \sigma < \lambda$, we have

$$n(r_n, \Omega(\alpha + \varepsilon, \beta - \varepsilon), \frac{1}{f-a}) > r_n^\sigma > r_n^\omega, \quad n \geq n_0.$$

By this and

$$\begin{aligned} C_{\alpha+\frac{\varepsilon}{2}, \beta-\frac{\varepsilon}{2}}(r, \frac{1}{f-a}) &\geq 2\omega \sin(\frac{\omega\varepsilon}{2}) \frac{N(r, \Omega(\alpha + \varepsilon, \beta - \varepsilon), \frac{1}{f-a})}{r^\omega} \\ &\quad + 2\omega^2 \sin(\frac{\omega\varepsilon}{2}) \int_1^r \frac{N(t, \Omega(\alpha + \varepsilon, \beta - \varepsilon), \frac{1}{f-a})}{t^{\omega+1}} dt, \end{aligned}$$

which can be found in [26, Lemma 2.2.2], we have

$$(2.3) \quad C_{\alpha+\frac{\varepsilon}{2}, \beta-\frac{\varepsilon}{2}}(r_n, \frac{1}{f-a}) > r_n^{\sigma-\omega}.$$

By using Lemma 2.1 and (2.3), we get

$$S_{\alpha+\frac{\varepsilon}{2}, \beta-\frac{\varepsilon}{2}}(r_n, f) > r_n^{\sigma-\omega}.$$

It follows from Lemmas 2.6 and 2.7 that

$$T(r_n, \Omega(\alpha + \frac{\varepsilon}{2}, \beta - \frac{\varepsilon}{2}), f) > r_n^\sigma.$$

Thus,

$$\limsup_{r \rightarrow \infty} \frac{\log T(r, \Omega(\alpha + \frac{\varepsilon}{2}, \beta - \frac{\varepsilon}{2}), f)}{\log r} > \sigma > \omega.$$

Noting ε is arbitrary small, hence lemma holds. \square

3. PROOF OF THEOREM 1.1

Suppose that $\rho(r)$ is a proximate order of meromorphic function f of infinite order, $g \in M(\rho(r))$ and that $\arg z = \theta \in [0, 2\pi)$ is a Borel direction of order $\rho(r)$ of f . For any given $\varepsilon > 0$, f and g satisfy (1.4) and (1.5) in the angular domain $\Omega(\theta - \varepsilon, \theta + \varepsilon) = \{z : \theta - \varepsilon \leq \arg z \leq \theta + \varepsilon\}$.

Firstly, we claim that $\arg z = \theta$ is also a Borel direction of order $\rho(r)$ of g . Since $\arg z = \theta$ is a Borel direction of order $\rho(r)$ of f , for above given ε , by using Lemmas 2.1, 2.3 and 2.4, then there exists a value a such that

$$\limsup_{r \rightarrow \infty} \frac{\log \bar{C}_{\theta-\varepsilon, \theta+\varepsilon}(r, \frac{1}{f-a})}{\rho(r) \log r} \geq 1.$$

Without loss of generality, we may assume that $a = a_1$. Thus,

$$\limsup_{r \rightarrow \infty} \frac{\log \bar{C}_{\theta-\varepsilon, \theta+\varepsilon}(r, \frac{1}{f-a_1})}{\rho(r) \log r} \geq 1.$$

It follows from (1.4) that

$$\limsup_{r \rightarrow \infty} \frac{\log \bar{C}_{\theta-\varepsilon, \theta+\varepsilon}(r, \frac{1}{g-a_1})}{\rho(r) \log r} \geq 1.$$

Therefore, we get

$$\limsup_{r \rightarrow \infty} \frac{\log S_{\theta-\varepsilon, \theta+\varepsilon}(r, g)}{\rho(r) \log r} \geq 1.$$

Combining this and $g \in M(\rho(r))$, we have

$$\limsup_{r \rightarrow \infty} \frac{\log S_{\theta-\varepsilon, \theta+\varepsilon}(r, g)}{\rho(r) \log r} = 1.$$

By using Lemma 2.4, we know that $\arg z = \theta$ is a Borel direction of order $\rho(r)$ of g .

In order to prove that $f = g$, we assume on the contrary to the assertion that $f \neq g$. Now we use the similar method of [23] to complete the proof. To this end, we consider two cases.

Case 1. We may assume that all a_i ($i = 1, 2, 3, 4, 5$) are finite. By using Lemma 2.1, we can obtain

$$(3.1) \quad 3S_{\theta-\varepsilon, \theta+\varepsilon}(r, f) \leq \sum_{i=1}^5 \bar{C}_{\theta-\varepsilon, \theta+\varepsilon}(r, \frac{1}{f-a_i}) + R_{\theta-\varepsilon, \theta+\varepsilon}(r, f),$$

and

$$(3.2) \quad 3S_{\theta-\varepsilon, \theta+\varepsilon}(r, g) \leq \sum_{i=1}^5 \bar{C}_{\theta-\varepsilon, \theta+\varepsilon}(r, \frac{1}{g-a_i}) + R_{\theta-\varepsilon, \theta+\varepsilon}(r, g).$$

From (1.4), we have

$$\begin{aligned}
 \sum_{i=1}^5 \bar{C}_{\theta-\varepsilon, \theta+\varepsilon}(r, \frac{1}{f-a_i}) &\leq C_{\theta-\varepsilon, \theta+\varepsilon}(r, \frac{1}{f-g}) \\
 &\leq S_{\theta-\varepsilon, \theta+\varepsilon}(r, \frac{1}{f-g}) \\
 (3.3) \quad &\leq S_{\theta-\varepsilon, \theta+\varepsilon}(r, f) + S_{\theta-\varepsilon, \theta+\varepsilon}(r, g) + O(1).
 \end{aligned}$$

Since $\arg z = \theta$ is a Borel direction of order $\rho(r)$ of f , by using Lemma 2.4, then we have

$$\limsup_{r \rightarrow \infty} \frac{\log S_{\theta-\varepsilon, \theta+\varepsilon}(r, f)}{\rho(r) \log r} = 1.$$

It follows from this and Lemma 2.3, we have

$$(3.4) \quad \limsup_{r \rightarrow \infty} \frac{R_{\theta-\varepsilon, \theta+\varepsilon}(r, f)}{S_{\theta-\varepsilon, \theta+\varepsilon}(r, f)} = 0.$$

Similarly, we have

$$(3.5) \quad \limsup_{r \rightarrow \infty} \frac{R_{\theta-\varepsilon, \theta+\varepsilon}(r, g)}{S_{\theta-\varepsilon, \theta+\varepsilon}(r, g)} = 0.$$

Combining (3.1)-(3.5), for sufficiently large r , we have

$$\begin{aligned}
 \sum_{i=1}^5 \bar{C}_{\theta-\varepsilon, \theta+\varepsilon}(r, \frac{1}{f-a_i}) &\leq (\frac{1}{3} + o(1)) \sum_{i=1}^5 \bar{C}_{\theta-\varepsilon, \theta+\varepsilon}(r, \frac{1}{f-a_i}) \\
 &\quad + (\frac{1}{3} + o(1)) \sum_{i=1}^5 \bar{C}_{\theta-\varepsilon, \theta+\varepsilon}(r, \frac{1}{g-a_i}).
 \end{aligned}$$

Therefore,

$$(\frac{2}{3} + o(1)) \sum_{i=1}^5 \bar{C}_{\theta-\varepsilon, \theta+\varepsilon}(r, \frac{1}{f-a_i}) \leq (\frac{1}{3} + o(1)) \sum_{i=1}^5 \bar{C}_{\theta-\varepsilon, \theta+\varepsilon}(r, \frac{1}{g-a_i}).$$

It follows that

$$\liminf_{r \rightarrow \infty} \sum_{i=1}^5 \bar{C}_{\theta-\varepsilon, \theta+\varepsilon}(r, \frac{1}{f-a_i}) / \sum_{i=1}^5 \bar{C}_{\theta-\varepsilon, \theta+\varepsilon}(r, \frac{1}{g-a_i}) \leq \frac{1}{2}.$$

This contradicts to (1.5), and hence $f = g$.

Case 2. If one of the values a_i ($i = 1, 2, 3, 4, 5$) is ∞ , without loss of generality, we may assume that $a_5 = \infty$. Take a finite value c such that $c \neq a_i$ ($i = 1, 2, 3, 4$) and set $F = \frac{1}{f-c}$, $G = \frac{1}{g-c}$, $b_i = \frac{1}{a_i-c}$ ($i = 1, 2, 3, 4$) and $b_5 = 0$, then F and G satisfy $\bar{E}(b_i, \Omega(\theta-\varepsilon, \theta+\varepsilon), F) \subseteq \bar{E}(b_i, \Omega(\theta-\varepsilon, \theta+\varepsilon), G)$ ($i = 1, 2, 3, 4, 5$), and

$$\liminf_{r \rightarrow \infty} \sum_{i=1}^5 \bar{C}_{\theta-\varepsilon, \theta+\varepsilon}(r, \frac{1}{F-b_i}) / \sum_{i=1}^5 \bar{C}_{\theta-\varepsilon, \theta+\varepsilon}(r, \frac{1}{G-b_i}) > \frac{1}{2}.$$

From Lemma 2.1, we also know that $S_{\theta-\varepsilon, \theta+\varepsilon}(r, F) = S_{\theta-\varepsilon, \theta+\varepsilon}(r, f) + O(1)$ and $S_{\theta-\varepsilon, \theta+\varepsilon}(r, G) = S_{\theta-\varepsilon, \theta+\varepsilon}(r, g) + O(1)$. From the previous proof, we know $F = G$. Therefore $f = g$. The proof is completed.

4. PROOFS OF THEOREMS 1.2 AND 1.3

Proof of Theorem 1.2. Suppose that f and g be two non-constant meromorphic functions of finite order in \mathbb{C} satisfying (1.6)-(1.8), $\Omega(\alpha, \beta) = \{z : \alpha \leq \arg z \leq \beta\}$ is an angular domain and $\omega = \frac{\pi}{\beta-\alpha}$. Set

$$(4.1) \quad \limsup_{\varepsilon \rightarrow 0^+} \limsup_{r \rightarrow \infty} \frac{\log T(r, \Omega(\alpha + \varepsilon, \beta - \varepsilon), f)}{\log r} = \lambda.$$

Firstly, we claim that

$$(4.2) \quad \limsup_{\varepsilon \rightarrow 0^+} \limsup_{r \rightarrow \infty} \frac{\log S_{\alpha+\varepsilon, \beta-\varepsilon}(r, g)}{\log r} \geq \lambda - \omega.$$

From (4.1), for any given $\varepsilon_1 \in (0, \frac{\lambda-\omega}{2})$, there exists at least some $\varepsilon_2 \in (0, \varepsilon_1)$ such that

$$\limsup_{r \rightarrow \infty} \frac{\log T(r, \Omega(\alpha + \varepsilon_2, \beta - \varepsilon_2), f)}{\log r} = \lambda' \geq \lambda - \varepsilon_1,$$

where $\lambda' (\leq \lambda)$ is a constant. It follows from Lemma 2.5 and (1.6) that

$$\limsup_{r \rightarrow \infty} \frac{\log S_{\alpha+\varepsilon_2, \beta-\varepsilon_2}(r, f)}{\log r} = \lambda' - \omega \geq \lambda - \omega - \varepsilon_1.$$

By using Lemmas 2.1 and 2.3, then there exists a value a such that

$$\limsup_{r \rightarrow \infty} \frac{\log \bar{C}_{\alpha+\varepsilon_2, \beta-\varepsilon_2}(r, \frac{1}{f-a})}{\log r} \geq \lambda - \omega - \varepsilon_1.$$

Without loss of generality, we may assume that $a = a_1$. Thus,

$$\limsup_{r \rightarrow \infty} \frac{\log \bar{C}_{\alpha+\varepsilon_2, \beta-\varepsilon_2}(r, \frac{1}{f-a_1})}{\log r} \geq \lambda - \omega - \varepsilon_1.$$

It follows from (1.7) that

$$\limsup_{r \rightarrow \infty} \frac{\log \bar{C}_{\alpha+\varepsilon_2, \beta-\varepsilon_2}(r, \frac{1}{g-a_1})}{\log r} \geq \lambda - \omega - \varepsilon_1.$$

Therefore, we get

$$\limsup_{r \rightarrow \infty} \frac{\log S_{\alpha+\varepsilon_2, \beta-\varepsilon_2}(r, g)}{\log r} \geq \lambda - \omega - \varepsilon_1.$$

Noting ε_1 is arbitrary and $\varepsilon_2 < \varepsilon_1$, so (4.2) holds.

We assume on the contrary to the assertion that $f \neq g$. We consider two cases.

Case 1. We may assume that all a_i ($i = 1, 2, 3, 4, 5$) are finite.

By arguing similar to that proof of Theorem 1.1, we can obtain the following inequalities,

$$(4.3) \quad 3S_{\alpha+\varepsilon, \beta-\varepsilon}(r, f) \leq \sum_{i=1}^5 \bar{C}_{\alpha+\varepsilon, \beta-\varepsilon}(r, \frac{1}{f-a_i}) + R_{\alpha+\varepsilon, \beta-\varepsilon}(r, f),$$

$$(4.4) \quad 3S_{\alpha+\varepsilon, \beta-\varepsilon}(r, g) \leq \sum_{i=1}^5 \bar{C}_{\alpha+\varepsilon, \beta-\varepsilon}(r, \frac{1}{g-a_i}) + R_{\alpha+\varepsilon, \beta-\varepsilon}(r, g),$$

$$(4.5) \quad \begin{aligned} \sum_{i=1}^5 \bar{C}_{\alpha+\varepsilon, \beta-\varepsilon}(r, \frac{1}{f-a_i}) &\leq C_{\alpha+\varepsilon, \beta-\varepsilon}(r, \frac{1}{f-g}) \\ &\leq S_{\alpha+\varepsilon, \beta-\varepsilon}(r, f) + S_{\alpha+\varepsilon, \beta-\varepsilon}(r, g) + O(1). \end{aligned}$$

By using (1.6), Lemmas 2.3 and 2.5, we get

$$(4.6) \quad \limsup_{r \rightarrow \infty} \frac{R_{\alpha+\varepsilon, \beta-\varepsilon}(r, f)}{S_{\alpha+\varepsilon, \beta-\varepsilon}(r, f)} = 0.$$

Similarly, it follows from (4.2) that

$$(4.7) \quad \limsup_{r \rightarrow \infty} \frac{R_{\alpha+\varepsilon, \beta-\varepsilon}(r, g)}{S_{\alpha+\varepsilon, \beta-\varepsilon}(r, g)} = 0.$$

Combining (4.3)-(4.7), for sufficiently large r , we have

$$(\frac{2}{3} + o(1)) \sum_{i=1}^5 \bar{C}_{\alpha+\varepsilon, \beta-\varepsilon}(r, \frac{1}{f-a_i}) \leq (\frac{1}{3} + o(1)) \sum_{i=1}^5 \bar{C}_{\alpha+\varepsilon, \beta-\varepsilon}(r, \frac{1}{g-a_i}).$$

It follows that

$$\liminf_{r \rightarrow \infty} \sum_{i=1}^5 \bar{C}_{\alpha+\varepsilon, \beta-\varepsilon}(r, \frac{1}{f-a_i}) / \sum_{i=1}^5 \bar{C}_{\alpha+\varepsilon, \beta-\varepsilon}(r, \frac{1}{g-a_i}) \leq \frac{1}{2}.$$

Noting $\varepsilon \rightarrow 0$, this contradicts to (1.8), and hence $f = g$.

Case 2. If one of the values a_i ($i = 1, 2, 3, 4, 5$) is ∞ , without loss of generality, we may assume that $a_5 = \infty$. By using similar way of the proof of Theorem 1.1, we can easily obtain $f = g$. The proof is completed. \square

Proof of Theorem 1.3. By Lemma 2.8, (1.9) implies (1.6). So combining Theorem 1.2 we get the conclusion of Theorem 1.3. \square

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ON AN INTERVAL-REPRESENTABLE GENERALIZED PSEUDO-CONVOLUTION BY MEANS OF THE INTERVAL-VALUED GENERALIZED FUZZY INTEGRAL AND THEIR PROPERTIES

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ABSTRACT. In this paper, we consider the generalized pseudo-convolution in the theory of probabilistic metric space and their properties which was introduced by Pap-Stajner (1999). Wu-Wang-Ma(1993) and Wu-Ma-Song(1995) studied the generalized fuzzy integral and their properties. Recently, Jang(2013) defined the interval-valued generalized fuzzy integral by using an interval-representable pseudo-multiplication. From the generalized fuzzy integral, we define a generalized pseudo-convolution by means of the generalized fuzzy integral and investigate their properties.

In particular, we also define an interval-representable generalized pseudo-convolution of interval-valued functions by means of the interval-valued generalized fuzzy integral and investigate their properties.

1. INTRODUCTION

Fang [8-10], Wu-Wang-Ma [35], Wu-Ma-Song [36], Xie-Fang [37] have studied the generalized fuzzy integral(for short, the (G) fuzzy integral) by using a pseudo-multiplication which is a generalization of fuzzy integrals in [5, 25, 26, 29, 31, 33, 39]. Pap-Stajner [28] introduced a notion of the generalized pseudo-convolution of functions based on pseudo-operations and proved their mathematical theories such as optimization, probabilistic metric spaces, and information theory

Many researchers [1,2,7,13-19, 21, 30, 34, 38, 40] have been studying various integrals of measurable multi-valued functions which are used for representing uncertain functions, for examples, the Aumann integral, the fuzzy integral, and the Choquet integral of measurable interval-valued functions in many different mathematical theories and their applications. Recently, Jang [20] defined the interval-valued generalized fuzzy integral (for short, the (IG) fuzzy integral) with respect to a fuzzy measure by using an interval-representable pseudo-multiplication of measurable interval-valued functions and investigated some convergence properties of them.

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The purpose of this study is to define the generalized pseudo-convolution of functions by means of the (G) fuzzy integral and to investigate some properties of them. In particular, we also define the interval-valued generalized pseudo-convolution of interval-valued functions by means of the (IG) fuzzy integral and to investigate some properties of them.

The paper is organized in five sections. In section 2, we list definitions and some properties of the generalized fuzzy integral with respect to a fuzzy measure by using generalized pseudo-multiplication and the interval-valued generalized fuzzy integral with respect to a fuzzy measure by using interval-representable generalized pseudo-multiplication. In section 3, we define the generalized pseudo-convolution of integrable nonnegative functions by means of the (G) fuzzy integral and investigate their properties. Furthermore, we give an example of the generalized pseudo-convolution of integrable nonnegative functions. In section 4, we define a interval-representable semigroup and the interval-valued generalized pseudo-convolution of integrable interval-valued functions by means of the (IG) fuzzy integral and investigate their properties. Furthermore, we give an example of the interval-valued generalized pseudo-convolution of integrable interval-valued functions. In section 5, we give a brief summary results and some conclusions.

2. DEFINITIONS AND PRELIMINARIES

In this section, we introduce some definitions and properties of a fuzzy measure, a pseudo-multiplication, a pseudo-addition, the (G) fuzzy integral with respect to a fuzzy measure by using a pseudo-multiplication of a measurable functions. Let X be a set and (X, \mathcal{A}) be a measurable space. Denote by $\mathfrak{F}(X)$ the set of all measurable nonnegative functions on X .

Definition 2.1. ([25, 26]) (1) A fuzzy measure $\mu : \mathcal{A} \rightarrow [0, \infty]$ is a set function satisfying

- (i) $\mu(\emptyset) = 0$
- (ii) $\mu(A) \leq \mu(B)$ whenever $A, B \in \mathcal{A}$ and $A \subset B$.

(2) A fuzzy measure μ is said to be finite if $\mu(X) < \infty$.

Definition 2.2. ([10, 33, 37]) (1) A binary operation $\oplus : [0, \infty]^2 \rightarrow [0, \infty]$ is called a pseudo-addition if it is non-decreasing in both components, associative, and 0 is its neutral element.

(2) A binary operation $\odot : [0, \infty]^2 \rightarrow [0, \infty]$ is called a pseudo-multiplication corresponding to \oplus if it satisfies the following axioms:

- (i) $a \odot b = b \odot a$,
- (ii) $a \odot (x \oplus y) = (a \odot x) \oplus (a \odot y)$,
- (iii) $a \leq b \implies a \odot x \leq b \odot x$,
- (iv) $a \odot x = 0 \iff a = 0$ or $x = 0$,
- (v) there exists a unit element, that is, $\exists e \in (0, \infty]$ such that $e \odot x = x$ for all $x \in [0, \infty]$,
- (vi) $a_n \rightarrow a \in (0, \infty)$ and $x_n \rightarrow x \in [0, \infty] \implies a_n \odot x_n \rightarrow a \odot x$ and $\lim_{a \rightarrow \infty} a \odot x = \infty \odot x$ for all $x \in (0, \infty]$.

Definition 2.3. ([20, 33, 37]) (1) Let (X, \mathcal{A}, μ) be a fuzzy measure space, $f \in \mathfrak{F}(X)$, and $A \in \mathcal{A}$. The (G) fuzzy integral with respect to a fuzzy measure μ by using a pseudo-multiplication \odot corresponding to the pseudo-addition $\oplus = \max(\text{maximum})$ of f on A is

defined by

$$(G) \int_A^\odot f d\mu = \sup_{\alpha > 0} \alpha \odot \mu_{A,f}(\alpha), \quad (1)$$

where $\mu_{A,f}(\alpha) = \mu(\{x \in A \mid f(x) \geq \alpha\})$ for all $\alpha \in (0, \infty)$.

(2) f is said to be integrable if $(G) \int_A^\odot f d\mu$ is finite.

Let $\mathfrak{F}(X)^*$ be the set of all nonnegative integrable functions on X . We consider the intervals, a standard interval-valued pseudo-multiplication, and an extended interval-valued pseudo-multiplication. Let $I(Y)$ be the set of all bounded closed intervals (intervals, for short) in Y as follows:

$$I(Y) = \{\bar{a} = [a_l, a_r] \mid a_l, a_r \in Y \text{ and } a_l \leq a_r\}, \quad (2)$$

where Y is $[0, \infty)$ or $[0, \infty]$. For any $a \in Y$, we define $a = [a, a]$. Obviously, $a \in I(Y)$ (see [4, 7, 16-21, 30, 34, 38-40]). Denote by $I\mathfrak{F}(X)$ the set of all measurable interval-valued functions on X .

Definition 2.4. ([20]) If $\bar{a} = [a_l, a_r], \bar{b} = [b_l, b_r], \bar{a}_n = [a_{ln}, a_{rn}], \bar{a}_\alpha = [a_{l\alpha}, a_{r\alpha}] \in I(Y)$ for all $n \in \mathbb{N}$ and $\alpha \in [0, \infty)$, and $k \in [0, \infty)$, then we define arithmetic, maximum, minimum, order, inclusion, superior, inferior operations as follows:

- (1) $\bar{a} + \bar{b} = [a_l + b_l, a_r + b_r]$,
- (2) $k\bar{a} = [ka_l, ka_r]$,
- (3) $\bar{a}\bar{b} = [a_lb_l, a_rb_r]$,
- (4) $\bar{a} \vee \bar{b} = [a_l \vee b_l, a_r \vee b_r]$,
- (5) $\bar{a} \wedge \bar{b} = [a_l \wedge b_l, a_r \wedge b_r]$,
- (6) $\bar{a} \leq \bar{b}$ if and only if $a_l \leq b_l$ and $a_r \leq b_r$,
- (7) $\bar{a} < \bar{b}$ if and only if $a_l \leq b_l$ and $a_l \neq b_l$,
- (8) $\bar{a} \subset \bar{b}$ if and only if $b_l \leq a_l$ and $a_r \leq b_r$,
- (9) $\sup_n \bar{a}_n = [\sup_n a_{nl}, \sup_n a_{nr}]$,
- (10) $\inf_n \bar{a}_n = [\inf_n a_{nl}, \inf_n a_{nr}]$,
- (11) $\sup_\alpha \bar{a}_\alpha = [\sup_\alpha a_{\alpha l}, \sup_\alpha a_{\alpha r}]$, and
- (12) $\inf_\alpha \bar{a}_\alpha = [\inf_\alpha a_{\alpha l}, \inf_\alpha a_{\alpha r}]$.

Definition 2.5. ([20]) (1) A mapping $\odot_I : I([0, \infty])^2 \longrightarrow I([0, \infty])$ is called a standard interval-valued pseudo-multiplication if there exist pseudo-multiplications \odot_l and \odot_r such that $x \odot_l y \leq x \odot_r y$ for all $x, y \in [0, \infty]$, and such that for all $\bar{a} = [a_l, a_r], \bar{b} = [b_l, b_r] \in I([0, \infty])$,

$$\bar{a} \odot_I \bar{b} = [a_l \odot_l b_l, a_r \odot_r b_r]. \quad (3)$$

Then \odot_l and \odot_r are called the representants of \odot_I .

(2) A mapping $\odot_{II} : I([0, \infty])^2 \longrightarrow I([0, \infty])$ is called an extended interval-valued pseudo-multiplication if there exists a pseudo-multiplication \odot such that for any $\bar{a} = [a_l, a_r], \bar{b} = [b_l, b_r] \in I([0, \infty])$,

$$\bar{a} \odot_{II} \bar{b} = [a_l \odot b_l, \max\{a_l \odot b_r, a_r \odot b_l\}]. \quad (4)$$

Then \odot is called the representant of \odot_{II} .

We also introduce the (IG) fuzzy integral with respect to a fuzzy measure by using two interval-representable pseudo-multiplications which are used to define the interval-valued generalized pseudo-convolution in the next section 4.

Definition 2.6. ([20]) Let (X, \mathcal{A}, μ) be a fuzzy measure space. (1) An interval-valued function $\bar{f} : X \rightarrow I([0, \infty) \setminus \{\emptyset\})$ is said to be measurable if for any open set $O \subset [0, \infty)$,

$$\bar{f}^{-1}(O) = \{x \in X \mid \bar{f} \cap O \neq \emptyset\} \in \mathcal{A}. \quad (5)$$

(2) If $\odot : I([0, \infty))^2 \rightarrow I([0, \infty))$ is an interval-representable pseudo-multiplication and $\bar{f} \in I\mathfrak{F}(X)$ and $A \in \mathcal{A}$, then the (IG) fuzzy integral with respect to μ by using \odot of \bar{f} on A is defined by

$$(IG) \int_A^{\odot} \bar{f} d\mu = \sup_{\alpha > 0} \alpha \odot \mu_{A, \bar{f}}(\alpha), \quad (6)$$

where $\mu_{A, \bar{f}}(\alpha) = [\mu_{A, f_l}(\alpha), \mu_{A, f_r}(\alpha)]$ for all $\alpha \in [0, \infty)$.

(3) \bar{f} is said to be integrable on A if

$$(IG) \int_A^{\odot} \bar{f} d\mu \in \mathcal{P}([0, \infty)) \setminus \{\emptyset\}, \quad (7)$$

where $\mathcal{P}(\mathbb{R}^+)$ is the set of all subsets of $[0, \infty)$.

Let $I\mathfrak{F}(X)^*$ be the set of all integrable interval-valued functions. We consider the following theorem which is used to investigate some characterizations of the interval-valued generalized pseudo-convolution by means of the (IG) fuzzy integral.

Theorem 2.1. (1) Let \odot_l and \odot_r be pseudo-multiplications on $[0, \infty]$ corresponding to a pseudo-addition $\oplus = \max$. If \odot_I is a standard interval-valued pseudo-multiplication, $A \in \mathcal{A}$, and $f \in I\mathfrak{F}(X)^*$, then we have

$$(IG) \int_A^{\odot_I} \bar{f} d\mu = \left[(G) \int_A^{\odot_l} f_l d\mu, (G) \int_A^{\odot_r} f_r d\mu \right]. \quad (8)$$

(2) Let $\odot : [0, \infty]^2 \rightarrow [0, \infty]$ be a pseudo-multiplication, $\bar{f} = [f_l, f_r] \in I\mathfrak{F}(X)^*$, and $A \in \mathcal{A}$. If \odot_{II} is an extended interval-valued pseudo-multiplication, then we have

$$(IG) \int_A^{\odot_{II}} \bar{f} d\mu = \left[(G) \int_A^{\odot} f_l d\mu, (G) \int_A^{\odot} f_r d\mu \right]. \quad (9)$$

3. THE GENERALIZED PSEUDO-CONVOLUTION ON $\mathfrak{F}(X)^*$

In this section, we consider a semigroup $([0, \infty), \otimes)$ and define the generalized pseudo-convolution on $\mathfrak{F}(X)^*$.

Definition 3.1. Let $f, h \in \mathfrak{F}(X)^*$ and $t \in [0, \infty)$. The generalized pseudo-convolution of f and h by means of the (G) fuzzy integral is defined by

$$(f * h)(t) = (G) \int_{[0,t]}^{\odot} f(t-u) \otimes h(u) d\mu(u). \quad (10)$$

Then we obtain the following basic properties and examples of the generalized pseudo-convolution of nonnegative measurable functions.

Theorem 3.1. (1) If $f, h \in \mathfrak{F}(X)^*$ and $t \in [0, \infty)$ and \otimes is a minimum operation(min) and $f(t-u) \leq h(u)$ for all $u \in [0, t]$, then we have

$$(f * h)(t) = \sup_{\alpha \in [0,t]} \alpha \odot \mu_{[0,t],f}(\alpha). \quad (11)$$

(2) If $f, h \in \mathfrak{F}(X)^*$ and $t \in [0, \infty)$ and \otimes is a multiplication operation(\cdot) and $f(x) = c$ for all $x \in [0, \infty)$, then we have

$$(f * h)(t) = \sup_{\alpha \in [0,t]} \alpha \odot \mu_{[0,t],h}\left(\frac{\alpha}{c}\right). \quad (12)$$

(3) If $f, h \in \mathfrak{F}(X)^*$ and $t \in [0, \infty)$ and $\{t-x|f(x) > 0\} \cap \{x|h(x) > 0\} = \emptyset$ and $a \otimes 0 = 0$ for all $a \in [0, t]$, then we have

$$(f * h)(t) = 0. \quad (13)$$

(4) If $f, h \in \mathfrak{F}(X)^*$ and $t \in [0, \infty)$ and \odot is a minimum operation(min) and $\mu(\{u \in [0, t]|f(t-u) \otimes h(u) > \alpha\}) = g(\alpha) \geq \alpha$ for all $\alpha \in [0, \infty)$, then we have

$$(f * h)(t) = t. \quad (14)$$

Proof.(1) Suppose that \otimes is a minimum operation(min) and $f(t-u) \leq h(u)$ for all $u \in [0, t]$. Then we have

$$\begin{aligned} \mu_{[0,t],f(t-\cdot) \otimes h(\cdot)}(\alpha) &= \mu(\{u \in [0, t] | \min\{f(t-u), h(u)\} > \alpha\}) \\ &= \mu(\{u \in [0, t] | h(u) > \alpha\}) = \mu_{[0,t],h}(\alpha). \end{aligned} \quad (15)$$

By (15), we have

$$\begin{aligned} (f * h)(t) &= (G) \int_{[0,t]}^{\odot} f(t-u) \otimes h(u) d\mu(u) \\ &= \sup_{\alpha \in [0,t]} \alpha \odot \mu_{[0,t],f(t-\cdot) \otimes h(\cdot)}(\alpha) \\ &= \sup_{\alpha \in [0,t]} \alpha \odot \mu_{[0,t],h}(\alpha). \end{aligned} \quad (16)$$

(2) Suppose that \otimes is a multiplication operation(\cdot) and $f(x) = c$ for all $x \in [0, \infty)$. Then we have

$$\begin{aligned} \mu_{[0,t],fh}(\alpha) &= \mu(\{u \in [0, t] | f(t-u)h(u) > \alpha\}) \\ &= \mu(\{u \in [0, t] | ch(u) > \alpha\}) \\ &= \mu\left(\left\{u \in [0, t] | h(u) > \frac{\alpha}{c}\right\}\right) = \mu_{[0,t],h}\left(\frac{\alpha}{c}\right). \end{aligned} \quad (17)$$

By (17), we have

$$(f * h)(t) = (G) \int_{[0,t]}^{\odot} f(t-u) \otimes h(u) d\mu(u)$$

$$\begin{aligned}
&= (G) \int_{[0,t]}^{\odot} ch(u) d\mu(u) \\
&= \sup_{\alpha \in [0,t]} \alpha \odot \mu_{[0,t],ch}(\alpha) \\
&= \sup_{\alpha \in [0,t]} \alpha \odot \mu_{[0,t],h} \left(\frac{\alpha}{c} \right). \tag{18}
\end{aligned}$$

(3) Suppose that $\{t-x|f(x) > 0\} \cap \{x|h(x) > 0\} = \emptyset$ and $u \otimes 0 = 0$ for all $u \in [0, t]$. Then we have

$$\begin{aligned}
\mu_{[0,t],f(t-\cdot) \otimes h(\cdot)}(\alpha) &= \mu(\{u \in [0, t] | f(t-u) \otimes h(u) > \alpha\}) \\
&= \mu(\emptyset) = 0. \tag{19}
\end{aligned}$$

By (19) and Definition 2.2 (2)(vi), we have

$$\begin{aligned}
(f * h)(t) &= \sup_{\alpha \in [0,t]} \alpha \odot \mu_{[0,t],f(t-\cdot) \otimes h(\cdot)}(\alpha) \\
&= \sup_{\alpha \in [0,t]} \alpha \odot 0 = 0. \tag{20}
\end{aligned}$$

(4) Suppose that \odot is a minimum operation(min) and $\mu(\{u \in [0, t] | f(t-u) \otimes h(u) > \alpha\}) = g(\alpha) \geq \alpha$ for all $\alpha \in [0, \infty)$. Then we have

$$\begin{aligned}
(f * h)(t) &= \sup_{\alpha \in [0,t]} \alpha \odot \mu_{[0,t],f(t-\cdot) \otimes h(\cdot)}(\alpha) \\
&= \sup_{\alpha \in [0,t]} \min\{\alpha, g(\alpha)\} \\
&= \sup_{\alpha \in [0,t]} \alpha = t. \tag{21}
\end{aligned}$$

Theorem 3.2. Let $([0, \infty), \otimes)$ be a semigroup and e be a unit element with respect to \otimes , that is, $e \otimes u = u$ for all $u \in [0, \infty)$. If $f \in \mathfrak{F}(X)^*$, then we have

$$(e * f)(t) = (G) \int_{[0,t]}^{\odot} f d\mu. \tag{22}$$

Proof. Since $(e \otimes f)(u) = e \otimes f(u) = f(u)$ for all $u \in [0, \infty)$, we have

$$\begin{aligned}
(e * f)(t) &= \sup_{\alpha \in [0,t]} \alpha \odot \mu_{[0,t],e \otimes f}(\alpha) \\
&= \sup_{\alpha \in [0,t]} \alpha \odot \mu_{[0,t],f}(\alpha) \\
&= (G) \int_{[0,t]}^{\odot} f d\mu. \tag{23}
\end{aligned}$$

Remark 3.3. A function $f : X \rightarrow [0, \infty)$ is an idempotent with respect to the generalized pseudo-convolution $*$ induced by semigroup $([0, \infty), \otimes)$ if and only if $f * f = f$. It is easy to see that if e is a unit element as in Theorem 3.3, that is, $f * e = f$ for all $f \in \mathfrak{F}(X)$, then we also have $e * e = e$. Therefore, e is an idempotent with respect to $*$.

Example 3.1. Let $u \odot v = \min\{u, v\}$ and $u \otimes v = u \cdot v$ for all $u, v \in [0, \infty)$, and $f(x) = 1$ and $h(x) = x^2$ for all $x \in [0, \infty)$, and m be the Lebesgue measure on $[0, \infty)$. If $\mu = m^2$, then clearly μ is a fuzzy measure. Thus, we have

$$\mu_{[0,t],f(t-\cdot) \otimes h(\cdot)}(\alpha) = \mu(\{u \in [0, t] | 1 \otimes u^2 > \alpha\})$$

$$= \mu([\sqrt{\alpha}, t]) = (t - \sqrt{\alpha})^2. \quad (24)$$

By (24), we have

$$\begin{aligned} (f * h)(t) &= \sup_{\alpha \in [0, t]} \min\{\alpha, (t - \sqrt{\alpha})^2\} \\ &= \frac{t^2}{4}. \end{aligned} \quad (25)$$

4. THE INTERVAL-VALUED GENERALIZED PSEUDO-CONVOLUTION ON $I\mathfrak{F}(X)^*$

In this section, we define a standard interval-valued semigroup $(I([0, \infty), \otimes)$ and the interval-representable generalized pseudo-convolution of interval-valued functions by means of the (IG) fuzzy integral on $I\mathfrak{F}(X)^*$.

Definition 4.1. A pair $(I([0, \infty), \otimes)$ is called a standard interval-valued semigroup if there exist two semigroups $([0, \infty), \otimes_l)$ and $([0, \infty), \otimes_r)$ such that

$$\bar{u} \otimes \bar{v} = [u_l \otimes_l v_l, u_r \otimes_r v_r], \quad (26)$$

for all $\bar{u} = [u_l, u_r], \bar{v} = [v_l, v_r] \in I([0, \infty))$.

Definition 4.2. Let $\bar{f}, \bar{h} \in I\mathfrak{F}(X)^*$ and $t \in [0, \infty)$. The interval-valued generalized pseudo-convolution of \bar{f} and \bar{h} by means of the (IG) fuzzy integral is defined by

$$(\bar{f} * \bar{h})(t) = (IG) \int_{[0, t]}^{\odot} \bar{f}(t - u) \otimes \bar{h}(u) d\mu(u) \quad (27)$$

where \odot is an interval-representable pseudo-multiplication.

Then we obtain the following basic properties and examples of the interval-valued generalized pseudo-convolution of measurable interval-valued functions.

Theorem 4.1. (1) Let \odot_l and \odot_r be pseudo-multiplications on $[0, \infty]$ corresponding to a pseudo-addition $\oplus = \max$. If \odot_I is a standard interval-valued pseudo-multiplication and $\bar{f} = [f_l, f_r], \bar{h} = [h_l, h_r] \in I\mathfrak{F}^*(X)$, $t \in [0, \infty]$ and

$$(\bar{f} *_1 \bar{f})(t) = (IG) \int_A^{\odot_I} \bar{f}(t - u) \otimes \bar{h}(u) d\mu(u), \quad (28)$$

then we have

$$(\bar{f} *_1 \bar{f})(t) = [(f_l *_1 h_l, f_r *_1 h_r)], \quad (29)$$

where $(f_l *_1 h_l)(t) = (G) \int_{[0, t]}^{\odot_l} f_l(t - u) \otimes_l h_l(u) d\mu(u)$ and $(f_r *_1 h_r)(t) = (G) \int_{[0, t]}^{\odot_r} f_r(t - u) \otimes_r h_r(u) d\mu(u)$.

(2) Let \odot be a pseudo-multiplications on $[0, \infty]$ corresponding to a pseudo-addition $\oplus = \max$. If \odot_{II} is an extended interval-valued pseudo-multiplication and $\bar{f} = [f_l, f_r], \bar{h} = [h_l, h_r] \in I\mathfrak{F}^*(X)$, $t \in [0, \infty]$ and

$$(\bar{f} *_2 \bar{f})(t) = (IG) \int_A^{\odot_{II}} \bar{f}(t - u) \otimes \bar{h}(u) d\mu(u), \quad (30)$$

then we have

$$(\bar{f} *_2 \bar{f})(t) = [(f_l *_2 h_l, f_r *_2 h_r)], \quad (31)$$

where $(f_l *_2 h_l)(t) = (G) \int_{[0,t]}^{\odot} f_l(t-u) \otimes_l h_l(u) d\mu(u)$ and $(f_r *_2 h_r)(t) = (G) \int_{[0,t]}^{\odot} f_r(t-u) \otimes_r h_r(u) d\mu(u)$.

Proof. (1) Since $\bar{f} \otimes \bar{h} = [f_l \otimes_l h_l, f_r \otimes_r h_r]$, by Theorem 2.7 (1), we have

$$\begin{aligned} (\bar{f} *_1 \bar{f})(t) &= (IG) \int_A^{\odot_I} \bar{f}(t-u) \otimes \bar{h}(u) d\mu(u) \\ &= \left[(G) \int_{[0,\infty]}^{\odot_l} f_l \otimes_l h_l d\mu, (G) \int_{[0,\infty]}^{\odot_r} f_r \otimes_r h_r d\mu \right] \\ &= [(f_l *_1 h_l, f_r *_1 h_r)]. \end{aligned}$$

(2) Since $\bar{f} \otimes \bar{h} = [f_l \otimes_l h_l, f_r \otimes_r h_r]$, by Theorem 2.7 (2), we have

$$\begin{aligned} (\bar{f} *_2 \bar{f})(t) &= (IG) \int_A^{\odot_{II}} \bar{f}(t-u) \otimes \bar{h}(u) d\mu(u) \\ &= \left[(G) \int_{[0,\infty]}^{\odot} f_l \otimes_l h_l d\mu, (G) \int_{[0,\infty]}^{\odot} f_r \otimes_r h_r d\mu \right] \\ &= [(f_l *_2 h_l, f_r *_2 h_r)]. \end{aligned}$$

Theorem 4.2. (1) If $\bar{f} = [f_l, f_r], \bar{h} = [h_l, h_r] \in I\mathfrak{F}(X)^*$ and $t \in [0, \infty)$, and $\otimes_l = \otimes_r$ are minimum operation(min) and $\bar{f}(t-u) \leq \bar{h}(u)$ for all $u \in [0, t]$, then we have

$$(\bar{f} *_1 \bar{h})(t) = \left[\sup_{\alpha \in [0,t]} \alpha \odot_l \mu_{[0,t],f_l}(\alpha), \sup_{\alpha \in [0,t]} \alpha \odot_r \mu_{[0,t],f_r}(\alpha) \right] \quad (32)$$

and

$$(\bar{f} *_2 \bar{h})(t) = \left[\sup_{\alpha \in [0,t]} \alpha \odot \mu_{[0,t],f_l}(\alpha), \sup_{\alpha \in [0,t]} \alpha \odot \mu_{[0,t],f_r}(\alpha) \right]. \quad (33)$$

(2) If $\bar{f} = [f_l, f_r], \bar{h} = [h_l, h_r] \in I\mathfrak{F}(X)^*$ and $t \in [0, \infty)$ and $\otimes_l = \otimes_r$ is multiplication operation(\cdot) and $\bar{f}(x) = [c, d] \in I([0, \infty))$ for all $x \in [0, \infty)$, then we have

$$(\bar{f} *_1 \bar{h})(t) = \left[\sup_{\alpha \in [0,t]} \alpha \odot_l \mu_{[0,t],h_l} \left(\frac{\alpha}{c} \right), \sup_{\alpha \in [0,t]} \alpha \odot_r \mu_{[0,t],f_r} \left(\frac{\alpha}{d} \right) \right] \quad (34)$$

and

$$(\bar{f} *_2 \bar{h})(t) = \left[\sup_{\alpha \in [0,t]} \alpha \odot \mu_{[0,t],h_l} \left(\frac{\alpha}{c} \right), \sup_{\alpha \in [0,t]} \alpha \odot \mu_{[0,t],f_r} \left(\frac{\alpha}{d} \right) \right]. \quad (35)$$

(3) If $\bar{f} = [f_l, f_r], \bar{h} = [h_l, h_r] \in I\mathfrak{F}(X)^*$ and $t \in [0, \infty)$ and $\{t-x|\bar{f}(x) > [0, 0]\} \cap \{x|\bar{h}(x) > [0, 0]\} = \emptyset$ and $\bar{a} \otimes [0, 0] = [0, 0]$ for all $\bar{a} \in I([0, t])$, then we have

$$(\bar{f} *_1 \bar{h})(t) = 0 \quad (36)$$

and

$$(\bar{f} *_2 \bar{h})(t) = 0. \quad (37)$$

(4) If $\bar{f} = [f_l, f_r]$, $\bar{h} = [h_l, h_r] \in I\mathfrak{F}(X)^*$ and $t \in [0, \infty)$ and $\mu(\{u \in [0, t] | f_l(t-u) \otimes_l h_l(u) > \alpha\}) = g_l(\alpha)$ and $\mu(\{u \in [0, t] | f_r(t-u) \otimes_r h_r(u) > \alpha\}) = g_r(\alpha)$ for all $\alpha \in [0, \infty)$, then we have

$$(\bar{f} *_1 \bar{h})(t) = [\sup_{\alpha \in [0, t]} \alpha \odot_l g_l(\alpha), \sup_{\alpha \in [0, t]} \alpha \odot_r g_r(\alpha)] \quad (38)$$

and

$$(\bar{f} *_2 \bar{h})(t) = [\sup_{\alpha \in [0, t]} \alpha \odot g_l(\alpha), \sup_{\alpha \in [0, t]} \alpha \odot g_r(\alpha)]. \quad (39)$$

Proof.(1) Suppose that $\otimes_l = \otimes_r$ are minimum operation(min) and $\bar{f}(t-u) \leq \bar{h}(u)$ for all $u \in [0, t]$. Then we have $f_l(t-u) \leq h_l(u)$ and $f_r(t-u) \leq h_r(u)$ for all $u \in [0, t]$. Thus, by Theorem 4.1(1) and Theorem 3.1 (1), we have

$$\begin{aligned} (\bar{f} *_1 \bar{h})(t) &= [(f_l *_1 h_l)(t), (f_r *_1 h_r)(t)] \\ &= \left[(G) \int_{[0, t]}^{\odot_l} f_l \otimes_l h_l d\mu, (G) \int_{[0, t]}^{\odot_r} f_r \otimes_r h_r d\mu \right] \\ &= \left[\sup_{\alpha \in [0, t]} \alpha \odot_l \mu_{[0, t], f_l}(\alpha), \sup_{\alpha \in [0, t]} \alpha \odot_r \mu_{[0, t], f_r}(\alpha) \right]. \end{aligned} \quad (40)$$

By Theorem 4.1(2) and Theorem 3.1 (1), we have

$$\begin{aligned} (\bar{f} *_2 \bar{h})(t) &= [(f_l *_2 h_l)(t), (f_r *_2 h_r)(t)] \\ &= \left[(G) \int_{[0, t]}^{\odot} f_l \otimes_l h_l d\mu, (G) \int_{[0, t]}^{\odot} f_r \otimes_r h_r d\mu \right] \\ &= \left[\sup_{\alpha \in [0, t]} \alpha \odot \mu_{[0, t], f_l}(\alpha), \sup_{\alpha \in [0, t]} \alpha \odot \mu_{[0, t], f_r}(\alpha) \right]. \end{aligned} \quad (41)$$

(2) Suppose that $\bar{f}(x) = [c, d] \in I([0, \infty))$ for all $x \in [0, \infty)$. By Theorem 3.1 (2) and Theorem 4.1 (1), we have

$$\begin{aligned} (\bar{f} *_1 \bar{h})(t) &= [(f_l *_1 h_l)(t), (f_r *_1 h_r)(t)] \\ &= \left[(G) \int_{[0, t]}^{\odot_l} f_l(t-u) \cdot h_l(u) d\mu(u), (G) \int_{[0, t]}^{\odot_r} f_r(t-u) \cdot h_r(u) d\mu(u) \right] \\ &= \left[(G) \int_{[0, t]}^{\odot_l} c \cdot h_l(u) d\mu(u), (G) \int_{[0, t]}^{\odot_r} d \cdot h_r(u) d\mu(u) \right] \\ &= \left[\sup_{\alpha \in [0, t]} \alpha \odot_l \mu_{[0, t], h_l} \left(\frac{\alpha}{c} \right), \sup_{\alpha \in [0, t]} \alpha \odot_r \mu_{[0, t], h_r} \left(\frac{\alpha}{d} \right) \right]. \end{aligned} \quad (42)$$

By Theorem 3.1 (2) and Theorem 4.1 (2), we have

$$\begin{aligned} (\bar{f} *_2 \bar{h})(t) &= [(f_l *_2 h_l)(t), (f_r *_2 h_r)(t)] \\ &= \left[(G) \int_{[0, t]}^{\odot} f_l(t-u) \cdot h_l(u) d\mu(u), (G) \int_{[0, t]}^{\odot} f_r(t-u) \cdot h_r(u) d\mu(u) \right] \\ &= \left[(G) \int_{[0, t]}^{\odot} c \cdot h_l(u) d\mu(u), (G) \int_{[0, t]}^{\odot} d \cdot h_r(u) d\mu(u) \right] \\ &= \left[\sup_{\alpha \in [0, t]} \alpha \odot \mu_{[0, t], h_l} \left(\frac{\alpha}{c} \right), \sup_{\alpha \in [0, t]} \alpha \odot \mu_{[0, t], h_r} \left(\frac{\alpha}{d} \right) \right]. \end{aligned} \quad (43)$$

(3) Suppose that $\{t - x | \bar{f}(x) > [0, 0]\} \cap \{x | \bar{h}(x) > [0, 0]\} = \emptyset$ and $\bar{a} \otimes [0, 0] = [0, 0]$ for all $\bar{a} \in I([0, t])$. Then we have that $\{t - x | f_l(x) > 0\} \cap \{x | h_l(x) > 0\} = \emptyset$ and $a_l \otimes 0 = 0$ for all $a_l \in [0, t]$, and $\{t - x | f_r(x) > 0\} \cap \{x | h_r(x) > 0\} = \emptyset$ and $a_r \otimes 0 = 0$ for all $a_r \in [0, t]$. By Theorem 3.1(3), we have

$$(f_l *_{1l} h_l)(t) = 0 \text{ and } (f_r *_{1r} h_r)(t) = 0 \quad (44)$$

and

$$(f_l *_{2l} h_l)(t) = 0 \text{ and } (f_r *_{2r} h_r)(t) = 0. \quad (45)$$

By (44) and Theorem 4.1(1), we have

$$(\bar{f} *_{1l} \bar{h})(t) = [(f_l *_{1l} h_l)(t), (f_r *_{1r} h_r)(t)] = 0. \quad (46)$$

By (45) and Theorem 4.1(2), we have

$$(\bar{f} *_{2l} \bar{h})(t) = [(f_l *_{2l} h_l)(t), (f_r *_{2r} h_r)(t)] = 0. \quad (47)$$

(4) Suppose that $f = [f_l, f_r], \bar{h} = [h_l, h_r] \in I\mathfrak{F}(X)$ and $t \in [0, \infty)$ and $\mu(\{u \in [0, t] | f_l(t - u) \otimes_l h_l(u) > \alpha\}) = g_l(\alpha)$ and $\mu(\{u \in [0, t] | f_r(t - u) \otimes_r h_r(u) > \alpha\}) = g_r(\alpha)$ for all $\alpha \in [0, \infty)$. By Theorem 3.1 (4), we have

$$(f_l *_{1l} h_l)(t) = \sup_{\alpha \in [0, t]} \alpha \odot_l g_l(\alpha) \text{ and } (f_r *_{1r} h_r)(t) = \sup_{\alpha \in [0, t]} \alpha \odot_r g_r(\alpha), \quad (48)$$

and

$$(f_l *_{2l} h_l)(t) = \sup_{\alpha \in [0, t]} \alpha \odot g_l(\alpha) \text{ and } (f_r *_{2r} h_r)(t) = \sup_{\alpha \in [0, t]} \alpha \odot g_r(\alpha). \quad (49)$$

By (48) and Theorem 4.1(1), we have

$$\begin{aligned} (\bar{f} *_{1l} \bar{h})(t) &= [(f_l *_{1l} h_l)(t), (f_r *_{1r} h_r)(t)] \\ &= [\sup_{\alpha \in [0, t]} \alpha \odot_l g_l(\alpha), \sup_{\alpha \in [0, t]} \alpha \odot_r g_r(\alpha)]. \end{aligned} \quad (50)$$

By (49) and Theorem 4.1(2), we have

$$\begin{aligned} (\bar{f} *_{2l} \bar{h})(t) &= [(f_l *_{2l} h_l)(t), (f_r *_{2r} h_r)(t)] \\ &= [\sup_{\alpha \in [0, t]} \alpha \odot g_l(\alpha), \sup_{\alpha \in [0, t]} \alpha \odot g_r(\alpha)]. \end{aligned} \quad (51)$$

Theorem 4.3. Let $(I([0, \infty)), \otimes = [\otimes_l, \otimes_r])$ be a standard interval-valued semigroup and e_l be a unit element with respect to \otimes_l and e_r be a unit element with respect to \otimes_r . If $\bar{f} \in I\mathfrak{F}(X)^*$, then we have

$$(\bar{e} *_{1l} \bar{f})(t) = (IG) \int_{[0, t]}^{\otimes_l} \bar{f} d\mu \quad (52)$$

and

$$(\bar{e} *_{2l} \bar{f})(t) = (IG) \int_{[0, t]}^{\otimes_{ll}} \bar{f} d\mu \quad (53)$$

where $\bar{e} = [e_l, e_r]$.

Proof. By Theorem 3.2, we have

$$(e_l *_{1l} f_l)(t) = (G) \int_{[0, t]}^{\odot_l} f_l d\mu \text{ and } (e_r *_{1r} f_r)(t) = (G) \int_{[0, t]}^{\odot_r} f_r d\mu \quad (54)$$

and

$$(e_l *_{2l} f_l)(t) = (G) \int_{[0,t]}^{\odot} f_l d\mu \text{ and } (e_r *_{2r} f_r)(t) = (G) \int_{[0,t]}^{\odot} f_r d\mu. \quad (55)$$

By Theorem 4.1(1) and (54), we have

$$\begin{aligned} (\bar{e} *_{1l} \bar{f})(t) &= [e_l *_{1l} f_l, e_r *_{1r} f_r] \\ &= \left[(G) \int_{[0,t]}^{\odot_l} e_l \otimes_l f_l d\mu, (G) \int_{[0,t]}^{\odot_r} e_r \otimes_r f_r d\mu \right] \\ &= \left[(G) \int_{[0,t]}^{\odot_l} f_l d\mu, (G) \int_{[0,t]}^{\odot_r} f_r d\mu \right] \\ &= (IG) \int_{[0,t]}^{\otimes_I} \bar{f} d\mu. \end{aligned} \quad (56)$$

By (55) and Theorem 4.1(2), we have

$$\begin{aligned} (\bar{e} *_{2l} \bar{f})(t) &= [e_l *_{2l} f_l, e_r *_{2r} f_r] \\ &= \left[(G) \int_{[0,t]}^{\odot} e_l \otimes_l f_l d\mu, (G) \int_{[0,t]}^{\odot} e_r \otimes_r f_r d\mu \right] \\ &= \left[(G) \int_{[0,t]}^{\odot} f_l d\mu, (G) \int_{[0,t]}^{\odot} f_r d\mu \right] \\ &= (IG) \int_{[0,t]}^{\otimes_{II}} \bar{f} d\mu. \end{aligned} \quad (57)$$

Remark 4.4. A function $\bar{f} : X \rightarrow I([0, \infty))$ is an interval-valued idempotent with respect to the standard interval-valued generalized pseudo-convolution $*_i$ (for $i = 1, 2$) induced by a standard interval-valued semigroup $(I([0, \infty)), \otimes)$ if and only if $\bar{f} *_i \bar{f} = \bar{f}$ for $i = 1, 2$. It is easy to see that if $\bar{e} = [e_l, e_r]$ is a unit element as in Theorem 4.2, that is, $\bar{f} *_i \bar{e} = \bar{f}$ for all $\bar{f} \in I\mathfrak{F}(X)^*$, then we also have $\bar{e} *_i \bar{e} = \bar{e}$ for $i = 1, 2$. Therefore, \bar{e} is an interval-valued idempotent with respect to $*_i$ for $i = 1, 2$.

Example 4.1. Suppose that $\odot_l = \odot_r = \odot$ and $u \odot v = \min\{u, v\}$ and $u \otimes_l v = u \otimes_r v = u \cdot v$ for all $u, v \in [0, \infty)$, and $\bar{f}(x) = [1, 2]$ and $\bar{h}(x) = [x^2, 2x^2]$ for all $x \in [0, \infty)$, and m be the Lebesgue measure on $[0, \infty)$. If $\mu = m^2$, then clearly μ is a fuzzy measure. Thus, we have

$$\begin{aligned} \mu_{[0,t], f_l(t-\cdot) \otimes_l h_l(\cdot)}(\alpha) &= \mu(\{u \in [0, t] | 1 \otimes u^2 > \alpha\}) \\ &= \mu([\sqrt{\alpha}, t]) = (t - \sqrt{\alpha})^2 \end{aligned} \quad (58)$$

and

$$\begin{aligned} \mu_{[0,t], f_r(t-\cdot) \otimes_r h_r(\cdot)}(\alpha) &= \mu(\{u \in [0, t] | 2 \otimes 2u^2 > \alpha\}) \\ &= \mu\left(\left[\frac{\sqrt{\alpha}}{2}, t\right]\right) = \left(t - \frac{\sqrt{\alpha}}{2}\right)^2. \end{aligned} \quad (59)$$

By (58) and Theorem 4.1(1), we have

$$\begin{aligned} &(\bar{f} *_{1l} \bar{h})(t) \\ &= \left[\sup_{\alpha \in [0,t]} \min \{ \alpha, \mu_{[0,t], f_l(t-\cdot) \otimes_l h_l(\cdot)}(\alpha) \}, \sup_{\alpha \in [0,t]} \min \{ \alpha, \mu_{[0,t], f_r(t-\cdot) \otimes_r h_r(\cdot)}(\alpha) \} \right] \end{aligned}$$

$$\begin{aligned}
&= \left[\sup_{\alpha \in [0, t]} \min \left\{ \alpha, (t - \sqrt{\alpha})^2 \right\}, \sup_{\alpha \in [0, t]} \min \left\{ \alpha, \left(t - \frac{\sqrt{\alpha}}{2} \right)^2 \right\} \right] \\
&= \left[\frac{t^2}{4}, 4t^2 \right].
\end{aligned}$$

5. CONCLUSIONS

This study was to define the generalized pseudo-convolution of integrable functions by means of the (G) fuzzy integral (see Definition 3.1) and to investigate some properties and an example of the generalized pseudo-convolution on $\mathfrak{F}(X)^*$ in Theorems 3.2, 3.3 and Example 3.1.

By using the concept of an interval-representable pseudo-multiplication (see Definitions 2.5 and 2.6), we can define a standard interval-valued semigroup (see Definition 4.1) and the interval-representable generalized pseudo-convolution on $I\mathfrak{F}(X)^*$ (see definition 4.2). From Theorems 4.3, 4.4, and 4.5, we investigate some characterizations of the interval-representable generalized pseudo-convolution of integrable interval-valued functions.

Furthermore, some applications of the interval-representable generalized pseudo-convolution are focused on various transform operations including pseudo-Laplace transform. For this reason, the future work can also be directed to interval-representable generalized pseudo-transform operations by means of the (IG) fuzzy integral.

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Fixed point and coupled fixed point theorems for generalized cyclic weak contractions in partially ordered probabilistic metric spaces

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Abstract. In this paper, we introduce the concept of new generalized cyclic weak contraction mappings and prove a class of fixed point theorems for such mappings in partially ordered probabilistic metric spaces. In addition, we also establish a coupled fixed point for mixed monotone mappings under contractive conditions in partially ordered probabilistic metric spaces. Our results extend and generalize Harjani *et al.* (Nonlinear anal. 71(2009)3403-3410) and Wu (Fixed Point Theory Appl. 2014(2014)49). Also, we introduce an example to support the validity of our results. Finally, an application of our results extends fixed point theorems for generalized weak contraction mappings in ordered metric spaces.

Keywords: Menger probabilistic metric space; partially ordered; cyclic weak contractions; fixed point

MR Subject Classification: 47H10, 34B15, 46S50

1 Introduction and preliminaries

Fixed point theory in metric spaces is an important branch of nonlinear analysis, which is closely related to the existence and uniqueness of solutions of differential and integral equations. The celebrated Banach's contraction mapping principle is one of the cornerstones in development of nonlinear analysis.

In the past years, Kirk and Srinivasan [1] presented fixed point theorems for mappings satisfying cyclical contractive conditions. Ran and Reurings [2] introduced fixed point theorems of Banach contraction operator in partially ordered metric spaces. Agarwal *et al.* [3] proved fixed point results of generalized contractive operators in partially ordered metric spaces; Harjani and Sadarangani [4] presented some fixed point theorems for weakly contractive mappings in complete metric spaces endowed with a partial order. Shatanwi [5] introduced nonlinear weakly C -contractive mappings in ordered metric spaces and proved some fixed point theorems. For more detail on fixed point theory and related results, we refer to [6-12] and the references therein.

In 1942, Menger [13] introduced the concept of probabilistic metric spaces, a number of authors have done considerable works on probabilistic metric spaces [14-19]. Recently, the extension of fixed point theory to generalized structures as partially ordered probabilistic metric spaces has received much attention (see, [20-22]).

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However, we rarely see any work about fixed point theorems for mappings under weakly contractive conditions in partially ordered probabilistic metric spaces.

The aim of this paper is to determine some fixed point theorems for generalized cyclic weak contractions in the framework of partially ordered probabilistic metric spaces. Also, we introduce an example to support the validity of our results. Our results extend and generalize the main results of [3-8,11-12].

We introduce some useful concepts and lemmas for the development of our results.

Let R denote the set of reals and R^+ the nonnegative reals. A mapping $F : R \rightarrow R^+$ is called a distribution function if it is nondecreasing and left continuous with $\inf_{t \in R} F(t) = 0$ and $\sup_{t \in R} F(t) = 1$. We will denote by D the set of all distribution functions and $D^+ = \{F \in D : F(t) = 0, t \leq 0\}$.

Let H denote the specific distribution function defined by

$$H(x) = \begin{cases} 0, & x \leq 0; \\ 1, & x > 0. \end{cases}$$

Definition 1.1 ([14]). The mapping $\Delta : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a triangular norm (for short, a t -norm) if the following conditions are satisfied:

- ($\Delta - 1$) $\Delta(a, 1) = a$, for all $a \in [0, 1]$;
- ($\Delta - 2$) $\Delta(a, b) = \Delta(b, a)$;
- ($\Delta - 3$) $\Delta(a, b) \leq \Delta(c, d)$, for $c \geq a, d \geq b$;
- ($\Delta - 4$) $\Delta(a, \Delta(b, c)) = \Delta(\Delta(a, b), c)$.

Two typical examples of continuous t -norm are $\Delta_1(a, b) = \max\{a + b - 1, 0\}$ and $\Delta_2(a, b) = ab$, for all $a, b \in [0, 1]$.

Definition 1.2 ([14]). A triplet (X, F, Δ) is called a Menger probabilistic metric space (for short, Menger PM-space), if X is a nonempty set, Δ is a t -norm and F is a mapping from $X \times X \rightarrow D^+$ satisfying the following conditions (for $x, y \in X$, we denote $F(x, y)$ by $F_{x,y}$):

- (MS-1) $F_{x,y}(t) = H(t)$, for all $t \in R$, if and only if $x = y$;
- (MS-2) $F_{x,y}(t) = F_{y,x}(t)$, for all $x, y \in X$ and $t \in R$;
- (MS-3) $F_{x,z}(s+t) \geq \Delta(F_{x,y}(s), F_{y,z}(t))$, for all $x, y, z \in X$ and $s, t \geq 0$.

Definition 1.3 ([15]). (X, F, Δ) is called a non-Archimedean Menger PM-space (shortly, a N.A Menger PM-space), if (X, F, Δ) is a Menger PM-space and Δ satisfies the following condition: for all $x, y, z \in X$ and $t_1, t_2 \geq 0$,

$$F_{x,z}(\max\{t_1, t_2\}) \geq \Delta(F_{x,y}(t_1), F_{y,z}(t_2)). \quad (1.1)$$

Definition 1.4 ([15]). A non-Archimedean Menger PM-space (X, F, Δ) is said to be type $(D)_g$ if there exists a $g \in \Omega$ such that

$$g(\Delta(s, t)) \leq g(s) + g(t),$$

for all $s, t \in [0, 1]$, where $\Omega = \{g : g : [0, 1] \rightarrow [0, \infty) \text{ is continuous, strictly decreasing, } g(1) = 0\}$.

Example 1.1 . (X, F, Δ) is a N.A Menger PM-space, and $\Delta \geq \Delta_1$, where $\Delta_1(s, t) = \max\{s + t - 1, 0\}$, then (X, F, Δ) is of $(D)_g$ -type for $g \in \Omega$ defined by $g(t) = 1 - t$.

Remark 1.1 Schweizer and Sklar [14] point out that if (X, F, Δ) is a Menger probabilistic metric space and Δ is continuous, then (X, F, Δ) is a Hausdorff topological space in the (ε, λ) -topology T , i.e., the family of sets $\{U_x(\varepsilon, \lambda) : \varepsilon > 0, \lambda \in (0, 1]\}$ ($x \in X$) is a basis of neighborhoods of a point x for T , where $U_x(\varepsilon, \lambda) = \{y \in X : F_{x,y}(\varepsilon) > 1 - \lambda\}$.

Lemma 1.1 ([15]). Let $\{x_n\}$ be a sequence in X such that $\lim_{n \rightarrow \infty} F_{x_n, x_{n+1}}(t) = 1$ for all $t > 0$. If the sequence $\{x_n\}$ is not a Cauchy sequence in X , then there exist $\varepsilon_0 > 0$, $t_0 > 0$ and two sequences $\{k(i)\}$, $\{m(i)\}$ of positive integers such that

- (1) $m(i) > k(i)$, and $m(i) \rightarrow \infty$ as $i \rightarrow \infty$;
- (2) $F_{x_{m(i)}, x_{k(i)}}(t_0) < 1 - \varepsilon_0$ and $F_{x_{m(i)-1}, x_{k(i)}}(t_0) \geq 1 - \varepsilon_0$, for $i = 1, 2, \dots$.

Definition 1.5 ([1]). Let X be a non-empty set, m be a positive integer, A_1, A_2, \dots, A_m be subsets of X , $Y = \bigcup_{i=1}^m A_i$ and a mapping $f : Y \rightarrow Y$. Then Y is said to be a cyclic representation of Y with respect to f , if

- (i) A_i , $i = 1, 2, \dots, m$, are nonempty closed sets;
- (ii) $f(A_1) \subseteq A_2, \dots, f(A_{m-1}) \subseteq A_m, f(A_m) \subseteq A_1$.

Example 1.2 Let $X = \mathbb{R}^+$. Let $A_1 = [0, 2]$, $A_2 = [\frac{1}{2}, \frac{3}{2}]$, $A_3 = [\frac{3}{4}, \frac{5}{4}]$, and $Y = \bigcup_{i=1}^3 A_i$. Defined $f : Y \rightarrow Y$ by $fx = \frac{1}{2} + \frac{1}{2}x$, for all $x \in Y$.

Clearly $Y = \bigcup_{i=1}^3 A_i$ is a cyclic representation of Y with respect to f .

Definition 1.6 ([9]). The function $h : [0, \infty) \rightarrow [0, \infty)$ is called an altering distance function, if the following properties are satisfied: (a) h is continuous and nondecreasing; (b) $h(t) = 0$ if and only if $t = 0$.

In [10], Bhasker and Lakshmikantham introduced the concepts of mixed monotone mappings and coupled fixed point.

Definition 1.7 ([10]). Let (X, \leq) be a partially ordered set and $A : X \times X \rightarrow X$. The mapping A is said to have the mixed monotone property if A is monotone nondecreasing in its first argument and is monotone nonincreasing in its second argument, that is, for any $x, y \in X$,

$$\begin{aligned} x_1, x_2 \in X, \quad x_1 \leq x_2 &\implies A(x_1, y) \leq A(x_2, y), \\ y_1, y_2 \in X, \quad y_2 \leq y_1 &\implies A(x, y_1) \leq A(x, y_2). \end{aligned}$$

Definition 1.8 ([10]). An element $(x, y) \in X^2$ is said to be a coupled fixed point of the mapping $A : X^2 \rightarrow X$ if $A(x, y) = x$ and $A(y, x) = y$.

For $\tilde{a} = (x, y), \tilde{b} = (u, v) \in X^2$, we introduce a distribution function \tilde{F} from X^2 into D^+ defined by

$$\tilde{F}_{\tilde{a}, \tilde{b}}(t) = \min\{F_{x,u}(t), F_{y,v}(t)\}, \text{ for all } t > 0.$$

In [20], Wu proved the following results:

Lemma 1.2 ([20]). If (X, F, Δ) is a complete Menger PM space, then (X^2, \tilde{F}, Δ) is also a complete Menger PM space.

In the section 3 of this paper, we establish some coupled point theorems under contractive conditions in partially ordered probabilistic metric spaces. The obtained results extend and generalized the main results of [20-22]. Finally, we also obtain the corresponding fixed point theorems for generalized weak contraction mapping in ordered metric spaces.

2 Fixed point theorems for generalized cyclic weak contractions

We start with the definition of generalized cyclic weak contraction mappings in probabilistic metric spaces.

Definition 2.1 Let (X, \leq) be a partially ordered set and (X, F, Δ) be a N.A Menger PM-space of type $(D)_g$. Let m be a positive integer, A_1, A_2, \dots, A_m be subsets of X , $Y = \cup_{i=1}^m A_i$. A mapping $T : X \rightarrow X$ is said to be a generalized cyclic weak contraction, if Y is a cyclic representation of Y with respect to T , $A_{m+1} = A_1$ and for $k \in \{1, 2, \dots, m\}$, and for all $x, y \in X$, $x \in A_k$ and $y \in A_{k+1}$ are comparable with

$$h(g(F_{Tx, Ty}(t))) \leq h(M_t(x, y)) - \phi(M_t(x, y)), \quad \text{for all } t > 0, \quad (2.1)$$

where $M_t(x, y) = \max\{g(F_{x, y}(t)), g(F_{x, Tx}(t)), g(F_{y, Ty}(t)), \frac{1}{2}[g(F_{x, Ty}(t)) + g(F_{y, Tx}(t))]\}$, h is a altering distance function, $\phi : [0, \infty) \rightarrow [0, \infty)$ is a continuous function such that $\phi(s) = 0$ if and only if $s = 0$.

Theorem 2.1 Let (X, \leq) be a partially ordered set and (X, F, Δ) be a complete N.A Menger PM-space of type $(D)_g$. Let m be a positive integer, A_1, A_2, \dots, A_m be subsets of X , $Y = \cup_{i=1}^m A_i$, $T : Y \rightarrow Y$ be a generalized cyclic weak contraction, and T be nondecreasing. Also assume that either

- (a) T is continuous or,
- (b) if a nondecreasing sequence $x_n \rightarrow x$, then $x_n \leq x$, for all $n \in N$.

If there exists $x_0 \in A_1$ such that $x_0 \leq Tx_0$, then T has a fixed point. Furthermore, the set of fixed points of T is well ordered if and only if T has a unique fixed point.

Proof. Since $T(A_1) \subseteq A_2$, there exists an $x_1 \in A_2$, such that $x_1 = Tx_0$. Since $T(A_2) \subseteq A_3$, there exists an $x_2 \in A_3$, such that $x_2 = Tx_1$. Continuing this process, we can construct a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$, for all $n \in N$, and there exists $i_n \in \{1, 2, \dots, m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$.

Since $x_0 \leq Tx_0 = x_1$ and T is nondecreasing, we have $Tx_0 \leq Tx_1$, that is, $x_1 \leq x_2$. By induction, we get that $x_0 \leq x_1 \leq \dots \leq x_n \leq \dots$, for all $n \in N$.

Without loss of generality, assume that $x_{n+1} \neq x_n$, for all $n \in N$ (otherwise, $x_{n+1} = Tx_n = x_n$, then the conclusion holds).

Since $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$ are comparable, for $i_n \in \{1, 2, \dots, m\}$, by inequality (2.1), we get

$$h[g(F_{x_{n+1}, x_n}(t))] \leq h[M_t(x_n, x_{n-1})] - \phi(M_t(x_n, x_{n-1})), \quad \text{for all } t > 0, \quad (2.2)$$

where

$$\begin{aligned}
M_t(x_n, x_{n-1}) &= \max\{g(F_{x_n, x_{n-1}}(t)), g(F_{x_n, x_{n-1}}(t)), g(F_{x_n, x_{n+1}}(t)), \frac{1}{2}g(F_{x_{n-1}, x_{n+1}}(t))\} \\
&\leq \max\{g(F_{x_n, x_{n-1}}(t)), g(F_{x_n, x_{n+1}}(t)), \frac{1}{2}g(\Delta(F_{x_{n-1}, x_n}(t), F_{x_n, x_{n+1}}(t)))\} \\
&\leq \max\{g(F_{x_n, x_{n-1}}(t)), g(F_{x_n, x_{n+1}}(t)), \frac{1}{2}[g(F_{x_{n-1}, x_n}(t)) + g(F_{x_n, x_{n+1}}(t))]\} \\
&= \max\{g(F_{x_n, x_{n-1}}(t)), g(F_{x_n, x_{n+1}}(t))\} = M_t(x_n, x_{n-1}).
\end{aligned}$$

Suppose that $M_t(x_n, x_{n-1}) = g(F_{x_n, x_{n+1}}(t))$, by (2.2), we have

$$h[g(F_{x_{n+1}, x_n}(t))] \leq h[g(F_{x_n, x_{n+1}}(t))] - \phi(g(F_{x_n, x_{n+1}}(t))), \quad \text{for all } t > 0,$$

which implies that $\phi(g(F_{x_n, x_{n+1}}(t))) = 0$. Thus, $g(F_{x_n, x_{n+1}}(t)) = 0$, that is, $F_{x_n, x_{n+1}}(t) = 1$ for all $t > 0$. Then $x_n = x_{n+1}$, which is in contradiction to $x_n \neq x_{n+1}$, for any $n \in N$.

Hence, $M_t(x_n, x_{n-1}) = g(F_{x_n, x_{n-1}}(t))$, it follows from (2.2) that

$$h[g(F_{x_{n+1}, x_n}(t))] \leq h[g(F_{x_n, x_{n-1}}(t))] - \phi(g(F_{x_n, x_{n-1}}(t))) \leq h[g(F_{x_n, x_{n-1}}(t))], \quad \forall t > 0, \quad (2.3)$$

Since h is nondecreasing, it follows from (2.3) that $\{g(F_{x_{n+1}, x_n}(t))\}$ is a decreasing sequence, for every $t > 0$. Hence, there exists $r_t \geq 0$ such that $\lim_{n \rightarrow \infty} g(F_{x_{n+1}, x_n}(t)) = r_t$.

By using the continuities of h and ϕ , letting $n \rightarrow \infty$ in (2.3), we get $h(r_t) \leq h(r_t) - \phi(r_t)$, which implies that $\phi(r_t) = 0$. Then $r_t = 0$, that is, $\lim_{n \rightarrow \infty} g(F_{x_{n+1}, x_n}(t)) = 0$ and $\lim_{n \rightarrow \infty} F_{x_{n+1}, x_n}(t) = 1$, for all $t > 0$.

In the sequel, we will prove that $\{x_n\}$ is Cauchy sequence. To prove this fact, we first prove the following claim.

Claim: for every $t > 0$, $\varepsilon > 0$, there exists $n_0 \in N$, such that $p, q \geq n_0$ with $p - q \equiv 1 \pmod{m}$ then $F_{x_p, x_q}(t) > 1 - \varepsilon$, that is, $g(F_{x_p, x_q}(t)) < g(1 - \varepsilon)$.

In fact, suppose to the contrary, there exist $t_0 > 0$ and $\varepsilon_0 > 0$, such that for any $n \in N$, we can find $p(n) > q(n) \geq n$ with $p(n) - q(n) \equiv 1 \pmod{m}$ satisfying $F_{x_{p(n)}, x_{q(n)}}(t_0) \leq 1 - \varepsilon_0$, that is, $g(F_{x_{p(n)}, x_{q(n)}}(t_0)) \geq g(1 - \varepsilon_0)$.

Now, we take $n > 2m$. Then corresponding to $q(n) \geq n$, we can choose $p(n)$ in such a way that it is the smallest integer with $p(n) > q(n)$ satisfying $p(n) - q(n) \equiv 1 \pmod{m}$ and $g(F_{x_{p(n)}, x_{q(n)}}(t_0)) \geq g(1 - \varepsilon_0)$. Therefore, $g(F_{x_{p(n)-m}, x_{q(n)}}(t_0)) < g(1 - \varepsilon_0)$. Using the non-Archimedean Menger triangular inequality and Definition 1.5, we have

$$\begin{aligned}
g(1 - \varepsilon_0) &\leq g(F_{x_{q(n)}, x_{p(n)}}(t_0)) \leq g(\Delta(F_{x_{q(n)}, x_{q(n)+1}}(t_0), F_{x_{q(n)+1}, x_{p(n)}}(t_0))) \\
&\leq g(F_{x_{q(n)}, x_{q(n)+1}}(t_0)) + g(F_{x_{q(n)+1}, x_{p(n)}}(t_0)) \\
&\leq g(F_{x_{q(n)}, x_{q(n)+1}}(t_0)) + g(F_{x_{q(n)+1}, x_{p(n)+1}}(t_0)) + g(F_{x_{p(n)+1}, x_{p(n)}}(t_0)) \\
&\leq 2g(F_{x_{q(n)}, x_{q(n)+1}}(t_0)) + g(F_{x_{q(n)}, x_{p(n)+1}}(t_0)) + g(F_{x_{p(n)+1}, x_{p(n)}}(t_0)) \\
&\leq 2g(F_{x_{q(n)}, x_{q(n)+1}}(t_0)) + g(F_{x_{q(n)}, x_{p(n)}}(t_0)) + 2g(F_{x_{p(n)+1}, x_{p(n)}}(t_0)) \\
&\leq 2g(F_{x_{q(n)}, x_{q(n)+1}}(t_0)) + g(F_{x_{q(n)}, x_{p(n)-m}}(t_0)) + g(F_{x_{p(n)-m}, x_{p(n)}}(t_0)) + 2g(F_{x_{p(n)+1}, x_{p(n)}}(t_0)) \\
&\leq 2g(F_{x_{q(n)}, x_{q(n)+1}}(t_0)) + g(1 - \varepsilon_0) + \sum_{i=1}^m g(F_{x_{p(n)-i}, x_{p(n)-i+1}}(t_0)) + 2g(F_{x_{p(n)+1}, x_{p(n)}}(t_0)). \quad (2.4)
\end{aligned}$$

Since $\lim_{n \rightarrow \infty} g(F_{x_{n+1}, x_n}(t)) = 0$ for all $t > 0$, letting $n \rightarrow \infty$ in (2.4), we have

$$\begin{aligned} g(1 - \varepsilon_0) &= \lim_{n \rightarrow \infty} g(F_{x_{q(n)}, x_{p(n)}}(t_0)) = \lim_{n \rightarrow \infty} g(F_{x_{q(n)+1}, x_{p(n)}}(t_0)) \\ &= \lim_{n \rightarrow \infty} g(F_{x_{q(n)+1}, x_{p(n)+1}}(t_0)) = \lim_{n \rightarrow \infty} g(F_{x_{q(n)}, x_{p(n)+1}}(t_0)). \end{aligned} \quad (2.5)$$

By $p(n) - q(n) \equiv 1 \pmod{m}$, we know that $x_{p(n)}$ and $x_{q(n)}$ lie in different adjacently labeled sets A_i and A_{i+1} , for $1 \leq i \leq m$. Using the fact that T is a generalized cyclic weak contraction, we have

$$h[g(F_{x_{q(n)+1}, x_{p(n)+1}}(t_0))] = h[g(F_{Tx_{q(n)}, Tx_{p(n)}}(t_0))] \leq h[M_{t_0}(x_{q(n)}, x_{p(n)})] - \phi(M_{t_0}(x_{q(n)}, x_{p(n)})), \quad (2.6)$$

where

$$\begin{aligned} M_{t_0}(x_{q(n)}, x_{p(n)}) &= \max\{g(F_{x_{q(n)}, x_{p(n)}}(t_0)), g(F_{x_{q(n)}, x_{q(n)+1}}(t_0)), g(F_{x_{p(n)}, x_{p(n)+1}}(t_0)), \\ &\quad \frac{1}{2}[g(F_{x_{q(n)}, x_{p(n)+1}}(t_0)) + g(F_{x_{p(n)}, x_{q(n)+1}}(t_0))]\}. \end{aligned}$$

By (2.5), we have $\lim_{n \rightarrow \infty} M_{t_0}(x_{q(n)}, x_{p(n)}) = \max\{g(1 - \varepsilon_0), 0, 0, \frac{1}{2}[g(1 - \varepsilon_0) + g(1 - \varepsilon_0)]\} = g(1 - \varepsilon_0)$. According to the continuities of h and ϕ , letting $n \rightarrow \infty$ in (2.6), we get

$$h[g(1 - \varepsilon_0)] \leq h[g(1 - \varepsilon_0)] - \phi(g(1 - \varepsilon_0)).$$

Thus, $\phi(g(1 - \varepsilon)) = 0$, that is $g(1 - \varepsilon_0) = 0$. Then $\varepsilon_0 = 0$, which is in contradiction to $\varepsilon_0 > 0$.

Therefore, our claim is proved. In the sequel, we will prove that $\{x_n\}$ is Cauchy sequence.

By the continuity of g and $g(1) = 0$, we have $\lim_{a \rightarrow 0^+} g(1 - a\varepsilon) = 0$, for any given $\varepsilon > 0$. Since g is strictly decreasing, then there exists $a > 0$ such that $g(1 - a\varepsilon) \leq \frac{g(1 - \varepsilon)}{2}$.

For any given $t > 0$, $\varepsilon > 0$, there exists $a > 0$ such that $g(1 - a\varepsilon) \leq \frac{g(1 - \varepsilon)}{2}$. By the claim, we find $n_0 \in N$ such that if $p, q \geq n_0$ with $p - q \equiv 1 \pmod{m}$, then

$$F_{x_p, x_q}(t) > 1 - a\varepsilon, \quad \text{and} \quad g(F_{x_p, x_q}(t)) < g(1 - a\varepsilon) \leq \frac{g(1 - \varepsilon)}{2}. \quad (2.7)$$

Since $\lim_{n \rightarrow \infty} g(F_{x_{n+1}, x_n}(t)) = 0$, we also find $n_1 \in N$ such that

$$g(F_{x_{n+1}, x_n}(t)) \leq \frac{g(1 - \varepsilon)}{2m}, \quad (2.8)$$

for any $n > n_1$.

Suppose that $r, s \geq \max\{n_0, n_1\}$ and $s > r$. Then there exists $k \in \{1, 2, \dots, m\}$ such that $s - r \equiv k \pmod{m}$. Therefore, $s - r + j \equiv 1 \pmod{m}$, for $j = m - k + 1, j \in \{0, 1, \dots, m - 1\}$. So, we have

$$g(F_{x_r, x_s}(t)) \leq g(F_{x_r, x_{s+j}}(t)) + g(F_{x_{s+j}, x_{s+j-1}}(t)) + \dots + g(F_{x_{s+1}, x_s}(t)).$$

From (2.7), (2.8) and the last inequality, we get

$$g(F_{x_r, x_s}(t)) < \frac{g(1 - \varepsilon)}{2} + j \cdot \frac{g(1 - \varepsilon)}{2m} \leq \frac{g(1 - \varepsilon)}{2} + \frac{g(1 - \varepsilon)}{2} = g(1 - \varepsilon). \quad (2.9)$$

Since g is strictly decreasing, by (2.9), we obtain $F_{x_r, x_s}(t) > 1 - \varepsilon$. Therefore $\{x_n\}$ is Cauchy sequence.

Since X is a complete PM-space, $Y = \cup_{i=0}^m A_i$ is closed, then Y also is a complete space. Thus there exists $x^* \in Y$ such that $x_n \rightarrow x^*$. As $Y = \cup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T , then the sequence $\{x_n\}$ has infinite terms in each A_i for $i \in \{1, 2, \dots, m\}$.

First, suppose that $x^* \in A_i$, then $Tx^* \in A_{i+1}$, and we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $x_{n_k} \in A_{i-1}$ (the existence of this subsequence is guaranteed by above mentioned comment).

Case (a): If T is continuous. Since $\lim_{n \rightarrow \infty} x_n = x^*$, we have $Tx^* = x^*$.

Case (b): If it satisfies a nondecreasing sequence $x_n \rightarrow x^*$, such that $x_n \leq x^*$, then $x_{n_k} \in A_{i-1}$ and $x^* \in A_i$ are comparable. By (2.1), we have

$$h[g(F_{x_{n_k}+1, Tx^*}(t))] = h[g(F_{Tx_{n_k}, Tx^*}(t))] \leq h[M_t(x_{n_k}, x^*)] - \phi(M_t(x_{n_k}, x^*)), \quad (2.10)$$

where

$$M_t(x_{n_k}, x^*) = \max\{g(F_{x_{n_k}, x^*}(t)), g(F_{x_{n_k}, x_{n_k}+1}(t)), g(F_{x^*, Tx^*}(t)), \frac{1}{2}[g(F_{x_{n_k}, Tx^*}(t)) + g(F_{x_{n_k}+1, x^*}(t))]\}.$$

Let G_0 be the set of all the discontinuous points of $F_{x^*, Tx^*}(t)$. Since g , h , and ϕ are continuous, we obtain that G_0 also is the set of all the discontinuous points of $g(F_{x^*, Tx^*}(t))$, $h[g(F_{x^*, Tx^*}(t))]$ and $\phi(g(F_{x^*, Tx^*}(t)))$. Moreover, we know that G_0 is a countable set. Let $G = R^+ \setminus G_0$. When $t \in G \setminus \{0\}$ (t is a continuity point of $F_{x^*, Tx^*}(t)$), we have

$$\lim_{k \rightarrow \infty} M_t(x_{n_k}, x^*) = \max\{0, 0, g(F_{x^*, Tx^*}(t)), \frac{1}{2}[g(F_{x^*, Tx^*}(t)) + 0]\} = g(F_{x^*, Tx^*}(t)).$$

Letting $n \rightarrow \infty$ in (2.10), we get

$$h[g(F_{x^*, Tx^*}(t))] \leq h[g(F_{x^*, Tx^*}(t))] - \phi(g(F_{x^*, Tx^*}(t))).$$

Thus, $\phi(g(F_{x^*, Tx^*}(t))) = 0$, that is, $g(F_{x^*, Tx^*}(t)) = 0$. Then

$$F_{x^*, Tx^*}(t) = H(t), \quad \text{for all } t \in G. \quad (2.11)$$

When $t \in G_0$ with $t > 0$, by the density of real numbers, there exist $t_1, t_2 \in G$ such that $0 < t_1 < t < t_2$. Since the distribution is nondecreasing, we have

$$1 = H(t_1) = F_{x^*, Tx^*}(t_1) \leq F_{x^*, Tx^*}(t) \leq F_{x^*, Tx^*}(t_2) = 1.$$

This shows that, for all $t \in G_0$ with $t > 0$,

$$F_{x^*, Tx^*}(t) = H(t). \quad (2.12)$$

Combining (2.11) with (2.12), we have $F_{x^*, Tx^*}(t) = H(t)$, for all $t > 0$, that is, $Tx^* = x^*$.

Hence, in all case, we have $Tx^* = x^*$.

Finally, we prove the uniqueness of the fixed point under the additional conditions. In fact, suppose that there exist $x^*, y^* \in Y$ such that $Tx^* = x^*$, $Ty^* = y^*$, then we have $x^*, y^* \in \cap_{i=1}^m A_i$.

Since the set of fixed points of T is well ordered, we have $x^* \in A_i$ and $y^* \in A_{i+1}$ are comparable. By (2, 1), we have

$$h[g(F_{x^*,y^*}(t))] \leq h[M_t(x^*, y^*)] - \phi(M_t(x^*, y^*)), \quad \text{for all } t > 0,$$

where

$$M_t(x^*, y^*) = \max\{g(F_{x^*,y^*}(t)), g(F_{x^*,x^*}(t)), g(F_{y^*,y^*}(t)), \frac{1}{2}[g(F_{x^*,y^*}(t)) + g(F_{x^*,y^*}(t))]\} = g(F_{x^*,y^*}(t)).$$

Thus, $\phi(g(F_{x^*,y^*}(t))) = 0$, that is, $g(F_{x^*,y^*}(t)) = 0$. Hence, $F_{x^*,y^*}(t) = 1$, for all $t > 0$. Then $x^* = y^*$.

Remark 2.1 Theorem 2.1 generalizes and extends Theorem 2.1 in [6] and Theorem 2.4 in [7].

Corollary 2.1 Let (X, \leq) be a partially ordered set and (X, F, Δ) be a complete N.A Menger PM-space, $T : X \rightarrow X$ be a nondecreasing mapping. Suppose that for comparable $x, y \in X$, we have

$$g(F_{Tx,Ty}(t)) \leq \Phi(\max\{g(F_{x,y}(t)), g(F_{x,Tx}(t)), g(F_{y,Ty}(t)), \frac{1}{2}[g(F_{x,Ty}(t)) + g(F_{y,Tx}(t))]\}),$$

for all $t > 0$, where $\Phi : [0, \infty) \rightarrow [0, \infty)$ is a continuous function, $\Phi(t) < t$, for $t > 0$ and $\Phi(0) = 0$. Also assume that either

(a) T is continuous or, (b) if a nondecreasing sequence $x_n \rightarrow x$, then $x_n \leq x$, for all $n \in N$.

If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point. Furthermore, the set of fixed points of T is well ordered if and only if T has a unique fixed point.

Proof. Taking $h(x) = x$ and $\Phi(t) = t - \phi(t)$ in Theorem 2.1, we can easily obtain the above corollary.

Corollary 2.2 Let (X, \leq) be a partially ordered set and (X, F, Δ) be a complete N.A Menger PM-space, $T : X \rightarrow X$ be a nondecreasing mapping. Suppose that for comparable $x, y \in X$, we have

$$F_{Tx,Ty}(t) \geq \psi(\min\{F_{x,y}(t), F_{x,Tx}(t), F_{y,Ty}(t), \frac{1}{2}[F_{x,Ty}(t) + F_{y,Tx}(t)]\}), \quad \text{for all } t > 0,$$

where $\psi : [0, 1] \rightarrow [0, 1]$ is a continuous function, $t < \psi(t) < 1$ for $t \in [0, 1)$, $\psi(t) = 1$ if and only if $t = 1$. Also assume that either

(a) T is continuous or, (b) if a nondecreasing sequence $x_n \rightarrow x$, then $x_n \leq x$, for all $n \in N$.

If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point. Furthermore, the set of fixed points of T is well ordered if and only if T has a unique fixed point.

Proof. Taking $h(x) = x$ and $g(t) = 1 - t$, $\psi(t) = t + \phi(1 - t)$ in Theorem 2.1, we can easily obtain the above corollary.

Remark 2.2 Corollary 2.2 generalizes and extends Theorem 2.1 in [22].

Now, we give an example to demonstrate Theorem 2.1.

Example 2.1 . Let $X = R^+$, $\Delta_1(a, b) = \max\{a + b - 1, 0\}$, F be defined by

$$F_{x,y}(t) = \begin{cases} 0, & t \leq 0, \\ \frac{\min\{x,y\}}{\max\{x,y\}}, & 0 < t \leq 1, \\ 1, & t > 1. \end{cases}$$

for all $x, y \in X$. Then, for every given $x, y \in X$, it is easy to verify that $F_{x,y}$ is a distribution function and (X, F, Δ) is a complete N.A Menger PM-space.

In fact, $(MS - 1)$ and $(MS - 2)$ are easy to check. To prove inequality (1.1). We consider the case:

Case 1. If $t_1 > 1$ or $t_2 > 1$, then $F_{x,z}(\max\{t_1, t_2\}) \geq \Delta_1(F_{x,y}(t_1), F_{y,z}(t_2))$, for any $x, y, z \in X$.

Case 2. If $0 < t_1, t_2 \leq 1$ and $x \leq y \leq z$, for $x, y, z \in R^+$, then

$$F_{x,z}(\max\{t_1, t_2\}) - \Delta_1(F_{x,y}(t_1), F_{y,z}(t_2)) = \frac{x}{z} + 1 - \left(\frac{x}{y} + \frac{y}{z}\right) = \frac{(y-x)(z-y)}{yz} \geq 0.$$

Case 3. If $0 < t_1, t_2 \leq 1$ and $y \leq x \leq z$, for $x, y, z \in R^+$, then

$$F_{x,z}(\max\{t_1, t_2\}) - \Delta_1(F_{x,y}(t_1), F_{y,z}(t_2)) = \frac{x}{z} + 1 - \left(\frac{y}{x} + \frac{y}{z}\right) = \frac{(x+z)(x-y)}{xz} \geq 0.$$

Case 4. If $0 < t_1, t_2 \leq 1$ and $x \leq z \leq y$, for $x, y, z \in R^+$, then

$$F_{x,z}(\max\{t_1, t_2\}) - \Delta_1(F_{x,y}(t_1), F_{y,z}(t_2)) = \frac{x}{z} + 1 - \left(\frac{x}{y} + \frac{z}{y}\right) = \frac{(x+z)(y-z)}{yz} \geq 0.$$

Hence, in all case, we have $F_{x,z}(\max\{t_1, t_2\}) \geq \Delta_1(F_{x,y}(t_1), F_{y,z}(t_2))$, for all $t_1, t_2 \in R^+$, that is, 1.1 holds.

Suppose that $A_1 = [0, 1]$, $A_2 = [\frac{1}{2}, 1]$, $A_3 = [\frac{3}{4}, 1]$, and $Y = \bigcup_{i=1}^3 A_i$. Let $f : Y \rightarrow Y$ and $fx = \frac{1}{2} + \frac{1}{2}x$, for all $x \in Y$,

Clearly $Y = \bigcup_{i=1}^3 A_i$ is a cyclic representation of Y with respect to f .

We next prove that it satisfies the conditions of Theorem 2.1, where $h(x) = \frac{1}{2}x$, $\phi(x) = \frac{1}{6}x$, and $g(t) = 1 - t$. By the definitions of F , g , h and ϕ , we only need to prove that

$$F_{fx, fy}(t) \geq Q_t(x, y) + \frac{1}{3}(1 - Q_t(x, y)) = \frac{2}{3}Q_t(x, y) + \frac{1}{3}, \quad (2.13)$$

where $Q_t(x, y) = \min\{F_{x,y}(t), F_{x,Tx}(t), F_{y,Ty}(t), \frac{1}{2}[F_{x,Ty}(t) + F_{y,Tx}(t)]\}$.

Since $fx = \frac{1}{2} + \frac{1}{2}x$. If $0 \leq x \leq y$, for $x, y \in [0, 1]$, then we have

$$Q_t(x, y) = \min\{F_{x,y}(t), F_{x,Tx}(t), F_{y,Ty}(t), \frac{1}{2}[F_{x,Ty}(t) + F_{y,Tx}(t)]\} \leq F_{x,y}(t) = \frac{x}{y}.$$

Hence, we consider the following two cases:

Case 1. If $0 < t \leq 1$, we have

$$F_{fx, fy}(t) - \frac{2}{3}Q_t(x, y) - \frac{1}{3} \geq \frac{x+1}{y+1} - \frac{2x}{3y} - \frac{1}{3} = \frac{(2-y)(y-x)}{3(y+1)y} \geq 0,$$

which implies that (2.13) holds.

Case 2. If $t > 1$, by the definition of F , we have

$$F_{fx, fy}(t) - \frac{2}{3}Q_t(x, y) - \frac{1}{3} = 0,$$

which implies that (2.13) holds.

Hence, in all case, we obtain that (2.13) holds.

Thus, all hypotheses of Theorem 2.1 are satisfied, and we deduce that f has a unique fixed point in Y . Here, $x = 1$ is the unique fixed point of f .

3 Coupled fixed point theorems in partially ordered probabilistic metric spaces

In the section, we will apply the Corollary 2.2 in the Section 2 to prove the coupled fixed point theorems under contractive conditions in partially ordered probabilistic metric spaces.

Lemma 3.1 If (X, F, Δ) is a N.A Menger PM space, then (X^2, \tilde{F}, Δ) is also a N.A Menger PM space.

Proof. It is sufficient to prove that, for $\tilde{a} = (x, y), \tilde{b} = (u, v), \tilde{c} = (p, q) \in X^2$,

$$\tilde{F}_{\tilde{a}, \tilde{c}}(\max\{t_1, t_2\}) \geq \Delta(\tilde{F}_{\tilde{a}, \tilde{b}}(t_1), \tilde{F}_{\tilde{b}, \tilde{c}}(t_2)),$$

for all $t_1, t_2 \geq 0$. In fact, for all $\tilde{a} = (x, y), \tilde{b} = (u, v), \tilde{c} = (p, q) \in X^2$ and $t_1, t_2 \geq 0$ we have

$$\begin{aligned} \tilde{F}_{\tilde{a}, \tilde{c}}(\max\{t_1, t_2\}) &= \min\{F_{x,p}(\max\{t_1, t_2\}), F_{y,q}(\max\{t_1, t_2\})\} \\ &\geq \min\{\Delta(F_{x,u}(t_1), F_{u,p}(t_2)), \Delta(F_{y,v}(t_1), F_{v,q}(t_2))\} \\ &\geq \Delta(\min\{F_{x,u}(t_1), F_{y,v}(t_1)\}, \min\{F_{u,p}(t_2), F_{v,q}(t_2)\}) \\ &= \Delta(\tilde{F}_{\tilde{a}, \tilde{b}}(t_1), \tilde{F}_{\tilde{b}, \tilde{c}}(t_2)). \end{aligned}$$

The proof is complete.

Theorem 3.1 Let (X, \leq) be a partially ordered set and (X, F, Δ) be a complete N.A Menger PM-space, $A : X \times X \rightarrow X$ be a mapping satisfying the mixed monotone property on X . Suppose that for all $x, y, u, v \in X$, $x \leq u$ and $v \leq y$, we have

$$\begin{aligned} F_{A(x,y), A(u,v)}(t) &\geq \psi(\min\{F_{x,u}(t), F_{y,v}(t), F_{x,A(x,y)}(t), F_{u,A(u,v)}(t), F_{y,A(y,x)}(t), F_{v,A(v,u)}(t), \\ &\quad \frac{1}{2}[\min\{F_{x,A(u,v)}(t), F_{y,A(v,u)}(t)\} + \min\{F_{u,A(x,y)}(t), F_{v,A(y,x)}(t)\}]\}), \end{aligned}$$

for all $t > 0$, where $\psi : [0, 1] \rightarrow [0, 1]$ is a continuous function, $t < \psi(t) < 1$ for $t \in [0, 1)$, $\psi(t) = 1$ if and only if $t = 1$. Also assume that either

- (a) A is continuous or,
- (b) if a nondecreasing sequence $x_n \rightarrow x$, then $x_n \leq x$, for all $n \in N$;

If a nonincreasing sequence $x_n \rightarrow x$, then $y \leq y_n$, for all $n \in N$.

If there exist $x_0, y_0 \in X$ such that $x_0 \leq A(x_0, y_0)$ and $A(y_0, x_0) \leq y_0$, then A has a coupled fixed point, that is, there exist $p, q \in X$ such that $A(p, q) = p$ and $A(q, p) = q$.

Proof. Let $\tilde{X} = X \times X$, for $\tilde{a} = (x, y), \tilde{b} = (u, v) \in \tilde{X}$, we introduce the order \preceq as

$$\tilde{a} \preceq \tilde{b} \text{ if and only if } x \leq u, v \leq y.$$

It follows from Lemma 1.2 and Lemma 3.1 that $(X, \preceq, \tilde{F}, \Delta)$ is also a complete partially ordered N.A Menger PM-space, where

$$\tilde{F}_{\tilde{a}, \tilde{b}}(t) = \min\{F_{x,u}(t), F_{y,v}(t)\}.$$

The self-mapping $T : \tilde{X} \rightarrow \tilde{X}$ is given by

$$T\tilde{a} = (A(x, y), A(y, x)) \text{ for all } \tilde{a} = (x, y) \in \tilde{X}.$$

Then a coupled point of A is a fixed point of T and vice versa.

If $\tilde{a} \preceq \tilde{b}$, then $x \leq u$ and $v \leq y$. Noting the mixed monotone property of A , we see that $A(x, y) \leq A(u, v)$ and $A(v, u) \leq A(y, x)$, then $T\tilde{a} \preceq T\tilde{b}$. Thus T is a nondecreasing mapping with respect to the order \preceq on \tilde{X} .

On the other hand, for all $t > 0$ and $\tilde{a} = (x, y), \tilde{b} = (u, v) \in \tilde{X}$ with $\tilde{a} \preceq \tilde{b}$, we have

$$\begin{aligned} F_{A(x,y), A(u,v)}(t) &\geq \psi(\min\{F_{x,u}(t), F_{y,v}(t), F_{x,A(x,y)}(t), F_{u,A(u,v)}(t), F_{y,A(y,x)}(t), F_{v,A(v,u)}(t) \\ &\quad \frac{1}{2}[\min\{F_{x,A(u,v)}(t), F_{y,A(v,u)}(t)\} + \min\{F_{u,A(x,y)}(t), F_{v,A(y,x)}(t)\}]) \\ &= \psi(\min\{\min\{F_{x,u}(t), F_{y,v}(t)\}, \min\{F_{x,A(x,y)}(t), F_{y,A(y,x)}(t)\}, \min\{F_{u,A(u,v)}(t), F_{v,A(v,u)}(t)\}, \\ &\quad \frac{1}{2}[\min\{F_{x,A(u,v)}(t), F_{y,A(v,u)}(t)\} + \min\{F_{u,A(x,y)}(t), F_{v,A(y,x)}(t)\}]) \\ &= \psi(\min\{\tilde{F}_{\tilde{a},\tilde{b}}(t), \tilde{F}_{\tilde{a},T\tilde{a}}(t), \tilde{F}_{\tilde{b},T\tilde{b}}(t), \frac{1}{2}[\tilde{F}_{\tilde{a},T\tilde{b}}(t) + \tilde{F}_{T\tilde{a},\tilde{b}}(t)]\}) \end{aligned}$$

Similarly, $F_{A(y,x), A(v,u)}(t) \geq \psi(\min\{\tilde{F}_{\tilde{a},\tilde{b}}(t), \tilde{F}_{\tilde{a},T\tilde{a}}(t), \tilde{F}_{\tilde{b},T\tilde{b}}(t), \frac{1}{2}[\tilde{F}_{\tilde{a},T\tilde{b}}(t) + \tilde{F}_{T\tilde{a},\tilde{b}}(t)]\})$. Thus,

$$F_{T\tilde{a},T\tilde{b}}(t) \geq \psi(\min\{\tilde{F}_{\tilde{a},\tilde{b}}(t), \tilde{F}_{\tilde{a},T\tilde{a}}(t), \tilde{F}_{\tilde{b},T\tilde{b}}(t), \frac{1}{2}[\tilde{F}_{\tilde{a},T\tilde{b}}(t) + \tilde{F}_{T\tilde{a},\tilde{b}}(t)]\}).$$

Also, there exists an $\tilde{x}_0 = (x_0, y_0) \in \tilde{X}$ such that $\tilde{x}_0 \preceq T\tilde{x}_0 = (A(x_0, y_0), A(y_0, x_0))$.

If a nondecreasing monotone sequence $\{\tilde{x}_n\} = \{(x_n, y_n)\}$ in \tilde{X} tends to $\tilde{x} = (x, y)$, then $\tilde{x}_n = (x_n, y_n) \preceq (x_{n+1}, y_{n+1}) = \tilde{x}_{n+1}$, that is, $x_n \leq x_{n+1}$ and $y_{n+1} \leq y_n$. Thus $\{x_n\}$ is nondecreasing sequence tending to x and $\{y_n\}$ a nonincreasing sequence tending to y . Thus $x_n \leq x$ and $y \leq y_n$ for all $n \in N$. This implies $\tilde{x}_n \preceq \tilde{x}$. Obviously, the continuity of A implies the continuity of T .

Therefore, all hypotheses of Corollary 2.2 are satisfied. Following Corollary 2.2, we deduce that A has a coupled point, that is, there exist $p, q \in \tilde{X}$ such that $A(p, q) = p$ and $A(q, p) = q$.

Remark 3.1 Theorem 3.1 generalizes and extends Theorem 71 in [21] and Corollary 2.1 in [22].

4 An application

In this section, using the Theorem 2.1, we establish some fixed results for generalized weak contractions in partially ordered metric spaces.

Theorem 4.1 Let (X, d, \leq) be an ordered complete metric space, $T : X \rightarrow X$ be a nondecreasing mapping. Suppose that for comparable $x, y \in X$, we have

$$d(Tx, Ty) \leq M(x, y) - \varphi(M(x, y)), \quad \forall t > 0, \quad (4.1)$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}$, $\varphi : [0, \infty) \rightarrow [0, \infty)$ is a continuous function, $\frac{\varphi(s)}{t} \geq \varphi(\frac{s}{t})$, for all $t > 0$, and $\varphi(s) = 0$ if and only if $s = 0$. Also assume that either

(a) T is continuous or, (b) if a nondecreasing sequence $x_n \rightarrow x$, then $x_n \leq x$, for all $n \in N$.

If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point. Furthermore, the set of fixed points of T is well ordered if and only if T has a unique fixed point.

Proof. Let (X, F, Δ_2) be the induced N.A Menger PM-space, where F is defined by $F_{x,y}(t) = e^{-\frac{d(x,y)}{t}}$, for $t > 0$, $x, y \in X$. We can easily prove that a sequence $\{x_n\}$ in X converges in the metric d to a point $x^* \in X$ if and if only $\{x_n\}$ in (X, F, Δ_2) τ -converges to x^* . Let $g \in \Omega$, where $g(t) = 1 - t$. Since (X, d) is a complete metric space, then (X, F, Δ_2) is a τ -complete N.A Menger PM-space of type $(D)_g$.

For $x, y \in X$, x and y are comparable, by (4.1), for $t > 0$, we have

$$\begin{aligned} 1 - e^{-\frac{d(Tx, Ty)}{t}} &\leq 1 - e^{-\frac{M(x, y)}{t} + \frac{\varphi(M(x, y))}{t}} \\ &\leq 1 - e^{-\frac{M(x, y)}{t} + \varphi(\frac{M(x, y)}{t})} \\ &= 1 - e^{-\frac{M(x, y)}{t}} - e^{-\frac{M(x, y)}{t}} [e^{\varphi(\frac{M(x, y)}{t})} - 1]. \end{aligned} \quad (4.2)$$

Let $\phi : [0, 1) \rightarrow [0, +\infty)$, where $\phi(u) = [1 - u][e^{\varphi(\ln \frac{1}{1-u})} - 1]$, for $u \in [0, 1]$. Since φ is continuous and $\varphi^{-1}(0) = 0$, then ϕ also is continuous and $\phi^{-1}(0) = 0$.

Since $\phi(1 - e^{-\frac{M(x, y)}{t}}) = e^{-\frac{M(x, y)}{t}} [e^{\varphi(\frac{M(x, y)}{t})} - 1]$, $g(s) = 1 - s$, and $F_{x,y}(t) = e^{-\frac{d(x,y)}{t}}$, by (4.2), we get

$$g(F_{Tx, Ty}(t)) \leq M_t(x, y) - \phi(M_t(x, y)),$$

for $t > 0$, where $M_t(x, y) = \max\{g(F_{x,y}(t)), g(F_{x, Tx}(t)), g(F_{y, Ty}(t)), \frac{1}{2}[g(F_{x, Ty}(t)) + g(F_{y, Tx}(t))]\}$.

Thus, all hypotheses of Theorem 2.1 are satisfied, when $h(s) = s$ and $m = 1$. Then the conclusion holds.

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Weak Galerkin finite element method for time dependent reaction-diffusion equation

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Abstract

We propose a weak Galerkin finite element procedure for time dependent reaction-diffusion equation by using weakly defined gradient operators over discontinuous functions with heterogeneous properties, in which the classical gradient operator is replaced by the discrete weak gradient. Numerical analysis and numerical experiments illustrate and confirm that our new method has effective numerical performances.

Mathematics subject classifications: 65M15, 65M60.

Keywords: Galerkin finite element methods, parabolic equation, weak gradient, error estimate, numerical experiment.

1. Introduction.

Time dependent reaction-diffusion equations are a large important class of equations. In this paper, we consider the following time dependent reaction-diffusion equation:

$$u_t + Au = f(x, t), \quad x \in \Omega, \quad 0 < t \leq T, \quad (1a)$$

$$u = u^0(x), \quad x \in \Omega, \quad t = 0, \quad (1b)$$

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with homogenous Dirichlet boundary condition, where Ω is a bounded region in R^2 , with a Lipschitz continuous boundary; $u_t = \frac{\partial u}{\partial t}$; and A is a second order elliptic differential operator:

$$Au \equiv -\nabla \cdot (a \nabla u) + cu,$$

where a and c are sufficiently smooth functions of x and satisfy $0 < a_* \leq a(x) \leq a^*$ and $c(x) \geq 0$ for fixed a_*, a^* . We define the following bilinear form

$$a(u, v) := \int_{\Omega} (a \nabla u \cdot \nabla v + cuv) dx. \quad (2)$$

It is obvious that there is a constant $\alpha_0 > 0$ such that

$$a(u, u) \geq \alpha_0 \|u\|_1^2, \quad \forall u \in H_0^1(\Omega). \quad (3)$$

The variational weak form to (1) is: find $u = u(x, t) \in L^2(0, T; H_0^1(\Omega))$, such that

$$(u_t, v) + a(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega), \quad t > 0, \quad (4a)$$

$$u(x, 0) = u^0(x), \quad x \in \Omega, \quad (4b)$$

where (\cdot, \cdot) denotes the inner product of $L^2(\Omega)$.

Many numerical methods for solving such problems have been developed, please see [3, 6, 7, 11, 12, 13, 16] and references in. In [5], a weak Galerkin finite element method (WG-FEM) was introduced and analyzed for parabolic equation based on a discrete weak gradient arising from local Raviart-Thomas (RT) elements [10]. Due to the use of RT elements, the WG finite element formulation of [5] was limited to finite element partitions of triangles for two dimensional problem. To overcome this, we presented a WG-FEM in [4] with a stabilization term for a diffusion equation without reaction term and derived optimal convergence rate in L^2 norm based on a dual argument technique for the solution of the WG-FEM. The WG-FEM was first introduced in [14] for solving second order elliptic problems. Later, the WG-FEMs were studied from implementation point of view in [8] and applied to solve the Helmholtz problem with high wave numbers in [9].

The purpose of this paper is to present a weak Galerkin (WG) finite element procedures using more flexible elements in arbitrary unstructured meshes for time dependent reaction-diffusion problem, and derive optimal convergence rate in the H^1 norm.

The outline of this article is as follows. In Section 3, we define the weak gradient and present semi-discrete and fully-discrete WG-FEMs for problem (1). In Section 4, we establish the optimal order error estimates in H^1 -norm to the WG-FEMs for the parabolic problem. Finally in Section 5 we give some numerical examples to verify the theory.

Throughout this paper, the notations of standard Sobolev spaces $L^2(\Omega)$, $H^k(\Omega)$ and associated norms $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_k = \|\cdot\|_{H^k(\Omega)}$ are adopted.

2. A weak gradient operator and its discrete approximation

Let T be any polygonal domain with interior T^0 and boundary ∂T . A *weak function* on the region T refers to a function $v = \{v_0, v_b\}$ such that $v_0 \in L^2(T)$ and $v_b \in H^{\frac{1}{2}}(\partial T)$. v_0 represents the value of v on T^0 and v_b represents that of v on ∂T . Note that v_b may not necessarily be related to the trace of v_0 on ∂T . Denote by $W(T)$ the space of weak function associated with T ; i.e.,

$$W(T) = \left\{ v = \{v_0, v_b\} : v_0 \in L^2(T), v_b \in H^{\frac{1}{2}}(\partial T) \right\}. \quad (5)$$

Definition 2.1. [14] *The dual of $L^2(T)$ can be identified with itself by using the standard L^2 inner product as action of linear functional. With a similar interpretation, for any $v \in W(T)$, the weak gradient of v is defined as a linear functional $\nabla_w v$ in the dual space of $H(\text{div}, T)$ whose action on each $q \in H(\text{div}, T)$ is given by*

$$(\nabla_w v, q)_T := - \int_T v_0 \nabla \cdot q dT + \int_{\partial T} v_b q \cdot \mathbf{n} ds, \quad (6)$$

where \mathbf{n} is the outer normal direction to ∂T .

Next, we introduce a discrete weak gradient operator by defining ∇_w in a polynomial subspace of $H(\text{div}, T)$. To this end, for any non-negative integer $r \geq 0$, denote by $P_r(T)$ the set of polynomials on T with degree no more than r . Let $V(K, r) \subset [P_r(T)]^2$ be a subspace of the space of vector-valued polynomials of degree r . A discrete weak gradient operator, denoted by $\nabla_{w,r}$, is defined so that $\nabla_{w,r} v \in V(T, r)$ is the unique solution of the following equation

$$(\nabla_{w,r} v, q)_T := - \int_T v_0 \nabla \cdot q dT + \int_{\partial T} v_b q \cdot \mathbf{n} ds, \quad \forall q \in V(T, r). \quad (7)$$

It is easy to know that $\nabla_{w,r}$ is a Galerkin-type approximation of the weak gradient operator ∇_w by using the polynomial space $V(T, r)$.

3. Weak Galerkin finite element methods

Let \mathcal{T}_h be a regular finite element grid on Ω with mesh size h . Assume that the partition \mathcal{T}_h is shape regular so that the routine inverse inequality in the finite element analysis holds true (see [2]). In the general spirit of the Galerkin procedure, we shall design a weak Galerkin finite element method for (4) by the following two basic principles: (1) replace $H^1(\Omega)$ by a space of discrete weak functions defined on the finite element partition \mathcal{T}_h and the boundary of triangular elements; and (2) replace the classical gradient operator by a discrete weak gradient operator ∇_w for weak functions on each triangle T .

For each $T \in \mathcal{T}_h$, denote by $P_j(T^0)$ the set of polynomials on T^0 , which is the interior of triangle T , with degree no more than j , and $P_l(\partial T)$ the set of polynomials on ∂T with degree no more than l (i.e., polynomials of degree l on each line segment of ∂T). A discrete weak function $v = \{v_0, v_b\}$ on T refers to a weak function $v = \{v_0, v_b\}$ such that $v_0 \in P_j(T^0)$ and $v_b \in P_l(\partial T)$ with $j \geq 0$ and $l \geq 0$. Denote this space by $W(T, j, l)$, i.e.,

$$W(T, j, l) := \{v = \{v_0, v_b\} : v_0 \in P_j(T^0), v_b \in P_l(\partial T)\}.$$

The corresponding FE space would be defined by matching $W(T, j, l)$ over all the triangles $T \in \mathcal{T}_h$ as

$$V_h := \{v = \{v_0, v_b\} : \{v_0, v_b\}|_T \in W(T, j, l), \forall T \in \mathcal{T}_h\}. \quad (8)$$

Denote by V_h^0 the subspace of V_h with zero boundary values on $\partial\Omega$; i.e.,

$$V_h^0 := \{v = \{v_0, v_b\} \in V_h, v_b|_{\partial T \cap \partial\Omega} = 0, \forall T \in \mathcal{T}_h\}. \quad (9)$$

According to (7), for each $v = \{v_0, v_b\} \in V_h^0$, the discrete weak gradient $\nabla_{w,r}v$ of v on each element T is given by the following equation:

$$\int_T \nabla_{w,r}v \cdot q dx = - \int_T v_0 \nabla \cdot q dx + \int_{\partial T} v_b q \cdot \mathbf{n} ds, \quad \forall q \in V(T, r). \quad (10)$$

For simplicity of notation, we shall drop the subscript r in the discrete weak gradient operator $\nabla_{w,r}$ from now on. Now, we define the semi-discrete weak Galerkin finite element scheme for (1) as: find $u_h = \{u_0, u_b\}(\cdot, t) \in V_h^0$ ($0 \leq t \leq T$) such that

$$(u_{h,t}, v) + a_w(u_h, v) = (f, v_0), \forall v = \{v_0, v_b\} \in V_h^0, t > 0, \quad (11a)$$

$$u_h(x, 0) = Q_h u^0(x), \quad x \in \Omega, \quad (11b)$$

where the bilinear form $a_w(\cdot, \cdot)$ is defined as

$$a_w(v, w) = \sum_{T \in \mathcal{T}_h} \int_T (a \nabla_w v \cdot \nabla_w w + c v_0 w_0) dx + \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle v_0 - v_b, w_0 - w_b \rangle_{\partial T}, \quad (12)$$

and $Q_h u = \{Q_0 u, Q_b u\}$ is the L^2 projection onto $P_j(T^0) \times P_l(\partial T)$. In other words, on each element T , the function $Q_0 u$ is defined as the L^2 projection of u on $P_j(T)$ and on ∂T , $Q_b u$ is the L^2 projection in $P_l(\partial T)$. Hereafter, we choose $l = j$.

Let $\{\varphi_i(x) : i = 1, 2, \dots, N\}$, where $N = \dim(V_h^0)$, be the bases of V_h^0 . For example, when $j = 0$ in $P_j(T)$, φ_i is a function which takes value one in the interior of triangle T of \mathcal{T}_h and zero everywhere else; and φ_i is a function that takes value one on the edge $e \in \partial T$ and zero everywhere else. Then (11) can be expressed as: find a solution of the form

$$u_h = \{u_0, u_b\} = \sum_{i=1}^N \mu_i(t) \varphi_i(x),$$

such that its coefficients $\mu_1(t), \mu_2(t), \dots, \mu_N$ satisfy

$$\sum_{i=1}^N \left[\frac{d\mu_i(t)}{dt} (\varphi_i, \varphi_j) + \mu_i a_w(\varphi_i, \varphi_j) \right] = (f, \varphi_j), \quad t > 0. \quad (13)$$

By Introducing the following matrix and vector notations:

$$\mathbf{M} = [m_{ij}] = [(\varphi_i, \varphi_j)], \quad \mathbf{K} = [k_{ij}] = [a_w(\varphi_i, \varphi_j)],$$

$$\mu = [\mu_1, \mu_2, \dots, \mu_N]^T, \quad \mathbf{F} = [(f, \varphi_1), (f, \varphi_2), \dots, (f, \varphi_N)]^T,$$

then (13) can be rewritten as

$$\mathbf{M} \frac{d\mu}{dt} + \mathbf{K} \mu = \mathbf{F}. \quad (14)$$

\mathbf{M} and \mathbf{K} are positive definite matrix. The ordinary differential equation (ODE) theory tells us that the semi-discrete WG scheme has a unique solution for any $f \in L^2(\Omega)$.

Define a norm $\|\cdot\|_{w,1}$ as

$$\|v\|_{w,1} := \sqrt{\sum_{T \in \mathcal{T}_h} (\|\nabla_w v\|_{0,T}^2 + \|v\|_{0,T}^2 + h_T^{-1} \|v_0 - v_b\|_{0,\partial T}^2)}, \quad (15)$$

which is a H^1 -equivalent norm for conventional finite element functions, since the presence of the $L^2(T)$ term renders the norm to be an equivalent H^1 norm for any H^1 function, regardless the value of their zeroth order traces on ∂T ; where $\|v\|_{0,T}^2 = \int_T v^2 dx$ and $\|v_0 - v_b\|_{0,\partial T}^2 = \int_{\partial T} (v_0 - v_b)^2 ds$. Moreover, the following *Poincaré* inequality holds true for functions in V_h^0 .

Lemma 3.1. *Assume that the finite element partition \mathcal{T}_h is shape regular. Then there exists a constant C independent of the mesh size h such that*

$$\|v\| \leq \|v\|_{w,1}, \forall v = \{v_0, v_b\} \in V_h^0. \quad (16)$$

Let us now return to our semi-discrete problem in the formulation (11). A basic stability inequality for problem (1) with $f = 0$, for simplicity, is as follows:

Theorem 3.1. *For the numerical solution to scheme (11) with initial setting (11b), there is a L^2 -stability as follows*

$$\frac{d}{dt} \int_{\Omega} u_h^2(x, t) dx \leq 0. \quad (17)$$

Proof. Taking $v = u_h$ in (11a), with $f = 0$, we get

$$(u_{h,t}(t), u_h(t)) + a_w(u_h(t), u_h(t)) = 0.$$

From the definition of bilinear form $a_w(\cdot, \cdot)$ in (12), we know that

$$a_w(u_h(t), u_h(t)) \geq 0.$$

Based on this fact,

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} u_h^2(t) dx = \frac{1}{2} \frac{d}{dt} (u_h(t), u_h(t)) = (u_{h,t}(t), u_h(t)) \leq 0.$$

This completes the proof. \square

Let τ denote the time step size, and $t_n = n\tau$ ($n = 0, 1, \dots$), $u_h^n := u_h(t_n) = \{u_0^n, u_b^n\}$. At time $t = t_n$, using backward difference quotient

$$\bar{\partial}_t u_h^n = (u_h^n - u_h^{n-1})/\tau$$

to approximate the differential quotient $u_{h,t}$ in the semi-discrete scheme (11), we get the fully-discrete WG-FE scheme: find $u_h^n = \{u_0^n, u_b^n\} \in V_h^0$ for $n = 1, 2, \dots$, such that

$$\sum_{T \in \mathcal{T}_h} (\bar{\partial}_t u_h^n, v_h)_T + a_w(u_h^n, v_h) = \sum_{T \in \mathcal{T}_h} (f^n, v_0)_T, \forall v_h = \{v_0, v_b\} \in V_h^0, \quad (18a)$$

$$u_h^0 = Q_h u^0(x). \quad (18b)$$

From (12), for $v, w \in V_h^0$, we get

$$a_w(v, v) \geq \alpha_0 \|v\|_{w,1}^2, \forall v \in V_h^0,$$

and

$$a_w(v, w) \leq C^* \|v\|_{w,1} \|w\|_{w,1},$$

which guarantees the existence and uniqueness of the solution $u_h^n = \{u_0^n, u_b^n\}$ to (18) for a given $u_h^{n-1} = \{u_0^{n-1}, u_b^{n-1}\}$.

4. Error estimate

In this section we will present a priori error estimates in H^1 -norm for the semi-discrete scheme (11) and fully-discrete scheme (18) for smooth solutions of (1).

For simplicity, we assume that diffusion coefficient a is piecewise constant with respect to the finite element partition \mathcal{T}_h . The corresponding results can be extended to the case of variable coefficients provided that the coefficient function a is sufficiently smooth.

Below we denote C (maybe with indices) as a positive constant depending solely on the exact solution, which may have different values in each occurrence.

4.1. Preliminaries

4.1.1. Sobolev space definitions and notations

Let Ω be any domain in R^2 . In this paper, we adopt the standard definition for the Sobolev space $W^{s,r}(\Omega)$, which consists of functions with (distributional) derivatives of order less than or equal to s in $L^r(\Omega)$ for $1 \leq r \leq +\infty$ and integer s . And their associated inner products $(\cdot, \cdot)_{s,r,\Omega}$, norms $\|\cdot\|_{s,r,\Omega}$, and seminorms $|\cdot|_{s,r,\Omega}$. Further, $\|\cdot\|_{\infty,\Omega}$ represents the norm on $L^\infty(\Omega)$, and $\|\cdot\|_{L^\infty([0,T];W^{s,r}(\Omega))}$ the norm on $L^\infty([0,T];W^{s,r}(\Omega))$. See Adams [1] for more details.

4.1.2. Properties of finite element space

In our analysis, we shall use two kinds of polynomial finite element spaces associated with each element $T \in \mathcal{T}_h$. one is a scalar polynomial space $P_k(T)$, in which the degree of polynomial is no more than k on T^0 and ∂T , and the other is the vector value polynomial space $[P_{k-1}(T)]^2$ which is used to define the discrete weak gradient ∇_w in (10). For convenience, we denote $[P_{k-1}(T)]^d$ by $G_{k-1}(T)$, which is called a local discrete gradient space.

In addition, we define the local L^2 -projection of the vector value function $\mathbf{w}(x)$ in this paper by $\mathcal{Q}_h \mathbf{w}(x)$. It is defined in each element $T \in \mathcal{T}_h$ as the unique vector value function in $G_{k-1}(T)$ such that

$$\int_T \mathcal{Q}_h \mathbf{w}(x) \cdot q(x) dx = \int_T \mathbf{w}(x) \cdot q(x) dx, \quad \forall q(x) \in G_{k-1}(T). \quad (19)$$

The following three lemmas are listed without any proof. Their proofs can be found in [14].

Lemma 4.1. *Let Q_h be the L^2 projection operator. Then, on each element $T \in \mathcal{T}_h$, we have the following relation*

$$\nabla_w(Q_h \phi) = \mathcal{Q}_h(\nabla \phi), \quad \forall \phi \in H^1(\Omega). \quad (20)$$

Lemma 4.2. *Let T be an element with $e \in \partial T$ is a portion of its boundary. For any function $\phi \in H^1(T)$, the following trace inequality is valid for general meshes (see [14] for details):*

$$\|\phi\|_e^2 \leq C(h_T^{-1} \|\phi\|_T^2 + h_T \|\nabla \phi\|_T^2). \quad (21)$$

Lemma 4.3. *Let \mathcal{T}_h be a finite element partition of domain Ω satisfying corresponding shape regularity assumptions as specified in [15]. Then, for any $\phi \in H^{k+1}(\Omega)$, we have*

$$\sum_{T \in \mathcal{T}_h} \|\phi - Q_0 \phi\|_T^2 + \sum_{T \in \mathcal{T}_h} h_T^2 \|\nabla(\phi - Q_0 \phi)\|_T^2 \leq Ch^{2(k+1)} \|\phi\|_{k+1}^2. \quad (22)$$

$$\sum_{T \in \mathcal{T}_h} \|a(\nabla \phi - \mathcal{Q}_h(\nabla \phi))\|_T^2 \leq Ch^{2k} \|\phi\|_{k+1}^2. \quad (23)$$

Lemma 4.4. *Assume that \mathcal{T}_h is shape regular. We have the following relation*

$$\left| \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle Q_0 w - Q_b w, v_0 - v_b \rangle_{\partial T} \right| \leq Ch^k \|w\|_{k+1} \|v\|_{w,1}, \quad (24)$$

and

$$\left| \sum_{T \in \mathcal{T}_h} \langle a(\nabla w - \mathcal{Q}_h \nabla w) \cdot \mathbf{n}, v_0 - v_b \rangle_{\partial T} \right| \leq Ch^k \|w\|_{k+1} \|v\|_{w,1} \quad (25)$$

for $\forall w \in H^{k+1}(\Omega)$ and $v = \{v_0, v_b\} \in V_h^0$.

4.2. Error estimate for semi-discrete WG scheme

In this section, we analyze semi-discrete WG scheme (11) first.

Theorem 4.1. *Let $u(x, t)$ and $u_h(x, t)$ be the solutions to the problem (1) and the semi-discrete WG scheme (11), respectively. Assume that the exact solution has a regularity such that $u, u_t \in H^{k+1}(\Omega)$. Then, there exists a constant C such that*

$$\|u - u_h\|_{w,1}^2 \leq C[\|u^0 - u_h^0\|_{w,1}^2 + h^{2k} \int_0^T (h^2 \|u_t\|_{k+1}^2 + \|u\|_{k+1}^2) dt]. \quad (26)$$

Proof Let

$$\rho = u - Q_h u, e = Q_h u - u_h. \quad (27)$$

where Q_h is the local L^2 -projection operator and $e = \{e_0, e_b\} = \{Q_0 u - u_0, Q_b u - u_b\}$. Then we have

$$u - u_h = \rho + e. \quad (28)$$

To estimate ρ , we apply Lemma 4.3 and 4.4. We start by estimating e . Since u and u_h satisfy (4) and (11) respectively, we have

$$(u_t - u_{h,t}, v) + a(u, v) - a_w(u_h, v) = 0, \quad \forall v \in V_h^0.$$

Further,

$$(u_t - Q_h u_t + Q_h u_t - u_{h,t}, v) + a(u, v) - a_w(u_h, v) = 0, \quad \forall v \in V_h^0,$$

i.e.,

$$(e_t, v) + a(u, v) - a_w(u_h, v) = -(\rho_t, v), \quad \forall v \in V_h^0. \quad (29)$$

In the following, we analyze the term $a(u, v) - a_w(u_h, v)$. Recalling the definitions of $a(u, v)$ and $a_w(u_h, v)$ and noting that $\sum_{T \in \mathcal{T}_h} \langle a \nabla u \cdot \mathbf{n}, v_b \rangle_{\partial T} = 0$,

we derive

$$\begin{aligned}
& a(u, v) - a_w(u_h, v) \\
&= \sum_{T \in \mathcal{T}_h} [(a \nabla u, \nabla v)_T + (cu, v)_T] - \sum_{T \in \mathcal{T}_h} \langle a \nabla u \cdot \mathbf{n}, v \rangle_{\partial T} \\
&\quad - \sum_{T \in \mathcal{T}_h} [(a \nabla_w u_h, \nabla_w v)_T + (cu_h, v)_T] - \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle u_0 - u_b, v_0 - v_b \rangle_{\partial T} . \\
&= \sum_{T \in \mathcal{T}_h} [(a \nabla u, \nabla v)_T + (cu, v)_T] - \sum_{T \in \mathcal{T}_h} \langle a \nabla u \cdot \mathbf{n}, v_0 - v_b \rangle_{\partial T} \\
&\quad - \sum_{T \in \mathcal{T}_h} [(a \nabla_w u_h, \nabla_w v)_T + (cu_h, v)_T] - \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle u_0 - u_b, v_0 - v_b \rangle_{\partial T}
\end{aligned}$$

Further, we have

$$\begin{aligned}
& a(u, v) - a_w(u_h, v) \\
&= \sum_{T \in \mathcal{T}_h} [(a \nabla u, \nabla v)_T - (a \nabla_w u_h, \nabla_w v)_T] \\
&\quad + \sum_{T \in \mathcal{T}_h} [(cu, v)_T - (cu_h, v)_T] - \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle u_0 - u_b, v_0 - v_b \rangle_{\partial T} \\
&\quad - \sum_{T \in \mathcal{T}_h} \langle a \nabla u \cdot \mathbf{n}, v_0 - v_b \rangle_{\partial T} \\
&= \sum_{T \in \mathcal{T}_h} [(a \nabla u, \nabla v)_T - (a \nabla_w Q_h u, \nabla_w v)_T \\
&\quad \quad \quad + (a \nabla_w Q_h u, \nabla_w v)_T - (a \nabla_w u_h, \nabla_w v)_T] \\
&\quad + \sum_{T \in \mathcal{T}_h} [(cu, v)_T - (cu_h, v)_T] - \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle u_0 - u_b, v_0 - v_b \rangle_{\partial T} \\
&\quad - \sum_{T \in \mathcal{T}_h} \langle a \nabla u \cdot \mathbf{n}, v_0 - v_b \rangle_{\partial T} .
\end{aligned} \tag{30}$$

From the definitions of the weak discrete gradient ∇_w and the projection \mathcal{Q}_h , as well as the expressions in (10) and (20), we get

$$\begin{aligned}
& (a \nabla_w Q_h u, \nabla_w v)_T = (a \mathcal{Q}_h(\nabla u), \nabla_w v)_T = (\nabla_w v, a \mathcal{Q}_h(\nabla u))_T \\
&= -(v_0, \nabla \cdot (a \mathcal{Q}_h(\nabla u)))_T + \langle v_b, a \mathcal{Q}_h(\nabla u) \cdot \mathbf{n} \rangle_{\partial T} \\
&= (\nabla v_0, a \mathcal{Q}_h(\nabla u))_T - \langle v_0 - v_b, a \mathcal{Q}_h(\nabla u) \cdot \mathbf{n} \rangle_{\partial T} \\
&= (a \nabla u, \nabla v_0)_T - \langle v_0 - v_b, a \mathcal{Q}_h(\nabla u) \cdot \mathbf{n} \rangle_{\partial T} .
\end{aligned} \tag{31}$$

Substituting (31) into (30) arrives at

$$\begin{aligned}
a(u, v) - a_w(u_h, v) &= \sum_{T \in \mathcal{T}_h} [(a \nabla_w e, \nabla_w v)_T + (c\rho, v)_T + (ce, v)_T] \\
&\quad - \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle u_0 - u_b, v_0 - v_b \rangle_{\partial T} \\
&\quad + \sum_{T \in \mathcal{T}_h} \langle a(\mathcal{Q}_h(\nabla u) - \nabla u) \cdot \mathbf{n}, v_0 - v_b \rangle_{\partial T} .
\end{aligned} \tag{32}$$

Combining (29) with (32) gives

$$\begin{aligned} (e_t, v) + \sum_{T \in \mathcal{T}_h} [(a \nabla_w e, \nabla_w v)_T + (c\rho, v)_T + (ce, v)_T] \\ - \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle u_0 - u_b, v_0 - v_b \rangle_{\partial T} \\ + \sum_{T \in \mathcal{T}_h} \langle a(\mathcal{Q}_h(\nabla u) - \nabla u) \cdot \mathbf{n}, v_0 - v_b \rangle_{\partial T} = -(\rho_t, v). \end{aligned} \quad (33)$$

Adding the term $\sum_{T \in \mathcal{T}_h} h_T^{-1} \langle Q_0 u - Q_b u, v_0 - v_b \rangle_{\partial T}$ to both sides of (33), we have

$$\begin{aligned} (e_t, v) + a_w(e, v) \\ = -(\rho_t, v) - \sum_{T \in \mathcal{T}_h} (c\rho, v)_T + \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle Q_0 u - Q_b u, v_0 - v_b \rangle_{\partial T} \\ + \sum_{T \in \mathcal{T}_h} \langle a(\nabla u - \mathcal{Q}_h(\nabla u)) \cdot \mathbf{n}, v_0 - v_b \rangle_{\partial T}. \end{aligned} \quad (34)$$

Choosing the test function $v = e_t$ in (34), we have

$$\begin{aligned} \|e_t\|_0^2 + \frac{1}{2} \frac{d}{dt} a_w(e, e) \\ = -(\rho_t, e_t) - \sum_{T \in \mathcal{T}_h} (c\rho, e_t)_T + \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle Q_0 u - Q_b u, e_{0,t} - e_{b,t} \rangle_{\partial T} \\ + \sum_{T \in \mathcal{T}_h} \langle a(\nabla u - \mathcal{Q}_h(\nabla u)) \cdot \mathbf{n}, e_{0,t} - e_{b,t} \rangle_{\partial T} \\ = -(\rho_t, e_t) - (c\rho, e_t) + \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle Q_0 u - Q_b u, e_{0,t} - e_{b,t} \rangle_{\partial T} \\ + \sum_{T \in \mathcal{T}_h} \langle a(\nabla u - \mathcal{Q}_h(\nabla u)) \cdot \mathbf{n}, e_{0,t} - e_{b,t} \rangle_{\partial T} \\ \equiv R_1 + R_2 + R_3 + R_4. \end{aligned} \quad (35)$$

We estimate each term of R_1, R_2, R_3 and R_4 , separately.

For R_3 and R_4 , we use Lemma 4.4, yielding:

$$|R_3| \leq Ch^k \|u\|_{k+1} \|e_t\|_{w,1}, \quad |R_4| \leq Ch^k \|u\|_{k+1} \|e_t\|_{w,1}. \quad (36)$$

The other two terms R_1 and R_2 can be bound by applying the Hölder inequality and Lemma 3.1, i.e.,

$$|R_1| = |-(\rho_t, e_t)| \leq C \|\rho_t\| \|e_t\| \leq C \|\rho_t\|_0^2 + \frac{1}{2} \|e_t\|_0^2. \quad (37)$$

$$|R_2| = |-(c\rho, e_t)| \leq C \|\rho\| \|e_t\| \leq C \|\rho\|_0^2 + \frac{1}{2} \|e_t\|_0^2. \quad (38)$$

Substituting (37), (38) and (36) into (35) leads to:

$$\frac{1}{2} \frac{d}{dt} a_w(e, e) \leq C(\|\rho\|^2 + \|\rho_t\|^2 + h^k \|u\|_{k+1} \|e_t\|_{w,1}) \quad (39)$$

Integrating (39) with respect to t from 0 to T , we have

$$\begin{aligned} & a_w(e(T), e(T)) - a_w(e(0), e(0)) \\ & \leq C[\int_0^T \|\rho\|^2 dt + \int_0^T \|\rho_t\|^2 dt + h^k \int_0^T \|u\|_{k+1} \|e_t\|_{w,1} dt] \\ & \leq C[\int_0^T \|\rho\|^2 dt + \int_0^T \|\rho_t\|^2 dt + h^{2k} \int_0^T \|u\|_{k+1}^2 dt + \int_0^T \|e_t\|_{w,1}^2 dt]. \end{aligned} \quad (40)$$

By virtue of Lemma 4.3,

$$\|\rho_t\|_0 = \|u_t - Q_h u_t\|_0 \leq Ch^{k+1} \|u_t\|_{k+1}. \quad (41)$$

A combination of (22) and (40)-(41) with Gronwall lemma leads to (26). \square

4.3. Error estimate for fully discrete WG scheme

Theorem 4.2. *Let u and $\{u_h^n\}$ be the solutions to the parabolic equation (1) and the fully discrete WG scheme (18), respectively. Then*

$$\begin{aligned} & \|u(t_n) - u_h^n\|_{w,1}^2 \\ & \leq C\{\|u^0 - u_h^0\|_{w,1}^2 + h^{2k}[(\|u^0\|_{k+1}^2 + \int_0^{t_n} \|u_t\|_{k+1}^2 dt) + \tau \sum_{i=1}^n \|u^i\|_{k+1}^2] \\ & \quad + \tau^2 \int_0^{t_n} \|u_{tt}\|_0^2 dt\}. \end{aligned} \quad (42)$$

Proof Set

$$\rho^n = u(t_n) - Q_h u(t_n), \quad e^n = Q_h u(t_n) - u_h^n,$$

then

$$u(t_n) - u_h^n = \rho^n + e^n \quad (43)$$

It follows from Lemma 4.3 that

$$\|\rho^n\|_{w,1} \leq C\|\rho^n\|_1 \leq Ch^k \|u(t_n)\|_{k+1} \leq Ch^k [\|u^0\|_{k+1} + \int_0^{t_n} \|u_\tau\|_{k+1} d\tau] \quad (44)$$

In (4a), we set $t = t_n$ we have

$$(u_t^n, v) + a(u^n, v) = (f^n, v), \quad (45)$$

where u_t^n denotes the value of derivative $\frac{\partial u(x,t)}{\partial t}$ at $t = t_n$, and similar definitions to u^n and f^n . Subtracting (18a) from (45), then we have

$$(u_t^n - \bar{\partial}_t u_h^n, v) + a(u^n, v) - a_w(u_h^n, v) = 0, \quad (46)$$

further

$$(\bar{\partial}_t e^n, v) + a(u^n, v) - a_w(u_h^n, v) = (\bar{\partial}_t Q_h u(t^n) - u_t^n, v). \quad (47)$$

For the term $a(u^n, v) - a_w(u_h^n, v)$, taking the same measures used in the analysis course of semi-discrete case, we have

$$\begin{aligned} & (\bar{\partial}_t e^n, v) + a_w(e^n, v) \\ &= (\bar{\partial}_t Q_h u(t^n) - u_t^n, v) - \sum_{T \in \mathcal{T}_h} (c\rho^n, v)_T \\ &+ \sum_{T \in \mathcal{T}_h} h_T^{-1} \langle Q_0 u^n - Q_b u^n, v_0 - v_b \rangle_{\partial T} \\ &+ \sum_{T \in \mathcal{T}_h} \langle a(\nabla u^n - \mathcal{Q}_h(\nabla u^n)) \cdot \mathbf{n}, v_0 - v_b \rangle_{\partial T}. \end{aligned} \quad (48)$$

Let LL_1, LL_2 be the two terms of the left hand side (LHS) of the equation (48) and RR_1, RR_2, RR_3, RR_4 be the four terms of the right hand side (RHS) of (48), respectively. Next, we choose the test function $v = \bar{\partial}_t e^n$ in (48), and estimate these six terms consecutively.

For the two terms LL_1, LL_2 of the LHS of the error equation, we have

$$|LL_1| = |(\bar{\partial}_t e^n, \bar{\partial}_t e^n)| = \|\bar{\partial}_t e^n\|_0^2. \quad (49)$$

Note that

$$a_w(e^n, e^n) \geq \alpha_0 \|e^n\|_{w,1}^2,$$

and

$$a_w(e^n, e^{n-1}) \leq C^* \|e^n\|_{w,1} \|e^{n-1}\|_{w,1}.$$

using the weighted *Hölder* inequality and choosing a suitable weight ϵ , such that $\epsilon < \alpha_0$ and

$$\begin{aligned} a_w(e^n, e^{n-1}) &\leq \epsilon \|e^n\|_{w,1}^2 + C \|e^{n-1}\|_{w,1}^2. \\ |LL_2| &\geq \frac{1}{\tau} [(\alpha_0 - \epsilon) \|e^n\|_{w,1}^2 - C \|e^{n-1}\|_{w,1}^2]. \end{aligned} \quad (50)$$

The terms RR_1 through RR_4 in the RHS of (48) are estimated as follows:

$$|RR_1| \leq C \|\bar{\partial}_t Q_h u(t^n) - u_t^n\|^2 + \frac{1}{2} \|\bar{\partial}_t e^n\|_0^2, |RR_2| \leq C \|\rho^n\|^2 + \frac{1}{2} \|\bar{\partial}_t e^n\|_0^2. \quad (51)$$

$$\begin{aligned} |RR_3| &\leq Ch^k \|u^n\|_{k+1} \|\bar{\partial}_t e^n\|_{w,1} \\ &\leq Ch^{2k} \|u^n\|_{k+1}^2 + \frac{C_1}{\tau} (\|e^n\|_{w,1}^2 + \|e^{n-1}\|_{w,1}^2), \\ |RR_4| &\leq Ch^k \|u^n\|_{k+1} \|\bar{\partial}_t e^n\|_{w,1} \\ &\leq Ch^{2k} \|u^n\|_{k+1}^2 + \frac{C_1}{\tau} (\|e^n\|_{w,1}^2 + \|e^{n-1}\|_{w,1}^2), \end{aligned} \quad (52)$$

where C_1 has to be less than $\frac{1}{2}(\alpha_0 - \epsilon)$. A combination of (48)-(52) leads to

$$\begin{aligned} &\|e^n\|_{w,1}^2 \\ &\leq \beta \|e^{n-1}\|_{w,1}^2 + C\tau (\|\bar{\partial}_t Q_h u(t^n) - u_t^n\|_0^2 + \|\rho^n\|_0^2 + h^{2k} \|u^n\|_{k+1}^2) \\ &\leq \beta \|e^0\|_{w,1}^2 + C\tau \sum_{i=1}^n (\|\bar{\partial}_t Q_h u(t^i) - u_t^i\|_0^2 + \|\rho^i\|_0^2 + h^{2k} \|u^i\|_{k+1}^2), \end{aligned} \quad (53)$$

where $\beta = \frac{2C_1 + C}{\alpha_0 - \epsilon - 2C_1}$.

Introducing $z^i = \bar{\partial}_t Q_h u(t^i) - u_t^i$, and writing $z^i = z_1^i + z_2^i$, where

$$z_1^i = \bar{\partial}_t Q_h u(t^i) - \bar{\partial}_t u(t^i) = \frac{1}{\tau} \int_{t_{i-1}}^{t_i} (Q_h - I) u_t dt,$$

and

$$z_2^i = \bar{\partial}_t u(t^i) - u_t(t^i) = -\frac{1}{\tau} \int_{t_{i-1}}^{t_i} (t - t_{i-1}) u_{tt} dt.$$

From Lemma 4.3,

$$\begin{aligned} \sum_{i=1}^n \|z_1^i\|_0^2 &\leq C\tau^{-2} \sum_{i=1}^n (\int_{t_{i-1}}^{t_i} Ch^{k+1} \|u_t\|_{k+1} dt)^2 \\ &\leq C\tau^{-1} h^{k+1} \int_0^{t_n} \|u_t\|_{k+1}^2 dt. \end{aligned} \quad (54)$$

Similarly

$$\sum_{i=1}^n \|z_2^i\|_0^2 \leq \sum_{i=1}^n (\int_{t_{i-1}}^{t_i} \|u_{tt}\|_0 dt)^2 = \tau \int_0^{t_n} \|u_{tt}\|_0^2 dt \quad (55)$$

Again by Lemma 4.3,

$$\begin{aligned} \|e^0\|_{w,1}^2 &= \|Q_h u^0 - u_h^0\|_{w,1}^2 = \|Q_h u^0 - u^0 + u^0 - u_h^0\|_{w,1}^2 \\ &\leq Ch^{2k} \|u^0\|_{k+1}^2 + \|u^0 - u_h^0\|_{w,1}^2. \end{aligned} \quad (56)$$

A combination of (53), (44) and (54)-(56) leads to (42). \square

5. Numerical Experiment

In this section, we give three numerical examples using scheme (18) and consider the following parabolic problem [11]

$$u_t - \operatorname{div}(\mathbf{D}\nabla u) = f, \quad \text{in } \Omega \times J, \quad (57)$$

with proper Dirichlet boundary and initial conditions. For simplicity, we let $D = 1, 10$; $\Omega = (0, 1) \times (0, 1)$ be unit square; and the time interval $J = (0, T)$ be $(0, 1)$, in all three numerical examples. One can determine the initial and boundary conditions and source term $f(x, t)$ according to the corresponding analytical solution of each example.

We construct triangular mesh as follows. Firstly, we partition the square domain $\Omega = (0, 1) \times (0, 1)$ into $N \times N$ sub-squares uniformly to obtain the square mesh. Secondly, we divide each square element into two triangles by the diagonal line with a negative slope so that we complete the constructing of triangular mesh.

In the first example, the analytical solution is

$$u = \sin(\pi x) \sin(\pi y) \exp(-t). \quad (58)$$

For a set of simulations, different mesh sizes $h = 1/N$ ($N = 4, 8, 16, 32, 64$) and different diffusion coefficients $D = 1$ and $D = 10$ are taken, and their corresponding discrete norms errors and convergence rates (CR) are listed in Table 1 for $D = 1$ and $D = 10$. Here $|||\cdot|||_{w,1}$ is defined as discrete version of the definition of (15) without the term $\|v\|_{0,T}^2$.

Table 1: Numerical results of the first example for $D = 1$ and $D = 10$.

	$D = 1$		$D = 10$	
h	$ u - u_h _{w,1}$	CR	$ u - u_h _{w,1}$	CR
2.5000e-01	1.6044e-01		1.2252e+00	
1.2500e-01	8.0594e-02	0.99	6.0921e-01	1.01
6.2500e-02	4.0329e-02	1.00	3.0412e-01	1.00
3.1250e-02	2.0165e-02	1.00	1.5200e-01	1.00
1.5625e-02	1.0082e-02	1.00	7.5990e-02	1.00

In the second example, the analytical solution is

$$u = x(1-x)y(1-y) \exp(x-y-t). \quad (59)$$

Numerical error results and CRs are listed in Table 2 for $D = 1$ and $D = 10$ based on the same triangular mesh as those of the first example.

Table 2: Numerical results of the second example for $D = 1$ and $D = 10$.

	$D = 1$		$D = 10$	
h	$ u - u_h _{w,1}$	CR	$ u - u_h _{w,1}$	CR
2.5000e-01	1.5858e-02		1.2052e-01	
1.2500e-01	7.9786e-03	0.99	5.9967e-02	1.01
6.2500e-02	3.9940e-03	1.00	2.9945e-02	1.00
3.1250e-02	1.9973e-03	1.00	1.4967e-02	1.00
1.5625e-02	9.9864e-04	1.00	7.4829e-03	1.00

In the third example, the analytical solution is

$$u = x(1-x)y(1-y)\exp(x+y+t). \quad (60)$$

Numerical error results and CRs are listed in Table 3 for $D = 1$ and $D = 10$ based on the same triangular mesh as those of the first example.

Table 3: Numerical results of the third example for $D = 1$ and $D = 10$.

	$D = 1$		$D = 10$	
h	$ u - u_h _{w,1}$	CR	$ u - u_h _{w,1}$	CR
2.5000e-01	3.1035e-01		2.3672e+00	
1.2500e-01	1.5916e-01	0.96	1.1977e+00	0.98
6.2500e-02	8.0078e-02	0.99	6.0062e-01	1.00
3.1250e-02	4.0099e-02	1.00	3.0052e-01	1.00
1.5625e-02	2.0056e-02	1.00	1.5029e-01	1.00

All three numerical examples show good agreement with the theoretical results in Section 4, which show that the WG-FEM (18) is stable and first order convergent in H^1 norm.

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Some fixed point theorems for generalized expansive mappings in cone metric spaces over Banach algebras

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Abstract: In this paper, we prove some fixed point theorems for expansive mappings in cone metric spaces over Banach algebras without the assumption of normality of cones. Moreover, we give some examples to support our results. Our results improve and generalize the recent results of Aage and Salunke(2011).

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Keywords: Generalized expansive mapping, Cone metric space over Banach algebra, Spectral radius

1 Introduction and Preliminaries

In 2007 Huang and Zhang[1] introduced cone metric space and proved some fixed point theorems of contractive mappings in such spaces. Since then, some authors proved lots of fixed point theorems for contractive or expansive mappings in cone metric spaces that expanded certain fixed point results in metric spaces (see [2-14]). However, recently, it is not an attractive topic since some authors have appealed to the equivalence of some metric and cone metric fixed point results (see [21-24]). Recently [13] introduced the concept of cone metric space with Banach algebra and obtained some fixed point theorems in such spaces. Moreover, the authors of [13] gave an example to illustrate that the non-equivalence of fixed point theorems between cone metric spaces over Banach algebras and metric spaces

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(in usual sense). As a result, it is necessary to further investigate fixed point theorems in cone metric spaces over Banach algebras. In this paper, we generalize the famous Banach expansive mapping theorems as follows:

Let (X, d) be a complete cone metric space over Banach algebra \mathcal{A} and K be a cone in \mathcal{A} . Suppose the mapping $T : X \rightarrow X$ is onto and satisfies the generalized expansive condition:

$$d(Tx, Ty) \succeq kd(x, y),$$

for all $x, y \in X$, where $k, k^{-1} \in K$ are generalized constants with $\rho(k^{-1}) < 1$. Then T has an unique fixed point in X .

Further, we give some other fixed point theorems for expansive mappings with generalized constants in cone metric spaces over Banach algebras. In addition, all cones are not necessarily normal ones. In these cases, our main results are not equivalent to those in metric spaces (see [7]).

For the sake of completeness, we introduce some basic concepts as follows:

Let \mathcal{A} be a Banach algebras with a unit e , and θ the zero element of \mathcal{A} . A nonempty closed convex subset K of \mathcal{A} is called a cone if and only if

- (i) $\{\theta, e\} \subset K$;
- (ii) $K^2 = KK \subset K, K \cap (-K) = \{\theta\}$;
- (iii) $\lambda K + \mu K \subset K$ for all $\lambda, \mu \geq 0$.

On this basis, we define a partial ordering \preceq with respect to K by $x \preceq y$ if and only if $y - x \in K$, we shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will indicate that $y - x \in \text{int}K$, where $\text{int}K$ stands for the interior of K . If $\text{int}K \neq \emptyset$, then K is called a solid cone. Write $\|\cdot\|$ as the norm on \mathcal{A} . A cone K is called normal if there is a number $M > 0$ such that for all $x, y \in \mathcal{A}$,

$$\theta \preceq x \preceq y \Rightarrow \|x\| \leq M\|y\|.$$

The least positive number satisfying above is called the normal constant of K . An element $x \in \mathcal{A}$ is said to be invertible if there is an element $y \in \mathcal{A}$ such that $yx = xy = e$. The inverse of x is denoted by x^{-1} . For more details, we refer to [10, 13].

In the following we always suppose that \mathcal{A} is a real Banach algebra with a unit e , K is a solid cone in \mathcal{A} and \preceq is a partial ordering with respect to K .

Definition 1.1([13]) Let X be a nonempty set and \mathcal{A} a Banach algebra. Suppose that the mapping $d : X \times X \rightarrow \mathcal{A}$ satisfies:

- (i) $\theta \prec d(x, y)$ for all $x, y \in X$ with $x \neq y$ and $d(x, y) = \theta$ if and only if $x = y$;
- (ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X , and (X, d) is called a cone metric space over Banach algebra \mathcal{A} .

Definition 1.2([2]) Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$ a sequence in X . Then

(i) $\{x_n\}$ converges to x whenever for every $c \in \mathcal{A}$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$, we denote this by $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$ (as $n \rightarrow \infty$).

(ii) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in \mathcal{A}$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

(iii) (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

Lemma 1.3 ([7]) Let $u, v, w \in \mathcal{A}$. If $u \preceq v$ and $v \ll w$, then $u \ll w$.

Lemma 1.4 ([7]) Let \mathcal{A} be a Banach algebra and $\{a_n\}$ a sequence in \mathcal{A} . If $a_n \rightarrow \theta$ ($n \rightarrow \infty$), then for any $c \gg \theta$, there exists N such that for all $n > N$, one has $a_n \ll c$.

Lemma 1.5 ([10]) Let \mathcal{A} be a Banach algebra with a unit e , $x \in \mathcal{A}$, then $\lim_{n \rightarrow \infty} \|x^n\|^{\frac{1}{n}}$ exists and the spectral radius $\rho(x)$ satisfies

$$\rho(x) = \lim_{n \rightarrow \infty} \|x^n\|^{\frac{1}{n}} = \inf \|x^n\|^{\frac{1}{n}}.$$

If $\rho(x) < |\lambda|$, then $\lambda e - x$ invertible in \mathcal{A} , moreover,

$$(\lambda e - x)^{-1} = \sum_{i=0}^{\infty} \frac{x^i}{\lambda^{i+1}},$$

where λ is a complex constant.

Lemma 1.6([10]) Let \mathcal{A} be a Banach algebra with a unit e , $a, b \in \mathcal{A}$. If a commutes with b , then

$$\rho(a + b) \leq \rho(a) + \rho(b), \quad \rho(ab) \leq \rho(a)\rho(b).$$

Lemma 1.7([20]) Let K be a cone in a Banach algebra \mathcal{A} and $k \in K$ be a given vector. Let $\{u_n\}$ be a sequence in K . If for each $c_1 \gg \theta$, there exists N_1 such that $u_n \ll c_1$ for all $n > N_1$, then for each $c_2 \gg \theta$, there exists N_2 such that $ku_n \ll c_2$ for all $n > N_2$.

Lemma 1.8([20]) If \mathcal{A} is a Banach algebra with a solid cone K and $\|x_n\| \rightarrow 0$ ($n \rightarrow \infty$), then for any $\theta \ll c$, there exists N such that for all $n > N$, we have $x_n \ll c$.

Remark 1.9 Let \mathcal{A} be a Banach algebra and $k \in \mathcal{A}$. If $\rho(k) < 1$, then $\lim_{n \rightarrow \infty} \|k^n\| = 0$.

2 Main results

In this section, we shall prove some fixed point theorems for expansive mappings in the setting of non-normal cone metric spaces over Banach algebras. Furthermore, we display two examples to support our main conclusions.

Theorem 2.1 Let (X, d) be a complete cone metric space over Banach algebra \mathcal{A} and K be a solid cone in \mathcal{A} . Suppose that the mapping $T : X \rightarrow X$ is onto and satisfies the expansive expansive condition:

$$d(Tx, Ty) \succeq kd(x, y) + ld(Tx, y), \quad (2.1)$$

for all $x, y \in X$, where $k, l, k^{-1} \in K$ are two generalized constants. If $e - l \in K$ and $\rho(k^{-1}) < 1$, then T has a fixed point in X .

Proof Since T is an onto mapping, for each $x_0 \in X$, there exists $x_1 \in X$ such that $Tx_1 = x_0$. Continuing this process, we can define $\{x_n\}$ by $x_n = Tx_{n+1}$ ($n = 0, 1, 2, \dots$). Without loss of generality, we assume $x_{n-1} \neq x_n$ for all $n \geq 1$. According to (2.1), we have

$$\begin{aligned} d(x_n, x_{n-1}) &= d(Tx_{n+1}, Tx_n) \\ &\succeq kd(x_{n+1}, x_n) + ld(Tx_{n+1}, x_n) \\ &= kd(x_{n+1}, x_n) + ld(x_n, x_n) \\ &= kd(x_{n+1}, x_n), \end{aligned}$$

then

$$d(x_{n+1}, x_n) \preceq k^{-1}d(x_n, x_{n-1}).$$

Letting $k^{-1} = h$ we get

$$d(x_{n+1}, x_n) \preceq hd(x_n, x_{n-1}) \preceq \dots \preceq h^n d(x_1, x_0).$$

So by the triangle inequality and $\rho(h) < 1$, for all $m > n$, we see

$$\begin{aligned} d(x_m, x_n) &\preceq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\preceq (h^{m-1} + h^{m-2} + \cdots + h^n) d(x_1, x_0) \\ &= (e + h + \cdots + h^{m-n-1}) h^n d(x_1, x_0) \\ &\preceq \left(\sum_{i=0}^{\infty} h^i \right) h^n d(x_1, x_0) \\ &= (e - h)^{-1} h^n d(x_1, x_0). \end{aligned}$$

By Lemma 1.8 and the fact that $\|(e - h)^{-1} h^n d(x_1, x_0)\| \rightarrow 0 (n \rightarrow \infty)$ (Because of Remark 1.9, $\|h^n\| \rightarrow 0 (n \rightarrow \infty)$), it follows that for any $c \in \mathcal{A}$ with $\theta \ll c$, there exists N such that for all $m > n > N$, we have

$$d(x_m, x_n) \preceq (e - h)^{-1} h^n d(x_1, x_0) \ll c,$$

which implies that $\{x_n\}$ is a Cauchy sequence.

By the completeness of X , there exists $x^* \in X$ such that $x_n \rightarrow x^* (n \rightarrow \infty)$. Consequently, we can find an $x^{**} \in X$ such that $Tx^{**} = x^*$. Now we show that $x^{**} = x^*$. In fact,

$$\begin{aligned} d(x^*, x_n) &= d(Tx^{**}, Tx_{n+1}) \\ &\succeq kd(x^{**}, x_{n+1}) + ld(Tx^{**}, x_{n+1}) \\ &= kd(x^{**}, x_{n+1}) + ld(x^*, x_{n+1}). \end{aligned}$$

Since

$$d(x^*, x_n) \preceq d(x^*, x_{n+1}) + d(x_{n+1}, x_n),$$

it follows that

$$kd(x^{**}, x_{n+1}) \preceq (e - l)d(x^*, x_{n+1}) + d(x_{n+1}, x_n).$$

Now, we have

$$d(x^{**}, x_{n+1}) \preceq k^{-1}((e - l)d(x^*, x_{n+1}) + d(x_{n+1}, x_n)).$$

Note that $x_n \rightarrow x^* (n \rightarrow \infty)$, by Lemma 1.7, it follows that for any $c \in \mathcal{A}$ with $\theta \ll c$, there exists N such that for any $n > N$, we have

$$k^{-1}((e - l)d(x^*, x_{n+1}) + d(x_{n+1}, x_n)) \ll c.$$

Thus

$$d(x^{**}, x_{n+1}) \ll c.$$

Since the limit of a convergent sequence in cone metric space over Banach algebra is unique, we get $x^{**} = x^*$, i.e., x^* is a fixed point of T .

Theorem 2.2 Let (X, d) be a complete cone metric space over Banach algebra \mathcal{A} and K be a solid cone in \mathcal{A} . Suppose that the mapping $T : X \rightarrow X$ is onto and satisfies the generalized expansive condition:

$$d(Tx, Ty) \succeq kd(x, y) + ld(x, Tx) + pd(y, Ty), \quad (2.2)$$

for all $x, y \in X$, where $k, l, p, e - p \in K$ are generalized constants with $(k + l)^{-1} \in K$ and $\rho[(k + l)^{-1}(e - p)] < 1$. Then T has a fixed point in X .

Proof Since T is an onto mapping, for each $x_0 \in X$, there exists $x_1 \in X$ such that $Tx_1 = x_0$. Continuing this process, we can define $\{x_n\}$ by $x_n = Tx_{n+1}$ ($n = 0, 1, 2, \dots$). Without loss of generality, we suppose $x_{n-1} \neq x_n$ for all $n \geq 1$. According to (2.2), we have

$$\begin{aligned} d(x_n, x_{n-1}) &= d(Tx_{n+1}, Tx_n) \\ &\succeq kd(x_{n+1}, x_n) + ld(x_{n+1}, Tx_{n+1}) + pd(x_n, Tx_n) \\ &= kd(x_{n+1}, x_n) + ld(x_{n+1}, x_n) + pd(x_n, x_{n-1}), \end{aligned}$$

which implies that

$$(k + l)d(x_n, x_{n+1}) \preceq (e - p)d(x_n, x_{n-1}).$$

Put $k + l = r$, then

$$rd(x_n, x_{n+1}) \preceq (e - p)d(x_n, x_{n-1}). \quad (2.3)$$

Since r is invertible, to multiply r^{-1} in both sides of (2.3), we have

$$d(x_n, x_{n+1}) \preceq hd(x_n, x_{n-1}),$$

where $h = (k + l)^{-1}(e - p)$. Note that $\rho(h) < 1$ and for all $m > n$,

$$\begin{aligned} d(x_m, x_n) &\preceq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\preceq (h^{m-1} + h^{m-2} + \cdots + h^n)d(x_1, x_0) \\ &= (e + h + \cdots + h^{m-n-1})h^n d(x_1, x_0) \\ &\preceq \left(\sum_{i=0}^{\infty} h^i \right) h^n d(x_1, x_0) \\ &= (e - h)^{-1} h^n d(x_1, x_0). \end{aligned}$$

As is shown in the proof of Theorem 2.1, it follows that $\{x_n\}$ is a Cauchy sequence. Then by the completeness of X , there exists $x^* \in X$ such that $x_n \rightarrow x^* (n \rightarrow \infty)$. Consequently, we can find a $x^{**} \in X$ such that $Tx^{**} = x^*$. Now we show that $x^{**} = x^*$. Indeed, Since

$$\begin{aligned} d(x^*, x_n) &= d(Tx^{**}, Tx_{n+1}) \\ &\succeq kd(x^{**}, x_{n+1}) + ld(x^{**}, Tx^{**}) + pd(x_{n+1}, Tx_{n+1}) \\ &= kd(x^{**}, x_{n+1}) + ld(x^{**}, x^*) + pd(x_{n+1}, x_n). \end{aligned}$$

Then

$$d(x^*, x_n) \succeq kd(x^{**}, x_{n+1}) + ld(x^{**}, x_{n+1}) - ld(x^*, x_{n+1}) + pd(x_{n+1}, x_n).$$

Note that

$$d(x^*, x_n) \preceq d(x^*, x_{n+1}) + d(x_{n+1}, x_n),$$

thus

$$d(x^*, x_{n+1}) + d(x_{n+1}, x_n) \succeq (k + l)d(x^{**}, x_{n+1}) - ld(x^*, x_{n+1}) + pd(x_{n+1}, x_n),$$

which implies that

$$(k + l)d(x^{**}, x_{n+1}) \preceq (e + l)d(x^*, x_{n+1}) + (e - p)d(x_{n+1}, x_n).$$

Since $k + l = r$ is invertible, we have

$$d(x^{**}, x_{n+1}) \preceq r^{-1}((e + l)d(x^*, x_{n+1}) + (e - p)d(x_{n+1}, x_n)).$$

Owing to $x_n \rightarrow x^* (n \rightarrow \infty)$, it follows by Lemma 1.7 that for any $c \in \mathcal{A}$ with $\theta \ll c$ there exists N such that for any $n > N$,

$$r^{-1}((e + l)d(x^*, x_{n+1}) + (e - p)d(x_{n+1}, x_n)) \ll c,$$

hence

$$d(x^{**}, x_{n+1}) \ll c.$$

Since the limit of a convergent sequence in cone metric space over Banach algebra is unique, we have $x^{**} = x^*$, i.e., x^* is a fixed point of T .

Corollary 2.3 Let (X, d) be a complete cone metric space over Banach algebra \mathcal{A} and K be a cone in \mathcal{A} . Suppose the mapping $T : X \rightarrow X$ is onto and satisfies the generalized expansive condition:

$$d(Tx, Ty) \succeq kd(x, y), \quad (2.4)$$

for all $x, y \in X$, where $k, k^{-1} \in K$ are generalized constants with $\rho(k^{-1}) < 1$. Then T has an unique fixed point in X .

Proof By using Theorem 2.1 and Theorem 2.2, letting $l = p = \theta$, we need to only prove the fixed point is unique. Indeed, if y^* is another fixed point of T , then

$$d(x^*, y^*) = d(Tx^*, Ty^*) \succeq kd(x^*, y^*),$$

that is,

$$d(x^*, y^*) \preceq k^{-1}d(x^*, y^*) = hd(x^*, y^*).$$

Thus

$$d(x^*, y^*) \preceq hd(x^*, y^*) \preceq h^2d(x^*, y^*) \preceq \cdots \preceq h^nd(x^*, y^*).$$

In view of $\|h^nd(x^*, y^*)\| \rightarrow 0 (n \rightarrow \infty)$, it establishes that for any $c \in \mathcal{A}$ with $\theta \ll c$, there exists N_2 such that for all $n > N_2$, we have

$$d(x^*, y^*) \preceq h^nd(x^*, y^*) \ll c,$$

so $d(x^*, y^*) = \theta$, which implies that $x^* = y^*$. Hence, the fixed point is unique.

Remark 2.4 Note that Corollary 2.3 only assumes that $\rho(k^{-1}) < 1$, which implies $\rho(k) > 1$, neither $k \succ e$ nor $\|k\| > 1$. This is a vital improvement.

Remark 2.5 Since we get the fixed point theorems in the setting of non-normal cone metric spaces over Banach algebras, our results are never equivalent to the fixed point

versions in metric spaces (see [7, 13]). The following examples illustrate our conclusions.

Example 2.6 Let $\mathcal{A} = C_{\mathbb{R}}^1[0, \frac{1}{4}]$ and define a norm on \mathcal{A} by $\|x\| = \|x\|_{\infty} + \|x'\|_{\infty}$ for $x \in \mathcal{A}$. Define multiplication in \mathcal{A} as just pointwise multiplication. Then \mathcal{A} is a Banach algebra with a unit $e = 1$. The set $K = \{x \in \mathcal{A} : x \geq 0\}$ is a non-normal cone in \mathcal{A} (see [7]). Let $X = \mathbb{R}$. Define $d : X \times X \rightarrow \mathcal{A}$ by $d(x, y)(t) = |x - y|e^t$, for all $t \in [0, \frac{1}{4}]$. Further, let $T : X \rightarrow X$ be a mapping defined by $Tx = 2x$ and let $k \in K$ define by $k(t) = \frac{4}{2t+3}$. By careful calculations one sees that all the conditions of Corollary 2.3 are fulfilled. The point $x = 0$ is the unique fixed point of the mapping T .

Example 2.7 Let $\mathcal{A} = \{a = (a_{ij})_{3 \times 3} \mid a_{ij} \in \mathbb{R}, 1 \leq i, j \leq 3\}$ and $\|a\| = \frac{1}{3} \sum_{1 \leq i, j \leq 3} |a_{ij}|$. Then the set $K = \{a \in \mathcal{A} \mid a_{ij} \geq 0, 1 \leq i, j \leq 3\}$ is a normal cone in \mathcal{A} . Let $X = \{1, 2, 3\}$. Define $d : X \times X \rightarrow \mathcal{A}$ by $d(1, 1) = d(2, 2) = d(3, 3) = \theta$ and

$$d(1, 2) = d(2, 1) = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{pmatrix},$$

$$d(1, 3) = d(3, 1) = \begin{pmatrix} 2 & 4 & 6 \\ 0 & 0 & 0 \\ 3 & 4 & 5 \end{pmatrix},$$

$$d(2, 3) = d(3, 2) = \begin{pmatrix} 2 & 4 & 6 \\ 0 & 0 & 0 \\ 3 & 4 & 5 \end{pmatrix}.$$

We find that (X, d) is a solid cone metric space over Banach algebra \mathcal{A} . Let $T : X \rightarrow X$ be a mapping defined by $T1 = 2, T2 = 1, T3 = 3$, and let $k, l, p \in K$ be defined by

$$k = \begin{pmatrix} \frac{4}{5} & 0 & 0 \\ 0 & \frac{4}{5} & 0 \\ 0 & 0 & \frac{4}{5} \end{pmatrix},$$

$$p = l = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{10} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $d(Tx, Ty) \succeq kd(x, y) + ld(x, Tx) + pd(y, Ty)$, where $k, l, p, e - p \in K$ are generalized constants. It is easy to prove that $\|e - k - l\| < 1$ and $\|(k + l)^{-1}(e - p)\| < 1$, which imply $\rho(e - k - l) < 1$ and $\rho[(k + l)^{-1}(e - p)] < 1$. Clearly, all conditions of Theorem 2.2 are

fulfilled. Hence T has a fixed point $x = 3$ in X .

Remark 2.8 It needs to emphasis that according to the expansive condition of [11, Theorem 2.1], we are easy to see that the mapping discussed is an injection, and the authors attempt to use [11, Example 2.7] to support this theorem. But unfortunately, this is impossible, since the mapping appearing in this example is not an injection at all. Therefore, it is unreasonable. Basing on the facts above, we may verify that Example 2.7 in this paper is reasonable. It is also interesting, since here we use matrixes as generalized constants.

Competing interests

The authors declare that there have no competing interests.

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ADDITIVE ρ -FUNCTIONAL INEQUALITIES IN FUZZY NORMED SPACES

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ABSTRACT. In this paper, we solve the following additive ρ -functional inequalities

$$N(f(x+y) - f(x) - f(y), t) \leq N\left(\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right), t\right) \quad (0.1)$$

and

$$N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y), t\right) \leq N(\rho(f(x+y) - f(x) - f(y)), t) \quad (0.2)$$

in fuzzy normed spaces, where ρ is a fixed real number with $|\rho| < 1$.

Using the fixed point method, we prove the Hyers-Ulam stability of the additive ρ -functional inequalities (0.1) and (0.2) in fuzzy Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

Katsaras [19] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [11, 23, 48]. In particular, Bag and Samanta [2], following Cheng and Mordeson [8], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [22]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [3].

We use the definition of fuzzy normed spaces given in [2, 27, 28] to investigate the Hyers-Ulam stability of additive ρ -functional inequalities in fuzzy Banach spaces.

Definition 1.1. [2, 27, 28, 29] Let X be a real vector space. A function $N : X \times \mathbb{R} \rightarrow [0, 1]$ is called a *fuzzy norm* on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,

(N_1) $N(x, t) = 0$ for $t \leq 0$;

(N_2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;

(N_3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;

(N_4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;

(N_5) $N(x, \cdot)$ is a non-decreasing function of \mathbb{R} and $\lim_{t \rightarrow \infty} N(x, t) = 1$.

(N_6) for $x \neq 0$, $N(x, \cdot)$ is continuous on \mathbb{R} .

The pair (X, N) is called a *fuzzy normed vector space*.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [26, 27].

Definition 1.2. [2, 27, 28, 29] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be *convergent* or *converge* if there exists an $x \in X$ such that $\lim_{n \rightarrow \infty} N(x_n - x, t) = 1$

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for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $N\text{-}\lim_{n \rightarrow \infty} x_n = x$.

Definition 1.3. [2, 27, 28, 29] Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \rightarrow Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X , then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \rightarrow Y$ is continuous at each $x \in X$, then $f : X \rightarrow Y$ is said to be *continuous* on X (see [3]).

The stability problem of functional equations originated from a question of Ulam [47] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [15] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [39] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [12] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

The functional equation $f\left(\frac{x+y}{2}\right) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$ is called the *Jensen equation*. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [7, 16, 18, 20, 21, 24, 35, 36, 37, 41, 42, 43, 44, 45, 46]).

Gilányi [13] showed that if f satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x-y)\| \leq \|f(x+y)\| \quad (1.1)$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x+y) + f(x-y).$$

See also [40]. Fechner [10] and Gilányi [14] proved the Hyers-Ulam stability of the functional inequality (1.1). Park, Cho and Han [34] investigated the Cauchy additive functional inequality

$$\|f(x) + f(y) + f(z)\| \leq \|f(x+y+z)\| \quad (1.2)$$

and the Cauchy-Jensen additive functional inequality

$$\|f(x) + f(y) + 2f(z)\| \leq \left\| 2f\left(\frac{x+y}{2} + z\right) \right\| \quad (1.3)$$

and proved the Hyers-Ulam stability of the functional inequalities (1.2) and (1.3) in Banach spaces.

Park [32, 33] defined additive ρ -functional inequalities and proved the Hyers-Ulam stability of the additive ρ -functional inequalities in Banach spaces and non-Archimedean Banach spaces.

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d : X \times X \rightarrow [0, \infty]$ is called a *generalized metric* on X if d satisfies

- (1) $d(x, y) = 0$ if and only if $x = y$;
- (2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.4. [4, 9] Let (X, d) be a complete generalized metric space and let $J : X \rightarrow X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

- (1) $d(J^n x, J^{n+1} x) < \infty$, $\forall n \geq n_0$;
- (2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;
- (3) y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;
- (4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [17] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [5, 6, 26, 30, 31, 37, 38]).

In Section 2, we solve the additive ρ -functional inequality (0.1) and prove the Hyers-Ulam stability of the additive ρ -functional inequality (0.1) in fuzzy Banach spaces by using the fixed point method.

In Section 3, we solve the additive ρ -functional inequality (0.2) and prove the Hyers-Ulam stability of the additive ρ -functional inequality (0.2) in fuzzy Banach spaces by using the fixed point method.

2. ADDITIVE ρ -FUNCTIONAL INEQUALITY (0.1)

In this section, we prove the Hyers-Ulam stability of the additive ρ -functional inequality (0.1) in fuzzy Banach spaces. Let ρ be a real number with $|\rho| < 1$. We need the following lemma to prove the main results.

Lemma 2.1. Let (Y, N) be a fuzzy normed vector spaces. Let $f : X \rightarrow Y$ be a mapping such that

$$N(f(x+y) - f(x) - f(y), t) \geq N\left(\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right), t\right) \quad (2.1)$$

for all $x, y \in X$ and all $t > 0$. Then f is Cauchy additive, i.e., $f(x+y) = f(x) + f(y)$ for all $x, y \in X$.

Proof. Assume that $f : X \rightarrow Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $N(f(0), t) = N(0, t) = 1$. So $f(0) = 0$.

Letting $y = x$ in (2.1), we get $N(f(2x) - 2f(x), t) \geq N(0, t) = 1$ and so $f(2x) = 2f(x)$ for all $x \in X$. Thus

$$f\left(\frac{x}{2}\right) = \frac{1}{2}f(x) \quad (2.2)$$

for all $x \in X$.

It follows from (2.1) and (2.2) that

$$\begin{aligned} N(f(x+y) - f(x) - f(y), t) &\geq N\left(\rho\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right), t\right) \\ &= N(\rho(f(x+y) - f(x) - f(y)), t) \\ &= N\left(f(x+y) - f(x) - f(y), \frac{t}{|\rho|}\right) \end{aligned}$$

for all $t > 0$. By (N_5) and (N_6) , $N(f(x+y) - f(x) - f(y), t) = 1$ for all $t > 0$. It follows from (N_2) that

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$. □

Theorem 2.2. Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function such that there exists an $L < 1$ with

$$\varphi(x, y) \leq \frac{L}{2} \varphi(2x, 2y)$$

for all $x, y \in X$. Let $f : X \rightarrow Y$ be an odd mapping satisfying

$$\begin{aligned} N(f(x+y) - f(x) - f(y), t) \\ \geq \min \left\{ N \left(\rho \left(2f \left(\frac{x+y}{2} \right) - f(x) - f(y) \right), t \right), \frac{t}{t + \varphi(x, y)} \right\} \end{aligned} \quad (2.3)$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines an additive mapping $A : X \rightarrow Y$ such that

$$N(f(x) - A(x), t) \geq \frac{(2 - 2L)t}{(2 - 2L)t + L\varphi(x, x)} \quad (2.4)$$

for all $x \in X$ and all $t > 0$.

Proof. Letting $y = x$ in (2.3), we get

$$N(f(2x) - 2f(x), t) \geq \frac{t}{t + \varphi(x, x)} \quad (2.5)$$

for all $x \in X$.

Consider the set

$$S := \{g : X \rightarrow Y\}$$

and introduce the generalized metric on S :

$$d(g, h) = \inf \left\{ \mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(x, x)}, \forall x \in X, \forall t > 0 \right\},$$

where, as usual, $\inf \emptyset = +\infty$. It is easy to show that (S, d) is complete (see [25, Lemma 2.1]).

Now we consider the linear mapping $J : S \rightarrow S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right), L\varepsilon t\right) \\ &= N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{2}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\varphi(x, x)} \\ &= \frac{t}{t + \varphi(x, x)} \end{aligned}$$

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for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (2.5) that

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), \frac{L}{2}t\right) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{2}$.

By Theorem 1.4, there exists a mapping $A : X \rightarrow Y$ satisfying the following:

(1) A is a fixed point of J , i.e.,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \quad (2.6)$$

for all $x \in X$. Since $f : X \rightarrow Y$ is odd, $A : X \rightarrow Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - A(x), \mu t) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$;

(2) $d(J^n f, A) \rightarrow 0$ as $n \rightarrow \infty$. This implies the equality

$$N\text{-}\lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \leq \frac{1}{1-L}d(f, Jf)$, which implies the inequality

$$d(f, A) \leq \frac{L}{2-2L}.$$

This implies that the inequality (2.4) holds.

By (2.3),

$$\begin{aligned} & N\left(2^n \left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right), 2^n t\right) \\ & \geq \min \left\{ N\left(\rho\left(2^{n+1} f\left(\frac{x+y}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right)\right), 2^n t\right), \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)} \right\} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\begin{aligned} & N\left(2^n \left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right), t\right) \\ & \geq \min \left\{ N\left(\rho\left(2^{n+1} f\left(\frac{x+y}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right)\right), t\right), \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n}{2^n} \varphi(x, y)} \right\} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n \rightarrow \infty} \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n}{2^n} \varphi(x, y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$N(A(x+y) - A(x) - A(y), t) \geq N\left(\rho\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y)\right), t\right)$$

for all $x, y \in X$ and all $t > 0$. By Lemma 2.1, the mapping $A : X \rightarrow Y$ is Cauchy additive, as desired. \square

Corollary 2.3. *Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with the norm $\| \cdot \|$. Let $f : X \rightarrow Y$ be an odd mapping satisfying*

$$N(f(x+y) - f(x) - f(y), t) \geq \min \left\{ N \left(\rho \left(2f \left(\frac{x+y}{2} \right) - f(x) - f(y) \right), t \right), \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \right\}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n \rightarrow \infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \rightarrow Y$ such that

$$N(f(x) - A(x), t) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 2\theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.2 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{1-p}$, and we get the desired result. \square

Theorem 2.4. *Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \rightarrow Y$ be an odd mapping satisfying (2.3). Then $A(x) := N\text{-}\lim_{n \rightarrow \infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \rightarrow Y$ such that

$$N(f(x) - A(x), t) \geq \frac{(2 - 2L)t}{(2 - 2L)t + \varphi(x, x)} \quad (2.7)$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2.

It follows from (2.5) that

$$N\left(f(x) - \frac{1}{2}f(2x), \frac{1}{2}t\right) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{1}{2}$. Hence

$$d(f, A) \leq \frac{1}{2 - 2L},$$

which implies that the inequality (2.7) holds.

The rest of the proof is similar to the proof of Theorem 2.2. \square

Corollary 2.5. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with the norm $\| \cdot \|$. Let $f : X \rightarrow Y$ be an odd mapping satisfying*

$$N(f(x+y) - f(x) - f(y), t) \geq \min \left\{ N \left(\rho \left(2f \left(\frac{x+y}{2} \right) - f(x) - f(y) \right), t \right), \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \right\}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n \rightarrow \infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \rightarrow Y$ such that

$$N(f(x) - A(x), t) \geq \frac{(2 - 2^p)t}{(2 - 2^p)t + 2\theta\|x\|^p}$$

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for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-1}$, and we get the desired result. \square

3. ADDITIVE ρ -FUNCTIONAL INEQUALITY (0.2)

In this section, we prove the Hyers-Ulam stability of the additive ρ -functional inequality (0.2) in fuzzy Banach spaces. Let ρ be a fuzzy number with $|\rho| < 1$.

Lemma 3.1. *Let (Y, N) be a fuzzy normed vector spaces. A mapping $f : X \rightarrow Y$ satisfies $f(0) = 0$ and*

$$N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y), t\right) \geq N(\rho(f(x+y) - f(x) - f(y)), t) \quad (3.1)$$

for all $x, y \in X$ and all $t > 0$. Then f is Cauchy additive, i.e., $f(x+y) = f(x) + f(y)$ for all $x, y \in X$.

Proof. Assume that $f : X \rightarrow Y$ satisfies (3.1).

Letting $y = 0$ in (3.1), we get $N(2f(\frac{x}{2}) - f(x), t) \geq N(0, t) = 1$ and so

$$f\left(\frac{x}{2}\right) = \frac{1}{2}f(x) \quad (3.2)$$

for all $x \in X$.

It follows from (3.1) and (3.2) that

$$\begin{aligned} N(f(x+y) - f(x) - f(y), t) &= N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y), t\right) \\ &\geq N(\rho(f(x+y) - f(x) - f(y)), t) \\ &= N\left(f(x+y) - f(x) - f(y), \frac{t}{|\rho|}\right) \end{aligned}$$

for all $t > 0$. By (N_5) and (N_6) , $N(f(x+y) - f(x) - f(y), t) = 1$ for all $t > 0$. It follows from (N_2) that

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$. \square

Theorem 3.2. *Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x, y) \leq \frac{L}{2}\varphi(2x, 2y)$$

for all $x, y \in X$. Let $f : X \rightarrow Y$ be an odd mapping satisfying

$$\begin{aligned} N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y), t\right) \\ \geq \min\left\{N(\rho(f(x+y) - f(x) - f(y)), t), \frac{t}{t + \varphi(x, y)}\right\} \end{aligned} \quad (3.3)$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n \rightarrow \infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \rightarrow Y$ such that

$$N(f(x) - A(x), t) \geq \frac{(1-L)t}{(1-L)t + \varphi(x, 0)} \quad (3.4)$$

for all $x \in X$ and all $t > 0$.

Proof. Since f is odd, $f(0) = 0$.

Letting $y = 0$ in (3.3), we get

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) = N\left(2f\left(\frac{x}{2}\right) - f(x), t\right) \geq \frac{t}{t + \varphi(x, 0)} \quad (3.5)$$

for all $x \in X$.

Consider the set

$$S := \{g : X \rightarrow Y\}$$

and introduce the generalized metric on S :

$$d(g, h) = \inf \left\{ \mu \in \mathbb{R}_+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(x, 0)}, \forall x \in X, \forall t > 0 \right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S, d) is complete (see [25, Lemma 2.1]).

Now we consider the linear mapping $J : S \rightarrow S$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(2g\left(\frac{x}{2}\right) - 2h\left(\frac{x}{2}\right), L\varepsilon t\right) \\ &= N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{2}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \varphi\left(\frac{x}{2}, 0\right)} \geq \frac{\frac{Lt}{2}}{\frac{Lt}{2} + \frac{L}{2}\varphi(x, 0)} \\ &= \frac{t}{t + \varphi(x, 0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (3.5) that

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), t\right) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq 1$.

By Theorem 1.4, there exists a mapping $A : X \rightarrow Y$ satisfying the following:

(1) A is a fixed point of J , i.e.,

$$A\left(\frac{x}{2}\right) = \frac{1}{2}A(x) \quad (3.6)$$

for all $x \in X$. Since $f : X \rightarrow Y$ is odd, $A : X \rightarrow Y$ is an odd mapping. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that A is a unique mapping satisfying (3.6) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - A(x), \mu t) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$;

(2) $d(J^n f, A) \rightarrow 0$ as $n \rightarrow \infty$. This implies the equality

$$N\text{-}\lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right) = A(x)$$

for all $x \in X$;

(3) $d(f, A) \leq \frac{1}{1-L}d(f, Jf)$, which implies the inequality

$$d(f, A) \leq \frac{1}{1-L}.$$

This implies that the inequality (3.4) holds.

By (3.3),

$$\begin{aligned} & N\left(2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right), 2^n t\right) \\ & \geq \min\left\{N\left(\rho\left(2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right), 2^n t\right), \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}\right\} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$\begin{aligned} & N\left(2^{n+1}f\left(\frac{x+y}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{y}{2^n}\right), t\right) \\ & \geq \min\left\{N\left(\rho\left(2^n\left(f\left(\frac{x+y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\right)\right), t\right), \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n}{2^n}\varphi(x, y)}\right\} \end{aligned}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n \rightarrow \infty} \frac{\frac{t}{2^n}}{\frac{t}{2^n} + \frac{L^n}{2^n}\varphi(x, y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$N\left(2A\left(\frac{x+y}{2}\right) - A(x) - A(y), t\right) \geq N(\rho(A(x+y) - A(x) - A(y)), t)$$

for all $x, y \in X$ and all $t > 0$. By Lemma 3.1, the mapping $A : X \rightarrow Y$ is Cauchy additive, as desired. \square

Corollary 3.3. Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with the norm $\|\cdot\|$. Let $f : X \rightarrow Y$ be an odd mapping satisfying

$$\begin{aligned} & N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y), t\right) \\ & \geq \min\left\{N(\rho(f(x+y) - f(x) - f(y)), t), \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}\right\} \end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n \rightarrow \infty} 2^n f(\frac{x}{2^n})$ exists for each $x \in X$ and defines an additive mapping $A : X \rightarrow Y$ such that

$$N(f(x) - A(x), t) \geq \frac{(2^p - 2)t}{(2^p - 2)t + 2^p \theta \|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.2 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{1-p}$, and we get the desired result. \square

Theorem 3.4. Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function such that there exists an $L < 1$ with

$$\varphi(x, y) \leq 2L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \rightarrow Y$ be an odd mapping satisfying (3.3). Then $A(x) := N\text{-}\lim_{n \rightarrow \infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \rightarrow Y$ such that

$$N(f(x) - A(x), t) \geq \frac{(1-L)t}{(1-L)t + L\varphi(x, 0)} \quad (3.7)$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.2.

It follows from (3.5) that

$$N\left(f(x) - \frac{1}{2}f(2x), Lt\right) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq L$. Hence

$$d(f, A) \leq \frac{L}{1-L},$$

which implies that the inequality (3.7) holds.

The rest of the proof is similar to the proof of Theorem 3.2. \square

Corollary 3.5. Let $\theta \geq 0$ and let p be a real number with $0 < p < 1$. Let X be a normed vector space with the norm $\|\cdot\|$. Let $f : X \rightarrow Y$ be an odd mapping satisfying

$$\begin{aligned} & N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y), t\right) \\ & \geq \min\left\{N(\rho(f(x+y) - f(x) - f(y)), t), \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}\right\} \end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Then $A(x) := N\text{-}\lim_{n \rightarrow \infty} \frac{1}{2^n} f(2^n x)$ exists for each $x \in X$ and defines an additive mapping $A : X \rightarrow Y$ such that

$$N(f(x) - A(x), t) \geq \frac{(2-2^p)t}{(2-2^p)t + 2^p\theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.4 by taking $\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 2^{p-1}$, and we get the desired result. \square

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A NOTE ON BARNES-TYPE BOOLE POLYNOMIALS WITH λ -PARAMETER

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ABSTRACT. In this paper, we consider Barnes-type Boole polynomials and give some formulae related to these polynomials.

1. INTRODUCTION

Let p be a fixed odd prime number. Throughout this paper, \mathbb{Z}_p , \mathbb{Q}_p and \mathbb{C}_p will denote the ring of p -adic integers, the field of p -adic rational numbers and the completion of the algebraic closure of \mathbb{Q}_p . The p -adic norm is normalized as $|p|_p = \frac{1}{p}$. Let $C(\mathbb{Z}_p)$ be the space of continuous functions on \mathbb{Z}_p . For $f \in C(\mathbb{Z}_p)$, the fermionic p -adic integral on \mathbb{Z}_p is defined by Kim as

$$(1.1) \quad I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} f(x) (-1)^x, \quad (\text{see [1-19, 21, 22]}).$$

From (1.1), we have

$$(1.2) \quad I_{-1}(f_n) + (-1)^{n-1} I_{-1}(f) = 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} f(l), \quad (\text{see [14]}).$$

As is well known, the Boole polynomials are given by the generating function

$$(1.3) \quad \frac{1}{(1+t)^\lambda + 1} (1+t)^x = \sum_{n=0}^{\infty} Bl_n(x | \lambda) \frac{t^n}{n!}, \quad (\text{see [10]}).$$

When $x = 0$, $Bl_n(\lambda) = Bl_n(0 | \lambda)$ are called the Boole numbers.

For $a_1, a_2, \dots, a_r \in \mathbb{C}_p$, the Barnes-type Euler polynomials are given by the generating function

$$(1.4) \quad \frac{2^r}{(e^{a_1 t} + 1)(e^{a_2 t} + 1) \cdots (e^{a_r t} + 1)} e^{xt} = \sum_{n=0}^{\infty} E_n(x | a_1, \dots, a_r) \frac{t^n}{n!}.$$

When $x = 0$, $E_n(a_1, \dots, a_r) = E_n(0 | a_1, \dots, a_r)$ are called the Barnes-type Euler numbers (see [12, 20]).

From (1.1), we can derive the following equation:

$$(1.5) \quad \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x)$$

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$$\begin{aligned}
&= \lim_{N \rightarrow \infty} \sum_{x=0}^{dp^N-1} f(x) (-1)^x \\
&= \lim_{N \rightarrow \infty} \sum_{a=0}^{d-1} \sum_{x=0}^{p^N-1} f(a+dx) (-1)^{a+x} \\
&= \sum_{a=0}^{d-1} (-1)^a \int_{\mathbb{Z}_p} f(a+dx) \mu_{-1}(x),
\end{aligned}$$

where $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$.

In [10], Kim-Kim derived the Witt-type formula for Boole polynomials which are given by

$$\begin{aligned}
(1.6) \quad & \frac{1}{2} \int_{\mathbb{Z}_p} (1+t)^{x+\lambda y} d\mu_0(y) \\
&= \frac{1}{(1+t)^\lambda + 1} (1+t)^x \\
&= \sum_{n=0}^{\infty} Bl_n(x | \lambda) \frac{t^n}{n!}.
\end{aligned}$$

In this paper, we consider Barnes-type Boole polynomials and give some formulae related to these polynomials.

2. BARNES-TYPE BOOLE POLYNOMIALS WITH λ -PARAMETER

Let $a_1, a_2, \dots, a_r \in \mathbb{C}_p$. Then, we consider the Barnes-type Boole polynomials which are given by the multivariate fermionic p -adic integral on \mathbb{Z}_p as follows:

$$\begin{aligned}
(2.1) \quad & \frac{1}{2^r} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{\lambda a_1 y_1 + \lambda a_2 y_2 + \cdots + \lambda a_r y_r + x} d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_r) \\
&= \prod_{l=1}^r \left(\frac{1}{1 + (1+t)^{\lambda a_l}} \right) (1+t)^x \\
&= \sum_{n=0}^{\infty} Bl_{n,\lambda}(x | a_1, \dots, a_r) \frac{t^n}{n!}.
\end{aligned}$$

Note that $Bl_{n,\lambda}^{(1)}(x | 1) = Bl_n(x | \lambda)$, ($n \geq 0$). When $x = 0$, $Bl_{n,\lambda}(a_1, \dots, a_r) = Bl_{n,\lambda}(0 | a_1, \dots, a_r)$ are called the Barnes-type Boole numbers.

From (2.1), we have

$$\begin{aligned}
(2.2) \quad & \frac{1}{2^r} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda a_1 y_1 + \cdots + \lambda a_r y_r + x)_n d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_r) \\
&= Bl_{n,\lambda}(x | a_1, \dots, a_r), \quad (n \geq 0),
\end{aligned}$$

where $(x)_n = x(x-1) \cdots (x-n+1)$.

We observe that

$$(2.3) \quad \frac{1}{2^r} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda a_1 y_1 + \cdots + \lambda a_r y_r + x)_n d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_r)$$

$$\begin{aligned}
&= \frac{1}{2^r} \sum_{l=0}^n S_1(n, l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda a_1 y_1 + \cdots + \lambda a_r y_r + x)^l d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_r) \\
&= \frac{1}{2^r} \sum_{l=0}^n S_1(n, l) \lambda^l \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(a_1 y_1 + \cdots + a_r y_r + \frac{x}{\lambda}\right)^l d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_r),
\end{aligned}$$

where $S_1(n, l)$ is the Stirling number of the first kind.

From (1.2), we have

$$\begin{aligned}
(2.4) \quad &\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(a_1 x_1 + \cdots + a_r x_r + x)t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&= \left(\frac{2^r}{(e^{a_1 t} + 1) \cdots (e^{a_r t} + 1)} \right) e^{xt} \\
&= \sum_{n=0}^{\infty} E_n(x | a_1, \dots, a_r) \frac{t^n}{n!}.
\end{aligned}$$

Thus, by (2.4), we get

$$\begin{aligned}
(2.5) \quad &\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (a_1 x_1 + \cdots + a_r x_r + x)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\
&= E_n(x | a_1, \dots, a_r), \quad (n \geq 0).
\end{aligned}$$

From (2.2) and (2.5), we obtain the following theorem.

Theorem 1. For $n \geq 0$, we have

$$\frac{1}{2^r} \sum_{l=0}^n S_1(n, l) \lambda^l E_l\left(\frac{x}{\lambda} \middle| a_1, \dots, a_r\right) = Bl_{n, \lambda}(x | a_1, \dots, a_r).$$

By (2.1), we get

$$\begin{aligned}
(2.6) \quad &\frac{1}{2^r} \prod_{l=1}^r \left(\frac{2}{e^{a_l t} + 1} \right) e^{\frac{x}{\lambda} t} \\
&= \sum_{n=0}^{\infty} Bl_{n, \lambda}(x | a_1, \dots, a_r) \frac{\left(e^{\frac{1}{\lambda} t} - 1\right)^n}{n!} \\
&= \sum_{m=0}^{\infty} \left(\lambda^{-m} \sum_{n=0}^m Bl_{n, \lambda}(x | a_1, \dots, a_r) S_2(m, n) \right) \frac{t^m}{m!},
\end{aligned}$$

where $S_2(m, n)$ is the Stirling number of the second kind.

By (1.4), we get

$$(2.7) \quad \prod_{l=1}^r \left(\frac{2}{e^{a_l t} + 1} \right) e^{\frac{x}{\lambda} t} = \sum_{m=0}^{\infty} E_m\left(\frac{x}{\lambda} \middle| a_1, \dots, a_r\right) \frac{t^m}{m!}.$$

Therefore, by (2.6) and (2.7), we obtain the following theorem.

Theorem 2. For $m \geq 0$, we have

$$\begin{aligned}
&\lambda^m E_m\left(\frac{x}{\lambda} \middle| a_1, \dots, a_r\right) \\
&= 2^r \sum_{n=0}^m Bl_{n, \lambda}(x | a_1, \dots, a_r) S_2(m, n).
\end{aligned}$$

From (1.5), we have

$$\begin{aligned}
 (2.8) \quad & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{\lambda a_1 y_1 + \cdots + \lambda a_r y_r + x} d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_r) \\
 &= \sum_{k_1, \dots, k_r=0}^{d-1} (-1)^{k_1 + \cdots + k_r} \\
 & \quad \times \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{\lambda a_1(k_1 + dy_1) + \cdots + \lambda a_r(k_r + dy_r) + x} d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_r) \\
 &= \sum_{k_1, \dots, k_r=0}^{d-1} (-1)^{k_1 + \cdots + k_r} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{\lambda d \left(\frac{a_1 k_1 + \cdots + a_r k_r + \frac{x}{\lambda}}{d} + a_1 y_1 + \cdots + a_r y_r \right)} \\
 & \quad \times d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_r) \\
 &= 2^r \sum_{k_1, \dots, k_r=0}^{d-1} (-1)^{k_1 + \cdots + k_r} \sum_{n=0}^{\infty} Bl_{n, \lambda d}(\lambda a_1 k_1 + \cdots + \lambda a_r k_r + x \mid a_1, \dots, a_r) \frac{t^n}{n!} \\
 &= 2^r \sum_{n=0}^{\infty} \left(\sum_{k_1, \dots, k_r=0}^{d-1} (-1)^{k_1 + \cdots + k_r} Bl_{n, \lambda d}(\lambda a_1 k_1 + \cdots + \lambda a_r k_r + x \mid a_1, \dots, a_r) \right) \frac{t^n}{n!},
 \end{aligned}$$

where $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$.

From (2.8), we have

$$\begin{aligned}
 (2.9) \quad & \sum_{n=0}^{\infty} \left(\sum_{k_1, \dots, k_r=0}^{d-1} (-1)^{k_1 + \cdots + k_r} Bl_{n, \lambda d}(\lambda a_1 k_1 + \cdots + \lambda a_r k_r + x \mid a_1, \dots, a_r) \right) \frac{t^n}{n!} \\
 &= \frac{1}{2^r} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{\lambda a_1 y_1 + \cdots + \lambda a_r y_r + x} d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_r) \\
 &= \sum_{n=0}^{\infty} Bl_{n, \lambda}(x \mid a_1, \dots, a_r) \frac{t^n}{n!}.
 \end{aligned}$$

By comparing the coefficients on the both sides of (2.9), we obtain the following equation:

Theorem 3. For $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, $n \geq 0$, we have

$$\begin{aligned}
 & Bl_{n, \lambda}(x \mid a_1, \dots, a_r) \\
 &= \sum_{k_1, \dots, k_r=0}^{d-1} (-1)^{k_1 + \cdots + k_r} Bl_{n, \lambda d}(\lambda a_1 k_1 + \cdots + \lambda a_r k_r + x \mid a_1, \dots, a_r).
 \end{aligned}$$

Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$. From (1.2), we have

$$(2.10) \quad \int_{\mathbb{Z}_p} e^{a_1(y_1+d)t} d\mu_{-1}(y_1) + \int_{\mathbb{Z}_p} e^{a_1 y_1 t} d\mu_{-1}(y) = 2 \sum_{l=0}^{d-1} (-1)^l e^{a_1 l t}.$$

Thus, by (2.10), we get

$$(2.11) \quad \int_{\mathbb{Z}_p} e^{a_1 y_1 t} d\mu_{-1}(y) = \frac{2}{e^{a_1 dt} + 1} \sum_{l=0}^{d-1} (-1)^l e^{a_1 l t}.$$

From (2.11), we can derive

$$(2.12) \quad \begin{aligned} & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(a_1 y_1 + a_2 y_2 + \cdots + a_r y_r + x)t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\ &= \sum_{l_1, \dots, l_r=0}^{d-1} (-1)^{l_1 + \cdots + l_r} \prod_{l=1}^r \left(\frac{2}{e^{a_l dt} + 1} \right) e^{(a_1 l_1 + \cdots + a_r l_r + x)t} \\ &= \sum_{n=0}^{\infty} d^n \sum_{l_1, \dots, l_r=0}^{d-1} (-1)^{l_1 + \cdots + l_r} E_n \left(\frac{a_1 l_1 + \cdots + a_r l_r + x}{d} \middle| a_1, \dots, a_r \right) \frac{t^n}{n!}. \end{aligned}$$

From (2.12) and (2.4), we get

$$\begin{aligned} & E_n(x \mid a_1, \dots, a_r) \\ &= d^n \sum_{l_1, \dots, l_r=0}^{d-1} (-1)^{l_1 + \cdots + l_r} E_n \left(\frac{a_1 l_1 + \cdots + a_r l_r + x}{d} \middle| a_1, \dots, a_r \right), \end{aligned}$$

where $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$.

On the other hand,

$$(2.13) \quad \begin{aligned} & \int_{\mathbb{Z}_p} (1+t)^{\lambda a_1 (d+y_1)} d\mu_{-1}(y) + \int_{\mathbb{Z}_p} (1+t)^{\lambda a_1 y_1} d\mu_{-1}(y) \\ &= 2 \sum_{l_1=0}^{d-1} (-1)^{l_1} (1+t)^{\lambda a_1 l_1}, \end{aligned}$$

where $d \in \mathbb{N}$ such that $d \equiv 1 \pmod{2}$.

By (2.13), we get

$$(2.14) \quad \begin{aligned} & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+t)^{\lambda a_1 y_1 + \cdots + \lambda a_r y_r + x} d\mu_{-1}(y_1) \cdots d\mu_{-1}(y_r) \\ &= \prod_{l=1}^r \frac{2}{1 + (1+t)^{\lambda a_l d}} \sum_{l_1, \dots, l_r=0}^{d-1} (-1)^{l_1 + \cdots + l_r} (1+t)^{\lambda a_1 l_1 + \cdots + \lambda a_r l_r + x} \\ &= 2^r \sum_{m=0}^{\infty} Bl_{m, \lambda d}(a_1, \dots, a_r) \frac{t^m}{m!} \\ & \quad \times \sum_{k=0}^{\infty} \left(\sum_{l_1, \dots, l_r=0}^{d-1} (-1)^{l_1 + \cdots + l_r} (\lambda a_1 l_1 + \cdots + \lambda a_r l_r + x)_k \right) \frac{t^k}{k!} \\ &= 2^r \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k} \sum_{l_1, \dots, l_r=0}^{d-1} (-1)^{l_1 + \cdots + l_r} (\lambda a_1 l_1 + \cdots + \lambda a_r l_r + x)_k Bl_{n-k, \lambda d}(a_1, \dots, a_r) \right) \frac{t^n}{n!}. \end{aligned}$$

From (2.9) and (2.14), we note that

$$(2.15) \quad \begin{aligned} & Bl_n(x \mid a_1, \dots, a_r) \\ &= \sum_{k=0}^n \binom{n}{k} \sum_{l_1, \dots, l_r=0}^{d-1} (-1)^{l_1+\dots+l_r} (x + \lambda a_1 l_1 + \dots + \lambda a_r l_r)_k \\ & \quad \times Bl_{n-k, \lambda d}(a_1, \dots, a_r), \end{aligned}$$

where $n \geq 0$ and $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$.

Therefore, by (2.15), we obtain the following theorem.

Theorem 4. For $n \geq 0$ and $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, we have

$$\begin{aligned} & Bl_n(x \mid a_1, \dots, a_r) \\ &= \sum_{k=0}^n \binom{n}{k} \sum_{l_1, \dots, l_r=0}^{d-1} (-1)^{l_1+\dots+l_r} (x + \lambda a_1 l_1 + \dots + \lambda a_r l_r)_k \\ & \quad \times Bl_{n-k, \lambda d}(a_1, \dots, a_r). \end{aligned}$$

From (2.14), we have

$$(2.16) \quad \begin{aligned} & \frac{1}{2^r} \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} (1+t)^{\lambda a_1 y_1 + \dots + \lambda a_r y_r} d\mu_{-1}(y_1) \dots d\mu_{-1}(y_r) \\ &= \sum_{l_1, \dots, l_r=0}^{d-1} (-1)^{l_1+\dots+l_r} \left(\prod_{i=1}^r \frac{1}{1 + (1+t)^{\lambda a_i d}} \right) (1+t)^{\lambda a_1 l_1 + \dots + \lambda a_r l_r} \\ &= \sum_{n=0}^{\infty} \left(\sum_{l_1, \dots, l_r=0}^{d-1} (-1)^{l_1+\dots+l_r} Bl_{n, \lambda d}(\lambda a_1 l_1 + \dots + \lambda a_r l_r \mid a_1, \dots, a_r) \right) \frac{t^n}{n!}, \end{aligned}$$

and

$$(2.17) \quad \begin{aligned} & \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} (1+t)^{\lambda a_1 y_1 + \dots + \lambda a_r y_r} d\mu_{-1}(y_1) \dots d\mu_{-1}(y_r) \\ &= 2^r \sum_{n=0}^{\infty} Bl_{n, \lambda}(a_1, \dots, a_r) \frac{t^n}{n!}. \end{aligned}$$

Therefore, by (2.16) and (2.17), we obtain the following theorem.

Theorem 5. For $n \geq 0$, $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, we have

$$Bl_{n, \lambda}(a_1, \dots, a_r) = \sum_{l_1, \dots, l_r=0}^{d-1} (-1)^{l_1+\dots+l_r} Bl_{n, \lambda d}(\lambda a_1 l_1 + \dots + \lambda a_r l_r \mid a_1, \dots, a_r).$$

By replacing t by $e^{\frac{1}{\lambda}t} - 1$ in (2.14), we get

$$(2.18) \quad \begin{aligned} & \int_{\mathbb{Z}_p} \dots \int_{\mathbb{Z}_p} e^{(a_1 y_1 + \dots + a_r y_r + \frac{x}{\lambda})t} d\mu_{-1}(y_1) \dots d\mu_{-1}(y_r) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k} \sum_{l_1, \dots, l_r=0}^{d-1} (-1)^{l_1+\dots+l_r} (\lambda a_1 l_1 + \dots + \lambda a_r l_r + x)_k \right) \end{aligned}$$

$$\begin{aligned}
& \times Bl_{n-k, \lambda d}(a_1, \dots, a_r) \frac{1}{n!} \left(e^{\frac{1}{\lambda} t} - 1 \right)^n \\
& = \sum_{m=0}^{\infty} \lambda^{-m} \left(\sum_{n=0}^m \sum_{k=0}^n \sum_{l_1, \dots, l_r=0}^{d-1} \binom{n}{k} S_2(m, n) (-1)^{l_1 + \dots + l_r} \right. \\
& \quad \left. \times (\lambda a_1 l_1 + \dots + \lambda a_r l_r + x)_k Bl_{n-k, \lambda d}(a_1, \dots, a_r) \right) \frac{t^m}{m!}
\end{aligned}$$

Thus, by (2.18), we get

$$\begin{aligned}
(2.19) \quad & \lambda^m E_m \left(\frac{x}{\lambda} \middle| a_1, \dots, a_r \right) \\
& = \sum_{n=0}^m \sum_{k=0}^n \sum_{l_1, \dots, l_r=0}^{d-1} \binom{n}{k} S_2(m, n) (-1)^{l_1 + \dots + l_r} \\
& \quad \times (\lambda a_1 l_1 + \dots + \lambda a_r l_r + x)_k Bl_{n-k, \lambda d}(a_1, \dots, a_r),
\end{aligned}$$

where $m \geq 0$ and $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$.

Therefore, by (2.19), we obtain the following theorem.

Theorem 6. For $m \geq 0$, $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$, we have

$$\begin{aligned}
& \lambda^m E_m \left(\frac{x}{\lambda} \middle| a_1, \dots, a_r \right) \\
& = \sum_{n=0}^m \sum_{k=0}^n \sum_{l_1, \dots, l_r=0}^{d-1} \binom{n}{k} S_2(m, n) (-1)^{l_1 + \dots + l_r} \\
& \quad \times (\lambda a_1 l_1 + \dots + \lambda a_r l_r + x)_k Bl_{n-k, \lambda d}(a_1, \dots, a_r).
\end{aligned}$$

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PRODUCT-TYPE OPERATORS FROM WEIGHTED BERGMAN-ORLICZ SPACES TO BLOCH-ORLICZ SPACES

HONG-BIN BAI AND ZHI-JIE JIANG

ABSTRACT. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk, φ an analytic self-map of \mathbb{D} and ψ an analytic function on \mathbb{D} . Let D be the differentiation operator and $W_{\varphi,\psi}$ the weighted composition operator. The boundedness and compactness of the product-type operators $DW_{\varphi,\psi}$ from the weighted Bergman-Orlicz spaces to the Bloch-Orlicz spaces on \mathbb{D} are characterized.

1. INTRODUCTION

Let \mathbb{C} be the complex plane, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk and $H(\mathbb{D})$ the class of all analytic functions on \mathbb{D} . Let φ be an analytic self-map of \mathbb{D} and $\psi \in H(\mathbb{D})$. Weighted composition operator $W_{\varphi,\psi}$ on $H(\mathbb{D})$ is defined by

$$W_{\varphi,\psi}f(z) = \psi(z)f(\varphi(z)), \quad z \in \mathbb{D}.$$

If $\psi \equiv 1$, the operator is reduced to, so called, composition operator and usually denoted by C_φ . If $\varphi(z) = z$, it is reduced to, so called, multiplication operator and usually denoted by M_ψ . A standard problem is to provide function theoretic characterizations when φ and ψ induce a bounded or compact weighted composition operator. Composition operators and weighted composition operators between various spaces of holomorphic functions on different domains have been studied in many papers, see, for example, [1, 3, 8, 11–15, 17, 19, 22, 26, 27, 31, 33–36, 40, 42, 48, 53, 55, 60] and the references therein.

Let D be the differentiation operator on $H(\mathbb{D})$, that is,

$$Df(z) = f'(z), \quad z \in \mathbb{D}.$$

Operator DC_φ has been studied, for example, in [6, 16, 18, 24, 25, 28, 41, 45, 50]. In [32] Sharma studied the operators $DM_\psi C_\varphi$ and $DC_\varphi M_\psi$ from Bergman spaces to Bloch type spaces. These operators on weighted Bergman spaces were also studied in [58] and [59] by Stević, Sharma and Bhat. If we consider the product-type operator $DW_{\varphi,\psi}$, it is clear that $DM_\psi C_\varphi = DW_{\varphi,\psi}$ and $DC_\varphi M_\psi = DW_{\varphi,\psi \circ \varphi}$. Quite recently, the present author has considered this operator in [7, 9]. For some other product-type operators, see, for example, [10, 20, 21, 23, 37–39, 43, 44, 46, 47, 51, 52, 54, 56, 61] and the references therein. This paper is devoted to characterizing the boundedness and compactness of the operators $DW_{\varphi,\psi}$ from the weighted Bergman-Orlicz spaces to the Bloch-Orlicz spaces.

Next we are ready to introduce the needed spaces and some facts in [30]. The function $\Phi \not\equiv 0$ is called a growth function, if it is a continuous and nondecreasing

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function from the interval $[0, \infty)$ onto itself. It is clear that these conditions imply that $\Phi(0) = 0$. It is said that the function Φ is of positive upper type (respectively, negative upper type), if there are $q > 0$ (respectively, $q < 0$) and $C > 0$ such that $\Phi(st) \leq Ct^q\Phi(s)$ for every $s > 0$ and $t \geq 1$. By \mathfrak{U}^q we denote the family of all growth functions Φ of positive upper type q ($q \geq 1$), such that the function $t \mapsto \Phi(t)/t$ is nondecreasing on $[0, \infty)$. It is said that function Φ is of positive lower type (respectively, negative upper type), if there are $r > 0$ (respectively, $r < 0$) and $C > 0$ such that $\Phi(st) \leq Ct^r\Phi(s)$ for every $s > 0$ and $0 < t \leq 1$. By \mathfrak{L}_r we denote the family of all growth functions Φ of positive lower type r ($0 < r \leq 1$), such that the function $t \mapsto \Phi(t)/t$ is nonincreasing on $[0, \infty)$. If $f \in \mathfrak{U}^q$, we will also assume that it is convex.

Let $dA(z) = \frac{1}{\pi}dxdy$ be the normalized Lebesgue measure on \mathbb{D} . For $\alpha > -1$, let $dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$ be the weighted Lebesgue measure on \mathbb{D} . Let Φ be a growth function. The weighted Bergman-Orlicz space $A_\alpha^\Phi(\mathbb{D}) := A_\alpha^\Phi$ is the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{A_\alpha^\Phi} = \int_{\mathbb{D}} \Phi(|f(z)|) dA_\alpha(z) < \infty.$$

On A_α^Φ is defined the following quasi-norm

$$\|f\|_{A_\alpha^\Phi}^{lux} = \inf \left\{ \lambda > 0 : \int_{\mathbb{D}} \Phi\left(\frac{|f(z)|}{\lambda}\right) dA_\alpha(z) \leq 1 \right\}.$$

If $\Phi \in \mathfrak{U}^q$ or $\Phi \in \mathfrak{L}_r$, then the quasi-norm on A_α^Φ is finite and called the Luxembourg norm. The classical weighted Bergman space A_α^p , $p > 0$, corresponds to $\Phi(t) = t^p$, consisting of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{A_\alpha^p}^p = \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) < \infty.$$

It is well known that for $p \geq 1$ it is a Banach space, while for $0 < p < 1$ it is a translation-invariant metric space with $d(f, g) = \|f - g\|_{A_\alpha^p}^p$. Moreover, if $\Phi \in \mathfrak{U}^s$, then $A_\alpha^{\Phi_p}$, where $\Phi_p(t) = \Phi(t^p)$, is a subspace of A_α^Φ ([30]).

Recently, the Bloch-Orlicz space was introduced in [29] by Ramos Fernández. More precisely, let Ψ be a strictly increasing convex function such that $\Psi(0) = 0$. From these conditions it follows that $\lim_{t \rightarrow \infty} \Psi(t) = \infty$. The Bloch-Orlicz space associated with the function Ψ , denoted by \mathcal{B}^Ψ , is the class of all $f \in H(\mathbb{D})$ such that

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) \Psi(\lambda |f'(z)|) < \infty$$

for some $\lambda > 0$ depending on f . On \mathcal{B}^Ψ Minkowski's functional

$$\|f\|_\Psi = \inf \left\{ k > 0 : S_\Psi\left(\frac{f'}{k}\right) \leq 1 \right\}$$

defines a seminorm, where

$$S_\Psi(f) = \sup_{z \in \mathbb{D}} (1 - |z|^2) \Psi(f(z)).$$

Moreover, \mathcal{B}^Ψ is a Banach space with the norm

$$\|f\|_{\mathcal{B}^\Psi} = |f(0)| + \|f\|_\Psi.$$

In fact, Ramos Fernández in [29] proved that \mathcal{B}^Ψ is isometrically equal to μ -Bloch space, where

$$\mu(z) = \frac{1}{\Psi^{-1}\left(\frac{1}{1-|z|^2}\right)}, \quad z \in \mathbb{D}.$$

Thus, for $f \in \mathcal{B}^\Psi$, we have

$$\|f\|_{\mathcal{B}^\Psi} = |f(0)| + \sup_{z \in \mathbb{D}} \mu(z) |f'(z)|.$$

We can study the operator $DW_{\varphi,\psi} : A_{\alpha}^{\Phi_p} \rightarrow \mathcal{B}^\Psi$ with the help of this equivalent norm. It is obviously seen that if $\Psi(t) = t^p$ with $p > 0$, then the space \mathcal{B}^Ψ coincides with the weighted Bloch space \mathcal{B}^α (see [62]), where $\alpha = 1/p$. Also, if $\Psi(t) = t \log(1+t)$ then \mathcal{B}^Ψ coincides with the Log-Bloch space (see [2]). For the generalization of Log-Bloch spaces, see, for example, [49, 57].

Let X and Y be topological vector spaces whose topologies are given by translation invariant metrics d_X and d_Y , respectively. It is said that a linear operator $L : X \rightarrow Y$ is metrically bounded if there exists a positive constant K such that

$$d_Y(Lf, 0) \leq K d_X(f, 0)$$

for all $f \in X$. When X and Y are Banach spaces, the metrical boundedness coincides with the usual definition of bounded operators between Banach spaces. Operator $L : X \rightarrow Y$ is said to be metrically compact if it maps bounded sets into relatively compact sets. When X and Y are Banach spaces, the metrical compactness coincides with the usual definition of compact operators between Banach spaces. Let $X = A_{\alpha}^{\Phi}$ and Y a Banach space. The norm of operator $L : X \rightarrow Y$ is defined by

$$\|L\|_{A_{\alpha}^{\Phi} \rightarrow Y} = \sup_{\|f\|_{A_{\alpha}^{\Phi}} \leq 1} \|Lf\|_Y$$

and often written by $\|L\|$.

Throughout this paper, an operator is bounded (respectively, compact), if it is metrically bounded (respectively, metrically compact). C will be a constant not necessary the same at each occurrence. The notation $a \lesssim b$ means that $a \leq Cb$ for some positive constant C . When $a \lesssim b$ and $b \lesssim a$, we write $a \simeq b$.

2. AUXILIARY RESULTS

In order to prove the compactness of the product-type operators, we need the following result which is similar to Proposition 3.11 in [4]. The details of the proof are omitted.

Lemma 2.1. *Let $p \geq 1$, $\alpha > -1$, and $\Phi \in \mathfrak{U}^s$ such that $\Phi_p \in \mathfrak{L}_r$. Then the bounded operator $DW_{\varphi,\psi} : A_{\alpha}^{\Phi_p} \rightarrow \mathcal{B}^\Psi$ is compact if and only if for every bounded sequence $\{f_n\}_{n \in \mathbb{N}}$ in $A_{\alpha}^{\Phi_p}$ such that $f_n \rightarrow 0$ uniformly on every compact subset of \mathbb{D} as $n \rightarrow \infty$, it follows that*

$$\lim_{n \rightarrow \infty} \|DW_{\varphi,\psi} f_n\|_{\mathcal{B}^\Psi} = 0.$$

We formulate the following two useful point estimates. For the first, see Lemma 2.4 in [30], and for the second, see Lemma 2.3 in [9].

Lemma 2.2. *Let $p \geq 1$, $\alpha > -1$ and $\Phi \in \mathfrak{U}^s$. Then for every $f \in A_{\alpha}^{\Phi_p}$ and $z \in \mathbb{D}$ we have*

$$|f(z)| \leq \Phi_p^{-1} \left(\left(\frac{4}{1-|z|^2} \right)^{\alpha+2} \right) \|f\|_{A_{\alpha}^{\Phi_p}}^{lux}.$$

Lemma 2.3. *Let $p \geq 1$, $\alpha > -1$, $\Phi \in \mathfrak{U}^s$ and $n \in \mathbb{N}$. Then there are two positive constants $C_n = C(\alpha, p, n)$ and $D_n = D(\alpha, p, n)$ independent of $f \in A_{\alpha}^{\Phi_p}$ and $z \in \mathbb{D}$ such that*

$$|f^{(n)}(z)| \leq \frac{C_n}{(1-|z|^2)^n} \Phi_p^{-1} \left(\frac{D_n}{(1-|z|^2)^{\alpha+2}} \right) \|f\|_{A_{\alpha}^{\Phi_p}}^{lux}.$$

We also need the following lemma which provides a class of useful test functions in space $A_{\alpha}^{\Phi_p}$ (see [9]).

Lemma 2.4. *Let $p > 0$, $\alpha > -1$ and $\Phi \in \mathfrak{U}^s$. Then for every $t \geq 0$ and $w \in \mathbb{D}$ the following function is in $A_{\alpha}^{\Phi_p}$*

$$f_{w,t}(z) = \Phi_p^{-1} \left(\left(\frac{C}{1-|w|^2} \right)^{\alpha+2} \right) \left(\frac{1-|w|^2}{1-\bar{w}z} \right)^{\frac{2(\alpha+2)}{p}+t},$$

where C is an arbitrary positive constant.

Moreover,

$$\sup_{w \in \mathbb{D}} \|f_{w,t}\|_{A_{\alpha}^{\Phi_p}}^{lux} \lesssim 1.$$

3. THE OPERATOR $DW_{\varphi,\psi} : A_{\alpha}^{\Phi_p} \rightarrow \mathcal{B}^{\Psi}$

First we characterize the boundedness of operator $DW_{\varphi,\psi} : A_{\alpha}^{\Phi_p} \rightarrow \mathcal{B}^{\Psi}$. We assume that $\Phi \in \mathfrak{U}^s$ such that $\Phi_p \in \mathfrak{L}_r$. Under this assumption, $A_{\alpha}^{\Phi_p}$ is a complete metric space (see [30]).

Theorem 3.1. *Let $p \geq 1$, $\alpha > -1$, and $\Phi \in \mathfrak{U}^s$ such that $\Phi_p \in \mathfrak{L}_r$. Then the following conditions are equivalent:*

- (i) *The operator $DW_{\varphi,\psi} : A_{\alpha}^{\Phi_p} \rightarrow \mathcal{B}^{\Psi}$ is bounded.*
- (ii) *Functions φ and ψ satisfy the following conditions:*

$$M_1 := \sup_{z \in \mathbb{D}} \mu(z) |\psi''(z)| \Phi_p^{-1} \left(\left(\frac{4}{1-|\varphi(z)|^2} \right)^{\alpha+2} \right) < \infty,$$

$$M_2 := \sup_{z \in \mathbb{D}} \frac{\mu(z)}{1-|\varphi(z)|^2} |\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)| \Phi_p^{-1} \left(\left(\frac{D_1}{1-|\varphi(z)|^2} \right)^{\alpha+2} \right) < \infty,$$

and

$$M_3 := \sup_{z \in \mathbb{D}} \frac{\mu(z)}{(1-|\varphi(z)|^2)^2} |\psi(z)| |\varphi'(z)|^2 \Phi_p^{-1} \left(\left(\frac{D_2}{1-|\varphi(z)|^2} \right)^{\alpha+2} \right) < \infty.$$

Moreover, if the operator $DW_{\varphi,\psi} : A_{\alpha}^{\Phi_p} \rightarrow \mathcal{B}^{\Psi}$ is nonzero and bounded, then

$$\|DW_{\varphi,\psi}\| \simeq 1 + M_1 + M_2 + M_3.$$

Proof. (i) \Rightarrow (ii). Suppose that $DW_{\varphi,\psi} : A_{\alpha}^{\Phi_p} \rightarrow \mathcal{B}^{\Psi}$ is bounded. For $w \in \mathbb{D}$, we choose the function

$$f_{1,\varphi(w)}(z) = c_0 \left(\frac{1 - |\varphi(w)|^2}{1 - \overline{\varphi(w)}z} \right)^{\frac{2(\alpha+2)}{p}} + c_1 \left(\frac{1 - |\varphi(w)|^2}{1 - \overline{\varphi(w)}z} \right)^{\frac{2(\alpha+2)}{p}+1} \\ + c_2 \left(\frac{1 - |\varphi(w)|^2}{1 - \overline{\varphi(w)}z} \right)^{\frac{2(\alpha+2)}{p}+2} - \left(\frac{1 - |\varphi(w)|^2}{1 - \overline{\varphi(w)}z} \right)^{\frac{2(\alpha+2)}{p}+3},$$

where

$$c_0 = \frac{2(\alpha+2)+3p}{2(\alpha+2)}, \quad c_1 = -\frac{6(\alpha+2)+9p}{2(\alpha+2)+p}, \quad c_2 = \frac{6(\alpha+2)+9p}{2(\alpha+2)+2p}.$$

By a direct calculation, we have

$$f'_{1,\varphi(w)}(\varphi(w)) = f''_{1,\varphi(w)}(\varphi(w)) = 0. \quad (1)$$

Using the function $f_{1,\varphi(w)}$, we define the function

$$f(z) = \Phi_p^{-1} \left(\left(\frac{4}{1 - |\varphi(w)|^2} \right)^{\alpha+2} \right) f_{1,\varphi(w)}(z).$$

Applying (1) to f' and f'' , we obtain

$$f'(\varphi(w)) = f''(\varphi(w)) = 0. \quad (2)$$

It is clear that

$$f(\varphi(w)) = C \Phi_p^{-1} \left(\left(\frac{4}{1 - |\varphi(w)|^2} \right)^{\alpha+2} \right), \quad (3)$$

where

$$C = \frac{2(\alpha+2)+3p}{2(\alpha+2)} - \frac{6(\alpha+2)+9p}{2(\alpha+2)+p} + \frac{6(\alpha+2)+9p}{2(\alpha+2)+2p} - 1 \neq 0.$$

By Lemma 2.4, $f \in A_{\alpha}^{\Phi_p}$ and $\|f\|_{A_{\alpha}^{\Phi_p}} \leq C$. By (2), (3) and the boundedness of $DW_{\varphi,\psi} : A_{\alpha}^{\Phi_p} \rightarrow \mathcal{B}^{\Psi}$,

$$\mu(w) |\psi''(w)| \Phi_p^{-1} \left(\left(\frac{4}{1 - |\varphi(w)|^2} \right)^{\alpha+2} \right) \leq C \|DW_{\varphi,\psi}\|, \quad (4)$$

which means that

$$M_1 = \sup_{z \in \mathbb{D}} \mu(z) |\psi''(z)| \Phi_p^{-1} \left(\left(\frac{4}{1 - |\varphi(z)|^2} \right)^{\alpha+2} \right) \leq C \|DW_{\varphi,\psi}\| < \infty. \quad (5)$$

Next we will prove $M_2 < \infty$. For this we consider the functions $f_1(z) = z$ and $f_2(z) \equiv 1$, respectively. Since the operator $DW_{\varphi,\psi} : A_{\alpha}^{\Phi_p} \rightarrow \mathcal{B}^{\Psi}$ is bounded, we have

$$\sup_{z \in \mathbb{D}} \mu(z) |\psi''(z) \varphi(z) + 2\psi'(z) \varphi'(z) + \psi(z) \varphi''(z)| \\ \leq \|DW_{\varphi,\psi} f_1\|_{\mathcal{B}^{\Psi}} \leq C \|DW_{\varphi,\psi}\| \quad (6)$$

and

$$\sup_{z \in \mathbb{D}} \mu(z) |\psi''(z)| \leq \|DW_{\varphi,\psi} f_2\|_{\mathcal{B}^{\Psi}} \leq C \|DW_{\varphi,\psi}\|. \quad (7)$$

By (6), (7) and the boundedness of φ ,

$$J_1 := \sup_{z \in \mathbb{D}} \mu(z) |\psi(z) \varphi''(z) + 2\psi'(z) \varphi'(z)| \leq C \|DW_{\varphi,\psi}\|. \quad (8)$$

For $w \in \mathbb{D}$, choose the function

$$f_{2,\varphi(w)}(z) = c_0 \left(\frac{1 - |\varphi(w)|^2}{1 - \varphi(w)z} \right)^{\frac{2(\alpha+2)}{p}} + c_1 \left(\frac{1 - |\varphi(w)|^2}{1 - \varphi(w)z} \right)^{\frac{2(\alpha+2)}{p}+1} \\ + c_2 \left(\frac{1 - |\varphi(w)|^2}{1 - \varphi(w)z} \right)^{\frac{2(\alpha+2)}{p}+2} - \left(\frac{1 - |\varphi(w)|^2}{1 - \varphi(w)z} \right)^{\frac{2(\alpha+2)}{p}+3},$$

where

$$c_1 = \frac{36p(\alpha+2)^2 + 78p^2(\alpha+2) + 36p^3}{[4(\alpha+2) + 2p][2(\alpha+2) + 2p][2(\alpha+2) + 3p]},$$

$$c_2 = \frac{4(\alpha+2)^2 + 42p(\alpha+2) + 36p^2}{[2(\alpha+2) + 2p][4(\alpha+2) + 6p]},$$

and

$$c_0 = 1 - c_1 - c_2.$$

From a calculation, we obtain

$$f_{2,\varphi(w)}(\varphi(w)) = f_{2,\varphi(w)}''(\varphi(w)) = 0. \quad (9)$$

Define the function

$$g(z) = \Phi_p^{-1} \left(\left(\frac{D_1}{1 - |\varphi(w)|^2} \right)^{\alpha+2} \right) f_{2,\varphi(w)}(z).$$

Then by (9),

$$g(\varphi(w)) = g''(\varphi(w)) = 0, \quad (10)$$

and by a direct calculation,

$$g'(\varphi(w)) = C \frac{\overline{\varphi(w)}}{1 - |\varphi(w)|^2} \Phi_p^{-1} \left(\left(\frac{D_1}{1 - |\varphi(w)|^2} \right)^{\alpha+2} \right), \quad (11)$$

where $C = c_1 + 2c_2 - 3$. Also by Lemma 2.4, $g \in A_{\alpha}^{\Phi_p}$ and $\|g\|_{A_{\alpha}^{\Phi_p}} \leq C$. Since $DW_{\varphi,\psi} : A_{\alpha}^{\Phi_p} \rightarrow \mathcal{B}^{\Psi}$ is bounded, we have

$$\mu(z)|(DW_{\varphi,\psi}g)'(z)| \leq C\|DW_{\varphi,\psi}\|, \quad (12)$$

for all $z \in \mathbb{D}$. By (10) and (11), letting $z = w$ in (12) gives

$$J(w) := \frac{\mu(w)|\varphi(w)|}{1 - |\varphi(w)|^2} |\psi(w)\varphi''(w) + 2\psi'(w)\varphi'(w)| \Phi_p^{-1} \left(\left(\frac{D_1}{1 - |\varphi(w)|^2} \right)^{\alpha+2} \right) \\ \leq C\|DW_{\varphi,\psi}\|. \quad (13)$$

Hence

$$\sup_{z \in \mathbb{D}} J(z) \leq C\|DW_{\varphi,\psi}\|. \quad (14)$$

For the fixed $\delta \in (0, 1)$, by (8)

$$\sup_{\{z \in \mathbb{D} : |\varphi(z)| \leq \delta\}} \frac{\mu(z)}{1 - |\varphi(z)|^2} |\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)| \Phi_p^{-1} \left(\left(\frac{D_1}{1 - |\varphi(z)|^2} \right)^{\alpha+2} \right) \\ \leq \frac{J_1}{1 - \delta^2} \Phi_p^{-1} \left(\left(\frac{D_1}{1 - \delta^2} \right)^{\alpha+2} \right) \leq C\|DW_{\varphi,\psi}\|, \quad (15)$$

and by (14)

$$\begin{aligned} & \sup_{\{z \in \mathbb{D}: |\varphi(z)| > \delta\}} \frac{\mu(z)}{1 - |\varphi(z)|^2} |\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)| \Phi_p^{-1} \left(\left(\frac{D_1}{1 - |\varphi(z)|^2} \right)^{\alpha+2} \right) \\ & \leq \frac{1}{\delta} \sup_{z \in \mathbb{D}} J(z) \leq C \|DW_{\varphi, \psi}\|. \end{aligned} \quad (16)$$

Consequently, it follows from (15) and (16) that

$$M_2 \leq C \|DW_{\varphi, \psi}\| < \infty. \quad (17)$$

Now we prove that $M_3 < \infty$. First taking the function $f(z) = z^2$, we have

$$\begin{aligned} & \sup_{z \in \mathbb{D}} \mu(z) |\psi''(z)\varphi(z)^2 + 4\psi'(z)\varphi'(z)\varphi(z) + 2\psi(z)\varphi''(z)\varphi(z) + 2\psi(z)\varphi'(z)^2| \\ & \leq \|DW_{\varphi, \psi} z^2\|_{\mathcal{B}^\Psi} \leq C \|DW_{\varphi, \psi}\| \end{aligned} \quad (18)$$

By (7) and the boundedness of φ , we obtain

$$\sup_{z \in \mathbb{D}} \mu(z) |\psi''(z)| |\varphi(z)|^2 \leq C \|DW_{\varphi, \psi}\|. \quad (19)$$

From (8), (18), (19) and the boundedness of φ , it follows that

$$J_2 := \sup_{z \in \mathbb{D}} \mu(z) |\psi(z)| |\varphi'(z)|^2 \leq C \|DW_{\varphi, \psi}\|. \quad (20)$$

For $w \in \mathbb{D}$, consider the function

$$\begin{aligned} f_{3, \varphi(w)}(z) &= c_0 \left(\frac{1 - |\varphi(w)|^2}{1 - \overline{\varphi(w)}z} \right)^{\frac{2(\alpha+2)}{p}} + c_1 \left(\frac{1 - |\varphi(w)|^2}{1 - \overline{\varphi(w)}z} \right)^{\frac{2(\alpha+2)}{p}+1} \\ &+ c_2 \left(\frac{1 - |\varphi(w)|^2}{1 - \overline{\varphi(w)}z} \right)^{\frac{2(\alpha+2)}{p}+2} - \left(\frac{1 - |\varphi(w)|^2}{1 - \overline{\varphi(w)}z} \right)^{\frac{2(\alpha+2)}{p}+3}, \end{aligned}$$

where

$$c_0 = \frac{2(\alpha+2)+p}{2(\alpha+2)+2p}, \quad c_1 = -\frac{3(\alpha+2)+4p}{\alpha+2+p}, \quad c_2 = \frac{6(\alpha+2)+7p}{2(\alpha+2)+2p}.$$

For the function $f_{3, \varphi(w)}$, we have

$$f_{3, \varphi(w)}(\varphi(w)) = f'_{3, \varphi(w)}(\varphi(w)) = 0. \quad (21)$$

For the function

$$h(z) = \Phi_p^{-1} \left(\left(\frac{D_2}{1 - |\varphi(w)|^2} \right)^{\alpha+2} \right) f_{3, \varphi(w)}(z),$$

it follows from (21) that

$$h(\varphi(w)) = h'(\varphi(w)) = 0. \quad (22)$$

By (21) and (22), the boundedness of the operator $DW_{\varphi, \psi} : A_{\alpha}^{\Phi_p} \rightarrow \mathcal{B}^\Psi$ gives

$$K(w) := \frac{\mu(w) |\varphi(w)|^2}{(1 - |\varphi(w)|^2)^2} |\psi(w)| |\varphi'(w)|^2 \Phi_p^{-1} \left(\left(\frac{D_2}{1 - |\varphi(w)|^2} \right)^{\alpha+2} \right) \leq C \|DW_{\varphi, \psi}\|.$$

This yields

$$\sup_{z \in \mathbb{D}} K(z) \leq C \|DW_{\varphi, \psi}\| < \infty. \quad (23)$$

For the fixed $\delta \in (0, 1)$, by (20) and (23) we respectively obtain

$$\begin{aligned} \sup_{\{z \in \mathbb{D}: |\varphi(z)| \leq \delta\}} \frac{\mu(z)}{(1 - |\varphi(z)|^2)^2} |\psi(z)| |\varphi'(z)|^2 \Phi_p^{-1} \left(\left(\frac{D_2}{1 - |\varphi(z)|^2} \right)^{\alpha+2} \right) \\ \leq \frac{J_2}{(1 - \delta^2)^2} \Phi_p^{-1} \left(\left(\frac{D_2}{1 - \delta^2} \right)^{\alpha+2} \right) \leq C \|DW_{\varphi, \psi}\| \end{aligned} \quad (24)$$

and

$$\begin{aligned} \sup_{\{z \in \mathbb{D}: |\varphi(z)| > \delta\}} \frac{\mu(z)}{(1 - |\varphi(z)|^2)^2} |\psi(z)| |\varphi'(z)|^2 \Phi_p^{-1} \left(\left(\frac{D_2}{1 - |\varphi(z)|^2} \right)^{\alpha+2} \right) \\ \leq \frac{1}{\delta^2} \sup_{z \in \mathbb{D}} K(z) \leq C \|DW_{\varphi, \psi}\|. \end{aligned} \quad (25)$$

So, by (24) and (25) we have

$$M_3 \leq C \|DW_{\varphi, \psi}\| < \infty. \quad (26)$$

(ii) \Rightarrow (i). By Lemmas 2.2 and 2.3, for all $f \in A_{\alpha}^{\Phi_p}$ we have

$$\begin{aligned} \|DW_{\varphi, \psi} f\|_{\mathcal{B}^{\Psi}} &= |(\psi \cdot f \circ \varphi)'(0)| + \sup_{z \in \mathbb{D}} \mu(z) |(\psi \cdot f \circ \varphi)''(z)| \\ &\leq |(\psi \cdot f \circ \varphi)'(0)| + \sup_{z \in \mathbb{D}} \mu(z) |\psi''(z)| |f(\varphi(z))| \\ &\quad + \sup_{z \in \mathbb{D}} \mu(z) |\psi(z) \varphi''(z) + 2\psi'(z) \varphi'(z)| |f'(\varphi(z))| \\ &\quad + \sup_{z \in \mathbb{D}} \mu(z) |\psi(z)| |\varphi'(z)|^2 |f''(\varphi(z))| \\ &\leq C(1 + M_1 + M_2 + M_3) \|f\|_{A_{\alpha}^{\Phi_p}}. \end{aligned} \quad (27)$$

From condition (ii) and (27), it follows that $DW_{\varphi, \psi} : A_{\alpha}^{\Phi_p} \rightarrow \mathcal{B}^{\Psi}$ is bounded.

Suppose that the operator $DW_{\varphi, \psi} : A_{\alpha}^{\Phi_p} \rightarrow \mathcal{B}^{\Psi}$ is nonzero and bounded. Then from the preceding inequalities (5), (17) and (26), we obtain

$$M_1 + M_2 + M_3 \lesssim \|DW_{\varphi, \psi}\|. \quad (28)$$

Since the operator $DW_{\varphi, \psi} : A_{\alpha}^{\Phi_p} \rightarrow \mathcal{B}^{\Psi}$ is nonzero, we have $\|DW_{\varphi, \psi}\| > 0$. From this, we can find a positive constant C such that $1 \leq C \|DW_{\varphi, \psi}\|$. This means that

$$1 \lesssim \|DW_{\varphi, \psi}\|. \quad (29)$$

Hence, combining (28) and (29) gives

$$1 + M_1 + M_2 + M_3 \lesssim \|DW_{\varphi, \psi}\|. \quad (30)$$

From (27), it is clear that

$$\|DW_{\varphi, \psi}\| \lesssim 1 + M_1 + M_2 + M_3. \quad (31)$$

So, from (30) and (31), we obtain the asymptotic expression of $\|DW_{\varphi, \psi}\|$. The proof is finished. \square

Remark 3.1. If $DW_{\varphi, \psi} : A_{\alpha}^{\Phi_p} \rightarrow \mathcal{B}^{\Psi}$ is a zero operator, then is obviously $\|DW_{\varphi, \psi}\| = 0$. Hence, the case is usually excluded from such considerations.

Now we characterize the compactness of operator $DW_{\varphi, \psi} : A_{\alpha}^{\Phi_p} \rightarrow \mathcal{B}^{\Psi}$.

Theorem 3.2. Let $p \geq 1$, $\alpha > -1$, and $\Phi \in \mathfrak{U}^s$ such that $\Phi_p \in \mathfrak{L}_r$. Then the following conditions are equivalent:

- (i) The operator $DW_{\varphi,\psi} : A_{\alpha}^{\Phi_p} \rightarrow \mathcal{B}^{\Psi}$ is compact.
(ii) Functions φ and ψ are such that $\psi' \in \mathcal{B}^{\Psi}$,

$$J_1 := \sup_{z \in \mathbb{D}} \mu(z) |\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)| < \infty,$$

$$J_2 := \sup_{z \in \mathbb{D}} \mu(z) |\psi(z)| |\varphi'(z)|^2 < \infty,$$

$$\lim_{|\varphi(z)| \rightarrow 1^-} \mu(z) |\psi''(z)| \Phi_p^{-1} \left(\left(\frac{4}{1 - |\varphi(z)|^2} \right)^{\alpha+2} \right) = 0,$$

$$\lim_{|\varphi(z)| \rightarrow 1^-} \frac{\mu(z)}{1 - |\varphi(z)|^2} |\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)| \Phi_p^{-1} \left(\left(\frac{D_1}{1 - |\varphi(z)|^2} \right)^{\alpha+2} \right) = 0,$$

and

$$\lim_{|\varphi(z)| \rightarrow 1^-} \frac{\mu(z)}{(1 - |\varphi(z)|^2)^2} |\psi(z)| |\varphi'(z)|^2 \Phi_p^{-1} \left(\left(\frac{D_2}{1 - |\varphi(z)|^2} \right)^{\alpha+2} \right) = 0.$$

Proof. (i) \Rightarrow (ii). Suppose that (i) holds. Then the operator $DW_{\varphi,\psi} : A_{\alpha}^{\Phi_p} \rightarrow \mathcal{B}^{\Psi}$ is bounded. In the proof of Theorem 3.1, we have obtained that $\psi' \in \mathcal{B}^{\Psi}$ and $J_1, J_2 < \infty$.

Next consider a sequence $\{\varphi(z_n)\}_{n \in \mathbb{N}}$ in \mathbb{D} such that $|\varphi(z_n)| \rightarrow 1^-$ as $n \rightarrow \infty$. If such sequence does not exist, then condition (ii) obviously holds. Using this sequence, we define the functions

$$f_n(z) = \Phi_p^{-1} \left(\left(\frac{4}{1 - |\varphi(z_n)|^2} \right)^{\alpha+2} \right) f_{1,\varphi(z_n)}(z).$$

By Lemma 2.4, we know that the sequence $\{f_n\}_{n \in \mathbb{N}}$ is uniformly bounded in $A_{\alpha}^{\Phi_p}$. From the proof of Theorem 3.6 in [30], it follows that the sequence $\{f_n\}_{n \in \mathbb{N}}$ uniformly converges to zero on any compact subset of \mathbb{D} as $n \rightarrow \infty$. Hence by Lemma 2.1,

$$\lim_{n \rightarrow \infty} \|DW_{\varphi,\psi} f_n\|_{\mathcal{B}^{\Psi}} = 0.$$

From this, (2) and (3), we have

$$\lim_{n \rightarrow \infty} \mu(z_n) |\psi''(z_n)| \Phi_p^{-1} \left(\left(\frac{4}{1 - |\varphi(z_n)|^2} \right)^{\alpha+2} \right) = 0.$$

By using the sequence of functions

$$g_n(z) = \Phi_p^{-1} \left(\left(\frac{D_1}{1 - |\varphi(z_n)|^2} \right)^{\alpha+2} \right) f_{2,\varphi(z_n)}(z),$$

similar to the above, we obtain

$$\lim_{n \rightarrow \infty} \frac{\mu(z_n)}{1 - |\varphi(z_n)|^2} |\psi(z_n)\varphi''(z_n) + 2\psi'(z_n)\varphi'(z_n)| \Phi_p^{-1} \left(\left(\frac{D_1}{1 - |\varphi(z_n)|^2} \right)^{\alpha+2} \right) = 0.$$

Also, by using sequence of functions

$$h_n(z) = \Phi_p^{-1} \left(\left(\frac{D_2}{1 - |\varphi(z_n)|^2} \right)^{\alpha+2} \right) f_{3,\varphi(z_n)}(z),$$

we obtain

$$\lim_{n \rightarrow \infty} \frac{\mu(z_n)}{(1 - |\varphi(z_n)|^2)^2} |\psi(z_n)| |\varphi'(z_n)|^2 \Phi_p^{-1} \left(\left(\frac{D_2}{1 - |\varphi(z_n)|^2} \right)^{\alpha+2} \right) = 0.$$

The proof of the implication is finished.

(ii) \Rightarrow (i). We first check that $DW_{\varphi,\psi} : A_{\alpha}^{\Phi_p} \rightarrow \mathcal{B}^{\Psi}$ is bounded. For this we observe that condition (ii) implies that for every $\varepsilon > 0$, there is an $\eta \in (0, 1)$ such that

$$L_1(z) := \mu(z)|\psi''(z)|\Phi_p^{-1}\left(\left(\frac{4}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right) < \varepsilon, \quad (32)$$

$$L_2(z) := \frac{\mu(z)}{1-|\varphi(z)|^2}|\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)|\Phi_p^{-1}\left(\left(\frac{D_1}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right) < \varepsilon, \quad (33)$$

and

$$L_3(z) := \frac{\mu(z)}{(1-|\varphi(z)|^2)^2}|\psi(z)||\varphi'(z)|^2\Phi_p^{-1}\left(\left(\frac{D_2}{1-|\varphi(z)|^2}\right)^{\alpha+2}\right) < \varepsilon, \quad (34)$$

for any $z \in K = \{z \in \mathbb{D} : |\varphi(z)| > \eta\}$. Then since $\psi' \in \mathcal{B}^{\Psi}$ and by (32), we have

$$M_1 = \sup_{z \in \mathbb{D}} L_1(z) \leq \sup_{z \in \mathbb{D} \setminus K} L_1(z) + \sup_{z \in K} L_1(z) \leq \|\psi'\|_{\mathcal{B}^{\Psi}} \Phi_p^{-1}\left(\left(\frac{4}{1-\eta^2}\right)^{\alpha+2}\right) + \varepsilon.$$

By (33) and $J_1 < \infty$, we obtain

$$M_2 = \sup_{z \in \mathbb{D}} L_2(z) \leq \sup_{z \in \mathbb{D} \setminus K} L_2(z) + \sup_{z \in K} L_2(z) \leq \frac{J_1}{1-\eta^2} \Phi_p^{-1}\left(\left(\frac{D_1}{1-\eta^2}\right)^{\alpha+2}\right) + \varepsilon.$$

By (34) and $J_2 < \infty$, it follows that $M_3 < \infty$. So by Theorem 3.1, $DW_{\varphi,\psi} : A_{\alpha}^{\Phi_p} \rightarrow \mathcal{B}^{\Psi}$ is bounded.

To prove that the operator $DW_{\varphi,\psi} : A_{\alpha}^{\Phi_p} \rightarrow \mathcal{B}^{\Psi}$ is compact, by Lemma 2.1 we just need to prove that, if $\{f_n\}_{n \in \mathbb{N}}$ is a sequence in $A_{\alpha}^{\Phi_p}$ such that $\|f_n\|_{A_{\alpha}^{\Phi_p}} \leq M$ and $f_n \rightarrow 0$ uniformly on any compact subset of \mathbb{D} as $n \rightarrow \infty$, then

$$\lim_{n \rightarrow \infty} \|DW_{\varphi,\psi} f_n\|_{\mathcal{B}^{\Psi}} = 0.$$

For any $\varepsilon > 0$ and the above η , we have, by using again the condition (ii), Lemma 2.2 and Lemma 2.3,

$$\begin{aligned} \sup_{z \in \mathbb{D}} \mu(z) |(DW_{\varphi,\psi} f_n)'(z)| &= \sup_{z \in \mathbb{D}} \mu(z) |(\psi \cdot f_n \circ \varphi)''(z)| \leq \sup_{z \in \mathbb{D}} \mu(z) |\psi''(z)| |f_n(\varphi(z))| \\ &\quad + \sup_{z \in \mathbb{D}} \mu(z) |\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)| |f_n'(\varphi(z))| + \sup_{z \in \mathbb{D}} \mu(z) |\psi(z)||\varphi'(z)|^2 |f_n''(\varphi(z))| \\ &\leq \sup_{z \in \mathbb{D} \setminus K} \mu(z) |\psi''(z)| |f_n(\varphi(z))| + \sup_{z \in K} \mu(z) |\psi''(z)| |f_n(\varphi(z))| \\ &\quad + \sup_{z \in \mathbb{D} \setminus K} \mu(z) |\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)| |f_n'(\varphi(z))| \\ &\quad + \sup_{z \in K} \mu(z) |\psi(z)\varphi''(z) + 2\psi'(z)\varphi'(z)| |f_n'(\varphi(z))| \\ &\quad + \sup_{z \in \mathbb{D} \setminus K} \mu(z) |\psi(z)||\varphi'(z)|^2 |f_n''(\varphi(z))| + \sup_{z \in K} \mu(z) |\psi(z)||\varphi'(z)|^2 |f_n''(\varphi(z))| \\ &\leq K_n + M \sup_{z \in K} L_1(z) + M \sup_{z \in K} L_2(z) + M \sup_{z \in K} L_3(z) \\ &\leq K_n + 3M\varepsilon, \end{aligned}$$

where

$$K_n = \|\psi'\|_{\mathcal{B}^{\Psi}} \sup_{\{z: |z| \leq \eta\}} |f_n(z)| + \sum_{i=1}^2 J_i \sup_{\{z: |z| \leq \eta\}} |f_n^{(i)}(z)|.$$

Hence

$$\begin{aligned}\|DW_{\varphi,\psi}f_n\|_{\mathcal{B}^\Psi} &\leq K_n + 3M\varepsilon + |(\psi \cdot f_n \circ \varphi)'(0)| \\ &= K_n + 3M\varepsilon + |\psi'(0)f_n(\varphi(0)) + \psi(0)f'_n(\varphi(0))\varphi'(0)|.\end{aligned}\quad (35)$$

It is easy to see that, when $\{f_n\}_{n \in \mathbb{N}}$ uniformly converges to zero on any compact subset of \mathbb{D} , $\{f'_n\}_{n \in \mathbb{N}}$ and $\{f''_n\}_{n \in \mathbb{N}}$ also do as $n \rightarrow \infty$. From this, we obtain $K_n \rightarrow 0$ as $n \rightarrow \infty$. Since $\{z : |z| \leq \eta\}$ and $\{\varphi(0)\}$ are compact subsets of \mathbb{D} , letting $n \rightarrow \infty$ in (35) gives

$$\lim_{n \rightarrow \infty} \|DW_{\varphi,\psi}f_n\|_{\mathcal{B}^\Psi} = 0.$$

From Lemma 2.1, it follows that the operator $DW_{\varphi,\psi} : A_{\alpha}^{\Phi_p} \rightarrow \mathcal{B}^\Psi$ is compact. The proof is finished. \square

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Lyapunov inequalities of linear Hamiltonian systems on time scales

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Abstract In this paper, we establish several Lyapunov-type inequalities for the following linear Hamiltonian systems

$$x^\Delta(t) = -A(t)x(\sigma(t)) - B(t)y(t), \quad y^\Delta(t) = C(t)x(\sigma(t)) + A^T(t)y(t)$$

on the time scale interval $[a, b]_{\mathbb{T}} \equiv [a, b] \cap \mathbb{T}$ for some $a, b \in \mathbb{T}$, where B and C are real $n \times n$ symmetric matrix-valued functions on $[a, b]_{\mathbb{T}}$ with B being semi-positive definite, A is real $n \times n$ matrix-valued function on $[a, b]_{\mathbb{T}}$ with $I + \mu(t)A$ being invertible, and x, y are real vector-valued functions on $[a, b]_{\mathbb{T}}$.

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1. Introduction

In 1990, Hilger introduced in [9] the theory of time scales with one goal being the unified treatment of differential equations (the continuous case) and difference equations (the discrete case). A time scale \mathbb{T} is an arbitrary nonempty closed subset of the real numbers \mathbb{R} , which has the topology that it inherits from the standard topology on \mathbb{R} . The two most popular examples are \mathbb{R} and the integers \mathbb{Z} . The study of dynamic equations on time scales reveals such discrepancies, and helps avoid proving results twice-once for differential equations and once again for difference equations. Not only can the theory of dynamic equations unify the theories of differential equations and difference equations, but also extends these classical cases to cases “in between”, e.g., to the so-called q -difference equations when $\mathbb{T} = \{1, q, q^2, \dots, q^n, \dots\}$, which has important applications in quantum theory (see [11]). For the time scale calculus, and some related basic concepts, and the basic notions connected to time scales, we refer the readers to the books by Bohner and Peterson [2,3] for further details.

In this paper, we study Lyapunov-type inequalities for the following linear Hamiltonian

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systems

$$x^\Delta(t) = -A(t)x(\sigma(t)) - B(t)y(t), \quad y^\Delta(t) = C(t)x(\sigma(t)) + A^T(t)y(t), \quad (1.1)$$

on the time scale interval $[a, b]_{\mathbb{T}} \equiv [a, b] \cap \mathbb{T}$ for some $a, b \in \mathbb{T}$, where B and C are real $n \times n$ symmetric matrix-valued functions on $[a, b]_{\mathbb{T}}$ with B being semi-positive definite, A is real $n \times n$ matrix-valued function on $[a, b]_{\mathbb{T}}$ with $I + \mu(t)A$ being invertible, and x, y are real vector-valued functions on $[a, b]_{\mathbb{T}}$.

When $n = 1$, (1.1) reduces to

$$x^\Delta(t) = \alpha(t)x(\sigma(t)) + \beta(t)y(t), \quad y^\Delta(t) = -\gamma(t)x(\sigma(t)) - \alpha(t)y(t) \quad (1.2)$$

on an arbitrary time scale \mathbb{T} , where $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are real-valued rd-continuous functions defined on \mathbb{T} with $\beta(t) \geq 0$ for any $t \in \mathbb{T}$.

In [10], Jiang and Zhou obtained some interesting Lyapunov-type inequalities.

Theorem 1.1^[10] Suppose that for any $t \in \mathbb{T}$,

$$1 - \mu(t)\alpha(t) > 0, \quad \beta(t) > 0, \quad \gamma(t) > 0,$$

and let $a, b \in \mathbb{T}^k$ with $\sigma(a) < b$. Assume that (1.2) has a real solution $(x(t), y(t))$ such that $x(a)x(\sigma(a)) < 0$, and $x(b)x(\sigma(b)) < 0$. Then the inequality

$$\int_a^b |\alpha(t)| \Delta(t) + \left[\int_a^{\sigma(b)} \beta(t) \Delta(t) \int_a^b \gamma(t) \Delta(t) \right]^{1/2} > 1 \quad (1.3)$$

holds.

Theorem 1.2^[10] Suppose that for any $t \in \mathbb{T}$,

$$1 - \mu(t)\alpha(t) > 0, \quad \beta(t) > 0,$$

and let $a, b \in \mathbb{T}^k$ with $\sigma(a) < b$. Assume that (1.2) has a real solution $(x(t), y(t))$ such that $x(a)x(\sigma(a)) < 0$, and $x(\sigma(b)) = 0$. Then the inequality

$$\int_{\sigma(a)}^b |\alpha(t)| \Delta(t) + \left[\int_{\sigma(a)}^{\sigma(b)} \beta(t) \Delta(t) \int_a^b \gamma^+(t) \Delta(t) \right]^{1/2} > 1 \quad (1.4)$$

holds, where $\gamma^+(t) = \max\{\gamma(t), 0\}$.

In [8], He et al. obtained the following Lyapunov-type inequality.

Theorem 1.3^[8] Suppose for any $t \in \mathbb{T}$,

$$1 - \mu(t)\alpha(t) > 0,$$

and let $a, b \in \mathbb{T}^k$ with $\sigma(a) \leq b$. Assume that (1.2) has a real solution $(x(t), y(t))$ such that $x(t)$ has generalized zeros at end-points a and b and $x(t)$ is not identically zero on $[a, b]_{\mathbb{T}} \equiv \{t \in \mathbb{T} : a \leq t \leq b\}$, i.e.,

$$x(a) = 0 \text{ or } x(a)x(\sigma(a)) < 0; \quad x(b) = 0 \text{ or } x(b)x(\sigma(b)) < 0; \quad \max_{t \in [a, b]_{\mathbb{T}}} |x(t)| > 0.$$

Then the inequality

$$\int_a^b |\alpha(t)| \Delta(t) + \left[\int_a^{\sigma(b)} \beta(t) \Delta(t) \int_a^b \gamma^+(t) \Delta(t) \right]^{1/2} \geq 2 \quad (1.5)$$

holds.

For some other related results on Lyapunov-type inequality, see, for example, [1,4-6,8,10,12-16].

2. Preliminaries and some lemmas

For any $x \in \mathbb{R}^n$ and any $A \in \mathbb{R}^{n \times n}$ (the space of real $n \times n$ matrices), denote by

$$|x| = \sqrt{x^T x} \quad \text{and} \quad |A| = \max_{x \in \mathbb{R}^n, |x|=1} |Ax|$$

the Euclidean norm of x and the matrix norm of A respectively, where C^T is the transpose of a $n \times m$ matrix C . It is easy to show

$$|Ax| \leq |A||x|$$

for any $x \in \mathbb{R}^n$ and any $A \in \mathbb{R}^{n \times n}$. Denote by $\mathbb{R}_s^{n \times n}$ the space of all symmetric real $n \times n$ matrices. For $A \in \mathbb{R}_s^{n \times n}$, we say that A is semi-positive definite (resp. positive definite), written as $A \geq 0$ (resp. $A > 0$), if $x^T A x \geq 0$ (resp. $x^T A x > 0$) for all $x \in \mathbb{R}^n$. If A is semi-positive definite (resp. positive definite), then there exists a unique semi-positive definite matrix (resp. positive definite matrix), written as \sqrt{A} , such that $[\sqrt{A}]^2 = A$.

In this paper, we study Lyapunov-type inequalities of (1.1) which admits some solution $(x(t), y(t))$ satisfying

$$x(a) = x(b) = 0 \quad \text{and} \quad \max_{t \in [a, b]_{\mathbb{T}}} |x(t)| > 0, \quad (2.1)$$

where $a, b \in \mathbb{T}$ with $\sigma(a) < b$, $A, B, C \in C_{rd}(\mathbb{T}, \mathbb{R}^{n \times n})$ are $n \times n$ -matrix-valued functions on \mathbb{T} with $I + \mu(t)A$ being invertible, $B, C \in \mathbb{R}_s^{n \times n}$ and $B \geq 0$. we first introduce the following notions and lemmas.

A partition of $[a, b]_{\mathbb{T}}$ is any finite ordered subset $P = \{t_0, t_1, \dots, t_n\} \subset [a, b]_{\mathbb{T}}$ with $a = t_0 < t_1 < \dots < t_n = b$. For given $\delta > 0$, we denote by $\mathcal{P}_\delta([a, b]_{\mathbb{T}})$ the set of all partitions $P : a = t_0 < t_1 < \dots < t_n = b$ that possess the property: for every $i \in \{1, 2, \dots, n\}$, either $t_i - t_{i-1} \leq \delta$ or $t_i - t_{i-1} > \delta$ and $\sigma(t_i) = t_{i-1}$.

Definition 2.1^[7] Let f be a bounded function on $[a, b]_{\mathbb{T}}$, and let $P : a = t_0 < t_1 < \dots < t_n = b$ be a partition of $[a, b]_{\mathbb{T}}$. In each interval $[t_{i-1}, t_i]_{\mathbb{T}}$ ($1 \leq i \leq n$), choose an arbitrary point ξ_i and form the sum

$$S(P, f) = \sum_{i=1}^n f(\xi_i)(t_i - t_{i-1}).$$

We say that f is Δ -integrable from a to b (or on $[a, b]_{\mathbb{T}}$) if there exists a constant number I with the following property: for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|S(P, f) - I| < \varepsilon$$

for every $P \in \mathcal{P}_\delta([a, b]_\mathbb{T})$ independent of the way in which we choose $\xi_i \in [t_{i-1}, t_i]_\mathbb{T}$ ($1 \leq i \leq n$).

It is easily seen that such a constant number I is unique. The number I , written as $\int_a^b f(t) \Delta t$, is called the Δ -integral of f from a to b .

Remark 2.2 In [7], Guseinov showed that if there exists $F : \mathbb{T} \rightarrow \mathbb{R}$ such that $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^k$, then

$$\int_a^b f(t) \Delta t = F(b) - F(a), \text{ for any } a, b \in \mathbb{T}.$$

Lemma 2.3 Let $a_i, b_i, c_i \in \mathbb{R}$ ($i \in \{1, 2, \dots, n\}$) with $c_i \geq 0$. Then

$$\left(\sum_{i=1}^n a_i c_i \right)^2 + \left(\sum_{i=1}^n b_i c_i \right)^2 \leq \left[\sum_{i=1}^n \sqrt{a_i^2 + b_i^2} c_i \right]^2. \quad (2.2)$$

Proof. Since $2a_i b_i a_j b_j \leq b_i^2 a_j^2 + b_j^2 a_i^2$ for any $i, j \in \{1, 2, \dots, n\}$, we have

$$a_i c_i a_j c_j + b_i c_i b_j c_j \leq \sqrt{a_i^2 + b_i^2} c_i \sqrt{a_j^2 + b_j^2} c_j,$$

which implies

$$\sum_{i=1}^n \sum_{j=1}^n (a_i c_i a_j c_j + b_i c_i b_j c_j) \leq \sum_{i=1}^n \sum_{j=1}^n \sqrt{a_i^2 + b_i^2} c_i \sqrt{a_j^2 + b_j^2} c_j.$$

That is

$$\left(\sum_{i=1}^n a_i c_i \right)^2 + \left(\sum_{i=1}^n b_i c_i \right)^2 \leq \left[\sum_{i=1}^n \sqrt{a_i^2 + b_i^2} c_i \right]^2.$$

This completes the proof of Lemma 2.3

Lemma 2.4 Let $f, g, f^2 + g^2$ be Δ -integrable from a to b . Then

$$\left[\int_a^b f(t) \Delta t \right]^2 + \left[\int_a^b g(t) \Delta t \right]^2 \leq \left[\int_a^b \sqrt{f^2(t) + g^2(t)} \Delta t \right]^2. \quad (2.3)$$

Proof. By Definition 2.1, for any $\varepsilon > 0$ there exists $\delta_i > 0$ ($i = 1, 2, 3$) such that

$$|S(P_1, f) - \int_a^b f(t) \Delta t| < \varepsilon, \quad (2.4)$$

$$|S(P_2, g) - \int_a^b g(t) \Delta t| < \varepsilon \quad (2.5)$$

and

$$|S(P_3, \sqrt{f^2(t) + g^2(t)}) - \int_a^b \sqrt{f^2(t) + g^2(t)} \Delta t| < \varepsilon \quad (2.6)$$

for every $P_i \in \mathcal{P}_{\delta_i}([a, b]_\mathbb{T})$. Let $P = P_1 \cup P_2 \cup P_3$ ($\in \cap_{i=1}^3 \mathcal{P}_{\delta_i}([a, b]_\mathbb{T})$) : $a = t_0 < t_1 < \dots < t_n = b$

and choose an arbitrary point $\xi_i \in [t_{i-1}, t_i]$. Then from (2.4)-(2.6) and Lemma 2.3 we have

$$\begin{aligned}
 \left[\int_a^b f(t) \Delta t \right]^2 + \left[\int_a^b g(t) \Delta t \right]^2 &\leq [|S(P, f)| + \varepsilon]^2 + [|S(P, g)| + \varepsilon]^2 \\
 &= [|\sum_{i=1}^n f(\xi_i)(t_i - t_{i-1})| + \varepsilon]^2 + [|\sum_{i=1}^n g(\xi_i)(t_i - t_{i-1})| + \varepsilon]^2 \\
 &\leq [\sum_{i=1}^n f(\xi_i)(t_i - t_{i-1})]^2 + [\sum_{i=1}^n g(\xi_i)(t_i - t_{i-1})]^2 \\
 &\quad + 2\varepsilon \left[\left| \int_a^b f(t) \Delta t \right| + \left| \int_a^b g(t) \Delta t \right| + 3\varepsilon \right] \\
 &\leq [\sum_{i=1}^n \sqrt{f^2(\xi_i) + g^2(\xi_i)}(t_i - t_{i-1})]^2 \\
 &\quad + 2\varepsilon \left[\left| \int_a^b f(t) \Delta t \right| + \left| \int_a^b g(t) \Delta t \right| + 3\varepsilon \right] \\
 &\leq \left[\int_a^b \sqrt{f^2(t) + g^2(t)} \Delta t + \varepsilon \right]^2 \\
 &\quad + 2\varepsilon \left[\left| \int_a^b f(t) \Delta t \right| + \left| \int_a^b g(t) \Delta t \right| + 3\varepsilon \right].
 \end{aligned}$$

Let $\varepsilon \rightarrow 0$, we obtain (2.3). This completes the proof of Lemma 2.4.

Corollary 2.5 Let $a, b \in \mathbb{T}$ with $a < b$ and $f_1(t), f_2(t), \dots, f_n(t)$ be Δ -integrable on $[a, b]_{\mathbb{T}}$. write $x(t) = (f_1(t), f_2(t), \dots, f_n(t))$. Then

$$\left| \int_a^b x(t) \Delta t \right| = \left\{ \sum_{i=1}^n \left(\int_a^b f_i(t) \Delta t \right)^2 \right\}^{\frac{1}{2}} \leq \int_a^b \left\{ \sum_{i=1}^n f_i^2(t) \right\}^{\frac{1}{2}} \Delta t = \int_a^b |x(t)| \Delta t. \quad (2.7)$$

Proof. By Lemma 2.4, we know that (2.7) holds when $n = 2$. Assume that (2.7) holds when $n = k \geq 2$, that is

$$\sum_{i=1}^k \left(\int_a^b f_i(t) \Delta t \right)^2 \leq \left[\int_a^b \left\{ \sum_{i=1}^k f_i^2(t) \right\}^{\frac{1}{2}} \Delta t \right]^2.$$

Then

$$\begin{aligned}
 \left[\int_a^b \left\{ \sum_{i=1}^{k+1} f_i^2(t) \right\}^{\frac{1}{2}} \Delta t \right]^2 &= \left\{ \int_a^b \left\{ f_{k+1}^2(t) + \left[\left(\sum_{i=1}^k f_i^2(t) \right)^{\frac{1}{2}} \right]^2 \right\}^{\frac{1}{2}} \Delta t \right\}^2 \\
 &\geq \left(\int_a^b f_{k+1}(t) \Delta t \right)^2 + \left[\int_a^b \left\{ \sum_{i=1}^k f_i^2(t) \right\}^{\frac{1}{2}} \Delta t \right]^2 \\
 &\geq \sum_{i=1}^{k+1} \left(\int_a^b f_i(t) \Delta t \right)^2.
 \end{aligned}$$

This completes the proof of Corollary 2.5.

Lemma 2.6^[2] (Cauchy-Schwarz inequality) Let $a, b \in \mathbb{T}$ and $f, g \in C_{rd}(\mathbb{T}, \mathbb{R})$. Then

$$\int_a^b |f(t)g(t)| \Delta(t) \leq \left\{ \int_a^b f^2(t) \Delta(t) \cdot \int_a^b g^2(t) \Delta(t) \right\}^{\frac{1}{2}}. \quad (2.8)$$

Lemma 2.7^[2] Suppose that $A \in C_{rd}(\mathbb{T}, \mathbb{R}^{n \times n})$ with $I + \mu(t)A$ being invertible and $f \in C_{rd}(\mathbb{T}, \mathbb{R}^n)$. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$. Then the initial value problem

$$x^\Delta(t) = -A(t)x(\sigma(t)) + f(t), \quad x(t_0) = x_0$$

has a unique solution $x : \mathbb{T} \rightarrow \mathbb{R}^n$. Moreover, this solution is given by

$$x(t) = e_{\Theta A}(t, t_0)x_0 + \int_{t_0}^t e_{\Theta A}(t, \tau)f(\tau)\Delta\tau. \quad (2.9)$$

Lemma 2.8 Let $C \in \mathbb{R}_s^{n \times n}$. Then for any $C_1 \in \mathbb{R}_s^{n \times n}$ with $C_1 \geq C$ (i.e., $C_1 - C \geq 0$), we have

$$(x^\sigma)^T C x^\sigma \leq |C_1| |x^\sigma|^2, x \in \mathbb{R}^n. \quad (2.10)$$

Proof. For $C, C_1 \in \mathbb{R}_s^{n \times n}$ with $C_1 \geq C$, we have $C_1 - C \geq 0$. Then for all $x \in \mathbb{R}^n$, we obtain $(x^\sigma)^T (C_1 - C)x^\sigma \geq 0$. Thus

$$\begin{aligned} (x^\sigma)^T C x^\sigma &\leq (x^\sigma)^T C_1 x^\sigma \leq |x^\sigma| |C_1 x^\sigma| \\ &\leq |x^\sigma| |C_1| |x^\sigma| = |C_1| |x^\sigma|^2. \end{aligned}$$

This completes the proof of Lemma 2.8.

3. Main results and proofs

Denote

$$\xi(\sigma(t)) = \int_a^{\sigma(t)} |B(s)| |e_{\Theta A}(\sigma(t), s)|^2 \Delta s \quad (3.1)$$

and

$$\eta(\sigma(t)) = \int_{\sigma(t)}^b |B(s)| |e_{\Theta A}(\sigma(t), s)|^2 \Delta s. \quad (3.2)$$

Theorem 3.1 Let $a, b \in \mathbb{T}$ with $\sigma(a) < b$. If (1.1) has a solution $(x(t), y(t))$ satisfying (2.1) on the interval $[a, b]_{\mathbb{T}}$, then for any $C_1 \in \mathbb{R}_s^{n \times n}$ with $C_1(t) \geq C(t)$, one has the following inequality

$$\int_a^b \frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t)) + \eta(\sigma(t))} |C_1(t)| \Delta t \geq 1. \quad (3.3)$$

Proof. At first let us notice that any solution $(x(t), y(t))$ of (1.1) satisfies the following equality

$$\begin{aligned} (y^T(t)x(t))^\Delta &= (y^T(t))^\Delta x^\sigma(t) + y^T(t)x^\Delta(t) \\ &= (x^\sigma(t))^T y^\Delta(t) + y^T(t)x^\Delta(t) \\ &= (x^\sigma(t))^T C(t)x^\sigma(t) - y^T(t)B(t)y(t). \end{aligned} \quad (3.4)$$

By integrating (3.4) from a to b and taking into account that $x(a) = x(b) = 0$, one has

$$\int_a^b y^T(t)B(t)y(t) \Delta t = \int_a^b (x^\sigma(t))^T C(t)x^\sigma(t) \Delta t.$$

Moreover, since $B(t)$ is semi-positive definite, we have

$$y^T(t)B(t)y(t) \geq 0, \quad t \in [a, b]_{\mathbb{T}}.$$

If

$$y^T(t)B(t)y(t) \equiv 0, \quad t \in [a, b]_{\mathbb{T}}$$

then

$$B(t)y(t) = 0.$$

Thus the first equation of (1.1) would read as

$$x^\Delta(t) = -A(t)x(\sigma(t)), \quad x(a) = 0.$$

By Lemma 2.7, it follows

$$x(t) = e_{\Theta A}(t, a) \cdot 0 = 0,$$

a contradiction with (2.1). Hence we have that

$$\int_a^b y^T(t)B(t)y(t) \Delta t = \int_a^b (x^\sigma)^T(t)C(t)x^\sigma(t) \Delta t > 0, \quad (3.5)$$

and for $t \in [a, b]_{\mathbb{T}}$, let $t_0 = a$ and $t_0 = b$, from Lemma 2.7, we obtain

$$x(t) = - \int_a^t e_{\Theta A}(t, \tau)B(\tau)y(\tau)\Delta\tau = - \int_b^t e_{\Theta A}(t, \tau)B(\tau)y(\tau)\Delta\tau. \quad (3.6)$$

Which follows that for $t \in [a, b)_{\mathbb{T}}$,

$$x^\sigma(t) = - \int_a^{\sigma(t)} e_{\Theta A}(\sigma(t), \tau)B(\tau)y(\tau)\Delta\tau = + \int_{\sigma(t)}^b e_{\Theta A}(\sigma(t), \tau)B(\tau)y(\tau)\Delta\tau. \quad (3.7)$$

Note that for $a \leq \tau \leq \sigma(t) \leq b$,

$$\begin{aligned} |e_{\Theta A}(\sigma(t), \tau)B(\tau)y(\tau)| &\leq |e_{\Theta A}(\sigma(t), \tau)||B(\tau)y(\tau)| \\ &= |e_{\Theta A}(\sigma(t), \tau)|\{y^T(\tau)B^T(\tau)B(\tau)y(\tau)\}^{\frac{1}{2}} \\ &= |e_{\Theta A}(\sigma(t), \tau)|\{(\sqrt{B(\tau)}y(\tau))^T B(\tau)\sqrt{B(\tau)}y(\tau)\}^{\frac{1}{2}} \\ &\leq |e_{\Theta A}(\sigma(t), \tau)|\{\|\sqrt{B(\tau)}y(\tau)\| \|B(\tau)\| \|\sqrt{B(\tau)}y(\tau)\|\}^{\frac{1}{2}} \\ &= |e_{\Theta A}(\sigma(t), \tau)| \|B(\tau)\|^{\frac{1}{2}} (y^T(\tau)B(\tau)y(\tau))^{\frac{1}{2}}. \end{aligned}$$

Then from Corollary 2.5 and Lemma 2.6 we obtain

$$\begin{aligned} |x^\sigma(t)| &= \left| \int_a^{\sigma(t)} e_{\Theta A}(\sigma(t), \tau)B(\tau)y(\tau)\Delta\tau \right| \\ &\leq \int_a^{\sigma(t)} |e_{\Theta A}(\sigma(t), \tau)B(\tau)y(\tau)|\Delta\tau \\ &\leq \int_a^{\sigma(t)} |e_{\Theta A}(\sigma(t), \tau)| \|B(\tau)\|^{\frac{1}{2}} (y^T(\tau)B(\tau)y(\tau))^{\frac{1}{2}} \Delta\tau \\ &\leq \left(\int_a^{\sigma(t)} |e_{\Theta A}(\sigma(t), \tau)|^2 \|B(\tau)\| \Delta\tau \right)^{\frac{1}{2}} \left(\int_a^{\sigma(t)} y^T(\tau)B(\tau)y(\tau)\Delta\tau \right)^{\frac{1}{2}}, \end{aligned}$$

that is

$$|x^\sigma(t)|^2 \leq \xi(\sigma(t)) \int_a^{\sigma(t)} y^T(\tau)B(\tau)y(\tau)\Delta\tau. \quad (3.8)$$

Similarly, by letting $\eta(\sigma(t))$ be as in (3.2), for $a \leq \sigma(t) \leq \tau \leq b$, we have

$$|x^\sigma(t)|^2 \leq \eta(\sigma(t)) \int_{\sigma(t)}^b y^T(\tau)B(\tau)y(\tau)\Delta\tau. \quad (3.9)$$

It follows from (3.8) and (3.9) that

$$\eta(\sigma(t))\xi(\sigma(t)) \int_a^{\sigma(t)} y^T(\tau)B(\tau)y(\tau)\Delta\tau \geq |x^\sigma(t)|^2\eta(\sigma(t))$$

and

$$\eta(\sigma(t))\xi(\sigma(t)) \int_{\sigma(t)}^b y^T(\tau)B(\tau)y(\tau)\Delta\tau \geq |x^\sigma(t)|^2\xi(\sigma(t)).$$

Thus

$$|x^\sigma(t)|^2 \leq \frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t)) + \eta(\sigma(t))} \int_a^b y^T(\tau)B(\tau)y(\tau)\Delta\tau.$$

By Lemma 2.8 we see

$$\begin{aligned} \int_a^b |C_1(t)||x^\sigma(t)|^2\Delta t &\leq \int_a^b (|C_1(t)| \frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t)) + \eta(\sigma(t))} \int_a^b y^T(\tau)B(\tau)y(\tau)\Delta\tau) \Delta t \\ &= \int_a^b |C_1(t)| \frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t)) + \eta(\sigma(t))} \Delta t \int_a^b y^T(\tau)B(\tau)y(\tau)\Delta\tau \\ &= \int_a^b |C_1(t)| \frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t)) + \eta(\sigma(t))} \Delta t \int_a^b (x^\sigma(t))^T C(t) x^\sigma(t) \Delta t \\ &\leq \int_a^b |C_1(t)| \frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t)) + \eta(\sigma(t))} \Delta t \int_a^b |C_1(t)||x^\sigma(t)|^2 \Delta t. \end{aligned}$$

Since

$$\int_a^b |C_1(t)||x^\sigma(t)|^2 \Delta t \geq \int_a^b (x^\sigma)^T(t)C(t)x^\sigma(t) \Delta t = \int_a^b y^T(t)B(t)y(t)\Delta t > 0,$$

we get

$$\int_a^b \frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t)) + \eta(\sigma(t))} |C_1(t)| \Delta t \geq 1.$$

This completes the proof of Theorem 3.1.

Theorem 3.2 Let $a, b \in \mathbb{T}$ with $\sigma(a) < b$. If (1.1) has a solution $(x(t), y(t))$ satisfying (2.1) on the interval $[a, b]_{\mathbb{T}}$, then for any $C_1 \in \mathbb{R}_s^{n \times n}$ with $C_1(t) \geq C(t)$, one has the following inequality

$$\int_a^b |C_1(t)| \left\{ \int_a^b |B(s)||e_{\Theta A}(\sigma(t), s)|^2 \Delta s \right\} \Delta t \geq 4. \quad (3.10)$$

Proof. Note

$$\frac{\xi(\sigma(t))\eta(\sigma(t))}{\xi(\sigma(t)) + \eta(\sigma(t))} \leq \frac{\xi(\sigma(t)) + \eta(\sigma(t))}{4}.$$

It follows from (3.3) that

$$\int_a^b \frac{\xi(\sigma(t)) + \eta(\sigma(t))}{4} |C_1(t)| \Delta t \geq 1.$$

Combining (3.1) and (3.2), we obtain

$$\int_a^b \left(\int_a^b |B(s)||e_{\Theta A}(\sigma(t), s)|^2 \Delta s |C_1(t)| \right) \Delta t \geq 4.$$

That is

$$\int_a^b |C_1(t)| \left\{ \int_a^b |B(s)| |e_{\Theta A}(\sigma(t), s)|^2 \Delta s \right\} \Delta t \geq 4.$$

This completes the proof of Theorem 3.2.

Theorem 3.3 Let $a, b \in \mathbb{T}$ with $\sigma(a) < b$. If (1.1) has a solution $(x(t), y(t))$ satisfying (2.1) on the interval $[a, b]_{\mathbb{T}}$, then for any $C_1 \in \mathbb{R}_s^{n \times n}$ with $C_1(t) \geq C(t)$, one has the following inequality

$$\int_a^b |A(t)| \Delta t + \left(\int_a^b |\sqrt{B(t)}|^2 \Delta t \right)^{1/2} \left(\int_a^b |C_1(t)| \Delta t \right)^{1/2} \geq 2. \quad (3.11)$$

Proof. From the proof of Theorem 3.1, we have

$$\int_a^b y^T(t) B(t) y(t) \Delta t = \int_a^b (x^\sigma(t))^T C(t) x^\sigma(t) \Delta t.$$

It follows from the first equation of (1.1) that for all $a \leq t \leq b$, we get

$$\begin{aligned} x(t) &= \int_a^t (-A(\tau) x^\sigma(\tau) - B(\tau) y(\tau)) \Delta \tau \\ x(t) &= \int_t^b (A(\tau) x^\sigma(\tau) + B(\tau) y(\tau)) \Delta \tau. \end{aligned}$$

Thus, from Corollary 2.5, Lemma 2.6 and Lemma 2.8 we obtain

$$\begin{aligned} |x(t)| &= \frac{1}{2} \left[\left| \int_a^t (A(\tau) x^\sigma(\tau) + B(\tau) y(\tau)) \Delta \tau \right| + \left| \int_t^b (A(\tau) x^\sigma(\tau) + B(\tau) y(\tau)) \Delta \tau \right| \right] \\ &\leq \frac{1}{2} \left[\int_a^t |A(\tau) x^\sigma(\tau) + B(\tau) y(\tau)| \Delta \tau + \int_t^b |A(\tau) x^\sigma(\tau) + B(\tau) y(\tau)| \Delta \tau \right] \\ &\leq \frac{1}{2} \left[\int_a^b (|A(\tau) x^\sigma(\tau)| + |B(\tau) y(\tau)|) \Delta \tau \right] \\ &\leq \frac{1}{2} \left[\int_a^b |A(\tau)| |x^\sigma(\tau)| \Delta \tau + \int_a^b |\sqrt{B(\tau)}| |\sqrt{B(\tau)} y(\tau)| \Delta \tau \right] \\ &\leq \frac{1}{2} \left[\int_a^b |A(t)| |x^\sigma(t)| \Delta t + \left(\int_a^b |\sqrt{B(t)}|^2 \Delta t \right)^{1/2} \left(\int_a^b |\sqrt{B(t)} y(t)|^2 \Delta t \right)^{1/2} \right] \\ &= \frac{1}{2} \left[\int_a^b |A(t)| |x^\sigma(t)| \Delta t + \left(\int_a^b |\sqrt{B(t)}|^2 \Delta t \right)^{1/2} \left(\int_a^b (\sqrt{B(t)} y(t))^T \sqrt{B(t)} y(t) \Delta t \right)^{1/2} \right] \\ &= \frac{1}{2} \left[\int_a^b |A(t)| |x^\sigma(t)| \Delta t + \left(\int_a^b |\sqrt{B(t)}|^2 \Delta t \right)^{1/2} \left(\int_a^b (x^\sigma)^T(t) C(t) (x^\sigma(t)) \Delta t \right)^{1/2} \right] \\ &\leq \frac{1}{2} \left[\int_a^b |A(t)| |x^\sigma(t)| \Delta t + \left(\int_a^b |\sqrt{B(t)}|^2 \Delta t \right)^{1/2} \left(\int_a^b |C_1(t)| |x^\sigma(t)|^2 \Delta t \right)^{1/2} \right]. \end{aligned}$$

Denote $M = \max_{a \leq t \leq b} |x(t)| > 0$, then

$$M \leq \frac{1}{2} \left[\int_a^b |A(t)| M \Delta t + \left(\int_a^b |\sqrt{B(t)}|^2 \Delta t \right)^{1/2} \left(\int_a^b |C_1(t)| M^2 \Delta t \right)^{1/2} \right]. \quad (3.12)$$

Thus inequality (3.11) follows from (3.12). This completes the proof of Theorem 3.3.

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Error analysis of distributed algorithm for large scale data classification *

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Abstract

The distributed algorithm is an important and basic approach, and it is usually used for large scale data processing. This paper aims to error analysis of distributed algorithm for large scale data classification generated from Tikhonov regularization schemes associated with varying Gaussian kernels and convex loss functions. The main goal is to provide fast convergence rates for the excess misclassification error. The number of subsets randomly divided from a large scale datasets is determined to guarantee that the distributed algorithm have lower time complexity and memory complexity.

Keywords: Distributed algorithm; Classification; Large scale data; Generalization error

Mathematics Subject Classification: 68T05, 68P30.

1 Introduction

In [11], a binary classification problem, which is generated from Tikhonov regularization schemes with general convex loss functions and varying Gaussian kernels, was studied well. This paper addresses error analysis of distributed algorithm for the classification with large scale datasets. For ease of description, we first introduce some concepts and notations. Most of them are the same as that of [11].

We denote the input space by a compact subset X of \mathbb{R}^p . To represent the two classes, we write the output space $Y = \{-1, 1\}$. Clearly, classification algorithms produce binary classifiers $\mathcal{C} : X \rightarrow Y$, and the prediction power of such classifier \mathcal{C} can be measured by using its misclassification error defined by

$$\mathcal{R}(\mathcal{C}) = \text{Prob}(\mathcal{C}(x) \neq y) = \int_X P(y \neq \mathcal{C}(x)|x) d\rho_x,$$

where ρ is a probability distribution on $Z := X \times Y$, ρ_x is the marginal distribution of ρ on X , and $P(y|x)$ is the conditional distribution at $x \in X$. So-called Bayes rule is the classifier minimizing $\mathcal{R}(\mathcal{C})$, and is given by

$$f_c(x) = \begin{cases} 1, & \text{if } P(y = 1|x) \geq P(y = -1|x), \\ -1, & \text{otherwise.} \end{cases}$$

So the excess misclassification error $\mathcal{R}(\mathcal{C}) - \mathcal{R}(f_c)$ of a classifier \mathcal{C} can be used to measure the performance of the classifier \mathcal{C} .

In this paper we consider classifiers \mathcal{C}_f induced by real-valued functions $f : X \rightarrow \mathbb{R}$, which is defined by

$$\mathcal{C}_f = \text{sgn}(f)(x) = \begin{cases} 1, & \text{if } f(x) \geq 0, \\ -1, & \text{otherwise.} \end{cases}$$

The real-valued functions are generated from Tikhonov regularization schemes associated with general convex loss functions and varying Gaussian kernels.

Now we give a definition for loss function [11].

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Definition 1.1. (see [11]) We say $\varphi : \mathbb{R} \rightarrow \mathbb{R}_+$ is a classifying loss (function) if it is convex, differentiable at 0 with $\varphi'(0) < 0$, and the smallest zero of φ is 1.

For details of such loss function, we refer reader to Cucker and Zhou [4].

The function on $X \times X$ given by $K^\sigma(x, x') = \exp\left\{-\frac{|x-x'|}{2\sigma^2}\right\}$ is called the Gaussian kernel with variance $\sigma > 0$. From [1], this function can be used to define a reproducing kernel Hilbert space (RKHS). We denote the RKHS by \mathcal{H}_σ .

From [10] and [5], the Tikhonov regularization scheme with the loss φ , Gaussian kernel K^σ , and a sample $\mathbf{z} = \{(x_i, y_i)\}_{i=1}^n \in Z^n$ can be defined as the solution $f_{\mathbf{z}}$ of the following minimization problem

$$f_{\mathbf{z}} = \operatorname{argmin}_{f \in \mathcal{H}_\sigma} \left\{ \frac{1}{m} \sum_{i=1}^n \varphi(y_i f(x_i)) + \lambda \|f\|_{\mathcal{H}_\sigma}^2 \right\}, \quad (1.1)$$

where $\lambda > 0$ is called the regularization parameter. The regularizing function in terms of the generalization error \mathcal{E}^φ is defined as

$$\tilde{f}_{\sigma, \lambda} := \arg \min_{f \in \mathcal{H}_\sigma} \{ \mathcal{E}^\varphi(f) + \lambda \|f\|_{\mathcal{H}_\sigma}^2 \}, \quad \text{where } \mathcal{E}^\varphi(f) = \int_Z \varphi(yf(x)) d\rho.$$

This function was used in Zhang [13], De Vito et al. [6], and Yao [12]. Zhou and Xiang [11] constructed a function (denoted by $f_{\sigma, \lambda}$) which works better than $\tilde{f}_{\sigma, \lambda}$ due to the special approximation ability of varying Gaussian kernels. The construction of $f_{\sigma, \lambda}$ is done under a Sobolev smoothness condition of a measurable function f_ρ^φ minimizing \mathcal{E}^φ , i.e., for almost everywhere $x \in X$,

$$f_\rho^\varphi(x) = \operatorname{argmin}_{t \in \mathbb{R}} \int_Y \varphi(yt) d\rho(y|x) = \operatorname{argmin}_{t \in \mathbb{R}} \{ \varphi(t)P(y=1|x) + \varphi(-t)P(y=-1|x) \}.$$

The constructed function $f_{\sigma, \lambda}$ was used to estimate the excess misclassification error in [11]. The following Lemma 2.2 is a key result in [11], which will be employed as a base of our proof.

We will use the concept of Sobolev space with index $s > 0$ and denote the space by $H^s(\mathbb{R}^p)$. In fact, the space is consisted by all functions in $L^2(\mathbb{R}^p)$ with the finite semi-norm

$$|f|_{H^s(\mathbb{R}^p)} = \left\{ (2\pi)^{-n} \int_{\mathbb{R}^p} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi \right\}^{\frac{1}{2}},$$

where \hat{f} is the Fourier transform of f defined for $f \in L^1(\mathbb{R}^p)$ as $\hat{f}(\xi) = \int_{\mathbb{R}^p} f(x) e^{-ix\xi} dx$.

It was proved in Chen et al. [3] and Bartlett et al. [2] that

$$\mathcal{R}(\operatorname{sgn}(f)) - \mathcal{R}(f_c) \leq c_\varphi \sqrt{\mathcal{E}^\varphi(f) - \mathcal{E}^\varphi(f_\rho^\varphi)} \quad (1.2)$$

holds for some $c_\varphi > 0$.

Although the statistical aspects of (1.1) are well investigated, the computation of (1.1) can be complicated for large data with size N . For example, in a standard implementation [9], it requires costs $\mathcal{O}(N^3)$ and $\mathcal{O}(N^2)$ in time and memory, respectively. Such scaling are prohibitive when the sample size is large.

In this work, we study a decomposition-based learning approach for large datasets, which is also called distributed algorithm for large datasets. Recently, the approach has attracted more attentions of researchers, and more results have been explored, such as McDonald et al. [8] for perceptron-based algorithms, Kleiner et al. [7] for bootstrap, and Zhang et al. [14] for parametric smooth convex optimization problems. The aim of this paper is to study the binary classification error of the distributed algorithm with varying λ and σ for general loss functions. For this purpose, we first describe the distributed algorithm [15].

We are given N samples $(x_1, y_1), \dots, (x_N, y_N)$ drawn independent identically distributed (i.i.d.) according to the distribution ρ on $Z = X \times Y$. Rather than solving the problem (1.1) on all N samples, we execute the following three steps: (1) Divide the set of samples $\{(x_1, y_1), \dots, (x_N, y_N)\}$ randomly and evenly into m disjoint subsets $S_1, \dots, S_m \subset Z$, and each

S_i has $n = \frac{N}{m}$ samples; (2) For each $i = 1, 2, \dots, m$, compute the local estimate

$$\hat{f}_i := \operatorname{argmin}_{f \in \mathcal{H}_\sigma} \left\{ \frac{1}{n} \sum_{(x,y) \in S_i} \varphi(yf(x)) + \lambda \|f\|_{\mathcal{H}_\sigma}^2 \right\};$$

(3) Average together the local estimates and output $\bar{f} = \frac{1}{m} \sum_{i=1}^m \hat{f}_i$.

Our aim is to estimate the error $\mathcal{R}(\operatorname{sgn}(\bar{f})) - \mathcal{R}(f_c)$. However, from (1.2), we only need to estimate $\mathcal{E}^\varphi(\bar{f}) - \mathcal{E}^\varphi(f_\rho^\varphi)$. The following section presents some results to bound $\mathcal{E}^\varphi(\bar{f}) - \mathcal{E}^\varphi(f_\rho^\varphi)$ and $\mathcal{R}(\operatorname{sgn}(\bar{f})) - \mathcal{R}(f_c)$. When solving each \hat{f}_i , similarly to [11], we take $\lambda = \lambda(n) = n^{-\gamma}$, $\sigma = \sigma(n) = \lambda^\zeta = n^{-\gamma\zeta}$, for some $\gamma, \zeta > 0$.

2 Main results

Lemma 2.1. *We have $\mathcal{E}^\varphi(\bar{f}) - \mathcal{E}^\varphi(f_\rho^\varphi) \leq \frac{1}{m} \sum_{i=1}^m (\mathcal{E}^\varphi(\hat{f}_i) - \mathcal{E}^\varphi(f_\rho^\varphi))$.*

Proof. Due to the convexity of φ , we have

$$\mathcal{E}^\varphi(\bar{f}) = \int_Z \varphi(y\bar{f}(x)) \, d\rho \leq \int_Z \frac{1}{m} \sum_{i=1}^m \varphi(y\hat{f}_i(x)) \, d\rho = \frac{1}{m} \sum_{i=1}^m \int_Z \varphi(y\hat{f}_i(x)) \, d\rho = \frac{1}{m} \sum_{i=1}^m \mathcal{E}^\varphi(\hat{f}_i).$$

$$\text{So } \mathcal{E}^\varphi(\bar{f}) - \mathcal{E}^\varphi(f_\rho^\varphi) \leq \frac{1}{m} \sum_{i=1}^m (\mathcal{E}^\varphi(\hat{f}_i) - \mathcal{E}^\varphi(f_\rho^\varphi)). \quad \square$$

Now in order to bound $\mathcal{E}^\varphi(\bar{f}) - \mathcal{E}^\varphi(f_\rho^\varphi)$, we only need to estimate $\mathcal{E}^\varphi(\hat{f}_i) - \mathcal{E}^\varphi(f_\rho^\varphi)$ for each i . In fact, the results for each i are the same because \hat{f}_i ($i = 1, 2, \dots, m$) are i.i.d., and share the same properties. We take Xiang and Zhou's approach [11] and make some modifications.

Lemma 2.2. (see [11]) *Assume that for some $s > 0$,*

$$f_\rho^\varphi = \tilde{f}_\rho^\varphi|_X \text{ for some } \tilde{f}_\rho^\varphi \in H^s(\mathbb{R}^p) \cap L^\infty(\mathbb{R}^p) \text{ and } \frac{d\rho_X}{dx} \in L^2(X). \quad (2.1)$$

Then we can find functions $\{f_{\sigma,\lambda} \in \mathcal{H}_\sigma : 0 < \sigma \leq 1, \lambda > 0\}$ such that

$$\|f_{\sigma,\lambda}\|_{L^\infty(X)} \leq \tilde{A}, \quad (2.2)$$

$$\mathcal{D}(\sigma, \lambda) := \mathcal{E}^\varphi(f_{\sigma,\lambda}) - \mathcal{E}^\varphi(f_\rho^\varphi) + \lambda \|f_{\sigma,\lambda}\|_{\mathcal{H}_\sigma}^2 \leq \tilde{A}(\sigma^s + \lambda\sigma^{-p})$$

for $0 < \sigma \leq 1$, $\lambda > 0$, where $\tilde{A} \geq 1$ is a constant independent of σ and λ .

Using the method of error decomposition of [11], we easily obtain the following Lemma 2.3.

Lemma 2.3. *Let φ be a classifying loss function, we have*

$$\mathcal{E}^\varphi(f_i) - \mathcal{E}^\varphi(f_\rho^\varphi) \leq \mathcal{D}(\sigma, \lambda) + \mathbf{S}_Z(f_{\sigma,\lambda}) - \mathbf{S}_Z(\hat{f}_i), \quad (2.3)$$

where $\mathbf{S}_Z(f)$ is defined for any f by $\mathbf{S}_Z(f) = [\mathcal{E}_Z^\varphi(f) - \mathcal{E}_Z^\varphi(f_\rho^\varphi)] - [\mathcal{E}^\varphi(f) - \mathcal{E}^\varphi(f_\rho^\varphi)]$, and $\mathcal{E}_Z^\varphi(f) = \frac{1}{n} \sum_{(x,y) \in S_i} \varphi(yf(x))$.

We also need the following Definition 2.1.

Definition 2.1. (see [11]) *A variancing power $\tau = \tau_{\varphi,\rho}$ of the pair (φ, ρ) is the maximal number τ in $[0, 1]$ such that for any $\tilde{B} \geq 1$, there exists $C_1 = C_1(\tilde{B})$ satisfying*

$$\mathbb{E}[\varphi(yf(x)) - \varphi(yf_\rho^\varphi(x))]^2 \leq C_1[\mathcal{E}^\varphi(f) - \mathcal{E}^\varphi(f_\rho^\varphi)]^\tau \quad \forall f : X \rightarrow [-\tilde{B}, \tilde{B}], \quad (2.4)$$

where $\mathbb{E}\xi$ denotes the expected value of ξ .

The following Lemma 2.4 is to bound the second term of (2.3).

Lemma 2.4. (see [11]) Suppose \tilde{A} and $f_{\sigma,\lambda}$ are as in Lemma 2.2, $\tau = \tau_{\varphi,\rho}$ and $C_1 = C_1(\tilde{A})$ are as in Definition 2.1. Then for any $0 < \delta < 1$, with confidence $1 - \frac{\delta}{2}$, we have

$$\mathcal{S}_{\mathbf{Z}}(f_{\sigma,\lambda}) \leq 2 \left(\|\varphi\|_{C[-\tilde{A},\tilde{A}]} + C_1 \right) \ln \frac{2}{\delta} n^{-\frac{1}{2-\tau}} + (\mathcal{E}^\varphi(f_{\sigma,\lambda}) - \mathcal{E}^\varphi(f_\rho^\varphi)).$$

To bound the third term of (2.3), $-\mathcal{S}_{\mathbf{Z}}(\hat{f}_i)$, we need the following Lemma 2.5, Lemma 2.6, and Lemma 2.7.

Lemma 2.5. For any $\lambda > 0$, we have $\|\hat{f}_i\|_{\mathcal{H}_\sigma} \leq \sqrt{\varphi(0)/\lambda}$.

The proof is easy by taking $f = 0$ in the definition of \hat{f}_i , referring to De Vito et al. [6].

The next Lemma 2.6 is from Cucker and Zhou [4].

Lemma 2.6. (see [4]) Let $0 \leq \tau \leq 1$, $c, B \geq 0$, and \mathcal{G} be a set of functions on Z such that for every $g \in \mathcal{G}$, $\mathbb{E}(g) \geq 0$, $\|g - \mathbb{E}(g)\|_\infty \leq B$ and $\mathbb{E}(g^2) \leq c(\mathbb{E}(g))^\tau$. Then for all $\varepsilon > 0$,

$$\text{Prob}_{\mathbf{Z} \in Z^n} \left\{ \sup_{g \in \mathcal{G}} \frac{\mathbb{E}(g) - \frac{1}{n} \sum_{i=1}^n f(z_i)}{\sqrt{(\mathbb{E}(g))^\tau + \varepsilon^\tau}} > 4\varepsilon^{1-\frac{\tau}{2}} \right\} \leq \mathcal{N}(\mathcal{G}, \varepsilon) \exp \left\{ -\frac{n\varepsilon^{2-\tau}}{2(c + \frac{1}{3}B\varepsilon^{1-\tau})} \right\},$$

where $\mathcal{N}(\mathcal{G}, \varepsilon)$ denotes the covering number to be the minimal $\ell \in \mathbb{N}$ such that there exist ℓ disks in \mathcal{G} with radius ε covering \mathcal{G} .

Note that if $\|f\|_{\mathcal{H}_\sigma} \leq \sqrt{\varphi(0)/\lambda}$, then $\|f\|_\infty \leq C_\sigma \sqrt{\varphi(0)/\lambda}$. From the above Lemma 2.6, we obtain the following Lemma 2.7.

Lemma 2.7. Let $\tau = \tau_{\varphi,\rho}$ with $\tilde{B} = C_\sigma \sqrt{\varphi(0)/\lambda}$ and $C_1 = C_1(\tilde{B})$ in Definition 2.1. For any $\varepsilon > 0$, there holds

$$\begin{aligned} \text{Prob}_{\mathbf{Z} \in Z^n} \left\{ \sup_{\|f\|_{\mathcal{H}_\sigma} \leq \sqrt{\varphi(0)/\lambda}} \frac{[\mathcal{E}^\varphi(f) - \mathcal{E}^\varphi(f_\rho^\varphi)] - [\mathcal{E}_z^\varphi(f) - \mathcal{E}_z^\varphi(f_\rho^\varphi)]}{\sqrt{(\mathcal{E}^\varphi(f) - \mathcal{E}^\varphi(f_\rho^\varphi))^\tau + \varepsilon^\tau}} \leq 4\varepsilon^{1-\frac{\tau}{2}} \right\} \geq \\ 1 - \mathcal{N} \left(B_1, \frac{\varepsilon\sqrt{\lambda}}{D_1\sqrt{\varphi(0)}} \right) \exp \left\{ -\frac{n\varepsilon^{2-\tau}}{2C_1 + \frac{4}{3}D_2\varepsilon^{1-\tau}} \right\}, \end{aligned}$$

where $D_1 = \max\{|\varphi'_+(-\tilde{B})|, |\varphi'_-(\tilde{B})|\}$, and $D_2 = \max\{\varphi(-1), \|\varphi\|_{C[-\tilde{B},\tilde{B}]}\}$.

Proof. We apply the above Lemma 2.6 to the function set

$$\mathcal{G} = \left\{ \varphi(yf(x)) - \varphi(yf_\rho^\varphi(x)) : \|f\|_{\mathcal{H}_\sigma} \leq \sqrt{\varphi(0)/\lambda} \right\},$$

and see that each function $g \in \mathcal{G}$ satisfies $\mathbb{E}(g^2) \leq c(\mathbb{E}(g))^\tau$ for $c = C_1$. Obviously $\|g\|_\infty \leq D_2 := \max\{\varphi(-1), \|\varphi\|_{C[-\tilde{B},\tilde{B}]}\}$, so $\|g - \mathbb{E}(g)\|_\infty \leq B := 2D_2$. To draw our conclusion, we only need to bound the covering number $\mathcal{N}(\mathcal{G}, \varepsilon)$. To do so, note that for f_1 and f_2 satisfying $\|f\|_{\mathcal{H}_\sigma} \leq \sqrt{\varphi(0)/\lambda}$ and $(x, y) \in Z$, we have

$$\begin{aligned} & |\{\varphi(yf_1(x)) - \varphi(yf_\rho^\varphi(x))\} - \{\varphi(yf_2(x)) - \varphi(yf_\rho^\varphi(x))\}| \\ &= |\varphi(yf_1(x)) - \varphi(yf_2(x))| \leq D_1 \|f_1 - f_2\|_\infty. \end{aligned}$$

Therefore, $\mathcal{N}(\mathcal{G}, \varepsilon) \leq \mathcal{N}(B_{\sqrt{\varphi(0)/\lambda}, \frac{\varepsilon}{D_1}}) = \mathcal{N}(B_1, \frac{\varepsilon\sqrt{\lambda}}{D_1\sqrt{\varphi(0)}})$, where $B_{\sqrt{\varphi(0)/\lambda}}$ denotes the ball with radius $\sqrt{\varphi(0)/\lambda}$ in \mathcal{H}_σ . The statement is proved. \square

Let $\varepsilon^*(n, \lambda, \sigma, \delta)$ denote the smallest positive number ε satisfying

$$1 - \mathcal{N} \left(B_1, \frac{\varepsilon\sqrt{\lambda}}{D_1\sqrt{\varphi(0)}} \right) \exp \left\{ -\frac{n\varepsilon^{2-\tau}}{2C_1 + \frac{4}{3}D_2\varepsilon^{1-\tau}} \right\} \geq 1 - \frac{\delta}{2}.$$

Then we have the following proposition.

Proposition 2.1. Let $\sigma = \lambda^\zeta$ with $0 < \zeta < \frac{1}{p}$ (Noting p is the dimension of X), s be as in Lemma 2.2, and $f_{\sigma,\lambda} \in \mathcal{H}_\sigma$ satisfy (2.2). For any $0 < \delta < 1$, with confidence at least $1 - \delta$, we have

$$\mathcal{E}^\varphi(\hat{f}_i) - \mathcal{E}^\varphi(f_\rho^\varphi) \leq 8\tilde{A}\lambda^{\min\{s\zeta, 1-p\zeta\}} + 40\varepsilon^*(n, \lambda, \sigma, \delta) + 4(\|\varphi\|_{C[-\tilde{A}, \tilde{A}]} + C_1) \ln \frac{2}{\delta} n^{-\frac{1}{2-\tau}}. \quad (2.5)$$

Proof. Xiang and Zhou [11] (see Proposition 2 in [11]) have proved that for any $0 < \delta < 1$, with confidence at least $1 - \delta$,

$$\mathcal{E}^\varphi(\hat{f}_i) - \mathcal{E}^\varphi(f_\rho^\varphi) \leq 4\mathcal{D}(\sigma, \lambda) + 40\varepsilon^*(n, \lambda, \sigma, \delta) + 4(\|\varphi\|_{C[-\tilde{A}, \tilde{A}]} + C_1) \ln \frac{2}{\delta} n^{-\frac{1}{2-\tau}}.$$

With Lemma 2.2 and $\sigma = \lambda^\zeta$, we have $\mathcal{D} \leq \tilde{A}(\lambda^{s\zeta} + \lambda^{1-p\zeta}) \leq 2\tilde{A}\lambda^{\min\{s\zeta, 1-p\zeta\}}$. So Proposition 2.1 is proved. \square

To get the more explicit bound, we need the following Lemma 2.8 to bound $\varepsilon^*(m, \lambda, \sigma, \delta)$. It can be proved via the same method as in [11].

Lemma 2.8. Let $0 \leq \tau \leq 1$, $\lambda = n^{-\gamma}$ and $\sigma = \lambda^\zeta$ with $\gamma > 0$ and $0 < \zeta < \frac{1}{2\gamma(p+1)}$. Then we have

$$\varepsilon^*(m, \lambda, \sigma, \delta) \leq C_2 n^{-\frac{1-2\gamma\zeta(p+1)}{2-\tau} \ln \frac{2}{\delta}}. \quad (2.6)$$

From Proposition 2.1 and Lemma 2.8, we have the following Proposition 2.2.

Proposition 2.2. Let $\sigma = \lambda^\zeta$ and $\lambda = n^{-\gamma}$ for some $0 < \zeta < \frac{1}{p}$ and $0 < \gamma < \frac{1}{2\zeta(p+1)}$. If (2.1) is valid for some $s > 0$, then for any $0 < \delta < 1$, with confidence $1 - \delta$ we have

$$\mathcal{E}^\varphi(\hat{f}_i) - \mathcal{E}^\varphi(f_\rho^\varphi) \leq \tilde{C} n^{-\theta} \ln \frac{2}{\delta}, \quad (2.7)$$

where

$$\theta = \min \left\{ s\zeta\gamma, \gamma(1-p\zeta), \frac{1-2\gamma\zeta(p+1)}{2-\tau} \right\}, \quad (2.8)$$

and \tilde{C} is a constant independent of n and δ .

Proof. Applying the bound for ε^* from Lemma 2.8 on Proposition 2.1, with confidence at least $1 - \delta$, we have

$$\mathcal{E}^\varphi(\hat{f}_i) - \mathcal{E}^\varphi(f_\rho^\varphi) \leq 8\tilde{A}\lambda^{\min\{s\zeta, 1-p\zeta\}} + 40C_2 n^{-\frac{1-2\gamma\zeta(p+1)}{2-\tau} \ln \frac{2}{\delta}} + 4(\|\varphi\|_{C[-\tilde{A}, \tilde{A}]} + C_1) \ln \frac{2}{\delta} n^{-\frac{1}{2-\tau}}.$$

Putting $\lambda = n^{-\gamma}$ into the above formula, we easily see that $\mathcal{E}^\varphi(\hat{f}_i) - \mathcal{E}^\varphi(f_\rho^\varphi) \leq \tilde{C} n^{-\theta} \ln \frac{2}{\delta}$. Here θ is given by (2.8) and \tilde{C} is the constant independent of n and δ given by $\tilde{C} = 8\tilde{A} + 40C_2 + 4(\|\varphi\|_{C[-\tilde{A}, \tilde{A}]} + C_1)$. \square

Now we can obtain our main result to bound $\mathcal{E}^\varphi(\bar{f}) - \mathcal{E}^\varphi(f_\rho^\varphi)$.

Theorem 2.1. Under the condition of Proposition 2.2, for any $0 < \delta < 1$, with confidence $1 - \delta$ we have

$$\mathcal{E}^\varphi(\bar{f}) - \mathcal{E}^\varphi(f_\rho^\varphi) \leq \tilde{C} n^{-\theta} \ln \frac{2m}{\delta}, \quad (2.9)$$

where θ and \tilde{C} are as in Proposition 2.2.

Proof. From Proposition 2.2, for any $\delta > 0$, with confidence $1 - \frac{\delta}{m}$, $\mathcal{E}^\varphi(\hat{f}_i) - \mathcal{E}^\varphi(f_\rho^\varphi) \leq \tilde{C} n^{-\theta} \ln \frac{2m}{\delta}$. From Lemma 2.1,

$$\begin{aligned} \text{Prob} \left\{ \mathcal{E}^\varphi(\bar{f}) - \mathcal{E}^\varphi(f_\rho^\varphi) \leq \tilde{C} n^{-\theta} \ln \frac{2m}{\delta} \right\} &\geq \text{Prob} \left\{ \frac{1}{m} \sum_{i=1}^m \left(\mathcal{E}^\varphi(\hat{f}_i) - \mathcal{E}^\varphi(f_\rho^\varphi) \right) \leq \tilde{C} n^{-\theta} \ln \frac{2m}{\delta} \right\} \\ &\geq \text{Prob} \left\{ \bigcap_{i=1}^m \left\{ \mathcal{E}^\varphi(\hat{f}_i) - \mathcal{E}^\varphi(f_\rho^\varphi) \leq \tilde{C} n^{-\theta} \ln \frac{2m}{\delta} \right\} \right\} \geq 1 - m \times \frac{\delta}{m} = 1 - \delta. \end{aligned}$$

\square

Remark 2.1. Given N , we take $n = m^a$, i.e. $m = N^{\frac{1}{a+1}}$ and $n = N^{\frac{a}{a+1}}$. We easily see that the above bound $\frac{1}{m^a} \ln \frac{2m}{\delta} \rightarrow 0 (m \rightarrow \infty)$ for all $a > 0$.

As mentioned in Introduction, the Tikhonov regularization scheme for all N samples have time complexity $\mathcal{O}(N^3)$ and memory complexity $\mathcal{O}(N^2)$. Now we can determine m (also n) to guarantee that the distributed algorithm have lower time complexity and memory complexity.

Corollary 2.1. For any $k < 3$, the time complexity of the distributed algorithm is less than $\mathcal{O}(N^k)$ if and only if $m > N^{\frac{3-k}{2}}$.

Proof. Let $n = m^a$, i.e. $m = N^{\frac{1}{a+1}}$. The time complexity is $m \cdot \mathcal{O}(n^3) = \mathcal{O}(m^{3a+1}) = \mathcal{O}(N^{\frac{3a+1}{a+1}})$. For $k < 3$, to ensure $\frac{3a+1}{a+1} < k$, it only needs $a < \frac{k-1}{3-k}$. So $m = N^{\frac{1}{a+1}} > N^{\frac{3-k}{2}}$. \square

For memory complexity, we have a similar result as follows.

Corollary 2.2. For any $k < 2$, the memory complexity of the distributed algorithm is less than $\mathcal{O}(N^k)$ if and only if $m > N^{2-k}$.

Due to (1.2), we have

$$\textbf{Theorem 2.2. } \mathcal{R}(\text{sgn}(\bar{f})) - \mathcal{R}(f_c) \leq c_\varphi \sqrt{\tilde{C} n^{-\theta} \ln \frac{2m}{\delta}}.$$

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Korovkin type statistical approximation theorem for a function of two variables

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Abstract. In this paper, we prove a Korovkin type approximation theorem for a function of two variables by using the notion of convergence in the Pringsheim's sense and statistical convergence of double sequences. We also display an example in support of our results.

Keywords and phrases: Double sequence; statistical convergence; positive linear operator; Korovkin type approximation theorem.

AMS subject classification (2000): 41A10, 41A25, 41A36, 40A30, 40G15.

1. Introduction and preliminaries

The concept of statistical convergence for sequences of real numbers was introduced by Fast [8] and further studied Fridy [9] and many others.

Let $K \subseteq \mathbb{N}$ and $K_n = \{k \leq n : k \in K\}$. Then the *natural density* of K is defined by $\delta(K) = \lim_n n^{-1}|K_n|$ if the limit exists, where $|K_n|$ denotes the cardinality of K_n .

A sequence $x = (x_k)$ of real numbers is said to be *statistically convergent* to L provided that for every $\epsilon > 0$ the set $K_\epsilon := \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ has natural density zero, i.e. for each $\epsilon > 0$,

$$\lim_n \frac{1}{n} |\{j \leq n : |x_j - L| \geq \epsilon\}| = 0.$$

By the convergence of a double sequence we mean the convergence in the Pringsheim's sense [20]. A double sequence $x = (x_{jk})$ is said to be *Pringsheim's convergent* (or *P-convergent*) if for given $\epsilon > 0$ there exists an integer N such that $|x_{jk} - \ell| < \epsilon$ whenever $j, k > N$. In this case, ℓ is called the Pringsheim limit of $x = (x_{jk})$ and it is written as $P - \lim x = \ell$.

A double sequence $x = (x_{jk})$ is said to be *bounded* if there exists a positive number M such that $|x_{jk}| < M$ for all j, k .

Note that, in contrast to the case for single sequences, a convergent double sequence need not be bounded.

The idea of statistical convergence for double sequences was introduced and studied by Moricz [17] and Mursaleen and Edely [18], independently in the same year and further studied in [15].

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers and let $K(m, n) = \{(j, k) : j \leq m, k \leq n\}$. Then the *double natural density* of the set K is defined as

$$P\text{-}\lim_{m,n} \frac{|K(m, n)|}{mn} = \delta_2(K)$$

provided that the sequence $(|K(m, n)|/mn)$ has a limit in Pringsheim's sense.

For example, let $K = \{(i^2, j^2) : i, j \in \mathbb{N}\}$. Then

$$\delta_2(K) = P\text{-}\lim_{m,n} \frac{|K(m, n)|}{mn} \leq P\text{-}\lim_{m,n} \frac{\sqrt{m}\sqrt{n}}{mn} = 0,$$

i.e. the set K has double natural density zero, while the set $\{(i, 2j) : i, j \in \mathbb{N}\}$ has double natural density $\frac{1}{2}$.

A real double sequence $x = (x_{jk})$ is said to be *statistically convergent* to the number L if for each $\epsilon > 0$, the set

$$\{(j, k), j \leq m \text{ and } k \leq n : |x_{jk} - L| \geq \epsilon\}$$

has double natural density zero. In this case we write $st_2\text{-}\lim_{j,k \rightarrow \infty} x_{jk} = L$.

Remark 1.1. Note that if $x = (x_{jk})$ is P -convergent then it is statistically convergent but not conversely. See the following example.

Example 1.1. The double sequence $w = (w_{jk})$ defined by

$$w_{jk} = \begin{cases} 1 & , \text{ if } j \text{ and } k \text{ are squares;} \\ 0 & , \text{ otherwise } \end{cases} \quad (1.1.1)$$

Then w is statistically convergent to zero but not P -convergent.

Let $C[a, b]$ be the space of all functions f continuous on $[a, b]$ equipped with the norm

$$\|f\|_{C[a,b]} := \sup_{x \in [a,b]} |f(x)|, \quad f \in C[a, b].$$

The classical Korovkin approximation theorem states as follows (cf. [10], [13]):

Let (T_n) be a sequence of positive linear operators from $C[a, b]$ into $C[a, b]$. Then $\lim_n \|T_n(f, x) - f(x)\|_{C[a, b]} = 0$, for all $f \in C[a, b]$ if and only if $\lim_n \|T_n(f_i, x) - f_i(x)\|_{C[a, b]} = 0$, for $i = 0, 1, 2$, where $f_0(x) = 1$, $f_1(x) = x$ and $f_2(x) = x^2$.

Korovkin type approximation theorems are also proved for different summability methods to replace the ordinary convergence, e.g. [4], [7], [11], [14], [16] etc..

Quite recently, such type of approximation theorems are proved in [1], [2], [3], [6] and [19] for functions of two variables by using almost convergence and statistical convergence of double sequences, respectively. For single sequences, Boyanov and Veselinov [2] have proved the Korovkin theorem on $C[0, \infty)$ by using the test functions $1, e^{-x}, e^{-2x}$. In this paper, we extend the result of Boyanov and Veselinov for functions of two variables by using the notion of Pringsheim's convergence and statistical convergence of double sequences.

2. Main result

Let $C(I^2)$ be the Banach space with the uniform norm $\| \cdot \|$ of all real-valued two dimensional continuous functions on $I \times I$, where $I = [0, \infty)$; provided that $\lim_{(x, y) \rightarrow (\infty, \infty)} f(x, y)$ is finite. Suppose that $T_{m, n} : C(I^2) \rightarrow C(I^2)$. We write $T_{m, n}(f; x, y)$ for $T_{m, n}(f(s, t); x, y)$; and we say that T is a positive operator if $T(f; x, y) \geq 0$ for all $f(x, y) \geq 0$.

The following result is an extension of Boyanov and Veselinov theorem [5] for functions of two variables.

Theorem 2.1. Let $(T_{j, k})$ be a double sequence of positive linear operators from $C(I^2)$ into $C(I^2)$. Then for all $f \in C(I^2)$

$$P\text{-}\lim_{j, k \rightarrow \infty} \|T_{j, k}(f; x, y) - f(x, y)\| = 0. \quad (2.1.0)$$

if and only if

$$P\text{-}\lim_{j, k \rightarrow \infty} \|T_{j, k}(1; x, y) - 1\| = 0, \quad (2.1.1)$$

$$P\text{-}\lim_{j, k \rightarrow \infty} \|T_{j, k}(e^{-s}; x, y) - e^{-x}\| = 0, \quad (2.1.2)$$

$$P\text{-}\lim_{j, k \rightarrow \infty} \|T_{j, k}(e^{-t}; x, y) - e^{-y}\| = 0, \quad (2.1.3)$$

$$P\text{-}\lim_{j, k \rightarrow \infty} \|T_{j, k}(e^{-2s} + e^{-2t}; x, y) - (e^{-2x} + e^{-2y})\| = 0. \quad (2.1.4)$$

Proof. Since each $1, e^{-x}, e^{-y}, e^{-2x} + e^{-2y}$ belongs to $C(I^2)$, conditions (2.1.1)-(2.1.4) follow immediately from (2.1.0). Let $f \in C(I^2)$. There exist a constant $M > 0$ such that $|f(x, y)| \leq M$ for each $(x, y) \in I^2$. Therefore,

$$|f(s, t) - f(x, y)| \leq 2M, \quad -\infty < s, t, x, y < \infty. \quad (2.1.5)$$

It is easy to prove that for a given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(s, t) - f(x, y)| < \varepsilon, \quad (2.1.6)$$

whenever $|e^{-s} - e^{-x}| < \delta$ and $|e^{-t} - e^{-y}| < \delta$ for all $(x, y) \in I^2$.

Using (2.1.5), (2.1.6), putting $\psi_1 = \psi_1(s, x) = (e^{-s} - e^{-x})^2$ and $\psi_2 = \psi_2(t, y) = (e^{-t} - e^{-y})^2$, we get

$$|f(s, t) - f(x, y)| < \varepsilon + \frac{2M}{\delta^2}(\psi_1 + \psi_2), \quad \forall |s - x| < \delta \text{ and } |t - y| < \delta.$$

This is,

$$-\varepsilon - \frac{2M}{\delta^2}(\psi_1 + \psi_2) < f(s, t) - f(x, y) < \varepsilon + \frac{2M}{\delta^2}(\psi_1 + \psi_2).$$

Now, operating $T_{j,k}(1; x, y)$ to this inequality since $T_{j,k}(f; x, y)$ is monotone and linear.

We obtain

$$\begin{aligned} T_{j,k}(1; x, y) \left(-\varepsilon - \frac{2M}{\delta^2}(\psi_1 + \psi_2) \right) &< T_{j,k}(1; x, y)(f(s, t) - f(x, y)) \\ &< T_{j,k}(1; x, y) \left(\varepsilon + \frac{2M}{\delta^2}(\psi_1 + \psi_2) \right). \end{aligned}$$

Note that x and y are fixed and so $f(x, y)$ is constant number. Therefore

$$\begin{aligned} -\varepsilon T_{j,k}(1; x, y) - \frac{2M}{\delta^2} T_{j,k}(\psi_1 + \psi_2; x, y) &< T_{j,k}(f; x, y) - f(x, y) T_{j,k}(1; x, y) \\ &< \varepsilon T_{j,k}(1; x, y) + \frac{2M}{\delta^2} T_{j,k}(\psi_1 + \psi_2; x, y). \end{aligned} \quad (2.1.7)$$

But

$$\begin{aligned} T_{j,k}(f; x, y) - f(x, y) &= T_{j,k}(f; x, y) - f(x, y) T_{j,k}(1; x, y) + f(x, y) T_{j,k}(1; x, y) - f(x, y) \\ &= [T_{j,k}(f; x, y) - f(x, y) T_{j,k}(1; x, y)] + f(x, y) [T_{j,k}(1; x, y) - 1]. \end{aligned} \quad (2.1.8)$$

Using (2.1.7) and (2.1.8), we have

$$T_{j,k}(f; x, y) - f(x, y) < \varepsilon T_{j,k}(1; x, y) + \frac{2M}{\delta^2} T_{j,k}(\psi_1 + \psi_2; x, y) + f(x, y) (T_{j,k}(1; x, y) - 1). \quad (2.1.9)$$

Now

$$T_{j,k}(\psi_1 + \psi_2; x, y) = T_{j,k}((e^{-s} - e^{-x})^2 + (e^{-t} - e^{-y})^2; x, y)$$

$$\begin{aligned}
 &= T_{j,k}(e^{-2s} - 2e^{-s}e^{-x} + e^{-2x} + e^{-2t} - 2e^{-t}e^{-y} + e^{-2y}; x, y) \\
 &= T_{j,k}(e^{-2s} + e^{-2t}; x, y) - 2e^{-x}T_{j,k}(e^{-s}; x, y) - 2e^{-y}T_{j,k}(e^{-t}; x, y) \\
 &\quad + (e^{-2x} + e^{-2y})T_{j,k}(1; x, y) \\
 &= [T_{j,k}(e^{-2s} + e^{-2t}; x, y) - (e^{-2x} + e^{-2y})] - 2e^{-x}[T_{j,k}(e^{-s}; x, y) - e^{-x}] \\
 &\quad - 2e^{-y}[T_{j,k}(e^{-t}; x, y) - e^{-y}] + (e^{-2x} + e^{-2y})[T_{j,k}(1; x, y) - 1].
 \end{aligned}$$

Using (2.1.9), we obtain

$$\begin{aligned}
 T_{j,k}(f; x, y) - f(x, y) &< \varepsilon T_{j,k}(1; x, y) + \frac{2M}{\delta^2} \{ [T_{j,k}((e^{-2s} + e^{-2t}); x, y) - (e^{-2x} + e^{-2y})] \\
 &\quad - 2e^{-x}[T_{j,k}(e^{-s}; x, y) - e^{-x}] - 2e^{-y}[T_{j,k}(e^{-t}; x, y) - e^{-y}] \\
 &\quad + (e^{-2x} + e^{-2y})[T_{j,k}(1; x, y) - 1] \} + f(x, y)(T_{j,k}(1; x, y) - 1) \\
 &= \varepsilon [T_{j,k}(1; x, y) - 1] + \varepsilon + \frac{2M}{\delta^2} \{ [T_{j,k}((e^{-2s} + e^{-2t}); x, y) - (e^{-2x} + e^{-2y})] \\
 &\quad - 2e^{-x}[T_{j,k}(e^{-s}; x, y) - e^{-x}] - 2e^{-y}[T_{j,k}(e^{-t}; x, y) - e^{-y}] \\
 &\quad + (e^{-2x} + e^{-2y})[T_{j,k}(1; x, y) - 1] \} + f(x, y)(T_{j,k}(1; x, y) - 1).
 \end{aligned}$$

Since ε is arbitrary, we can write

$$\begin{aligned}
 T_{j,k}(f; x, y) - f(x, y) &\leq \varepsilon [T_{j,k}(1; x, y) - 1] + \frac{2M}{\delta^2} \{ [T_{j,k}((e^{-2s} + e^{-2t}); x, y) - (e^{-2x} + e^{-2y})] \\
 &\quad - 2e^{-x}[T_{j,k}(e^{-s}; x, y) - e^{-x}] - 2e^{-y}[T_{j,k}(e^{-t}; x, y) - e^{-y}] \\
 &\quad + (e^{-2x} + e^{-2y})[T_{j,k}(1; x, y) - 1] \} + f(x, y)(T_{j,k}(1; x, y) - 1).
 \end{aligned}$$

Therefore

$$\begin{aligned}
 |T_{j,k}(f; x, y) - f(x, y)| &\leq \varepsilon + (\varepsilon + M) |T_{j,k}(1; x, y) - 1| + \frac{2M}{\delta^2} |e^{-2x} + e^{-2y}| |T_{j,k}(1; x, y) - 1| \\
 &\quad + \frac{2M}{\delta^2} |T_{j,k}(e^{-2s} + e^{-2t}; x, y) - (e^{-2x} + e^{-2y})| \\
 &\quad + \frac{4M}{\delta^2} |e^{-x}| |T_{j,k}(e^{-s}; x, y) - e^{-x}| + \frac{4M}{\delta^2} |e^{-y}| |T_{j,k}(e^{-t}; x, y) - e^{-y}| \\
 &\leq \varepsilon + (\varepsilon + M + \frac{4M}{\delta^2}) |T_{j,k}(1; x, y) - 1| + \frac{2M}{\delta^2} |e^{-2x} + e^{-2y}| |T_{j,k}(1; x, y) - 1| \\
 &\quad + \frac{2M}{\delta^2} |T_{j,k}(e^{-2s} + e^{-2t}; x, y) - (e^{-2x} + e^{-2y})| \\
 &\quad + \frac{4M}{\delta^2} |T_{j,k}(e^{-s}; x, y) - e^{-x}| + \frac{4M}{\delta^2} |T_{j,k}(e^{-t}; x, y) - e^{-y}|. \tag{2.1.10}
 \end{aligned}$$

since $|e^{-x}|, |e^{-y}| \leq 1$ for all $x, y \in I$. Now, taking $\sup_{(x,y) \in I^2}$, we get

$$\left\| T_{j,k}(f; x, y) - f(x, y) \right\| \leq \varepsilon + K \left(\left\| T_{j,k}(1; x, t) - 1 \right\| \right)$$

$$\begin{aligned}
& + \left\| T_{j,k}(e^{-s}; x, y) - e^{-x} \right\| + \left\| T_{j,k}(e^{-t}; x, y) - e^{-y} \right\| \\
& + \left\| T_{j,k}(e^{-2s} + e^{-2t}; x, y) - (e^{-2x} + e^{-2y}) \right\|, \quad (2.1.11)
\end{aligned}$$

where where $K = \max\{\varepsilon + M + \frac{4M}{\delta^2}, \frac{4M}{\delta^2}, \frac{2M}{\delta^2}\}$. Taking P -lim as $j, k \rightarrow \infty$ and using (2.1.1), (2.1.2), (2.1.3), (2.1.4), we get

$$P - \lim_{p, q \rightarrow \infty} \left\| T_{j,k}(f; x, y) - f(x, y) \right\| = 0, \text{ uniformly in } m, n.$$

This completes the proof of the theorem.

3. Statistical version

In the following theorem we use the notion of statistical convergence of double sequences to generalize the above theorem. We also display an interesting example to show its importance.

Theorem 3.1. Let $(T_{j,k})$ be a double sequence of positive linear operators from $C(I^2)$ into $C(I^2)$. Then for all $f \in C(I^2)$

$$st_2 - \lim_{j, k \rightarrow \infty} \left\| T_{j,k}(f; x, y) - f(x, y) \right\| = 0. \quad (3.1.0)$$

if and only if

$$st_2 - \lim_{j, k \rightarrow \infty} \left\| T_{j,k}(1; x, y) - 1 \right\| = 0, \quad (3.1.1)$$

$$st_2 - \lim_{j, k \rightarrow \infty} \left\| T_{j,k}(e^{-s}; x, y) - e^{-x} \right\| = 0, \quad (3.1.2)$$

$$st_2 - \lim_{j, k \rightarrow \infty} \left\| T_{j,k}(e^{-t}; x, y) - e^{-y} \right\| = 0, \quad (3.1.3)$$

$$st_2 - \lim_{j, k \rightarrow \infty} \left\| T_{j,k}(e^{-2s} + e^{-2t}; x, y) - (e^{-2x} + e^{-2y}) \right\| = 0. \quad (3.1.4)$$

Proof. For a given $r > 0$ choose $\varepsilon > 0$ such that $\varepsilon < r$. Define the following sets

$$D := \{(j, k), j \leq m \text{ and } k \leq n : \left\| T_{j,k}(f; x, y) - f(x, y) \right\| \geq r\},$$

$$D_1 := \{(j, k), j \leq m \text{ and } k \leq n : \left\| T_{j,k}(1; x, y) - 1 \right\| \geq \frac{r - \varepsilon}{4K}\},$$

$$D_2 := \{(j, k), j \leq m \text{ and } k \leq n : \left\| T_{j,k}(e^{-s}; x, y) - e^{-x} \right\| \geq \frac{r - \varepsilon}{4K}\},$$

$$D_3 := \{(j, k), j \leq m \text{ and } k \leq n : \left\| T_{j,k}(e^{-t}; x, y) - e^{-y} \right\| \geq \frac{r - \varepsilon}{4K}\}.$$

$$D_4 := \{(j, k), j \leq m \text{ and } k \leq n : \left\| T_{j,k}(e^{-2s} + e^{-2t}; x, y) - (e^{-2x} + e^{-2y}) \right\| \geq \frac{r - \varepsilon}{4K}\}.$$

Then from (2.1.11), we see that $D \subset D_1 \cup D_2 \cup D_3 \cup D_4$ and therefore $\delta_2(D) \leq \delta_2(D_1) + \delta_2(D_2) + \delta_2(D_3) + \delta_2(D_4)$. Hence conditions (3.1.1)–(3.1.4) imply the condition (3.1.0).

This completes the proof of the theorem.

We show that the following double sequence of positive linear operators satisfies the conditions of Theorem 3.1 but does not satisfy the conditions of Theorem 2.1.

Example 3.2. Consider the sequence of classical Baskakov operators of two variables [12]

$$B_{m,n}(f; x, y) := \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} f\left(\frac{j}{m}, \frac{k}{n}\right) \binom{m-1+j}{j} \binom{n-1+k}{k} x^j (1+x)^{-m-j} y^k (1+y)^{-n-k};$$

where $0 \leq x, y < \infty$. Let $L_{m,n} : C(I^2) \rightarrow C(I^2)$ be defined by

$$L_{m,n}(f; x, y) = (1 + w_{mn}) B_{m,n}(f; x, y),$$

where the sequence (w_{mn}) is defined by (1.1.1). Since

$$B_{m,n}(1; x, y) = 1,$$

$$B_{m,n}(e^{-s}; x, y) = (1 + x - x e^{-\frac{1}{m}})^{-m},$$

$$B_{m,n}(e^{-t}; x, y) = (1 + y - y e^{-\frac{1}{n}})^{-n},$$

$$B_{m,n}(e^{-2s} + e^{-2t}; x, y) = (1 + x^2 - x^2 e^{-\frac{1}{m}})^{-m} + (1 + y^2 - y^2 e^{-\frac{1}{n}})^{-n},$$

we have that the sequence $(L_{m,n})$ satisfies the conditions (3.1.1), (3.2.2), (3.1.3) and (3.1.4). Hence by Theorem 3.1, we have

$$\text{st}_2\text{-}\lim_{m,n \rightarrow \infty} \|L_{m,n}(f; x, y) - f(x, y)\| = 0.$$

On the other hand, we get $L_{m,n}(f; 0, 0) = (1 + w_{mn})f(0, 0)$, since $B_{m,n}(f; 0, 0) = f(0, 0)$, and hence

$$\|L_{m,n}(f; x, y) - f(x, y)\| \geq |L_{m,n}(f; 0, 0) - f(0, 0)| = w_{mn}|f(0, 0)|.$$

We see that $(L_{m,n})$ does not satisfy the conditions of Theorem 2.1, since $P\text{-}\lim_{m,n \rightarrow \infty} w_{mn}$ does not exist.

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Advanced Fractional Taylor's formulae

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Abstract

Here are presented five new advanced fractional Taylor's formulae under as weak as possible assumptions.

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1 Introduction

In [3] we proved

Theorem 1 *Let $f, f', \dots, f^{(n)}; g, g'$ be continuous functions from $[a, b]$ (or $[b, a]$) into \mathbb{R} , $n \in \mathbb{N}$. Assume that $(g^{-1})^{(k)}$, $k = 0, 1, \dots, n$, are continuous functions. Then it holds*

$$f(b) = f(a) + \sum_{k=1}^{n-1} \frac{(f \circ g^{-1})^{(k)}(g(a))}{k!} (g(b) - g(a))^k + R_n(a, b), \quad (1)$$

where

$$\begin{aligned} R_n(a, b) &:= \frac{1}{(n-1)!} \int_a^b (g(b) - g(s))^{n-1} (f \circ g^{-1})^{(n)}(g(s)) g'(s) ds \\ &= \frac{1}{(n-1)!} \int_{g(a)}^{g(b)} (g(b) - t)^{n-1} (f \circ g^{-1})^{(n)}(t) dt. \end{aligned} \quad (2)$$

Remark 2 *Let g be strictly increasing and $g \in AC([a, b])$ (absolutely continuous functions). Set $g([a, b]) = [c, d]$, where $c, d \in \mathbb{R}$, i.e. $g(a) = c$, $g(b) = d$, and call $l := f \circ g^{-1}$.*

Assume that $l \in AC^n([c, d])$ (i.e. $l^{(n-1)} \in AC([c, d])$).

[Obviously here it is implied that $f \in C([a, b])$.]

Furthermore assume that $(f \circ g^{-1})^{(n)} \in L_\infty([c, d])$. [By this very last assumption, the function $(g(b) - t)^{n-1} (f \circ g^{-1})^{(n)}(t)$ is integrable over $[c, d]$. Since $g \in AC([a, b])$ and it is increasing, by [9] the function $(g(b) - g(s))^{n-1} (f \circ g^{-1})^{(n)}(g(s)) g'(s)$ is integrable on $[a, b]$, and again by [9], (2) is valid in this general setting.] Clearly (1) is now valid under these general assumptions.

2 Results

We need

Lemma 3 Let g be strictly increasing and $g \in AC([a, b])$. Assume that $(f \circ g^{-1})^{(m)}$ is Lebesgue measurable function over $[c, d]$. Then

$$\left\| (f \circ g^{-1})^{(m)} \right\|_{\infty, [c, d]} \leq \left\| (f \circ g^{-1})^{(m)} \circ g \right\|_{\infty, [a, b]}, \quad (3)$$

where $(f \circ g^{-1})^{(m)} \circ g \in L_\infty([a, b])$.

Proof. We observe by definition of $\|\cdot\|_\infty$ that:

$$\left\| (f \circ g^{-1})^{(m)} \circ g \right\|_{\infty, [a, b]} = \quad (4)$$

$$\inf \left\{ M : m \left\{ t \in [a, b] : \left| \left((f \circ g^{-1})^{(m)} \circ g \right) (t) \right| > M \right\} = 0 \right\},$$

where m is the Lebesgue measure.

Because g is absolutely continuous and strictly increasing function on $[a, b]$, by [11], p. 108, exercise 14, we get that

$$\begin{aligned} m \left\{ z \in [c, d] : \left| (f \circ g^{-1})^{(m)}(z) \right| > M \right\} &= \\ m \left\{ g(t) \in [c, d] : \left| (f \circ g^{-1})^{(m)}(g(t)) \right| > M \right\} &= \\ m \left(g \left(\left\{ t \in [a, b] : \left| (f \circ g^{-1})^{(m)}(g(t)) \right| > M \right\} \right) \right) &= 0, \end{aligned}$$

given that

$$m \left\{ t \in [a, b] : \left| \left((f \circ g^{-1})^{(m)} \circ g \right) (t) \right| > M \right\} = 0.$$

Therefore each M of (4) fulfills

$$M \in \left\{ L : m \left\{ z \in [c, d] : \left| (f \circ g^{-1})^{(m)}(z) \right| > L \right\} = 0 \right\}. \quad (5)$$

The last implies (3). ■

We give

Definition 4 (see also [10, p. 99]) The left and right fractional integrals, respectively, of a function f with respect to given function g are defined as follows:

Let $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$. Here $g \in AC([a, b])$ and is strictly increasing, $f \in L_\infty([a, b])$. We set

$$(I_{a+;g}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) f(t) dt, \quad x \geq a, \quad (6)$$

where Γ is the gamma function, clearly $(I_{a+;g}^\alpha f)(a) = 0$, $I_{a+;g}^0 f := f$ and

$$(I_{b-;g}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) f(t) dt, \quad x \leq b, \quad (7)$$

clearly $(I_{b-;g}^\alpha f)(b) = 0$, $I_{b-;g}^0 f := f$.

When g is the identity function id , we get that $I_{a+;id}^\alpha = I_{a+}^\alpha$, and $I_{b-;id}^\alpha = I_{b-}^\alpha$, the ordinary left and right Riemann-Liouville fractional integrals, where

$$(I_{a+}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x \geq a, \quad (8)$$

$(I_{a+}^\alpha f)(a) = 0$ and

$$(I_{b-}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x \leq b, \quad (9)$$

$(I_{b-}^\alpha f)(b) = 0$.

In [5], we proved

Lemma 5 Let $g \in AC([a, b])$ which is strictly increasing and f Borel measurable in $L_\infty([a, b])$. Then $f \circ g^{-1}$ is Lebesgue measurable, and

$$\|f\|_{\infty,[a,b]} \geq \|f \circ g^{-1}\|_{\infty,[g(a),g(b)]}, \quad (10)$$

i.e. $(f \circ g^{-1}) \in L_\infty([g(a), g(b)])$.

If additionally $g^{-1} \in AC([g(a), g(b)])$, then

$$\|f\|_{\infty,[a,b]} = \|f \circ g^{-1}\|_{\infty,[g(a),g(b)]}. \quad (11)$$

Remark 6 We proved ([5]) that

$$(I_{a+;g}^\alpha f)(x) = \left(I_{g(a)+}^\alpha (f \circ g^{-1}) \right)(g(x)), \quad x \geq a \quad (12)$$

and

$$(I_{b-;g}^\alpha f)(x) = \left(I_{g(b)-}^\alpha (f \circ g^{-1}) \right)(g(x)), \quad x \leq b. \quad (13)$$

It is well known that, if f is a Lebesgue measurable function, then there exists f^* a Borel measurable function, such that $f = f^*$, a.e. Also it holds $\|f\|_\infty = \|f^*\|_\infty$, and $\int \dots f \dots dx = \int \dots f^* \dots dx$.

Of course a Borel measurable function is a Lebesgue measurable function.

Thus, by Lemma 5, we get

$$\|f\|_{\infty, [a, b]} = \|f^*\|_{\infty, [a, b]} \geq \|f^* \circ g^{-1}\|_{\infty, [g(a), g(b)]}. \quad (14)$$

We observe the following:

Let $\alpha, \beta > 0$, then

$$\begin{aligned} \left(I_{a+;g}^\beta \left(I_{a+;g}^\alpha f \right) \right) (x) &= \left(I_{a+;g}^\beta \left(I_{a+;g}^\alpha f^* \right) \right) (x) = \\ I_{g(a)+}^\beta \left(\left(I_{a+;g}^\alpha f^* \right) \circ g^{-1} \right) (g(x)) &= I_{g(a)+}^\beta \left(I_{g(a)+}^\alpha \left(f^* \circ g^{-1} \right) \circ g \circ g^{-1} \right) (g(x)) = \\ &= \left(I_{g(a)+}^\beta I_{g(a)+}^\alpha \left(f^* \circ g^{-1} \right) \right) (g(x)) \stackrel{(by [8], p. 14)}{=} \\ &= \left(I_{g(a)+}^{\beta+\alpha} f^* \circ g^{-1} \right) (g(x)) = \left(I_{a+;g}^{\beta+\alpha} f^* \right) (x) = \left(I_{a+;g}^{\beta+\alpha} f \right) (x) \text{ a.e.} \end{aligned} \quad (15)$$

The last is true for all x , if $\alpha + \beta \geq 1$ or $f \in C([a, b])$.

We have proved the semigroup composition property

$$\left(I_{a+;g}^\alpha I_{a+;g}^\beta f \right) (x) = \left(I_{a+;g}^{\alpha+\beta} f \right) (x) = \left(I_{a+;g}^\beta I_{a+;g}^\alpha f \right) (x), \quad x \geq a, \quad (16)$$

a.e., which is true for all x , if $\alpha + \beta \geq 1$ or $f \in C([a, b])$.

Similarly we get

$$\begin{aligned} \left(I_{b-;g}^\beta \left(I_{b-;g}^\alpha f \right) \right) (x) &= \left(I_{b-;g}^\beta \left(I_{b-;g}^\alpha f^* \right) \right) (x) = \\ I_{g(b)-}^\beta \left(\left(I_{b-;g}^\alpha f^* \right) \circ g^{-1} \right) (g(x)) &= I_{g(b)-}^\beta \left(I_{g(b)-}^\alpha \left(f^* \circ g^{-1} \right) \circ g \circ g^{-1} \right) (g(x)) = \\ &= I_{g(b)-}^\beta \left(I_{g(b)-}^\alpha \left(f^* \circ g^{-1} \right) \right) (g(x)) \stackrel{(by [1])}{=} \\ &= \left(I_{g(b)-}^{\beta+\alpha} f^* \circ g^{-1} \right) (g(x)) = \left(I_{b-;g}^{\beta+\alpha} f^* \right) (x) = \left(I_{b-;g}^{\beta+\alpha} f \right) (x) \text{ a.e.,} \end{aligned} \quad (17)$$

true for all $x \in [a, b]$, if $\alpha + \beta \geq 1$ or $f \in C([a, b])$.

We have proved the semigroup property that

$$\left(I_{b-;g}^\alpha I_{b-;g}^\beta f \right) (x) = \left(I_{b-;g}^{\alpha+\beta} f \right) (x) = \left(I_{b-;g}^\beta I_{b-;g}^\alpha f \right) (x), \text{ a.e., } x \leq b, \quad (18)$$

which is true for all $x \in [a, b]$, if $\alpha + \beta \geq 1$ or $f \in C([a, b])$.

From now on without loss of generality, within integrals we may assume that $f = f^*$, and we mean that $f = f^*$, a.e.

We make

Definition 7 Let $\alpha > 0$, $[\alpha] = n$, $\lceil \cdot \rceil$ the ceiling of the number. Again here $g \in AC([a, b])$ and strictly increasing. We assume that $(f \circ g^{-1})^{(n)} \circ g \in L_\infty([a, b])$. We define the left generalized g -fractional derivative of f of order α as follows:

$$(D_{a+;g}^\alpha f)(x) := \frac{1}{\Gamma(n-\alpha)} \int_a^x (g(x) - g(t))^{n-\alpha-1} g'(t) (f \circ g^{-1})^{(n)}(g(t)) dt, \quad (19)$$

$x \geq a$.

If $\alpha \notin \mathbb{N}$, by [6], we have that $D_{a+;g}^\alpha f \in C([a, b])$.

We see that

$$\left(I_{a+;g}^{n-\alpha} \left((f \circ g^{-1})^{(n)} \circ g \right) \right)(x) = (D_{a+;g}^\alpha f)(x), \quad x \geq a. \quad (20)$$

We set

$$D_{a+;g}^n f(x) := \left((f \circ g^{-1})^{(n)} \circ g \right)(x), \quad (21)$$

$$D_{a+;g}^0 f(x) = f(x), \quad \forall x \in [a, b]. \quad (22)$$

When $g = id$, then

$$D_{a+;g}^\alpha f = D_{a+;id}^\alpha f = D_{*a}^\alpha f, \quad (23)$$

the usual left Caputo fractional derivative.

We make

Remark 8 Under the assumption that $(f \circ g^{-1})^{(n)} \circ g \in L_\infty([a, b])$, which could be considered as Borel measurable within integrals, we obtain

$$\begin{aligned} (I_{a+;g}^\alpha D_{a+;g}^\alpha f)(x) &= \left(I_{a+;g}^\alpha \left(I_{a+;g}^{n-\alpha} \left((f \circ g^{-1})^{(n)} \circ g \right) \right) \right)(x) = \\ &= \left(I_{a+;g}^{\alpha+n-\alpha} \left((f \circ g^{-1})^{(n)} \circ g \right) \right)(x) = I_{a+;g}^n \left((f \circ g^{-1})^{(n)} \circ g \right)(x) = \\ &= \frac{1}{(n-1)!} \int_a^x (g(x) - g(t))^{n-1} g'(t) \left((f \circ g^{-1})^{(n)} \circ g \right)(t) dt. \end{aligned} \quad (24)$$

We have proved that

$$\begin{aligned} (I_{a+;g}^\alpha D_{a+;g}^\alpha f)(x) &= \frac{1}{(n-1)!} \int_a^x (g(x) - g(t))^{n-1} g'(t) (f \circ g^{-1})^{(n)}(g(t)) dt \\ &= R_n(a, x), \quad \forall x \geq a, \end{aligned} \quad (25)$$

see (2).

But also it holds

$$\begin{aligned} R_n(a, x) &= (I_{a+;g}^\alpha D_{a+;g}^\alpha f)(x) = \\ &= \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) (D_{a+;g}^\alpha f)(t) dt, \quad x \geq a. \end{aligned} \quad (26)$$

We have proved the following g -left fractional generalized Taylor's formula:

Theorem 9 *Let g be strictly increasing function and $g \in AC([a, b])$. We assume that $(f \circ g^{-1}) \in AC^n([g(a), g(b)])$, where $\mathbb{N} \ni n = \lceil \alpha \rceil$, $\alpha > 0$. Also we assume that $(f \circ g^{-1})^{(n)} \circ g \in L_\infty([a, b])$. Then*

$$f(x) = f(a) + \sum_{k=1}^{n-1} \frac{(f \circ g^{-1})^{(k)}(g(a))}{k!} (g(x) - g(a))^k + \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) (D_{a+;g}^\alpha f)(t) dt, \quad \forall x \in [a, b]. \quad (27)$$

Calling $R_n(a, x)$ the remainder of (27), we get that

$$R_n(a, x) = \frac{1}{\Gamma(\alpha)} \int_{g(a)}^{g(x)} (g(x) - z)^{\alpha-1} ((D_{a+;g}^\alpha f) \circ g^{-1})(z) dz, \quad \forall x \in [a, b]. \quad (28)$$

Remark 10 *By [6], $R_n(a, x)$ is a continuous function in $x \in [a, b]$. Also, by [9], change of variable in Lebesgue integrals, (28) is valid.*

By [3] we have

Theorem 11 *Let $f, f', \dots, f^{(n)}; g, g'$ be continuous from $[a, b]$ into \mathbb{R} , $n \in \mathbb{N}$. Assume that $(g^{-1})^{(k)}$, $k = 0, 1, \dots, n$, are continuous. Then*

$$f(x) = f(b) + \sum_{k=1}^{n-1} \frac{(f \circ g^{-1})^{(k)}(g(b))}{k!} (g(x) - g(b))^k + R_n(b, x), \quad (29)$$

where

$$R_n(b, x) := \frac{1}{(n-1)!} \int_b^x (g(x) - g(s))^{n-1} (f \circ g^{-1})^{(n)}(g(s)) g'(s) ds \quad (30)$$

$$= \frac{1}{(n-1)!} \int_{g(b)}^{g(x)} (g(x) - t)^{n-1} (f \circ g^{-1})^{(n)}(t) dt, \quad \forall x \in [a, b]. \quad (31)$$

Notice that (29), (30) and (31) are valid under more general weaker assumptions, as follows: g is strictly increasing and $g \in AC([a, b])$, $(f \circ g^{-1}) \in AC^n([g(a), g(b)])$, and $(f \circ g^{-1})^{(n)} \in L_\infty([g(a), g(b)])$.

We make

Definition 12 Here we assume that $(f \circ g^{-1})^{(n)} \circ g \in L_\infty([a, b])$, where $N \ni n = \lceil \alpha \rceil$, $\alpha > 0$. We define the right generalized g -fractional derivative of f of order α as follows:

$$(D_{b-;g}^\alpha f)(x) := \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (g(t) - g(x))^{n-\alpha-1} g'(t) (f \circ g^{-1})^{(n)}(g(t)) dt, \quad (32)$$

all $x \in [a, b]$.

If $\alpha \notin \mathbb{N}$, by [7], we get that $(D_{b-;g}^\alpha f) \in C([a, b])$.

We see that

$$I_{b-;g}^{n-\alpha} \left((-1)^n (f \circ g^{-1})^{(n)} \circ g \right)(x) = (D_{b-;g}^\alpha f)(x), \quad a \leq x \leq b. \quad (33)$$

We set

$$\begin{aligned} D_{b-;g}^n f(x) &= (-1)^n \left((f \circ g^{-1})^{(n)} \circ g \right)(x), \\ D_{b-;g}^0 f(x) &= f(x), \quad \forall x \in [a, b]. \end{aligned} \quad (34)$$

When $g = id$, then

$$D_{b-;g}^\alpha f(x) = D_{b-;id}^\alpha f(x) = D_{b-}^\alpha f, \quad (35)$$

the usual right Caputo fractional derivative.

We make

Remark 13 Furthermore it holds

$$\begin{aligned} (I_{b-;g}^\alpha D_{b-;g}^\alpha f)(x) &= \left(I_{b-;g}^\alpha I_{b-;g}^{n-\alpha} \left((-1)^n (f \circ g^{-1})^{(n)} \circ g \right) \right)(x) = \\ &= \left(I_{b-;g}^n \left((-1)^n (f \circ g^{-1})^{(n)} \circ g \right) \right)(x) = (-1)^n \left(I_{b-;g}^n \left((f \circ g^{-1})^{(n)} \circ g \right) \right)(x) = \end{aligned} \quad (36)$$

$$\begin{aligned} &= \frac{(-1)^n}{(n-1)!} \int_x^b (g(t) - g(x))^{n-1} g'(t) \left((f \circ g^{-1})^{(n)} \circ g \right)(t) dt = \\ &= \frac{(-1)^{2n}}{(n-1)!} \int_b^x (g(x) - g(t))^{n-1} g'(t) \left((f \circ g^{-1})^{(n)} \circ g \right)(t) dt = \\ &= \frac{1}{(n-1)!} \int_b^x (g(x) - g(t))^{n-1} g'(t) \left((f \circ g^{-1})^{(n)} \circ g \right)(t) dt = R_n(b, x), \end{aligned} \quad (37)$$

as in (30).

That is

$$\begin{aligned} R_n(b, x) &= (I_{b-;g}^\alpha D_{b-;g}^\alpha f)(x) = \\ &= \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) (D_{b-;g}^\alpha f)(t) dt, \quad \text{all } a \leq x \leq b. \end{aligned} \quad (38)$$

We have proved the g -right generalized fractional Taylor's formula:

Theorem 14 *Let g be strictly increasing function and $g \in AC([a, b])$. We assume that $(f \circ g^{-1}) \in AC^n([g(a), g(b)])$, where $\mathbb{N} \ni n = [\alpha]$, $\alpha > 0$. Also we assume that $(f \circ g^{-1})^{(n)} \circ g \in L_\infty([a, b])$. Then*

$$f(x) = f(b) + \sum_{k=1}^{n-1} \frac{(f \circ g^{-1})^{(k)}(g(b))}{k!} (g(x) - g(b))^k + \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) (D_{b-;g}^\alpha f)(t) dt, \quad \text{all } a \leq x \leq b. \quad (39)$$

Calling $R_n(b, x)$ the remainder in (39), we get that

$$R_n(b, x) = \frac{1}{\Gamma(\alpha)} \int_{g(x)}^{g(b)} (z - g(x))^{\alpha-1} ((D_{b-;g}^\alpha f) \circ g^{-1})(z) dz, \quad \forall x \in [a, b]. \quad (40)$$

Remark 15 *By [7], $R_n(b, x)$ is a continuous function in $x \in [a, b]$. Also, by [9], change of variable in Lebesgue integrals, (40) is valid.*

Basics 16 *The right Riemann-Liouville fractional integral of order $\alpha > 0$, $f \in L_1([a, b])$, $a < b$, is defined as follows:*

$$I_{b-}^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (z - x)^{\alpha-1} f(z) dz, \quad \forall x \in [a, b]. \quad (41)$$

$$I_{b-}^0 := I \text{ (the identity operator).}$$

Let $\alpha, \beta \geq 0$, $f \in L_1([a, b])$. Then, by [1], we have

$$I_{b-}^\alpha I_{b-}^\beta f = I_{b-}^{\alpha+\beta} f = I_{b-}^\beta I_{b-}^\alpha f, \quad (42)$$

valid a.e. on $[a, b]$. If $f \in C([a, b])$ or $\alpha + \beta \geq 1$, then the last identity is true on all of $[a, b]$.

The right Caputo fractional derivative of order $\alpha > 0$, $m = [\alpha]$, $f \in AC^m([a, b])$ is defined as follows:

$$D_{b-}^\alpha f(x) := (-1)^m I_{b-}^{m-\alpha} f^{(m)}(x), \quad (43)$$

that is

$$D_{b-}^\alpha f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z - x)^{m-\alpha-1} f^{(m)}(z) dz, \quad \forall x \in [a, b], \quad (44)$$

with $D_{b-}^m f(x) := (-1)^m f^{(m)}(x)$.

By [1], we have the following right fractional Taylor's formula:

Let $f \in AC^m([a, b])$, $x \in [a, b]$, $\alpha > 0$, $m = \lceil \alpha \rceil$, then

$$f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!} (x-b)^k = \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} D_{b-}^{\alpha} f(z) dz = \quad (45)$$

$$\begin{aligned} (I_{b-}^{\alpha} D_{b-}^{\alpha} f)(x) &= (-1)^m \left(I_{b-}^{\alpha} I_{b-}^{m-\alpha} f^{(m)} \right)(x) = (-1)^m \left(I_{b-}^m f^{(m)} \right)(x) = \\ &= (-1)^m \frac{1}{(m-1)!} \int_x^b (z-x)^{m-1} f^{(m)}(z) dz = \\ &= (-1)^m \frac{(-1)^m}{(m-1)!} \int_b^x (x-z)^{m-1} f^{(m)}(z) dz = \\ &= \frac{1}{(m-1)!} \int_b^x (x-z)^{m-1} f^{(m)}(z) dz. \end{aligned} \quad (46)$$

That is

$$\begin{aligned} (I_{b-}^{\alpha} D_{b-}^{\alpha} f)(x) &= (-1)^m \left(I_{b-}^m f^{(m)} \right)(x) = \\ f(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!} (x-b)^k &= \frac{1}{(m-1)!} \int_b^x (x-z)^{m-1} f^{(m)}(z) dz. \end{aligned} \quad (47)$$

We make

Remark 17 If $0 < \alpha \leq 1$, then $m = 1$, hence

$$\begin{aligned} (I_{b-}^{\alpha} D_{b-}^{\alpha} f)(x) &= f(x) - f(b) \\ &= \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} D_{b-}^{\alpha} f(z) dz =: (\psi_1). \end{aligned} \quad (48)$$

[Let $f' \in L_{\infty}([a, b])$, then by [4], we get that $D_{b-}^{\alpha} f \in C([a, b])$, $0 < \alpha < 1$, where

$$(D_{b-}^{\alpha} f)(x) = \frac{(-1)}{\Gamma(1-\alpha)} \int_x^b (z-x)^{-\alpha} f'(z) dz, \quad (49)$$

with $(D_{b-}^1 f)(x) = -f'(x)$.

Also $(z-x)^{\alpha-1} > 0$, over (x, b) , and

$$\int_x^b (z-x)^{\alpha-1} dz = \frac{(b-x)^{\alpha}}{\alpha} < \infty, \quad \text{for any } 0 < \alpha \leq 1, \quad (50)$$

thus $(z-x)^{\alpha-1}$ is integrable over $[x, b]$.

By the first mean value theorem for integration, when $0 < \alpha < 1$, we get that

$$\begin{aligned} (\psi_1) &= \frac{(D_{b-}^\alpha f)(\xi_x)}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} dz = \frac{(D_{b-}^\alpha f)(\xi_x)}{\Gamma(\alpha)} \frac{(b-x)^\alpha}{\alpha} \\ &= \frac{(D_{b-}^\alpha f)(\xi_x)}{\Gamma(\alpha+1)} (b-x)^\alpha, \quad \xi_x \in [x, b]. \end{aligned} \quad (51)$$

Thus, we obtain

$$f(x) - f(b) = \frac{(D_{b-}^\alpha f)(\xi_x)}{\Gamma(\alpha+1)} (b-x)^\alpha, \quad \xi_x \in [x, b], \quad (52)$$

where $f \in AC([a, b])$.

We have proved

Theorem 18 (*Right generalized mean value theorem*). Let $f \in AC([a, b])$, $f' \in L_\infty([a, b])$, $0 < \alpha < 1$. Then

$$f(x) - f(b) = \frac{(D_{b-}^\alpha f)(\xi_x)}{\Gamma(\alpha+1)} (b-x)^\alpha, \quad (53)$$

with $x \leq \xi_x \leq b$, where $x \in [a, b]$.

If $f \in C([a, b])$ and there exists $f'(x)$, for any $x \in (a, b)$, then

$$f(x) - f(b) = (-1) f'(\xi_x) (b-x), \quad (54)$$

equivalently,

$$f(b) - f(x) = f'(\xi_x) (b-x), \quad (55)$$

the usual mean value theorem.

We make

Remark 19 *In general: we notice the following*

$$\begin{aligned} |D_{b-}^\alpha f(x)| &\leq \frac{1}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} |f^{(m)}(z)| dz \\ & \quad (\text{assuming } f^{(m)} \in L_\infty([a, b])) \\ &\leq \frac{\|f^{(m)}\|_\infty}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} dz = \frac{\|f^{(m)}\|_\infty}{\Gamma(m-\alpha)} \frac{(b-x)^{m-\alpha}}{m-\alpha} \\ &= \frac{\|f^{(m)}\|_\infty}{\Gamma(m-\alpha+1)} (b-x)^{m-\alpha} \leq \frac{\|f^{(m)}\|_\infty}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}. \end{aligned} \quad (56)$$

So when $f^{(m)} \in L_\infty([a, b])$ we get that

$$D_{b-}^\alpha f(b) = 0, \text{ where } \alpha \notin \mathbb{N}, \quad (57)$$

and

$$\|D_{b-}^\alpha f\|_\infty \leq \frac{\|f^{(m)}\|_\infty}{\Gamma(m - \alpha + 1)} (b - a)^{m - \alpha}. \quad (58)$$

In particular when $f' \in L_\infty([a, b])$, $0 < \alpha < 1$, we have that

$$D_{b-}^\alpha f(b) = 0. \quad (59)$$

Notation 20 Denote by

$$D_{b-}^{n\alpha} := D_{b-}^\alpha D_{b-}^\alpha \dots D_{b-}^\alpha \quad (n \text{ times}), n \in \mathbb{N}. \quad (60)$$

Also denote by

$$I_{b-}^{n\alpha} := I_{b-}^\alpha I_{b-}^\alpha \dots I_{b-}^\alpha \quad (n \text{ times}), n \in \mathbb{N}. \quad (61)$$

We have

Theorem 21 Suppose that $D_{b-}^{n\alpha} f, D_{b-}^{(n+1)\alpha} f \in C([a, b])$, $0 < \alpha \leq 1$. Then

$$(I_{b-}^{n\alpha} D_{b-}^{n\alpha} f)(x) - (I_{b-}^{(n+1)\alpha} D_{b-}^{(n+1)\alpha} f)(x) = \frac{(b-x)^{n\alpha}}{\Gamma(n\alpha+1)} (D_{b-}^{n\alpha} f)(b). \quad (62)$$

Proof. By (42) we get that

$$\begin{aligned} & (I_{b-}^{n\alpha} D_{b-}^{n\alpha} f)(x) - (I_{b-}^{(n+1)\alpha} D_{b-}^{(n+1)\alpha} f)(x) = \\ & I_{b-}^{n\alpha} \left((D_{b-}^{n\alpha} f)(x) - (I_{b-}^\alpha D_{b-}^{(n+1)\alpha} f)(x) \right) = \\ & I_{b-}^{n\alpha} \left((D_{b-}^{n\alpha} f)(x) - ((I_{b-}^\alpha D_{b-}^\alpha) (D_{b-}^{n\alpha} f))(x) \right) \stackrel{(48)}{=} \\ & I_{b-}^{n\alpha} \left((D_{b-}^{n\alpha} f)(x) - (D_{b-}^{n\alpha} f)(x) + (D_{b-}^{n\alpha} f)(b) \right) = \\ & I_{b-}^{n\alpha} \left((D_{b-}^{n\alpha} f)(b) \right) = \frac{(b-x)^{n\alpha}}{\Gamma(n\alpha+1)} (D_{b-}^{n\alpha} f)(b). \end{aligned} \quad (63)$$

■

Remark 22 Suppose that $D_{b-}^{k\alpha} f \in C([a, b])$, for $k = 0, 1, \dots, n+1$; $0 < \alpha \leq 1$. By (62) we get that

$$\begin{aligned} & \sum_{i=0}^n \left((I_{b-}^{i\alpha} D_{b-}^{i\alpha} f)(x) - (I_{b-}^{(i+1)\alpha} D_{b-}^{(i+1)\alpha} f)(x) \right) = \\ & \sum_{i=0}^n \frac{(b-x)^{i\alpha}}{\Gamma(i\alpha+1)} (D_{b-}^{i\alpha} f)(b). \end{aligned} \quad (64)$$

That is

$$f(x) - \left(I_{b-}^{(n+1)\alpha} D_{b-}^{(n+1)\alpha} f \right)(x) = \sum_{i=0}^n \frac{(b-x)^{i\alpha}}{\Gamma(i\alpha+1)} (D_{b-}^{i\alpha} f)(b). \quad (65)$$

Hence it holds

$$\begin{aligned} f(x) &= \sum_{i=0}^n \frac{(b-x)^{i\alpha}}{\Gamma(i\alpha+1)} (D_{b-}^{i\alpha} f)(b) + \left(I_{b-}^{(n+1)\alpha} D_{b-}^{(n+1)\alpha} f \right)(x) = \\ &\quad \sum_{i=0}^n \frac{(b-x)^{i\alpha}}{\Gamma(i\alpha+1)} (D_{b-}^{i\alpha} f)(b) + R^*(x, b), \end{aligned} \quad (66)$$

where

$$R^*(x, b) := \frac{1}{\Gamma((n+1)\alpha)} \int_x^b (z-x)^{(n+1)\alpha-1} \left(D_{b-}^{(n+1)\alpha} f \right)(z) dz. \quad (67)$$

We see that (there exists $\xi_x \in [x, b]$:)

$$\begin{aligned} R^*(x, b) &= \frac{\left(D_{b-}^{(n+1)\alpha} f \right)(\xi_x)}{\Gamma((n+1)\alpha)} \int_x^b (z-x)^{(n+1)\alpha-1} dz = \\ &= \frac{\left(D_{b-}^{(n+1)\alpha} f \right)(\xi_x)}{\Gamma((n+1)\alpha)} \frac{(b-x)^{(n+1)\alpha}}{(n+1)\alpha} = \frac{\left(D_{b-}^{(n+1)\alpha} f \right)(\xi_x)}{\Gamma((n+1)\alpha+1)} (b-x)^{(n+1)\alpha}. \end{aligned} \quad (68)$$

We have proved the following right generalized fractional Taylor's formula:

Theorem 23 Suppose that $D_{b-}^k f \in C([a, b])$, for $k = 0, 1, \dots, n+1$, where $0 < \alpha \leq 1$. Then

$$f(x) = \sum_{i=0}^n \frac{(b-x)^{i\alpha}}{\Gamma(i\alpha+1)} (D_{b-}^{i\alpha} f)(b) + \quad (69)$$

$$\begin{aligned} &\frac{1}{\Gamma((n+1)\alpha)} \int_x^b (z-x)^{(n+1)\alpha-1} \left(D_{b-}^{(n+1)\alpha} f \right)(z) dz = \\ &\sum_{i=0}^n \frac{(b-x)^{i\alpha}}{\Gamma(i\alpha+1)} (D_{b-}^{i\alpha} f)(b) + \frac{\left(D_{b-}^{(n+1)\alpha} f \right)(\xi_x)}{\Gamma((n+1)\alpha+1)} (b-x)^{(n+1)\alpha}, \end{aligned} \quad (70)$$

where $\xi_x \in [x, b]$, with $x \in [a, b]$.

We make

Remark 24 Let $\alpha > 0$, $m = [\alpha]$, g is strictly increasing and $g \in AC([a, b])$. Call $l = f \circ g^{-1}$, $f : [a, b] \rightarrow \mathbb{R}$. Assume that $l \in AC^m([c, d])$ (i.e. $l^{(m-1)} \in AC([c, d])$) (where $g([a, b]) = [c, d]$, $c, d \in \mathbb{R} : g(a) = c$, $g(b) = d$; hence here f is continuous on $[a, b]$).

Assume also that $(f \circ g^{-1})^{(m)} \circ g \in L_\infty([a, b])$.

The right generalized g -fractional derivative of f of order α is defined as follows:

$$(D_{b-;g}^{\alpha} f)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (g(t) - g(x))^{m-\alpha-1} g'(t) (f \circ g^{-1})^{(m)}(g(t)) dt, \quad (71)$$

$$a \leq x \leq b.$$

We saw that

$$I_{b-;g}^{m-\alpha} \left((-1)^m (f \circ g^{-1})^{(m)} \circ g \right)(x) = (D_{b-;g}^{\alpha} f)(x), \quad a \leq x \leq b. \quad (72)$$

We proved earlier (37), (38), (39) that ($a \leq x \leq b$)

$$\begin{aligned} (I_{b-;g}^{\alpha} D_{b-;g}^{\alpha} f)(x) &= \\ \frac{1}{(m-1)!} \int_b^x (g(x) - g(t))^{m-1} g'(t) \left((f \circ g^{-1})^{(m)} \circ g \right)(t) dt &= \\ \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) (D_{b-;g}^{\alpha} f)(t) dt &= \\ f(x) - f(b) - \sum_{k=1}^{m-1} \frac{(f \circ g^{-1})^{(k)}(g(b))}{k!} (g(x) - g(b))^k. \end{aligned} \quad (73)$$

If $0 < \alpha \leq 1$, then $m = 1$, hence

$$\begin{aligned} (I_{b-;g}^{\alpha} D_{b-;g}^{\alpha} f)(x) &= f(x) - f(b) \\ &= \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) (D_{b-;g}^{\alpha} f)(t) dt \end{aligned} \quad (74)$$

(when $\alpha \in (0, 1)$, $D_{b-;g}^{\alpha} f$ is continuous on $[a, b]$ and)

$$= \frac{(D_{b-;g}^{\alpha} f)(\xi_x)}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) dt = \frac{(D_{b-;g}^{\alpha} f)(\xi_x)}{\Gamma(\alpha+1)} (g(b) - g(x))^{\alpha}, \quad (75)$$

where $\xi_x \in [x, b]$.

We have proved

Theorem 25 (right generalized g -mean value theorem). Let $0 < \alpha < 1$, and $f \circ g^{-1} \in AC([c, d])$, $(f \circ g^{-1})' \circ g \in L_{\infty}([a, b])$, where g strictly increasing, $g \in AC([a, b])$, $f : [a, b] \rightarrow \mathbb{R}$. Then

$$f(x) - f(b) = \frac{(D_{b-;g}^{\alpha} f)(\xi_x)}{\Gamma(\alpha+1)} (g(b) - g(x))^{\alpha}, \quad (76)$$

where $\xi_x \in [x, b]$, for $x \in [a, b]$.

Denote by

$$D_{b-;g}^{n\alpha} := D_{b-;g}^{\alpha} D_{b-;g}^{\alpha} \dots D_{b-;g}^{\alpha} \quad (n \text{ times}), \quad n \in \mathbb{N}. \quad (77)$$

Also denote by

$$I_{b-;g}^{n\alpha} := I_{b-;g}^{\alpha} I_{b-;g}^{\alpha} \dots I_{b-;g}^{\alpha} \quad (n \text{ times}). \quad (78)$$

Here to remind

$$(I_{b-;g}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) f(t) dt, \quad x \leq b. \quad (79)$$

We need

Theorem 26 Suppose that $F_k := D_{b-;g}^{k\alpha} f$, $k = n, n+1$, fulfill $F_k \circ g^{-1} \in AC([c, d])$, and $(F_k \circ g^{-1})' \circ g \in L_{\infty}([a, b])$, $0 < \alpha \leq 1$, $n \in \mathbb{N}$. Then

$$(I_{b-;g}^{n\alpha} D_{b-;g}^{n\alpha} f)(x) - (I_{b-;g}^{(n+1)\alpha} D_{b-;g}^{(n+1)\alpha} f)(x) = \frac{(g(b) - g(x))^{n\alpha}}{\Gamma(n\alpha + 1)} (D_{b-;g}^{n\alpha} f)(b). \quad (80)$$

Proof. By semigroup property of $I_{b-;g}^{\alpha}$, we get

$$\begin{aligned} (I_{b-;g}^{n\alpha} D_{b-;g}^{n\alpha} f)(x) - (I_{b-;g}^{(n+1)\alpha} D_{b-;g}^{(n+1)\alpha} f)(x) &= \\ (I_{b-;g}^{n\alpha} (D_{b-;g}^{n\alpha} f - I_{b-;g}^{\alpha} D_{b-;g}^{(n+1)\alpha} f))(x) &= \end{aligned} \quad (81)$$

$$\begin{aligned} (I_{b-;g}^{n\alpha} (D_{b-;g}^{n\alpha} f - (I_{b-;g}^{\alpha} D_{b-;g}^{\alpha} (D_{b-;g}^{n\alpha} f))))(x) &\stackrel{(74)}{=} \\ (I_{b-;g}^{n\alpha} (D_{b-;g}^{n\alpha} f - D_{b-;g}^{n\alpha} f + D_{b-;g}^{n\alpha} f(b)))(x) &= \\ (I_{b-;g}^{n\alpha} (D_{b-;g}^{n\alpha} f(b)))(x) &= (D_{b-;g}^{n\alpha} f(b)) (I_{b-;g}^{n\alpha} (1))(x) = \end{aligned} \quad (82)$$

[Notice that

$$\begin{aligned} (I_{b-;g}^{\alpha} 1)(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) dt = \\ \frac{1}{\Gamma(\alpha)} \frac{(g(b) - g(x))^{\alpha}}{\alpha} &= \frac{1}{\Gamma(\alpha + 1)} (g(b) - g(x))^{\alpha}. \end{aligned} \quad (83)$$

Thus we have

$$(I_{b-;g}^{\alpha} 1)(x) = \frac{(g(b) - g(x))^{\alpha}}{\Gamma(\alpha + 1)}. \quad (84)$$

Hence it holds

$$(I_{b-;g}^{2\alpha} 1)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (g(t) - g(x))^{\alpha-1} g'(t) \frac{(g(b) - g(t))^{\alpha}}{\Gamma(\alpha + 1)} dt =$$

$$\begin{aligned}
& \frac{1}{\Gamma(\alpha)\Gamma(\alpha+1)} \int_x^b (g(b)-g(t))^\alpha (g(t)-g(x))^{\alpha-1} g'(t) dt = \\
& \frac{1}{\Gamma(\alpha)\Gamma(\alpha+1)} \int_{g(x)}^{g(b)} (g(b)-z)^{(\alpha+1)-1} (z-g(x))^{\alpha-1} dz = \\
& \frac{1}{\Gamma(\alpha)\Gamma(\alpha+1)} \frac{\Gamma(\alpha+1)\Gamma(\alpha)}{\Gamma(2\alpha+1)} (g(b)-g(x))^{2\alpha} = \frac{1}{\Gamma(2\alpha+1)} (g(b)-g(x))^{2\alpha},
\end{aligned} \tag{85}$$

etc.]

$$= (D_{b-;g}^{n\alpha} f)(b) \frac{(g(b)-g(x))^{n\alpha}}{\Gamma(n\alpha+1)}, \tag{86}$$

proving the claim. ■

We make

Remark 27 Suppose that $F_k = D_{b-;g}^{k\alpha} f$, for $k = 0, 1, \dots, n+1$; are as in last Theorem 26, $0 < \alpha \leq 1$. By (80) we get

$$\begin{aligned}
& \sum_{i=0}^n \left((I_{b-;g}^{i\alpha} D_{b-;g}^{i\alpha} f)(x) - I_{b-;g}^{(i+1)\alpha} D_{b-;g}^{(i+1)\alpha} f(x) \right) = \\
& \sum_{i=0}^n \frac{(g(b)-g(x))^{i\alpha}}{\Gamma(i\alpha+1)} (D_{b-;g}^{i\alpha} f)(b).
\end{aligned} \tag{87}$$

That is

$$(notice that I_{b-;g}^0 f = D_{b-;g}^0 f = f)$$

$$f(x) - \left(I_{b-;g}^{(n+1)\alpha} D_{b-;g}^{(n+1)\alpha} f \right)(x) = \sum_{i=0}^n \frac{(g(b)-g(x))^{i\alpha}}{\Gamma(i\alpha+1)} (D_{b-;g}^{i\alpha} f)(b). \tag{88}$$

Hence

$$f(x) = \sum_{i=0}^n \frac{(g(b)-g(x))^{i\alpha}}{\Gamma(i\alpha+1)} (D_{b-;g}^{i\alpha} f)(b) + \left(I_{b-;g}^{(n+1)\alpha} D_{b-;g}^{(n+1)\alpha} f \right)(x) = \tag{89}$$

$$\sum_{i=0}^n \frac{(g(b)-g(x))^{i\alpha}}{\Gamma(i\alpha+1)} (D_{b-;g}^{i\alpha} f)(b) + R_g(x, b), \tag{90}$$

where

$$R_g(x, b) := \frac{1}{\Gamma((n+1)\alpha)} \int_x^b (g(t)-g(x))^{(n+1)\alpha-1} g'(t) \left(D_{b-;g}^{(n+1)\alpha} f \right)(t) dt. \tag{91}$$

(here $D_{b-;g}^{(n+1)\alpha} f$ is continuous over $[a, b]$)

Hence it holds

$$R_g(x, b) = \frac{\left(D_{b-;g}^{(n+1)\alpha} f\right)(\psi_x)}{\Gamma((n+1)\alpha)} \int_x^b (g(t) - g(x))^{(n+1)\alpha-1} g'(t) dt =$$

$$\frac{\left(D_{b-;g}^{(n+1)\alpha} f\right)(\psi_x)}{\Gamma((n+1)\alpha)} \frac{(g(b) - g(x))^{(n+1)\alpha}}{(n+1)\alpha} = \frac{\left(D_{b-;g}^{(n+1)\alpha} f\right)(\psi_x)}{\Gamma((n+1)\alpha + 1)} (g(b) - g(x))^{(n+1)\alpha}, \quad (92)$$

where $\psi_x \in [x, b]$.

We have proved the following g -right generalized modified Taylor's formula:

Theorem 28 Suppose that $F_k := D_{b-;g}^{k\alpha} f$, for $k = 0, 1, \dots, n+1$, fulfill: $F_k \circ g^{-1} \in AC([c, d])$ and $(F_k \circ g^{-1})' \circ g \in L_\infty([a, b])$, where $0 < \alpha \leq 1$. Then

$$f(x) = \sum_{i=0}^n \frac{(g(b) - g(x))^{i\alpha}}{\Gamma(i\alpha + 1)} (D_{b-;g}^{i\alpha} f)(b) +$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_x^b (g(t) - g(x))^{(n+1)\alpha-1} g'(t) \left(D_{b-;g}^{(n+1)\alpha} f\right)(t) dt = \quad (93)$$

$$\sum_{i=0}^n \frac{(g(b) - g(x))^{i\alpha}}{\Gamma(i\alpha + 1)} (D_{b-;g}^{i\alpha} f)(b) + \frac{\left(D_{b-;g}^{(n+1)\alpha} f\right)(\psi_x)}{\Gamma((n+1)\alpha + 1)} (g(b) - g(x))^{(n+1)\alpha}, \quad (94)$$

where $\psi_x \in [x, b]$, any $x \in [a, b]$.

We make

Remark 29 Let $\alpha > 0$, $m = \lceil \alpha \rceil$, g is strictly increasing and $g \in AC([a, b])$. Call $l = f \circ g^{-1}$, $f : [a, b] \rightarrow \mathbb{R}$. Assume $l \in AC^m([c, d])$ (i.e. $l^{(m-1)} \in AC([c, d])$) (where $g([a, b]) = [c, d]$, $c, d \in \mathbb{R} : g(a) = c, g(b) = d$, hence here f is continuous on $[a, b]$).

Assume also that $(f \circ g^{-1})^{(m)} \circ g \in L_\infty([a, b])$.

The left generalized g -fractional derivative of f of order α is defined as follows:

$$(D_{a+;g}^\alpha f)(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x (g(x) - g(t))^{m-\alpha-1} g'(t) (f \circ g^{-1})^{(m)}(g(t)) dt, \quad (95)$$

$x \geq a$.

If $\alpha \notin \mathbb{N}$, then $(D_{a+;g}^\alpha f) \in C([a, b])$.

We see that

$$\left(I_{a+;g}^{m-\alpha} \left((f \circ g^{-1})^{(m)} \circ g\right)\right)(x) = (D_{a+;g}^\alpha f)(x), \quad x \geq a. \quad (96)$$

We proved earlier (24), (25), (26), (27), that $(a \leq x \leq b)$

$$\begin{aligned} & (I_{a+;g}^\alpha D_{a+;g}^\alpha f)(x) = \\ & \frac{1}{(m-1)!} \int_a^x (g(x) - g(t))^{m-1} g'(t) \left((f \circ g^{-1})^{(m)} \circ g \right)(t) dt = \quad (97) \\ & \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) (D_{a+;g}^\alpha f)(t) dt = \end{aligned}$$

$$f(x) - f(a) - \sum_{k=1}^{m-1} \frac{(f \circ g^{-1})^{(k)}(g(a))}{k!} (g(x) - g(a))^k. \quad (98)$$

If $0 < \alpha \leq 1$, then $m = 1$, and then

$$\begin{aligned} & (I_{a+;g}^\alpha D_{a+;g}^\alpha f)(x) = f(x) - f(a) \quad (99) \\ & = \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) (D_{a+;g}^\alpha f)(t) dt \\ & \quad (\alpha \in (0,1) \text{ case}) \frac{(D_{a+;g}^\alpha f)(\xi_x)}{\Gamma(\alpha+1)} (g(x) - g(a))^\alpha, \quad (100) \end{aligned}$$

where $\xi_x \in [a, x]$, any $x \in [a, b]$.

We have proved

Theorem 30 (left generalized g -mean value theorem). Let $0 < \alpha < 1$ and $f \circ g^{-1} \in AC([c, d])$ and $(f \circ g^{-1})' \circ g \in L_\infty([a, b])$, where g strictly increasing, $g \in AC([a, b])$, $f : [a, b] \rightarrow \mathbb{R}$. Then

$$f(x) - f(a) = \frac{(D_{a+;g}^\alpha f)(\xi_x)}{\Gamma(\alpha+1)} (g(x) - g(a))^\alpha, \quad (101)$$

where $\xi_x \in [a, x]$, any $x \in [a, b]$.

Denote by

$$D_{a+;g}^{n\alpha} := D_{a+;g}^\alpha D_{a+;g}^\alpha \dots D_{a+;g}^\alpha \quad (n \text{ times}), n \in \mathbb{N}. \quad (102)$$

Also denote by

$$I_{a+;g}^{n\alpha} := I_{a+;g}^\alpha I_{a+;g}^\alpha \dots I_{a+;g}^\alpha \quad (n \text{ times}). \quad (103)$$

Here to remind

$$(I_{a+;g}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) f(t) dt, \quad x \geq a. \quad (104)$$

By convention $I_{a+;g}^0 = D_{a+;g}^0 = I$ (identity operator).

We give

Theorem 31 Suppose that $F_k := D_{a+;g}^{k\alpha} f$, $k = n, n+1$, fulfill $F_k \circ g^{-1} \in AC([c, d])$, and $(F_k \circ g^{-1})' \circ g \in L_\infty([a, b])$, $0 < \alpha \leq 1$, $n \in \mathbb{N}$. Then

$$(I_{a+;g}^{n\alpha} D_{a+;g}^{n\alpha} f)(x) - (I_{a+;g}^{(n+1)\alpha} D_{a+;g}^{(n+1)\alpha} f)(x) = \frac{(g(x) - g(a))^{n\alpha}}{\Gamma(n\alpha + 1)} (D_{a+;g}^{n\alpha} f)(a). \quad (105)$$

Proof. By semigroup property of $I_{a+;g}^\alpha$, we get

$$(I_{a+;g}^{n\alpha} D_{a+;g}^{n\alpha} f)(x) - (I_{a+;g}^{(n+1)\alpha} D_{a+;g}^{(n+1)\alpha} f)(x) = (I_{a+;g}^{n\alpha} (D_{a+;g}^{n\alpha} f - I_{a+;g}^\alpha D_{a+;g}^{(n+1)\alpha} f))(x) = \quad (106)$$

$$\begin{aligned} & (I_{a+;g}^{n\alpha} (D_{a+;g}^{n\alpha} f - (I_{a+;g}^\alpha D_{a+;g}^\alpha (D_{a+;g}^{n\alpha} f))))(x) \stackrel{(99)}{=} \\ & (I_{a+;g}^{n\alpha} (D_{a+;g}^{n\alpha} f - D_{a+;g}^{n\alpha} f + D_{a+;g}^{n\alpha} f(a)))(x) = \\ & (I_{a+;g}^{n\alpha} (D_{a+;g}^{n\alpha} f(a)))(x) = (D_{a+;g}^{n\alpha} f(a)) (I_{a+;g}^{n\alpha} (1))(x) = \end{aligned} \quad (107)$$

[notice that

$$\begin{aligned} (I_{a+;g}^\alpha 1)(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) dt \\ &= \frac{(g(x) - g(a))^\alpha}{\Gamma(\alpha + 1)}. \end{aligned} \quad (108)$$

Hence

$$(I_{a+;g}^{2\alpha} 1)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) \frac{(g(t) - g(a))^\alpha}{\Gamma(\alpha + 1)} dt = \quad (109)$$

$$\begin{aligned} & \frac{1}{\Gamma(\alpha) \Gamma(\alpha + 1)} \int_a^x (g(x) - g(t))^{\alpha-1} g'(t) (g(t) - g(a))^\alpha dt = \\ & \frac{1}{\Gamma(\alpha) \Gamma(\alpha + 1)} \int_{g(a)}^{g(x)} (g(x) - z)^{\alpha-1} (z - g(a))^{(\alpha+1)-1} dz = \\ & \frac{1}{\Gamma(\alpha) \Gamma(\alpha + 1)} \frac{\Gamma(\alpha) \Gamma(\alpha + 1)}{\Gamma(2\alpha + 1)} (g(x) - g(a))^{2\alpha}. \end{aligned}$$

That is

$$(I_{a+;g}^{2\alpha} 1)(x) = \frac{(g(x) - g(a))^{2\alpha}}{\Gamma(2\alpha + 1)}, \quad (110)$$

etc.]

$$= (D_{a+;g}^{n\alpha} f(a)) \frac{(g(x) - g(a))^{n\alpha}}{\Gamma(n\alpha + 1)}, \quad (111)$$

proving the claim. ■

Remark 32 Suppose that $F_k = D_{a+;g}^{k\alpha} f$, for $k = 0, 1, \dots, n+1$; are as in Theorem 31, $0 < \alpha \leq 1$. By (105) we get

$$\sum_{i=0}^n \left((I_{a+;g}^{i\alpha} D_{a+;g}^{i\alpha} f)(x) - I_{a+;g}^{(i+1)\alpha} D_{a+;g}^{(i+1)\alpha} f(x) \right) = \quad (112)$$

$$\sum_{i=0}^n \frac{(g(x) - g(a))^{i\alpha}}{\Gamma(i\alpha + 1)} (D_{a+;g}^{i\alpha} f)(a).$$

That is

$$f(x) - \left(I_{a+;g}^{(n+1)\alpha} D_{a+;g}^{(n+1)\alpha} f \right)(x) = \sum_{i=0}^n \frac{(g(x) - g(a))^{i\alpha}}{\Gamma(i\alpha + 1)} (D_{a+;g}^{i\alpha} f)(a).$$

Hence

$$f(x) = \sum_{i=0}^n \frac{(g(x) - g(a))^{i\alpha}}{\Gamma(i\alpha + 1)} (D_{a+;g}^{i\alpha} f)(a) + \left(I_{a+;g}^{(n+1)\alpha} D_{a+;g}^{(n+1)\alpha} f \right)(x) = \quad (113)$$

$$\sum_{i=0}^n \frac{(g(x) - g(a))^{i\alpha}}{\Gamma(i\alpha + 1)} (D_{a+;g}^{i\alpha} f)(a) + R_g(a, x), \quad (114)$$

where

$$R_g(a, x) := \frac{1}{\Gamma((n+1)\alpha)} \int_a^x (g(x) - g(t))^{(n+1)\alpha-1} g'(t) \left(D_{a+;g}^{(n+1)\alpha} f \right)(t) dt. \quad (115)$$

(there $D_{a+;g}^{(n+1)\alpha} f$ is continuous over $[a, b]$.)

Hence it holds

$$R_g(a, x) = \frac{\left(D_{a+;g}^{(n+1)\alpha} f \right)(\psi_x)}{\Gamma((n+1)\alpha)} \left(\int_a^x (g(x) - g(t))^{(n+1)\alpha-1} g'(t) dt \right) =$$

$$\frac{\left(D_{a+;g}^{(n+1)\alpha} f \right)(\psi_x)}{\Gamma((n+1)\alpha + 1)} (g(x) - g(a))^{(n+1)\alpha}, \quad (116)$$

where $\psi_x \in [a, x]$.

We have proved the following g -left generalized modified Taylor's formula:

Theorem 33 Suppose that $F_k := D_{a+;g}^{k\alpha} f$, for $k = 0, 1, \dots, n+1$, fulfill: $F_k \circ g^{-1} \in AC([c, d])$ and $(F_k \circ g^{-1})' \circ g \in L_\infty([a, b])$, where $0 < \alpha \leq 1$. Then

$$f(x) = \sum_{i=0}^n \frac{(g(x) - g(a))^{i\alpha}}{\Gamma(i\alpha + 1)} (D_{a+;g}^{i\alpha} f)(a) + \quad (117)$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_a^x (g(x) - g(t))^{(n+1)\alpha-1} g'(t) \left(D_{a+;g}^{(n+1)\alpha} f \right)(t) dt =$$

$$\sum_{i=0}^n \frac{(g(x) - g(a))^{i\alpha}}{\Gamma(i\alpha + 1)} \left(D_{a+;g}^{i\alpha} f \right)(a) + \frac{\left(D_{a+;g}^{(n+1)\alpha} f \right)(\psi_x)}{\Gamma((n+1)\alpha + 1)} (g(x) - g(a))^{(n+1)\alpha},$$
(118)

where $\psi_x \in [a, x]$, any $x \in [a, b]$.

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Generalized Canavati type Fractional Taylor's formulae

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Abstract

We present here four new generalized Canavati type fractional Taylor's formulae.

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1 Results

Let $g : [a, b] \rightarrow \mathbb{R}$ be a strictly increasing function. Let $f \in C^n([a, b])$, $n \in \mathbb{N}$. Assume that $g \in C^1([a, b])$, and $g^{-1} \in C^n([a, b])$. Call $l := f \circ g^{-1} : [g(a), g(b)] \rightarrow \mathbb{R}$. It is clear that $l, l', \dots, l^{(n)}$ are continuous functions from $[g(a), g(b)]$ into $f([a, b]) \subseteq \mathbb{R}$.

Let $\nu \geq 1$ such that $[\nu] = n$, $n \in \mathbb{N}$ as above, where $[\cdot]$ is the integral part of the number.

Clearly when $0 < \nu < 1$, $[\nu] = 0$. Next we follow [1], pp. 7-9.

I) Let $h \in C([g(a), g(b)])$, we define the left Riemann-Liouville fractional integral as

$$(J_\nu^{z_0} h)(z) := \frac{1}{\Gamma(\nu)} \int_{z_0}^z (z-t)^{\nu-1} h(t) dt, \quad (1)$$

for $g(a) \leq z_0 \leq z \leq g(b)$, where Γ is the gamma function; $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt$.

We set $J_0^{z_0} h = h$.

Let $\alpha := \nu - [\nu]$ ($0 < \alpha < 1$). We define the subspace $C_{g(x_0)}^\nu([g(a), g(b)])$ of $C^{[\nu]}([g(a), g(b)])$, where $x_0 \in [a, b]$:

$$C_{g(x_0)}^\nu([g(a), g(b)]) := \left\{ h \in C^{[\nu]}([g(a), g(b)]) : J_{1-\alpha}^{g(x_0)} h^{([\nu])} \in C^1([g(x_0), g(b)]) \right\}. \quad (2)$$

So let $h \in C_{g(x_0)}^\nu([g(a), g(b)])$; we define the left g -generalized fractional derivative of h of order ν , of Canavati type, over $[g(x_0), g(b)]$ as

$$D_{g(x_0)}^\nu h := \left(J_{1-\alpha}^{g(x_0)} h^{([\nu])} \right)' . \quad (3)$$

Clearly, for $h \in C_{g(x_0)}^\nu([g(a), g(b)])$, there exists

$$\left(D_{g(x_0)}^\nu h \right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x_0)}^z (z-t)^{-\alpha} h^{([\nu])}(t) dt, \quad (4)$$

for all $g(x_0) \leq z \leq g(b)$.

In particular, when $f \circ g^{-1} \in C_{g(x_0)}^\nu([g(a), g(b)])$ we have that

$$\left(D_{g(x_0)}^\nu (f \circ g^{-1}) \right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x_0)}^z (z-t)^{-\alpha} (f \circ g^{-1})^{([\nu])}(t) dt, \quad (5)$$

for all $g(x_0) \leq z \leq g(b)$. We have $D_{g(x_0)}^n (f \circ g^{-1}) = (f \circ g^{-1})^{(n)}$ and $D_{g(x_0)}^0 (f \circ g^{-1}) = f \circ g^{-1}$.

By Theorem 2.1, p. 8 of [1], we have for $f \circ g^{-1} \in C_{g(x_0)}^\nu([g(a), g(b)])$, where $x_0 \in [a, b]$ is fixed, that

(i) if $\nu \geq 1$, then

$$\begin{aligned} (f \circ g^{-1})(z) &= \sum_{k=0}^{[\nu]-1} \frac{(f \circ g^{-1})^{(k)}(g(x_0))}{k!} (z - g(x_0))^k + \\ &\quad \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^z (z-t)^{\nu-1} \left(D_{g(x_0)}^\nu (f \circ g^{-1}) \right)(t) dt, \end{aligned} \quad (6)$$

all $z \in [g(a), g(b)] : z \geq g(x_0)$,

(ii) if $0 < \nu < 1$, we get

$$(f \circ g^{-1})(z) = \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^z (z-t)^{\nu-1} \left(D_{g(x_0)}^\nu (f \circ g^{-1}) \right)(t) dt, \quad (7)$$

all $z \in [g(a), g(b)] : z \geq g(x_0)$.

We have proved the following left generalized g -fractional, of Canavati type, Taylor's formula:

Theorem 1 Let $f \circ g^{-1} \in C_{g(x_0)}^\nu([g(a), g(b)])$, where $x_0 \in [a, b]$ is fixed.

(i) if $\nu \geq 1$, then

$$\begin{aligned} f(x) - f(x_0) &= \sum_{k=1}^{[\nu]-1} \frac{(f \circ g^{-1})^{(k)}(g(x_0))}{k!} (g(x) - g(x_0))^k + \\ &\quad \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D_{g(x_0)}^\nu (f \circ g^{-1}) \right)(t) dt, \quad \text{all } x \in [a, b] : x \geq x_0, \end{aligned} \quad (8)$$

(ii) if $0 < \nu < 1$, we get

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D_{g(x_0)}^{\nu} (f \circ g^{-1}) \right) (t) dt, \quad \text{all } x \in [a, b] : x \geq x_0. \quad (9)$$

By the change of variable method, see [3], we may rewrite the remainder of (8), (9), as

$$\begin{aligned} & \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D_{g(x_0)}^{\nu} (f \circ g^{-1}) \right) (t) dt = \\ & \frac{1}{\Gamma(\nu)} \int_{x_0}^x (g(x) - g(s))^{\nu-1} \left(D_{g(x_0)}^{\nu} (f \circ g^{-1}) \right) (g(s)) g'(s) ds, \end{aligned} \quad (10)$$

all $x \in [a, b] : x \geq x_0$.

We may rewrite (9) as follows:

if $0 < \nu < 1$, we have

$$f(x) = \left(J_{\nu}^{g(x_0)} \left(D_{g(x_0)}^{\nu} (f \circ g^{-1}) \right) \right) (g(x)), \quad (11)$$

all $x \in [a, b] : x \geq x_0$.

II) Next we follow [2], pp. 345-348.

Let $h \in C([g(a), g(b)])$, we define the right Riemann-Liouville fractional integral as

$$(J_{z_0-}^{\nu} h)(z) := \frac{1}{\Gamma(\nu)} \int_z^{z_0} (t - z)^{\nu-1} h(t) dt, \quad (12)$$

for $g(a) \leq z \leq z_0 \leq g(b)$. We set $J_{z_0-}^0 h = h$.

Let $\alpha := \nu - [\nu]$ ($0 < \alpha < 1$). We define the subspace $C_{g(x_0)-}^{\nu}([g(a), g(b)])$ of $C^{[\nu]}([g(a), g(b)])$, where $x_0 \in [a, b]$:

$$\begin{aligned} & C_{g(x_0)-}^{\nu}([g(a), g(b)]) := \\ & \left\{ h \in C^{[\nu]}([g(a), g(b)]) : J_{g(x_0)-}^{1-\alpha} h^{([\nu])} \in C^1([g(x_0), g(b)]) \right\}. \end{aligned} \quad (13)$$

So let $h \in C_{g(x_0)-}^{\nu}([g(a), g(b)])$; we define the right g -generalized fractional derivative of h of order ν , of Canavati type, over $[g(a), g(x_0)]$ as

$$D_{g(x_0)-}^{\nu} h := (-1)^{n-1} \left(J_{g(x_0)-}^{1-\alpha} h^{([\nu])} \right)'. \quad (14)$$

Clearly, for $h \in C_{g(x_0)-}^{\nu}([g(a), g(b)])$, there exists

$$\left(D_{g(x_0)-}^{\nu} h \right) (z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_z^{g(x_0)} (t - z)^{-\alpha} h^{([\nu])}(t) dt, \quad (15)$$

for all $g(a) \leq z \leq g(x_0) \leq g(b)$.

In particular, when $f \circ g^{-1} \in C_{g(x_0)-}^\nu([g(a), g(b)])$ we have that

$$\left(D_{g(x_0)-}^\nu(f \circ g^{-1})\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_z^{g(x_0)} (t-z)^{-\alpha} (f \circ g^{-1})^{([\nu])}(t) dt, \quad (16)$$

for all $g(a) \leq z \leq g(x_0) \leq g(b)$.

We get that

$$\left(D_{g(x_0)-}^n(f \circ g^{-1})\right)(z) = (-1)^n (f \circ g^{-1})^{(n)}(z) \quad (17)$$

and $\left(D_{g(x_0)-}^0(f \circ g^{-1})\right)(z) = (f \circ g^{-1})(z)$, all $z \in [g(a), g(x_0)]$.

By Theorem 23.19, p. 348 of [2], we have for $f \circ g^{-1} \in C_{g(x_0)-}^\nu([g(a), g(b)])$, where $x_0 \in [a, b]$ is fixed, that

(i) if $\nu \geq 1$, then

$$(f \circ g^{-1})(z) = \sum_{k=0}^{[\nu]-1} \frac{(f \circ g^{-1})^{(k)}(g(x_0))}{k!} (z - g(x_0))^k + \quad (18)$$

$$\frac{1}{\Gamma(\alpha)} \int_z^{g(x_0)} (t-z)^{\nu-1} \left(D_{g(x_0)-}^\nu(f \circ g^{-1})\right)(t) dt,$$

all $z \in [g(a), g(b)] : z \leq g(x_0)$,

(ii) if $0 < \nu < 1$, we get

$$(f \circ g^{-1})(z) = \frac{1}{\Gamma(\nu)} \int_z^{g(x_0)} (t-z)^{\nu-1} \left(D_{g(x_0)-}^\nu(f \circ g^{-1})\right)(t) dt, \quad (19)$$

all $z \in [g(a), g(b)] : z \leq g(x_0)$.

We have proved the following right generalized g -fractional, of Canavati type, Taylor's formula:

Theorem 2 Let $f \circ g^{-1} \in C_{g(x_0)-}^\nu([g(a), g(b)])$, where $x_0 \in [a, b]$ is fixed.

(i) if $\nu \geq 1$, then

$$f(x) - f(x_0) = \sum_{k=1}^{[\nu]-1} \frac{(f \circ g^{-1})^{(k)}(g(x_0))}{k!} (g(x) - g(x_0))^k +$$

$$\frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t-g(x))^{\nu-1} \left(D_{g(x_0)-}^\nu(f \circ g^{-1})\right)(t) dt, \quad \text{all } a \leq x \leq x_0, \quad (20)$$

(ii) if $0 < \nu < 1$, we get

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t-g(x))^{\nu-1} \left(D_{g(x_0)-}^\nu(f \circ g^{-1})\right)(t) dt, \quad \text{all } a \leq x \leq x_0. \quad (21)$$

By change of variable, see [3], we may rewrite the remainder of (20), (21), as

$$\begin{aligned} & \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D_{g(x_0)-}^{\nu} (f \circ g^{-1}) \right) (t) dt = \\ & \frac{1}{\Gamma(\nu)} \int_x^{x_0} (g(s) - g(x))^{\nu-1} \left(D_{g(x_0)-}^{\nu} (f \circ g^{-1}) \right) (g(s)) g'(s) ds, \end{aligned} \quad (22)$$

all $a \leq x \leq x_0$.

We may rewrite (21) as follows:

if $0 < \nu < 1$, we have

$$f(x) = \left(J_{g(x_0)-}^{\nu} \left(D_{g(x_0)-}^{\nu} (f \circ g^{-1}) \right) \right) (g(x)), \quad (23)$$

all $a \leq x \leq x_0 \leq b$.

III) Denote by

$$D_{g(x_0)}^{m\nu} = D_{g(x_0)}^{\nu} D_{g(x_0)}^{\nu} \dots D_{g(x_0)}^{\nu} \quad (m\text{-times}), \quad m \in \mathbb{N}. \quad (24)$$

Also denote by

$$J_{m\nu}^{g(x_0)} = J_{\nu}^{g(x_0)} J_{\nu}^{g(x_0)} \dots J_{\nu}^{g(x_0)} \quad (m\text{-times}), \quad m \in \mathbb{N}. \quad (25)$$

We need

Theorem 3 Here $0 < \nu < 1$. Assume that $\left(D_{g(x_0)}^{m\nu} (f \circ g^{-1}) \right) \in C_{g(x_0)}^{\nu} ([g(a), g(b)])$, where $x_0 \in [a, b]$ is fixed. Assume also that $\left(D_{g(x_0)}^{(m+1)\nu} (f \circ g^{-1}) \right) \in C([g(x_0), g(b)])$. Then

$$\left(J_{m\nu}^{g(x_0)} D_{g(x_0)}^{m\nu} (f \circ g^{-1}) \right) (g(x)) - \left(J_{(m+1)\nu}^{g(x_0)} D_{g(x_0)}^{(m+1)\nu} (f \circ g^{-1}) \right) (g(x)) = 0, \quad (26)$$

for all $x_0 \leq x \leq b$.

Proof. We observe that $(l := f \circ g^{-1})$

$$\begin{aligned} & \left(J_{m\nu}^{g(x_0)} D_{g(x_0)}^{m\nu} (l) \right) (g(x)) - \left(J_{(m+1)\nu}^{g(x_0)} D_{g(x_0)}^{(m+1)\nu} (l) \right) (g(x)) = \\ & \left(J_{m\nu}^{g(x_0)} \left(D_{g(x_0)}^{m\nu} (l) - J_{\nu}^{g(x_0)} D_{g(x_0)}^{(m+1)\nu} (l) \right) \right) (g(x)) = \\ & \left(J_{m\nu}^{g(x_0)} \left(D_{g(x_0)}^{m\nu} (l) - \left(J_{\nu}^{g(x_0)} D_{g(x_0)}^{\nu} \right) \left(\left(D_{g(x_0)}^{m\nu} (l) \right) \circ g \circ g^{-1} \right) \right) \right) (g(x)) = \\ & \left(J_{m\nu}^{g(x_0)} \left(D_{g(x_0)}^{m\nu} (l) - \left(D_{g(x_0)}^{m\nu} (l) \right) \right) \right) (g(x)) = \left(J_{m\nu}^{g(x_0)} (0) \right) (g(x)) = 0. \end{aligned} \quad (27)$$

■

We make

Remark 4 Let $0 < \nu < 1$. Assume that $\left(D_{g(x_0)}^{i\nu} (f \circ g^{-1})\right) \in C_{g(x_0)}^\nu ([g(a), g(b)])$, $x_0 \in [a, b]$, for all $i = 0, 1, \dots, m$. Assume also that $\left(D_{g(x_0)}^{(m+1)\nu} (f \circ g^{-1})\right) \in C([g(x_0), g(b)])$. We have that

$$\sum_{i=0}^m \left[\left(J_{i\nu}^{g(x_0)} D_{g(x_0)}^{i\nu} (f \circ g^{-1})\right) (g(x)) - \left(J_{(i+1)\nu}^{g(x_0)} D_{g(x_0)}^{(i+1)\nu} (f \circ g^{-1})\right) (g(x)) \right] = 0. \quad (28)$$

Hence it holds

$$f(x) - \left(J_{(m+1)\nu}^{g(x_0)} D_{g(x_0)}^{(m+1)\nu} (f \circ g^{-1})\right) (g(x)) = 0, \quad (29)$$

for all $x_0 \leq x \leq b$.

That is

$$f(x) = \left(J_{(m+1)\nu}^{g(x_0)} D_{g(x_0)}^{(m+1)\nu} (f \circ g^{-1})\right) (g(x)), \quad (30)$$

for all $x_0 \leq x \leq b$.

We have proved the following modified and generalized left fractional Taylor's formula of Canavati type:

Theorem 5 Let $0 < \nu < 1$. Assume that $\left(D_{g(x_0)}^{i\nu} (f \circ g^{-1})\right) \in C_{g(x_0)}^\nu ([g(a), g(b)])$, $x_0 \in [a, b]$, for $i = 0, 1, \dots, m$. Assume also that $\left(D_{g(x_0)}^{(m+1)\nu} (f \circ g^{-1})\right) \in C([g(x_0), g(b)])$. Then

$$f(x) = \frac{1}{\Gamma((m+1)\nu)} \int_{g(x_0)}^{g(x)} (g(x) - z)^{(m+1)\nu-1} \left(D_{g(x_0)}^{(m+1)\nu} (f \circ g^{-1})\right) (z) dz \quad (31)$$

$$= \frac{1}{\Gamma((m+1)\nu)} \int_{x_0}^x (g(x) - g(s))^{(m+1)\nu-1} \left(D_{g(x_0)}^{(m+1)\nu} (f \circ g^{-1})\right) (g(s)) g'(s) ds,$$

all $x_0 \leq x \leq b$.

IV) Denote by

$$D_{g(x_0)-}^{m\nu} = D_{g(x_0)-}^\nu D_{g(x_0)-}^\nu \dots D_{g(x_0)-}^\nu \quad (m\text{-times}), \quad m \in \mathbb{N}. \quad (32)$$

Also denote by

$$J_{g(x_0)-}^{m\nu} = J_{g(x_0)-}^\nu J_{g(x_0)-}^\nu \dots J_{g(x_0)-}^\nu \quad (m\text{-times}), \quad m \in \mathbb{N}. \quad (33)$$

We need

Theorem 6 Here $0 < \nu < 1$. Assume that $\left(D_{g(x_0)-}^{m\nu} (f \circ g^{-1})\right) \in C_{g(x_0)-}^\nu ([g(a), g(b)])$, where $x_0 \in [a, b]$ is fixed. Assume also that $\left(D_{g(x_0)-}^{(m+1)\nu} (f \circ g^{-1})\right) \in C([g(a), g(x_0)])$. Then

$$\left(J_{g(x_0)-}^{m\nu} D_{g(x_0)-}^{m\nu} (f \circ g^{-1})\right) (g(x)) - \left(J_{g(x_0)-}^{(m+1)\nu} D_{g(x_0)-}^{(m+1)\nu} (f \circ g^{-1})\right) (g(x)) = 0, \quad (34)$$

for all $a \leq x \leq x_0$.

Proof. We observe that $(l := f \circ g^{-1})$

$$\begin{aligned} & \left(J_{g(x_0)-}^{m\nu} D_{g(x_0)-}^{m\nu} (l) \right) (g(x)) - \left(J_{g(x_0)-}^{(m+1)\nu} D_{g(x_0)-}^{(m+1)\nu} (l) \right) (g(x)) = \\ & \left(J_{g(x_0)-}^{m\nu} \left(D_{g(x_0)-}^{m\nu} (l) - J_{g(x_0)-}^{\nu} D_{g(x_0)-}^{(m+1)\nu} (l) \right) \right) (g(x)) = \\ & \left(J_{g(x_0)-}^{m\nu} \left(D_{g(x_0)-}^{m\nu} (l) - \left(J_{g(x_0)-}^{\nu} D_{g(x_0)-}^{\nu} \right) \left(D_{g(x_0)-}^{m\nu} (l) \circ g \circ g^{-1} \right) \right) \right) (g(x)) = \\ & \left(J_{g(x_0)-}^{m\nu} \left(D_{g(x_0)-}^{m\nu} (l) - D_{g(x_0)-}^{m\nu} (l) \right) \right) (g(x)) = J_{g(x_0)-}^{m\nu} (0) (g(x)) = 0. \end{aligned} \quad (35)$$

■

We make

Remark 7 Let $0 < \nu < 1$. Assume that $\left(D_{g(x_0)-}^{i\nu} (f \circ g^{-1}) \right) \in C_{g(x_0)-}^{\nu} ([g(a), g(b)])$, $x_0 \in [a, b]$, for all $i = 0, 1, \dots, m$. Assume also that $\left(D_{g(x_0)-}^{(m+1)\nu} (f \circ g^{-1}) \right) \in C([g(a), g(x_0)])$. We have that (by (34))

$$\sum_{i=0}^m \left[\left(J_{g(x_0)-}^{i\nu} D_{g(x_0)-}^{i\nu} (f \circ g^{-1}) \right) (g(x)) - \left(J_{g(x_0)-}^{(i+1)\nu} D_{g(x_0)-}^{(i+1)\nu} (f \circ g^{-1}) \right) (g(x)) \right] = 0. \quad (36)$$

Hence it holds

$$f(x) - \left(J_{g(x_0)-}^{(m+1)\nu} D_{g(x_0)-}^{(m+1)\nu} (f \circ g^{-1}) \right) (g(x)) = 0, \quad (37)$$

for all $a \leq x \leq x_0 \leq b$.

That is

$$f(x) = \left(J_{g(x_0)-}^{(m+1)\nu} D_{g(x_0)-}^{(m+1)\nu} (f \circ g^{-1}) \right) (g(x)), \quad (38)$$

for all $a \leq x \leq x_0 \leq b$.

We have proved the following modified and generalized right fractional Taylor's formula of Canavati type:

Theorem 8 Let $0 < \nu < 1$. Assume that $\left(D_{g(x_0)-}^{i\nu} (f \circ g^{-1}) \right) \in C_{g(x_0)-}^{\nu} ([g(a), g(b)])$, $x_0 \in [a, b]$, for all $i = 0, 1, \dots, m$. Assume also that $\left(D_{g(x_0)-}^{(m+1)\nu} (f \circ g^{-1}) \right) \in C([g(a), g(x_0)])$. Then

$$\begin{aligned} f(x) &= \frac{1}{\Gamma((m+1)\nu)} \int_{g(x)}^{g(x_0)} (z - g(x))^{(m+1)\nu-1} \left(D_{g(x_0)-}^{(m+1)\nu} (f \circ g^{-1}) \right) (z) dz \\ &= \frac{1}{\Gamma((m+1)\nu)} \int_x^{x_0} (g(s) - g(x))^{(m+1)\nu-1} \left(D_{g(x_0)-}^{(m+1)\nu} (f \circ g^{-1}) \right) (g(s)) g'(s) ds, \\ &\text{all } a \leq x \leq x_0 \leq b. \end{aligned} \quad (39)$$

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Properties on a subclass of univalent functions defined by using Sălăgean operator and Ruscheweyh derivative

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Abstract

In this paper we have introduced and studied the subclass $\mathcal{L}(d, \alpha, \beta)$ of univalent functions defined by the linear operator $L_\gamma^n f(z)$ defined by using the Ruscheweyh derivative $R^n f(z)$ and the Sălăgean operator $S^n f(z)$, as $L_\gamma^n : \mathcal{A} \rightarrow \mathcal{A}$, $L_\gamma^n f(z) = (1 - \gamma)R^n f(z) + \gamma S^n f(z)$, $z \in U$, where $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \dots, z \in U\}$ is the class of normalized analytic functions with $\mathcal{A}_1 = \mathcal{A}$. The main object is to investigate several properties such as coefficient estimates, distortion theorems, closure theorems, neighborhoods and the radii of starlikeness, convexity and close-to-convexity of functions belonging to the class $\mathcal{L}(d, \alpha, \beta)$.

Keywords: univalent function, Starlike functions, Convex functions, Distortion theorem.

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1 Introduction

Denote by U the unit disc of the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U .

Let $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \dots, z \in U\}$ with $\mathcal{A}_1 = \mathcal{A}$.

Definition 1.1 (Sălăgean [8]) For $f \in \mathcal{A}$, $n \in \mathbb{N}$, the operator S^n is defined by $S^n : \mathcal{A} \rightarrow \mathcal{A}$,

$$\begin{aligned} S^0 f(z) &= f(z), \quad S^1 f(z) = z f'(z), \quad \dots \\ S^{n+1} f(z) &= z (S^n f(z))', \quad z \in U. \end{aligned}$$

Remark 1.1 If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $S^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j$, for $z \in U$.

Definition 1.2 (Ruscheweyh [7]) For $f \in \mathcal{A}$, $n \in \mathbb{N}$, the operator R^n is defined by $R^n : \mathcal{A} \rightarrow \mathcal{A}$,

$$\begin{aligned} R^0 f(z) &= f(z), \quad R^1 f(z) = z f'(z), \quad \dots \\ (n+1) R^{n+1} f(z) &= z (R^n f(z))' + n R^n f(z), \quad z \in U. \end{aligned}$$

Remark 1.2 If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$, $z \in U$.

Definition 1.3 [1] Let $\gamma \geq 0$, $n \in \mathbb{N}$. Denote by L_γ^n the operator given by $L_\gamma^n : \mathcal{A} \rightarrow \mathcal{A}$, $L_\gamma^n f(z) = (1 - \gamma)R^n f(z) + \gamma S^n f(z)$, $z \in U$.

Remark 1.3 If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $L_\gamma^n f(z) = z + \sum_{j=2}^{\infty} \left(\gamma j^n + (1 - \gamma) \frac{(n+j-1)!}{n!(j-1)!} \right) a_j z^j$, $z \in U$.

This operator was studied also in [2], [3], [4], [5].

We follow the works of A.R. Juma and H. Ziraz .

Definition 1.4 Let the function $f \in \mathcal{A}$. Then $f(z)$ is said to be in the class $\mathcal{L}(d, \alpha, \beta)$ if it satisfies the following criterion:

$$\left| \frac{1}{d} \left(\frac{z(L_\gamma^n f(z))' + \alpha z^2 (L_\gamma^n f(z))''}{(1 - \alpha)L_\gamma^n f(z) + \alpha z (L_\gamma^n f(z))'} - 1 \right) \right| < \beta, \quad (1.1)$$

where $d \in \mathbb{C} - \{0\}$, $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, $z \in U$.

In this paper we shall first deduce a necessary and sufficient condition for a function $f(z)$ to be in the class $\mathcal{L}(d, \alpha, \beta)$. Then obtain the distortion and growth theorems, closure theorems, neighborhood and radii of univalent starlikeness, convexity and close-to-convexity of order δ , $0 \leq \delta < 1$, for these functions.

2 Coefficient Inequality

Theorem 2.1 *Let the function $f \in \mathcal{A}$. Then $f(z)$ is said to be in the class $\mathcal{L}(d, \alpha, \beta)$ if and only if*

$$\sum_{j=2}^{\infty} (1 + \alpha(j-1))(j-1 + \beta|d|) \left\{ \gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j \leq \beta|d|, \quad (2.1)$$

where $d \in \mathbb{C} - \{0\}$, $0 \leq \alpha \leq 1$, $0 < \beta \leq 1$, $z \in U$.

Proof. Let $f(z) \in \mathcal{L}(d, \alpha, \beta)$. Assume that inequality (2.1) holds true. Then we find that

$$\left| \frac{z(L_\gamma^n f(z))' + \alpha z^2 (L_\gamma^n f(z))''}{(1-\alpha)L_\gamma^n f(z) + \alpha z(L_\gamma^n f(z))'} - 1 \right| = \left| \frac{\sum_{j=2}^{\infty} (1+\alpha(j-1))(j-1) [\gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!}] a_j z^j}{z + \sum_{j=2}^{\infty} (1+\alpha(j-1)) [\gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!}] a_j z^j} \right| \leq$$

$$\frac{\sum_{j=2}^{\infty} (1+\alpha(j-1))(j-1) [\gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!}] a_j |z|^{j-1}}{1 - \sum_{j=2}^{\infty} (1+\alpha(j-1)) [\gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!}] a_j |z|^{j-1}} < \beta|d|.$$

Choosing values of z on real axis and letting $z \rightarrow 1^-$, we have

$$\sum_{j=2}^{\infty} (1 + \alpha(j-1))(j-1 + \beta|d|) \left[\gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right] a_j \leq \beta|d|. \text{ Conversely, assume that } f(z) \in \mathcal{L}(d, \alpha, \beta),$$

then we get the following inequality $\operatorname{Re} \left\{ \frac{z(L_\gamma^n f(z))' + \alpha z^2 (L_\gamma^n f(z))''}{(1-\alpha)L_\gamma^n f(z) + \alpha z(L_\gamma^n f(z))'} - 1 \right\} > -\beta|d|$,

$$\operatorname{Re} \left\{ \frac{z + \sum_{j=2}^{\infty} j(1+\alpha(j-1)) [\gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!}] a_j z^j}{z + \sum_{j=2}^{\infty} (1+\alpha(j-1)) [\gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!}] a_j z^j} - 1 + \beta|d| \right\} > 0$$

$$\operatorname{Re} \frac{\beta|d|z + \sum_{j=2}^{\infty} (1+\alpha(j-1))(j-1 + \beta|d|) [\gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!}] a_j z^j}{z + \sum_{j=2}^{\infty} (1+\alpha(j-1)) [\gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!}] a_j z^j} > 0. \text{ Since } \operatorname{Re}(-e^{i\theta}) \geq -|e^{i\theta}| = -1, \text{ the above inequality}$$

reduces to $\frac{\beta|d|r - \sum_{j=2}^{\infty} (1+\alpha(j-1))(j-1 + \beta|d|) [\gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!}] a_j r^j}{r - \sum_{j=2}^{\infty} (1+\alpha(j-1)) [\gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!}] a_j r^j} > 0$. Letting $r \rightarrow 1^-$ and by the mean value theorem we have desired inequality (2.1). This completes the proof of Theorem 2.1 ■

Corollary 2.2 *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{L}(d, \alpha, \beta)$. Then $a_j \leq \frac{\beta|d|}{(1+\alpha(j-1))(j-1 + \beta|d|) [\gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!}]}$, $j \geq 2$.*

3 Distortion Theorems

Theorem 3.1 *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{L}(d, \alpha, \beta)$. Then for $|z| = r < 1$, we have*

$$r - \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n\gamma + (1-\gamma)(n+1)]} r^2 \leq |f(z)| \leq r + \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n\gamma + (1-\gamma)(n+1)]} r^2.$$

The result is sharp for the function $f(z)$ given by $f(z) = z + \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n\gamma + (1-\gamma)(n+1)]} z^2$.

Proof. Given that $f(z) \in \mathcal{L}(d, \alpha, \beta)$, from the equation (2.1) and since $(1+\alpha)(1+\beta|d|) [2^n\gamma + (1-\gamma)(n+1)]$ is non decreasing and positive for $j \geq 2$, then we have $(1+\alpha)(1+\beta|d|) [2^n\gamma + (1-\gamma)(n+1)] \sum_{j=2}^{\infty} a_j \leq \sum_{j=2}^{\infty} (1+\alpha(j-1))(j-1 + \beta|d|) \left\{ \gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j \leq \beta|d|$, which is equivalent to,

$$\sum_{j=2}^{\infty} a_j \leq \frac{\beta|d|}{(1+\alpha)(1+\beta|d|) [2^n\gamma + (1-\gamma)(n+1)]}. \quad (3.1)$$

Using (3.1), we obtain for $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ that $|f(z)| \leq |z| + \sum_{j=2}^{\infty} a_j |z|^j \leq r + \sum_{j=2}^{\infty} a_j r^j \leq r + r^2 \sum_{j=2}^{\infty} a_j \leq r + \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n\gamma + (1-\gamma)(n+1)]} r^2$. Similarly, $|f(z)| \geq r^2 - \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n\gamma + (1-\gamma)(n+1)]} r^2$. This completes the proof of Theorem 3.1. ■

Theorem 3.2 *Let the function $f \in \mathcal{A}$ be in the class $\mathcal{L}(d, \alpha, \beta)$. Then for $|z| = r < 1$, we have*

$$-\frac{2\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n\gamma + (1-\gamma)(n+1)]} r \leq |f'(z)| \leq \frac{2\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n\gamma + (1-\gamma)(n+1)]} r.$$

The result is sharp for the function $f(z)$ given by $f(z) = z + \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n\gamma + (1-\gamma)(n+1)]} z^2$.

Proof. From (3.1) we obtain $f'(z) = 1 + \sum_{j=2}^{\infty} j a_j z^{j-1}$ and $|f'(z)| \leq 1 + \sum_{j=2}^{\infty} j a_j |z|^{j-1} \leq 1 + \sum_{j=2}^{\infty} j a_j r^{j-1} \leq 1 + \frac{2\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n\gamma+(1-\gamma)(n+1)]} r$. Similarly, $|f'(z)| \geq 1 - \frac{2\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n\gamma+(1-\gamma)(n+1)]} r$. This completes the proof of Theorem 3.2. ■

4 Closure Theorems

Theorem 4.1 Let the functions f_k , $k = 1, 2, \dots, m$, defined by

$$f_k(z) = z + \sum_{j=2}^{\infty} a_{j,k} z^j, \quad a_{j,k} \geq 0, \quad (4.1)$$

be in the class $\mathcal{L}(d, \alpha, \beta)$. Then the function $h(z)$ defined by $h(z) = \sum_{k=1}^m \mu_k f_k(z)$, $\mu_k \geq 0$, is also in the class $\mathcal{L}(d, \alpha, \beta)$, where $\sum_{k=1}^m \mu_k = 1$.

Proof. We can write $h(z) = \sum_{k=1}^m \mu_k z + \sum_{k=1}^m \sum_{j=2}^{\infty} \mu_k a_{j,k} z^j = z + \sum_{j=2}^{\infty} \sum_{k=1}^m \mu_k a_{j,k} z^j$. Furthermore, since the functions $f_k(z)$, $k = 1, 2, \dots, m$, are in the class $\mathcal{L}(d, \alpha, \beta)$, then from Theorem 2.1 we have $\sum_{j=2}^{\infty} (1 + \alpha(j-1))(j-1+\beta|d|) \left\{ \gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_{j,k} \leq \beta|d|$. Thus it is enough to prove that $\sum_{j=2}^{\infty} (1 + \alpha(j-1))(j-1+\beta|d|) \left\{ \gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} (\sum_{k=1}^m \mu_k a_{j,k}) = \sum_{k=1}^m \mu_k \sum_{j=2}^{\infty} (1 + \alpha(j-1))(j-1+\beta|d|) \left\{ \gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_{j,k} \leq \sum_{k=1}^m \mu_k \beta|d| = \beta|d|$. Hence the proof is complete. ■

Corollary 4.2 Let the functions f_k , $k = 1, 2$, defined by (4.1) be in the class $\mathcal{L}(d, \alpha, \beta)$. Then the function $h(z)$ defined by $h(z) = (1-\zeta)f_1(z) + \zeta f_2(z)$, $0 \leq \zeta \leq 1$, is also in the class $\mathcal{L}(d, \alpha, \beta)$.

Theorem 4.3 Let $f_1(z) = z$, and $f_j(z) = z + \frac{\beta|d|}{(1+\alpha(j-1))(j-1+\beta|d|) \left\{ \gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\}} z^j$, $j \geq 2$. Then the function $f(z)$ is in the class $\mathcal{L}(d, \alpha, \beta)$ if and only if it can be expressed in the form $f(z) = \mu_1 f_1(z) + \sum_{j=2}^{\infty} \mu_j f_j(z)$, where $\mu_1 \geq 0$, $\mu_j \geq 0$, $j \geq 2$ and $\mu_1 + \sum_{j=2}^{\infty} \mu_j = 1$.

Proof. Assume that $f(z)$ can be expressed in the form $f(z) = \mu_1 f_1(z) + \sum_{j=2}^{\infty} \mu_j f_j(z) = z + \sum_{j=2}^{\infty} \frac{\beta|d|}{(1+\alpha(j-1))(j-1+\beta|d|) \left\{ \gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\}} \mu_j z^j$. Thus $\sum_{j=2}^{\infty} \frac{(1+\alpha(j-1))(j-1+\beta|d|) \left\{ \gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\}}{\beta|d|} \mu_j = \sum_{j=2}^{\infty} \mu_j = 1 - \mu_1 \leq 1$. Hence $f(z) \in \mathcal{L}(d, \alpha, \beta)$.

Conversely, assume that $f(z) \in \mathcal{L}(d, \alpha, \beta)$.

Setting $\mu_j = \frac{(1+\alpha(j-1))(j-1+\beta|d|) \left\{ \gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\}}{\beta|d|} a_j$, since $\mu_1 = 1 - \sum_{j=2}^{\infty} \mu_j$. Thus $f(z) = \mu_1 f_1(z) + \sum_{j=2}^{\infty} \mu_j f_j(z)$. Hence the proof is complete. ■

Corollary 4.4 The extreme points of the class $\mathcal{L}(d, \alpha, \beta)$ are the functions $f_1(z) = z$, and $f_j(z) = z + \frac{\beta|d|}{(1+\alpha(j-1))(j-1+\beta|d|) \left\{ \gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\}} z^j$, $j \geq 2$.

5 Inclusion and Neighborhood Results

We define the δ - neighborhood of a function $f(z) \in \mathcal{A}$ by

$$N_{\delta}(f) = \{g \in \mathcal{A} : g(z) = z + \sum_{j=2}^{\infty} b_j z^j \text{ and } \sum_{j=2}^{\infty} j |a_j - b_j| \leq \delta\}. \quad (5.1)$$

In particular, for $e(z) = z$

$$N_{\delta}(e) = \{g \in \mathcal{A} : g(z) = z + \sum_{j=2}^{\infty} b_j z^j \text{ and } \sum_{j=2}^{\infty} j |b_j| \leq \delta\}. \quad (5.2)$$

Furthermore, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{L}^{\xi}(d, \alpha, \beta)$ if there exists a function $h(z) \in \mathcal{L}(d, \alpha, \beta)$ such that

$$\left| \frac{f(z)}{h(z)} - 1 \right| < 1 - \xi, \quad z \in U, \quad 0 \leq \xi < 1. \quad (5.3)$$

Theorem 5.1 If $\left\{ \gamma j^n + (1 - \gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} \geq [2^n \gamma + (1 - \gamma)(n + 1)]$, $j \geq 2$, and $\delta = \frac{2\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n \gamma + (1-\gamma)(n+1)]}$, then $\mathcal{L}(d, \alpha, \beta) \subset N_\delta(e)$.

Proof. Let $f \in \mathcal{L}(d, \alpha, \beta)$. Then in view of assertion (2.1) of Theorem 2.1 and the condition $\left\{ \gamma j^n + (1 - \gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} \geq [2^n \gamma + (1 - \gamma)(n + 1)]$ for $j \geq 2$, we get $(1+\alpha)(1+\beta|d|)[2^n \gamma + (1 - \gamma)(n + 1)] \sum_{j=2}^{\infty} a_j \leq \sum_{j=2}^{\infty} (1 + \alpha(j - 1))(j - 1 + \beta|d|) \left\{ \gamma j^n + (1 - \gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\} a_j \leq \beta|d|$, which implise

$$\sum_{j=2}^{\infty} a_j \leq \frac{\beta|d|}{(1 + \alpha)(1 + \beta|d|)[2^n \gamma + (1 - \gamma)(n + 1)]}. \quad (5.4)$$

Applying assertion (2.1) of Theorem 2.1 in conjunction with (5.4), we obtain $(1+\alpha)(1+\beta|d|)[2^n \gamma + (1 - \gamma)(n + 1)] \sum_{j=2}^{\infty} a_j \leq \beta|d|$, $2(1+\alpha)(1+\beta|d|)[2^n \gamma + (1 - \gamma)(n + 1)] \sum_{j=2}^{\infty} a_j \leq 2\beta|d|$ and $\sum_{j=2}^{\infty} j a_j \leq \frac{2\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n \gamma + (1-\gamma)(n+1)]} = \delta$, by virtue of (5.1), we have $f \in N_\delta(e)$.

This completes the proof of the Theorem 5.1. ■

Theorem 5.2 If $h \in \mathcal{L}(d, \alpha, \beta)$ and

$$\xi = 1 - \frac{\delta}{2} \frac{(1 + \alpha)(1 + \beta|d|)[2^n \gamma + (1 - \gamma)(n + 1)]}{(1 + \alpha)(1 + \beta|d|)[2^n \gamma + (1 - \gamma)(n + 1)] - \beta|d|}, \quad (5.5)$$

then $N_\delta(h) \subset \mathcal{L}^\xi(d, \alpha, \beta)$.

Proof. Suppose that $f \in N_\delta(h)$, we then find from (5.1) that $\sum_{j=2}^{\infty} j|a_j - b_j| \leq \delta$, which readily implies the following coefficient inequality

$$\sum_{j=2}^{\infty} |a_j - b_j| \leq \frac{\delta}{2}. \quad (5.6)$$

Next, since $h \in \mathcal{L}(d, \alpha, \beta)$ in the view of (5.4), we have

$$\sum_{j=2}^{\infty} b_j \leq \frac{\beta|d|}{(1 + \alpha)(1 + \beta|d|)[2^n \gamma + (1 - \gamma)(n + 1)]}. \quad (5.7)$$

Using (5.6) and (5.7), we get $\left| \frac{f(z)}{h(z)} - 1 \right| \leq \frac{\sum_{j=2}^{\infty} |a_j - b_j|}{1 - \sum_{j=2}^{\infty} b_j} \leq \frac{\delta}{2(1 - \frac{\beta|d|}{(1+\alpha)(1+\beta|d|)[2^n \gamma + (1-\gamma)(n+1)]})} \leq \frac{\delta}{2} \frac{(1+\alpha)(1+\beta|d|)[2^n \gamma + (1-\gamma)(n+1)]}{(1+\alpha)(1+\beta|d|)[2^n \gamma + (1-\gamma)(n+1)] - \beta|d|} = 1 - \xi$, provided that ξ is given by (5.5), thus by condition (5.3), $f \in \mathcal{L}^\xi(d, \alpha, \beta)$, where ξ is given by (5.5). ■

6 Radii of Starlikeness, Convexity and Close-to-Convexity

Theorem 6.1 Let the function $f \in \mathcal{A}$ be in the class $\mathcal{L}(d, \alpha, \beta)$. Then f is univalent starlike of order δ , $0 \leq \delta < 1$, in $|z| < r_1$, where $r_1 = \inf_j \left\{ \frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\left\{ \gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\}}{\beta|d|(1-\delta)} \right\}^{\frac{1}{j-1}}$. The result is sharp for the function $f(z)$ given by $f_j(z) = z + \frac{\beta|d|}{(1+\alpha(j-1))(j-1+\beta|d|)\left\{ \gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\}} z^j$, $j \geq 2$.

Proof. It suffices to show that $\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \delta$, $|z| < r_1$. Since $\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{\sum_{j=2}^{\infty} (j-1)a_j z^{j-1}}{1 + \sum_{j=2}^{\infty} a_j z^{j-1}} \right| \leq \frac{\sum_{j=2}^{\infty} (j-1)a_j |z|^{j-1}}{1 - \sum_{j=2}^{\infty} a_j |z|^{j-1}}$. To prove the theorem, we must show that $\frac{\sum_{j=2}^{\infty} (j-1)a_j |z|^{j-1}}{1 - \sum_{j=2}^{\infty} a_j |z|^{j-1}} \leq 1 - \delta$. It is equivalent to $\sum_{j=2}^{\infty} (j-1)a_j |z|^{j-1} \leq (1 - \delta)(1 + \sum_{j=2}^{\infty} a_j |z|^{j-1})$, using Theorem 2.1, we obtain $|z| \leq \left\{ \frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\left\{ \gamma j^n + (1-\gamma) \frac{(n+j-1)!}{n!(j-1)!} \right\}}{\beta|d|(1-\delta)} \right\}^{\frac{1}{j-1}}$. Hence the proof is complete. ■

Theorem 6.2 Let the function $f \in \mathcal{A}$ be in the class $\mathcal{L}(d, \alpha, \beta)$. Then f is univalent convex of order δ , $0 \leq \delta \leq 1$, in $|z| < r_2$, where $r_2 = \inf_j \left\{ \frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2(j-\delta)\beta|d|} \right\}^{\frac{1}{k-p}}$. The result is sharp for the function $f(z)$ given by

$$f_j(z) = z + \frac{\beta|d|}{(1+\alpha(j-1))(j-1+\beta|d|)\left\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}} z^j, \quad j \geq 2. \quad (6.1)$$

Proof. It suffices to show that $\left| \frac{zf''(z)}{f'(z)} \right| \leq 1 - \delta$, $|z| < r_2$. Since $\left| \frac{zf''(z)}{f'(z)} \right| = \left| \frac{\sum_{j=2}^{\infty} j(j-1)a_j z^{j-1}}{1 + \sum_{j=2}^{\infty} j a_j z^{j-1}} \right| \leq \frac{\sum_{j=2}^{\infty} j(j-1)a_j |z|^{j-1}}{1 - \sum_{j=2}^{\infty} j a_j |z|^{j-1}}$. To prove the theorem, we must show that $\frac{\sum_{j=2}^{\infty} j(j-1)a_j |z|^{j-1}}{1 - \sum_{j=2}^{\infty} j a_j |z|^{j-1}} \leq 1 - \delta$, and $\sum_{j=2}^{\infty} j(j-\delta)a_j |z|^{j-1} \leq 1 - \delta$, using Theorem 2.1, we obtain $|z|^{j-1} \leq \frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2(j-\delta)\beta|d|}$, or $|z| \leq \left\{ \frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2(j-\delta)\beta|d|} \right\}^{\frac{1}{j-1}}$. Hence the proof is complete. ■

Theorem 6.3 Let the function $f \in \mathcal{A}$ be in the class $\mathcal{L}(d, \alpha, \beta)$. Then f is univalent close-to-convex of order δ , $0 \leq \delta < 1$, in $|z| < r_3$, where $r_3 = \inf_j \left\{ \frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{j\beta|d|} \right\}^{\frac{1}{j-1}}$. The result is sharp for the function $f(z)$ given by (6.1).

Proof. It suffices to show that $|f'(z) - 1| \leq 1 - \delta$, $|z| < r_3$. Then $|f'(z) - 1| = \left| \sum_{j=2}^{\infty} j a_j z^{j-1} \right| \leq \sum_{j=2}^{\infty} j a_j |z|^{j-1}$. Thus $|f'(z) - 1| \leq 1 - \delta$ if $\sum_{j=2}^{\infty} \frac{j a_j}{1-\delta} |z|^{j-1} \leq 1$. Using Theorem 2.1, the above inequality holds true if $|z|^{j-1} \leq \frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{j\beta|d|}$ or $|z| \leq \left\{ \frac{(1-\delta)(1+\alpha(j-1))(j-1+\beta|d|)\{\gamma j^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{j\beta|d|} \right\}^{\frac{1}{j-1}}$. Hence the proof is complete. ■

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About some differential sandwich theorems using a multiplier transformation and Ruscheweyh derivative

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Abstract

In this work we study a new operator $IR_{\lambda,l}^{m,n}$ defined as the Hadamard product of the multiplier transformation $I(m, \lambda, l)$ and Ruscheweyh derivative R^n , given by $IR_{\lambda,l}^{m,n} : \mathcal{A} \rightarrow \mathcal{A}$, $IR_{\lambda,l}^{m,n} f(z) = (I(m, \lambda, l) * R^n) f(z)$ and $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \dots, z \in U\}$ is the class of normalized analytic functions with $\mathcal{A}_1 = \mathcal{A}$. The purpose of this paper is to derive certain subordination and superordination results involving the operator $IR_{\lambda,l}^{m,n}$ and we establish differential sandwich-type theorems.

Keywords: analytic functions, differential operator, differential subordination, differential superordination.

2010 Mathematical Subject Classification: 30C45.

1 Introduction

Let $\mathcal{H}(U)$ be the class of analytic function in the open unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{H}(a, n)$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$.

Let $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1} z^{n+1} + \dots, z \in U\}$ and $\mathcal{A} = \mathcal{A}_1$.

Let the functions f and g be analytic in U . We say that the function f is subordinate to g , written $f \prec g$, if there exists a Schwarz function w , analytic in U , with $w(0) = 0$ and $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$, for all $z \in U$. In particular, if the function g is univalent in U , the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $\psi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ and h be an univalent function in U . If p is analytic in U and satisfies the second order differential subordination

$$\psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z), \quad \text{for } z \in U, \quad (1.1)$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1.1). A dominant \tilde{q} that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of U .

Let $\psi : \mathbb{C}^2 \times U \rightarrow \mathbb{C}$ and h analytic in U . If p and $\psi(p(z), zp'(z), z^2 p''(z); z)$ are univalent and if p satisfies the second order differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2 p''(z); z), \quad z \in U, \quad (1.2)$$

then p is a solution of the differential superordination (1.2) (if f is subordinate to F , then F is called to be superordinate to f). An analytic function q is called a subordinant if $q \prec p$ for all p satisfying (1.2). An univalent subordinant \tilde{q} that satisfies $q \prec \tilde{q}$ for all subordinants q of (1.2) is said to be the best subordinant.

Miller and Mocanu [4] obtained conditions h , q and ψ for which the following implication holds $h(z) \prec \psi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) \prec p(z)$.

For two functions $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$ and $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$ analytic in the open unit disc U , the Hadamard product (or convolution) of $f(z)$ and $g(z)$, written as $(f * g)(z)$ is defined by $f(z) * g(z) = (f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j$.

Definition 1.1 ([1]) Let $\lambda, l \geq 0$ and $n, m \in \mathbb{N}$. Denote by $IR_{\lambda, l}^{m, n} : \mathcal{A} \rightarrow \mathcal{A}$ the operator given by the Hadamard product of the multiplier transformation $I(m, \lambda, l)$ and the Ruscheweyh derivative R^n , $IR_{\lambda, l}^{m, n} f(z) = (I(m, \lambda, l) * R^n) f(z)$, for any $z \in U$ and each nonnegative integers m, n .

Remark 1.1 If $f \in \mathcal{A}$ and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $IR_{\lambda, l}^{m, n} f(z) = z + \sum_{j=2}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^m \frac{(n+j-1)!}{n!(j-1)!} a_j^2 z^j$, $z \in U$.

Using simple computation one obtains the next result.

Proposition 1.1 [2] For $m, n \in \mathbb{N}$ and $\lambda, l \geq 0$ we have

$$(n+1)IR_{\lambda, l}^{m, n+1} f(z) - nIR_{\lambda, l}^{m, n} f(z) = z \left(IR_{\lambda, l}^{m, n} f(z) \right)' . \quad (1.3)$$

The purpose of this paper is to derive the several subordination and superordination results involving a differential operator. Furthermore, we studied the results of Selvaraj and Karthikeyan [6], Shanmugam, Ramachandran, Darus and Sivasubramanian [7] and Srivastava and Lashin [8].

In order to prove our subordination and superordination results, we make use of the following known results.

Definition 1.2 [5] Denote by Q the set of all functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where $E(f) = \{\zeta \in \partial U : \lim_{z \rightarrow \zeta} f(z) = \infty\}$, and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Lemma 1.1 [5] Let the function q be univalent in the unit disc U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that Q is starlike univalent in U and $\operatorname{Re} \left(\frac{zh'(z)}{Q(z)} \right) > 0$ for $z \in U$. If p is analytic with $p(0) = q(0)$, $p(U) \subseteq D$ and $\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$, then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 1.2 [3] Let the function q be convex univalent in the open unit disc U and ν and ϕ be analytic in a domain D containing $q(U)$. Suppose that $\operatorname{Re} \left(\frac{\nu'(q(z))}{\phi(q(z))} \right) > 0$ for $z \in U$ and $\psi(z) = zq'(z)\phi(q(z))$ is starlike univalent in U . If $p(z) \in \mathcal{H}[q(0), 1] \cap Q$, with $p(U) \subseteq D$ and $\nu(p(z)) + zp'(z)\phi(p(z))$ is univalent in U and $\nu(q(z)) + zq'(z)\phi(q(z)) \prec \nu(p(z)) + zp'(z)\phi(p(z))$, then $q(z) \prec p(z)$ and q is the best subordinant.

2 Main results

We begin with the following

Theorem 2.1 Let $\frac{IR_{\lambda, l}^{m, n+1} f(z)}{IR_{\lambda, l}^{m, n} f(z)} \in \mathcal{H}(U)$ and let the function $q(z)$ be analytic and univalent in U such that $q(z) \neq 0$, for all $z \in U$. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U . Let

$$\operatorname{Re} \left(\frac{\alpha + \mu}{\mu} + \frac{2\beta}{\mu} q(z) + \frac{zq''(z)}{q'(z)} \right) > 0, \quad (2.1)$$

for $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$, $z \in U$ and

$$\psi_{\lambda, l}^{m, n}(\alpha, \beta, \mu; z) := \mu(n+2) \frac{IR_{\lambda, l}^{m, n+2} f(z)}{IR_{\lambda, l}^{m, n} f(z)} + (\alpha - \mu) \frac{IR_{\lambda, l}^{m, n+1} f(z)}{IR_{\lambda, l}^{m, n} f(z)} + [\beta - \mu(n+1)] \left(\frac{IR_{\lambda, l}^{m, n+1} f(z)}{IR_{\lambda, l}^{m, n} f(z)} \right)^2. \quad (2.2)$$

If q satisfies the following subordination

$$\psi_{\lambda, l}^{m, n}(\alpha, \beta, \mu; z) \prec \alpha q(z) + \beta (q(z))^2 + \mu zq'(z), \quad (2.3)$$

for $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$, then $\frac{IR_{\lambda, l}^{m, n+1} f(z)}{IR_{\lambda, l}^{m, n} f(z)} \prec q(z)$, and q is the best dominant.

Proof. Let the function p be defined by $p(z) := \frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}$, $z \in U$, $z \neq 0$, $f \in \mathcal{A}$. We have $p'(z) = \frac{(IR_{\lambda,l}^{m,n+1}f(z))' IR_{\lambda,l}^{m,n}f(z) - IR_{\lambda,l}^{m,n+1}f(z)(IR_{\lambda,l}^{m,n}f(z))'}{(IR_{\lambda,l}^{m,n}f(z))^2} = \frac{(IR_{\lambda,l}^{m,n+1}f(z))'}{IR_{\lambda,l}^{m,n}f(z)} - \frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)} \cdot \frac{(IR_{\lambda,l}^{m,n}f(z))'}{IR_{\lambda,l}^{m,n}f(z)}$. Then $zp'(z) = \frac{z(IR_{\lambda,l}^{m,n+1}f(z))'}{IR_{\lambda,l}^{m,n}f(z)} - \frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)} \cdot \frac{z(IR_{\lambda,l}^{m,n}f(z))'}{IR_{\lambda,l}^{m,n}f(z)}$.

By using the identity (1.3), we obtain

$$zp'(z) = (n+2) \frac{IR_{\lambda,l}^{m,n+2}f(z)}{IR_{\lambda,l}^{m,n}f(z)} - (n+1) \left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)} \right)^2 - \frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}. \quad (2.4)$$

By setting $\theta(w) := \alpha w + \beta w^2$ and $\phi(w) := \mu$, it can be easily verified that θ is analytic in \mathbb{C} , ϕ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$.

Also, by letting $Q(z) = zq'(z)\phi(q(z)) = \mu zq'(z)$ and $h(z) = \theta(q(z)) + Q(z) = \alpha q(z) + \beta(q(z))^2 + \mu zq'(z)$, we find that $Q(z)$ is starlike univalent in U .

We have $h'(z) = (\alpha + \mu)q'(z) + 2\beta q(z)q'(z) + \mu zq''(z)$ and $\frac{zh'(z)}{Q(z)} = \frac{zh'(z)}{\mu zq'(z)} = \frac{\alpha + \mu}{\mu} + \frac{2\beta}{\mu}q(z) + \frac{zq''(z)}{q'(z)}$.

We deduce that $Re\left(\frac{zh'(z)}{Q(z)}\right) = Re\left(\frac{\alpha + \mu}{\mu} + \frac{2\beta}{\mu}q(z) + \frac{zq''(z)}{q'(z)}\right) > 0$.

By using (2.4), we obtain

$$\alpha p(z) + \beta(p(z))^2 + \mu zp'(z) = \mu(n+2) \frac{IR_{\lambda,l}^{m,n+2}f(z)}{IR_{\lambda,l}^{m,n}f(z)} + (\alpha - \mu) \frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)} + [\beta - \mu(n+1)] \left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)} \right)^2.$$

By using (2.3), we have $\alpha p(z) + \beta(p(z))^2 + \mu zp'(z) \prec \alpha q(z) + \beta(q(z))^2 + \mu zq'(z)$.

By an application of Lemma 1.1, we have $p(z) \prec q(z)$, $z \in U$, i.e. $\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)} \prec q(z)$, $z \in U$ and q is the best dominant. ■

Corollary 2.2 Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.1) holds. If $f \in \mathcal{A}$ and $\psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z) \prec \alpha \frac{1+Az}{1+Bz} + \beta \left(\frac{1+Az}{1+Bz} \right)^2 + \mu \frac{(A-B)z}{(1+Bz)^2}$, for $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$, $-1 \leq B < A \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)} \prec \frac{1+Az}{1+Bz}$, and $\frac{1+Az}{1+Bz}$ is the best dominant.

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$ in Theorem 2.1 we get the corollary. ■

Corollary 2.3 Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.1) holds. If $f \in \mathcal{A}$ and $\psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z) \prec \alpha \left(\frac{1+z}{1-z} \right)^\gamma + \beta \left(\frac{1+z}{1-z} \right)^{2\gamma} + \frac{2\mu\gamma z}{(1-z)^2} \left(\frac{1+z}{1-z} \right)^{\gamma-1}$, for $\alpha, \beta, \mu \in \mathbb{C}$, $0 < \gamma \leq 1$, $\mu \neq 0$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)} \prec \left(\frac{1+z}{1-z} \right)^\gamma$, and $\left(\frac{1+z}{1-z} \right)^\gamma$ is the best dominant.

Proof. Corollary follows by using Theorem 2.1 for $q(z) = \left(\frac{1+z}{1-z} \right)^\gamma$, $0 < \gamma \leq 1$. ■

Theorem 2.4 Let q be analytic and univalent in U such that $q(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ be starlike univalent in U . Assume that

$$Re\left(\frac{\alpha}{\mu}q'(z) + \frac{2\beta}{\mu}q(z)q'(z)\right) > 0, \text{ for } \alpha, \beta, \mu \in \mathbb{C}, \mu \neq 0. \quad (2.5)$$

If $f \in \mathcal{A}$, $\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)} \in \mathcal{H}[q(0), 1] \cap Q$ and $\psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z)$ is univalent in U , where $\psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z)$ is as defined in (2.2), then

$$\alpha q(z) + \beta(q(z))^2 + \mu zq'(z) \prec \psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z) \quad (2.6)$$

implies $q(z) \prec \frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}$, $z \in U$, and q is the best subdominant.

Proof. Let the function p be defined by $p(z) := \frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}$, $z \in U$, $z \neq 0$, $f \in \mathcal{A}$.

By setting $\nu(w) := \alpha w + \beta w^2$ and $\phi(w) := \mu$ it can be easily verified that ν is analytic in \mathbb{C} , ϕ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$.

Since $\frac{\nu'(q(z))}{\phi(q(z))} = \frac{q'(z)[\alpha+2\beta q(z)]}{\mu}$, it follows that $Re\left(\frac{\nu'(q(z))}{\phi(q(z))}\right) = Re\left(\frac{\alpha}{\mu}q'(z) + \frac{2\beta}{\mu}q(z)q'(z)\right) > 0$, for $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$.

By using (2.4) and (2.6) we obtain $\alpha q(z) + \mu(q(z))^2 + \mu z q'(z) \prec \alpha p(z) + \beta(p(z))^2 + \mu z p'(z)$.

Using Lemma 1.2, we have $q(z) \prec p(z) = \frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}$, $z \in U$, and q is the best subordinator. ■

Corollary 2.5 Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.5) holds. If $f \in \mathcal{A}$, $\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)} \in \mathcal{H}[q(0), 1] \cap Q$ and $\alpha \frac{1+A_1z}{1+B_1z} + \beta \left(\frac{1+A_1z}{1+B_1z}\right)^2 + \mu \frac{(A_1-B_1)z}{(1+B_1z)^2} \prec \psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z)$, for $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$, $-1 \leq B_1 < A_1 \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\frac{1+A_1z}{1+B_1z} \prec \frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}$, and $\frac{1+A_1z}{1+B_1z}$ is the best subordinator.

Proof. For $q(z) = \frac{1+A_1z}{1+B_1z}$, $-1 \leq B_1 < A_1 \leq 1$ in Theorem 2.4 we get the corollary. ■

Corollary 2.6 Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.5) holds. If $f \in \mathcal{A}$, $\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)} \in \mathcal{H}[q(0), 1] \cap Q$ and $\alpha \left(\frac{1+z}{1-z}\right)^\gamma + \beta \left(\frac{1+z}{1-z}\right)^{2\gamma} + \frac{2\mu\gamma z}{(1-z)^2} \left(\frac{1+z}{1-z}\right)^{\gamma-1} \prec \psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z)$, for $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$, $0 < \gamma \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\left(\frac{1+z}{1-z}\right)^\gamma \prec \frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}$, and $\left(\frac{1+z}{1-z}\right)^\gamma$ is the best subordinator.

Proof. For $q(z) = \left(\frac{1+z}{1-z}\right)^\gamma$, $0 < \gamma \leq 1$ in Theorem 2.4 we get the corollary. ■

Combining Theorem 2.1 and Theorem 2.4, we state the following sandwich theorem.

Theorem 2.7 Let q_1 and q_2 be analytic and univalent in U such that $q_1(z) \neq 0$ and $q_2(z) \neq 0$, for all $z \in U$, with $\frac{zq_1'(z)}{q_1(z)}$ and $\frac{zq_2'(z)}{q_2(z)}$ being starlike univalent. Suppose that q_1 satisfies (2.1) and q_2 satisfies (2.5). If $f \in \mathcal{A}$, $\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)} \in \mathcal{H}[q(0), 1] \cap Q$ and $\psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z)$ is as defined in (2.2) univalent in U , then $\alpha q_1(z) + \beta(q_1(z))^2 + \mu z q_1'(z) \prec \psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z) \prec \alpha q_2(z) + \beta(q_2(z))^2 + \mu z q_2'(z)$, for $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$, implies $q_1(z) \prec \frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)} \prec q_2(z)$, and q_1 and q_2 are respectively the best subordinator and the best dominant.

For $q_1(z) = \frac{1+A_1z}{1+B_1z}$, $q_2(z) = \frac{1+A_2z}{1+B_2z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 2.8 Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.1) and (2.5) hold. If $f \in \mathcal{A}$, $\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)} \in \mathcal{H}[q(0), 1] \cap Q$ and $\alpha \frac{1+A_1z}{1+B_1z} + \beta \left(\frac{1+A_1z}{1+B_1z}\right)^2 + \mu \frac{(A_1-B_1)z}{(1+B_1z)^2} \prec \psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z) \prec \alpha \frac{1+A_2z}{1+B_2z} + \beta \left(\frac{1+A_2z}{1+B_2z}\right)^2 + \mu \frac{(A_2-B_2)z}{(1+B_2z)^2}$, for $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\frac{1+A_1z}{1+B_1z} \prec \frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)} \prec \frac{1+A_2z}{1+B_2z}$, hence $\frac{1+A_1z}{1+B_1z}$ and $\frac{1+A_2z}{1+B_2z}$ are the best subordinator and the best dominant, respectively.

For $q_1(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_1}$, $q_2(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, where $0 < \gamma_1 < \gamma_2 \leq 1$, we have the following corollary.

Corollary 2.9 Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.1) and (2.5) hold. If $f \in \mathcal{A}$, $\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)} \in \mathcal{H}[q(0), 1] \cap Q$ and $\alpha \left(\frac{1+z}{1-z}\right)^{\gamma_1} + \beta \left(\frac{1+z}{1-z}\right)^{2\gamma_1} + \frac{2\mu\gamma_1 z}{(1-z)^2} \left(\frac{1+z}{1-z}\right)^{\gamma_1-1} \prec \psi_{\lambda,l}^{m,n}(\alpha, \beta, \mu; z) \prec \alpha \left(\frac{1+z}{1-z}\right)^{\gamma_2} + \beta \left(\frac{1+z}{1-z}\right)^{2\gamma_2} + \frac{2\mu\gamma_2 z}{(1-z)^2} \left(\frac{1+z}{1-z}\right)^{\gamma_2-1}$, for $\alpha, \beta, \mu \in \mathbb{C}$, $\mu \neq 0$, $0 < \gamma_1 < \gamma_2 \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\left(\frac{1+z}{1-z}\right)^{\gamma_1} \prec \frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, hence $\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $\left(\frac{1+z}{1-z}\right)^{\gamma_2}$ are the best subordinator and the best dominant, respectively.

We have also

Theorem 2.10 Let $\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^\delta \in \mathcal{H}(U)$, $f \in \mathcal{A}$, $z \in U$, $\delta \in \mathbb{C}$, $\delta \neq 0$, $m, n \in \mathbb{N}$, $\lambda, l \geq 0$ and let the function $q(z)$ be convex and univalent in U such that $q(0) = 1$, $z \in U$. Assume that

$$Re\left(1 + \frac{\xi}{\beta}q(z) + \frac{2\mu}{\beta}q^2(z) - z\frac{q'(z)}{q(z)} + z\frac{q''(z)}{q'(z)}\right) > 0, \quad (2.7)$$

for $\alpha, \xi, \mu, \beta \in \mathbb{C}$, $\beta \neq 0$, $z \in U$, and

$$\begin{aligned} \psi_{\lambda,l}^{m,n}(\delta, \alpha, \xi, \mu, \beta; z) &:= \alpha + \xi \left(\frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \right)^\delta + \mu \left(\frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \right)^{2\delta} \\ &\quad + \beta \delta (n+2) \frac{IR_{\lambda,l}^{m,n+2} f(z)}{IR_{\lambda,l}^{m,n+1} f(z)} - \beta \delta (n+1) \frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} - \beta \delta. \end{aligned} \quad (2.8)$$

If q satisfies the following subordination

$$\psi_{\lambda,l}^{m,n}(\delta, \alpha, \xi, \mu, \beta; z) \prec \alpha + \xi q(z) + \mu q^2(z) + \frac{\beta z q'(z)}{q(z)}, \quad (2.9)$$

for $\alpha, \xi, \mu, \beta \in \mathbb{C}$, $\beta \neq 0$, $z \in U$, then $\left(\frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \right)^\delta \prec q(z)$, $z \in U$, $\delta \in \mathbb{C}$, $\delta \neq 0$, and q is the best dominant.

Proof. Let the function p be defined by $p(z) := \left(\frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \right)^\delta$, $z \in U$, $z \neq 0$, $f \in \mathcal{A}$. The function p is analytic in U and $p(0) = 1$

$$\text{We have } zp'(z) = \delta \left(\frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \right)^\delta \left[\frac{z(IR_{\lambda,l}^{m,n+1} f(z))'}{IR_{\lambda,l}^{m,n} f(z)} - \frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \cdot \frac{z(IR_{\lambda,l}^{m,n} f(z))'}{IR_{\lambda,l}^{m,n} f(z)} \right].$$

By using the identity (1.3), we obtain

$$\frac{zp'(z)}{p(z)} = \delta (n+2) \frac{IR_{\lambda,l}^{m,n+2} f(z)}{IR_{\lambda,l}^{m,n+1} f(z)} - \delta (n+1) \frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)}. \quad (2.10)$$

By setting $\theta(w) := \alpha + \xi w + \mu w^2$ and $\phi(w) := \frac{\beta}{w}$, it can be easily verified that θ is analytic in \mathbb{C} , ϕ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$.

Also, by letting $Q(z) = zq'(z)\phi(q(z)) = \frac{\beta z q'(z)}{q(z)}$, we find that $Q(z)$ is starlike univalent in U .

Let $h(z) = \theta(q(z)) + Q(z) = \alpha + \xi q(z) + \mu q^2(z) + \frac{\beta z q'(z)}{q(z)}$.

We have $\operatorname{Re} \left(\frac{zh'(z)}{Q(z)} \right) = \operatorname{Re} \left(1 + \frac{\xi}{\beta} q(z) + \frac{2\mu}{\beta} q^2(z) - z \frac{q'(z)}{q(z)} + z \frac{q''(z)}{q'(z)} \right) > 0$.

By using (2.10), we obtain $\alpha + \xi p(z) + \mu (p(z))^2 + \beta \frac{zp'(z)}{p(z)} = \alpha + \xi \left(\frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \right)^\delta + \mu \left(\frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \right)^{2\delta} + \beta \delta (n+2) \frac{IR_{\lambda,l}^{m,n+2} f(z)}{IR_{\lambda,l}^{m,n+1} f(z)} - \beta \delta (n+1) \frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} - \beta \delta$.

By using (2.9), we have $\alpha + \xi p(z) + \mu (p(z))^2 + \beta \frac{zp'(z)}{p(z)} \prec \alpha + \xi q(z) + \mu q^2(z) + \frac{\beta z q'(z)}{q(z)}$.

From Lemma 1.1, we have $p(z) \prec q(z)$, $z \in U$, i.e. $\left(\frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \right)^\delta \prec q(z)$, $z \in U$, $\delta \in \mathbb{C}$, $\delta \neq 0$ and q is the best dominant. ■

Corollary 2.11 Let $q(z) = \frac{1+Az}{1+Bz}$, $z \in U$, $-1 \leq B < A \leq 1$, $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.7) holds. If $f \in \mathcal{A}$ and $\psi_{\lambda,l}^{m,n}(\delta, \alpha, \xi, \mu, \beta; z) \prec \alpha + \xi \frac{1+Az}{1+Bz} + \mu \left(\frac{1+Az}{1+Bz} \right)^2 + \beta \frac{(A-B)z}{(1+Az)(1+Bz)}$, for $\alpha, \xi, \mu, \beta, \delta \in \mathbb{C}$, $\beta, \delta \neq 0$, $-1 \leq B < A \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.8), then $\left(\frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \right)^\delta \prec \frac{1+Az}{1+Bz}$, and $\frac{1+Az}{1+Bz}$ is the best dominant.

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, in Theorem 2.10 we get the corollary. ■

Corollary 2.12 Let $q(z) = \left(\frac{1+z}{1-z} \right)^\gamma$, $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.7) holds. If $f \in \mathcal{A}$ and $\psi_{\lambda,l}^{m,n}(\delta, \alpha, \xi, \mu, \beta; z) \prec \alpha + \xi \left(\frac{1+z}{1-z} \right)^\gamma + \mu \left(\frac{1+z}{1-z} \right)^{2\gamma} + \frac{2\beta\gamma z}{1-z^2}$, for $\alpha, \xi, \mu, \beta, \delta \in \mathbb{C}$, $0 < \gamma \leq 1$, $\beta, \delta \neq 0$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.8), then $\left(\frac{IR_{\lambda,l}^{m,n+1} f(z)}{IR_{\lambda,l}^{m,n} f(z)} \right)^\delta \prec \left(\frac{1+z}{1-z} \right)^\gamma$, and $\left(\frac{1+z}{1-z} \right)^\gamma$ is the best dominant.

Proof. Corollary follows by using Theorem 2.10 for $q(z) = \left(\frac{1+z}{1-z}\right)^\gamma$, $0 < \gamma \leq 1$. ■

Theorem 2.13 Let q be convex and univalent in U such that $q(0) = 1$. Assume that

$$Re \left(\frac{\xi}{\beta} q(z) q'(z) + \frac{2\mu}{\beta} q^2(z) q'(z) \right) > 0, \text{ for } \alpha, \xi, \mu, \beta \in \mathbb{C}, \beta \neq 0. \quad (2.11)$$

If $f \in \mathcal{A}$, $\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^\delta \in \mathcal{H}[q(0), 1] \cap Q$ and $\psi_{\lambda,l}^{m,n}(\delta, \alpha, \xi, \mu, \beta; z)$ is univalent in U , where $\psi_{\lambda,l}^{m,n}(\delta, \alpha, \xi, \mu, \beta; z)$ is as defined in (2.8), then

$$\alpha + \xi q(z) + \mu q^2(z) + \frac{\beta z q'(z)}{q(z)} \prec \psi_{\lambda,l}^{m,n}(\delta, \alpha, \xi, \mu, \beta; z) \quad (2.12)$$

implies $q(z) \prec \left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^\delta$, $\delta \in \mathbb{C}$, $\delta \neq 0$, $z \in U$, and q is the best subdominant.

Proof. Let the function p be defined by $p(z) := \left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^\delta$, $z \in U$, $z \neq 0$, $\delta \in \mathbb{C}$, $\delta \neq 0$, $f \in \mathcal{A}$. The function p is analytic in U and $p(0) = 1$.

By setting $\nu(w) := \alpha + \xi w + \mu w^2$ and $\phi(w) := \frac{\beta}{w}$ it can be easily verified that ν is analytic in \mathbb{C} , ϕ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$.

Since $\frac{\nu'(q(z))}{\phi(q(z))} = \frac{\xi}{\beta} q(z) q'(z) + \frac{2\mu}{\beta} q^2(z) q'(z)$, it follows that $Re \left(\frac{\nu'(q(z))}{\phi(q(z))} \right) = Re \left(\frac{\xi}{\beta} q(z) q'(z) + \frac{2\mu}{\beta} q^2(z) q'(z) \right) > 0$, for $\alpha, \xi, \mu, \beta \in \mathbb{C}$, $\beta \neq 0$.

Now, by using (2.12) we obtain $\alpha + \xi q(z) + \mu q^2(z) + \frac{\beta z q'(z)}{q(z)} \prec \alpha + \xi p(z) + \mu p^2(z) + \frac{\beta z p'(z)}{p(z)}$, $z \in U$. From Lemma 1.2, we have $q(z) \prec p(z) = \left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^\delta$, $z \in U$, $\delta \in \mathbb{C}$, $\delta \neq 0$, and q is the best subdominant. ■

Corollary 2.14 Let $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, $z \in U$, $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.11) holds. If $f \in \mathcal{A}$, $\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^\delta \in \mathcal{H}[q(0), 1] \cap Q$, $\delta \in \mathbb{C}$, $\delta \neq 0$ and $\alpha + \xi \frac{1+Az}{1+Bz} + \mu \left(\frac{1+Az}{1+Bz}\right)^2 + \beta \frac{(A-B)z}{(1+Az)(1+Bz)} \prec \psi_{\lambda,l}^{m,n}(\delta, \alpha, \xi, \mu, \beta; z)$, for $\alpha, \xi, \mu, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.8), then $\frac{1+Az}{1+Bz} \prec \left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^\delta$, $\delta \in \mathbb{C}$, $\delta \neq 0$, and $\frac{1+Az}{1+Bz}$ is the best subdominant.

Proof. For $q(z) = \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, in Theorem 2.13 we get the corollary. ■

Corollary 2.15 Let $q(z) = \left(\frac{1+z}{1-z}\right)^\gamma$, $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.11) holds. If $f \in \mathcal{A}$, $\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^\delta \in \mathcal{H}[q(0), 1] \cap Q$ and $\alpha + \xi \left(\frac{1+z}{1-z}\right)^\gamma + \mu \left(\frac{1+z}{1-z}\right)^{2\gamma} + \frac{2\beta\gamma z}{1-z^2} \prec \psi_{\lambda,l}^{m,n}(\delta, \alpha, \xi, \mu, \beta; z)$, for $\alpha, \xi, \mu, \beta, \delta \in \mathbb{C}$, $0 < \gamma \leq 1$, $\beta, \delta \neq 0$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.8), then $\left(\frac{1+z}{1-z}\right)^\gamma \prec \left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^\delta$, and $\left(\frac{1+z}{1-z}\right)^\gamma$ is the best subdominant.

Proof. Corollary follows by using Theorem 2.13 for $q(z) = \left(\frac{1+z}{1-z}\right)^\gamma$, $0 < \gamma \leq 1$. ■

Combining Theorem 2.10 and Theorem 2.13, we state the following sandwich theorem.

Theorem 2.16 Let q_1 and q_2 be convex and univalent in U such that $q_1(z) \neq 0$ and $q_2(z) \neq 0$, for all $z \in U$. Suppose that q_1 satisfies (2.7) and q_2 satisfies (2.11). If $f \in \mathcal{A}$, $\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^\delta \in \mathcal{H}[q(0), 1] \cap Q$, $\delta \in \mathbb{C}$, $\delta \neq 0$ and $\psi_{\lambda,l}^{m,n}(\delta, \alpha, \xi, \mu, \beta; z)$ is as defined in (2.8) univalent in U , then $\alpha + \xi q_1(z) + \mu q_1^2(z) + \frac{\beta z q_1'(z)}{q_1(z)} \prec \psi_{\lambda,l}^{m,n}(\delta, \alpha, \xi, \mu, \beta; z) \prec \alpha + \xi q_2(z) + \mu q_2^2(z) + \frac{\beta z q_2'(z)}{q_2(z)}$, for $\alpha, \xi, \mu, \beta \in \mathbb{C}$, $\beta \neq 0$, implies $q_1(z) \prec \left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^\delta \prec q_2(z)$, $z \in U$, $\delta \in \mathbb{C}$, $\delta \neq 0$, and q_1 and q_2 are respectively the best subdominant and the best dominant.

For $q_1(z) = \frac{1+A_1z}{1+B_1z}$, $q_2(z) = \frac{1+A_2z}{1+B_2z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 2.17 Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.7) and (2.11) hold for $q_1(z) = \frac{1+A_1z}{1+B_1z}$ and $q_2(z) = \frac{1+A_2z}{1+B_2z}$, respectively. If $f \in \mathcal{A}$, $\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^\delta \in \mathcal{H}[q(0), 1] \cap Q$ and $\alpha + \xi \frac{1+A_1z}{1+B_1z} + \mu \left(\frac{1+A_1z}{1+B_1z}\right)^2 + \beta \frac{(A_1-B_1)z}{(1+A_1z)(1+B_1z)} \prec \psi_{\lambda,l}^{m,n}(\delta, \alpha, \xi, \mu, \beta; z) \prec \alpha + \xi \frac{1+A_2z}{1+B_2z} + \mu \left(\frac{1+A_2z}{1+B_2z}\right)^2 + \beta \frac{(A_2-B_2)z}{(1+A_2z)(1+B_2z)}$, $z \in U$, for $\alpha, \xi, \mu, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\frac{1+A_1z}{1+B_1z} \prec \left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^\delta \prec \frac{1+A_2z}{1+B_2z}$, $z \in U$, $\delta \in \mathbb{C}$, $\delta \neq 0$, hence $\frac{1+A_1z}{1+B_1z}$ and $\frac{1+A_2z}{1+B_2z}$ are the best subdominant and the best dominant, respectively.

For $q_1(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_1}$, $q_2(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, where $0 < \gamma_1 < \gamma_2 \leq 1$, we have the following corollary.

Corollary 2.18 Let $m, n \in \mathbb{N}$, $\lambda, l \geq 0$. Assume that (2.7) and (2.11) hold for $q_1(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $q_2(z) = \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, respectively. If $f \in \mathcal{A}$, $\left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^\delta \in \mathcal{H}[q(0), 1] \cap Q$ and $\alpha + \xi \left(\frac{1+z}{1-z}\right)^{\gamma_1} + \mu \left(\frac{1+z}{1-z}\right)^{2\gamma_1} + \frac{2\beta\gamma_1z}{1-z^2} \prec \psi_{\lambda,l}^{m,n}(\delta, \alpha, \xi, \mu, \beta; z) \prec \alpha + \xi \left(\frac{1+z}{1-z}\right)^{\gamma_2} + \mu \left(\frac{1+z}{1-z}\right)^{2\gamma_2} + \frac{2\beta\gamma_2z}{1-z^2}$, $z \in U$, for $\alpha, \xi, \mu, \beta \in \mathbb{C}$, $\beta \neq 0$, $0 < \gamma_1 < \gamma_2 \leq 1$, where $\psi_{\lambda,l}^{m,n}$ is defined in (2.2), then $\left(\frac{1+z}{1-z}\right)^{\gamma_1} \prec \left(\frac{IR_{\lambda,l}^{m,n+1}f(z)}{IR_{\lambda,l}^{m,n}f(z)}\right)^\delta \prec \left(\frac{1+z}{1-z}\right)^{\gamma_2}$, $z \in U$, $\delta \in \mathbb{C}$, $\delta \neq 0$, hence $\left(\frac{1+z}{1-z}\right)^{\gamma_1}$ and $\left(\frac{1+z}{1-z}\right)^{\gamma_2}$ are the best subdominant and the best dominant, respectively.

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Approximating fixed points with applications in fractional calculus

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Abstract

We approximate fixed points of some iterative methods on a generalized Banach space setting. Earlier studies such as [5, 6, 7, 12] require that the operator involved is Fréchet-differentiable. In the present study we assume that the operator is only continuous. This way we extend the applicability of these methods to include generalized fractional calculus and problems from other areas. Some applications include generalized fractional calculus involving the Riemann-Liouville fractional integral and the Caputo fractional derivative. Fractional calculus is very important for its applications in many applied sciences.

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1 Introduction

Many problems in Computational sciences can be formulated as an operator equation using Mathematical Modelling [7, 10, 13, 14, 15]. The fixed points of these operators can rarely be found in closed form. That is why most solution methods are usually iterative.

The semilocal convergence is, based on the information around an initial point, to give conditions ensuring the convergence of the method.

We present a semilocal convergence analysis for some iterative methods on a generalized Banach space setting to approximate fixed point or a zero of an operator. A generalized norm is defined to be an operator from a linear space into a partially order Banach space (to be precised in section 2). Earlier studies such as [5, 6, 7, 12] for Newton's method have shown that a more precise convergence analysis is obtained when compared to the real norm theory. However, the main assumption is that the operator involved is Fréchet-differentiable. This hypothesis limits the applicability of Newton's method. In the present study we only assume the continuity of the operator. This may be expand the applicability of these methods.

The rest of the paper is organized as follows: section 2 contains the basic concepts on generalized Banach spaces and auxiliary results on inequalities and fixed points. In section 3 we present the semilocal convergence analysis of these methods. Finally, in the concluding sections 4-5, we present special cases and applications in generalized fractional calculus.

2 Generalized Banach spaces

We present some standard concepts that are needed in what follows to make the paper as self contained as possible. More details on generalized Banach spaces can be found in [5, 6, 7, 12], and the references there in.

Definition 2.1 *A generalized Banach space is a triplet $(X, E, / \cdot /)$ such that*

- (i) X is a linear space over $\mathbb{R}(\mathbb{C})$.*
- (ii) $E = (E, K, \|\cdot\|)$ is a partially ordered Banach space, i.e.*
 - (ii₁) $(E, \|\cdot\|)$ is a real Banach space,*
 - (ii₂) E is partially ordered by a closed convex cone K ,*
 - (iii₃) The norm $\|\cdot\|$ is monotone on K .*
- (iii) The operator $/ \cdot / : X \rightarrow K$ satisfies*
 - $/x/ = 0 \Leftrightarrow x = 0$, $/\theta x/ = |\theta| /x/$,*
 - $/x + y/ \leq /x/ + /y/$ for each $x, y \in X$, $\theta \in \mathbb{R}(\mathbb{C})$.*
- (iv) X is a Banach space with respect to the induced norm $\|\cdot\|_i := \|\cdot\| \cdot / \cdot /$.*

Remark 2.2 *The operator $/ \cdot /$ is called a generalized norm. In view of (iii) and (ii₃) $\|\cdot\|_i$, is a real norm. In the rest of this paper all topological concepts will be understood with respect to this norm.*

Let $L(X^j, Y)$ stand for the space of j -linear symmetric and bounded operators from X^j to Y , where X and Y are Banach spaces. For X, Y partially

ordered $L_+(X^j, Y)$ stands for the subset of monotone operators P such that

$$0 \leq a_i \leq b_i \Rightarrow P(a_1, \dots, a_j) \leq P(b_1, \dots, b_j). \quad (2.1)$$

Definition 2.3 The set of bounds for an operator $Q \in L(X, X)$ on a generalized Banach space $(X, E, / \cdot /)$ the set of bounds is defined to be:

$$B(Q) := \{P \in L_+(E, E), /Qx/ \leq P/x/ \text{ for each } x \in X\}. \quad (2.2)$$

Let $D \subset X$ and $T : D \rightarrow D$ be an operator. If $x_0 \in D$ the sequence $\{x_n\}$ given by

$$x_{n+1} := T(x_n) = T^{n+1}(x_0) \quad (2.3)$$

is well defined. We write in case of convergence

$$T^\infty(x_0) := \lim(T^n(x_0)) = \lim_{n \rightarrow \infty} x_n. \quad (2.4)$$

We need some auxiliary results on inequations.

Lemma 2.4 Let $(E, K, \|\cdot\|)$ be a partially ordered Banach space, $\xi \in K$ and $M, N \in L_+(E, E)$.

(i) Suppose there exists $r \in K$ such that

$$R(r) := (M + N)r + \xi \leq r \quad (2.5)$$

and

$$(M + N)^k r \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (2.6)$$

Then, $b := R^\infty(0)$ is well defined satisfies the equation $t = R(t)$ and is the smaller than any solution of the inequality $R(s) \leq s$.

(ii) Suppose there exists $q \in K$ and $\theta \in (0, 1)$ such that $R(q) \leq \theta q$, then there exists $r \leq q$ satisfying (i).

Proof. (i) Define sequence $\{b_n\}$ by $b_n = R^n(0)$. Then, we have by (2.5) that $b_1 = R(0) = \xi \leq r \Rightarrow b_1 \leq r$. Suppose that $b_k \leq r$ for each $k = 1, 2, \dots, n$. Then, we have by (2.5) and the inductive hypothesis that $b_{n+1} = R^{n+1}(0) = R(R^n(0)) = R(b_n) = (M + N)b_n + \xi \leq (M + N)r + \xi \leq r \Rightarrow b_{n+1} \leq r$. Hence, sequence $\{b_n\}$ is bounded above by r . Set $P_n = b_{n+1} - b_n$. We shall show that

$$P_n \leq (M + N)^n r \text{ for each } n = 1, 2, \dots \quad (2.7)$$

We have by the definition of P_n and (2.6) that

$$\begin{aligned} P_1 &= R^2(0) - R(0) = R(R(0)) - R(0) \\ &= R(\xi) - R(0) = \int_0^1 R'(t\xi) \xi dt \leq \int_0^1 R'(\xi) \xi dt \end{aligned}$$

$$\leq \int_0^1 R'(r) r dt \leq (M + N) r,$$

which shows (2.7) for $n = 1$. Suppose that (2.7) is true for $k = 1, 2, \dots, n$. Then, we have in turn by (2.6) and the inductive hypothesis that

$$\begin{aligned} P_{k+1} &= R^{k+2}(0) - R^{k+1}(0) = R^{k+1}(R(0)) - R^{k+1}(0) = \\ &= R^{k+1}(\xi) - R^{k+1}(0) = R(R^k(\xi)) - R(R^k(0)) = \\ &= \int_0^1 R'(R^k(0) + t(R^k(\xi) - R^k(0))) (R^k(\xi) - R^k(0)) dt \leq \\ &= R'(R^k(\xi)) (R^k(\xi) - R^k(0)) = R'(R^k(\xi)) (R^{k+1}(0) - R^k(0)) \leq \\ &= R'(r) (R^{k+1}(0) - R^k(0)) \leq (M + N) (M + N)^k r = (M + N)^{k+1} r, \end{aligned}$$

which completes the induction for (2.7). It follows that $\{b_n\}$ is a complete sequence in a Banach space and as such it converges to some b . Notice that $R(b) = R\left(\lim_{n \rightarrow \infty} R^n(0)\right) = \lim_{n \rightarrow \infty} R^{n+1}(0) = b \Rightarrow b$ solves the equation $R(t) = t$. We have that $b_n \leq r \Rightarrow b \leq r$, where r a solution of $R(r) \leq r$. Hence, b is smaller than any solution of $R(s) \leq s$.

(ii) Define sequences $\{v_n\}, \{w_n\}$ by $v_0 = 0, v_{n+1} = R(v_n), w_0 = q, w_{n+1} = R(w_n)$. Then, we have that

$$\begin{aligned} 0 &\leq v_n \leq v_{n+1} \leq w_{n+1} \leq w_n \leq q, \\ w_n - v_n &\leq \theta^n (q - v_n) \end{aligned} \quad (2.8)$$

and sequence $\{v_n\}$ is bounded above by q . Hence, it converges to some r with $r \leq q$. We also get by (2.8) that $w_n - v_n \rightarrow 0$ as $n \rightarrow \infty \Rightarrow w_n \rightarrow r$ as $n \rightarrow \infty$. ■

We also need the auxiliary result for computing solutions of fixed point problems.

Lemma 2.5 *Let $(X, (E, K, \|\cdot\|), / \cdot /)$ be a generalized Banach space, and $P \in B(Q)$ be a bound for $Q \in L(X, X)$. Suppose there exists $y \in X$ and $q \in K$ such that*

$$Pq + /y/ \leq q \text{ and } P^k q \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (2.9)$$

Then, $z = T^\infty(0)$, $T(x) := Qx + y$ is well defined and satisfies: $z = Qz + y$ and $/z/ \leq P/z/ + /y/ \leq q$. Moreover, z is the unique solution in the subspace $\{x \in X \mid \exists \theta \in \mathbb{R} : \{x\} \leq \theta q\}$.

The proof can be found in [12, Lemma 3.2].

3 Semilocal convergence

Let $(X, (E, K, \|\cdot\|), / \cdot /)$ and Y be generalized Banach spaces, $D \subset X$ an open subset, $G : D \rightarrow Y$ a continuous operator and $A(\cdot) : D \rightarrow L(X, Y)$. A zero of operator G is to be determined by a method starting at a point $x_0 \in D$. The results are presented for an operator $F = JG$, where $J \in L(Y, X)$. The iterates are determined through a fixed point problem:

$$\begin{aligned} x_{n+1} &= x_n + y_n, \quad A(x_n) y_n + F(x_n) = 0 \\ \Leftrightarrow y_n &= T(y_n) := (I - A(x_n)) y_n - F(x_n). \end{aligned} \quad (3.1)$$

Let $U(x_0, r)$ stand for the ball defined by

$$U(x_0, r) := \{x \in X : /x - x_0/ \leq r\}$$

for some $r \in K$.

Next, we present the semilocal convergence analysis of method (3.1) using the preceding notation.

Theorem 3.1 *Let $F : D \subset X$, $A(\cdot) : D \rightarrow L(X, Y)$ and $x_0 \in D$ be as defined previously. Suppose:*

- (H₁) *There exists an operator $M \in B(I - A(x))$ for each $x \in D$.*
- (H₂) *There exists an operator $N \in L_+(E, E)$ satisfying for each $x, y \in D$*

$$/F(y) - F(x) - A(x)(y - x)/ \leq N /y - x/.$$

- (H₃) *There exists a solution $r \in K$ of*

$$R_0(t) := (M + N)t + /F(x_0)/ \leq t.$$

- (H₄) $U(x_0, r) \subseteq D$.

- (H₅) $(M + N)^k r \rightarrow 0$ as $k \rightarrow \infty$.

Then, the following hold:

- (C₁) *The sequence $\{x_n\}$ defined by*

$$x_{n+1} = x_n + T_n^\infty(0), \quad T_n(y) := (I - A(x_n))y - F(x_n) \quad (3.2)$$

is well defined, remains in $U(x_0, r)$ for each $n = 0, 1, 2, \dots$ and converges to the unique zero of operator F in $U(x_0, r)$.

- (C₂) *An a priori bound is given by the null-sequence $\{r_n\}$ defined by $r_0 := r$ and for each $n = 1, 2, \dots$*

$$r_n = P_n^\infty(0), \quad P_n(t) = Mt + Nr_{n-1}.$$

- (C₃) *An a posteriori bound is given by the sequence $\{s_n\}$ defined by*

$$s_n := R_n^\infty(0), \quad R_n(t) = (M + N)t + Na_{n-1},$$

$$b_n := /x_n - x_0/ \leq r - r_n \leq r,$$

where

$$a_{n-1} := /x_n - x_{n-1}/ \text{ for each } n = 1, 2, \dots$$

Proof. Let us define for each $n \in \mathbb{N}$ the statement:

(I_n) $x_n \in X$ and $r_n \in K$ are well defined and satisfy

$$r_n + a_{n-1} \leq r_{n-1}.$$

We use induction to show (I_n). The statement (I₁) is true: By Lemma 2.4 and (H₃), (H₅) there exists $q \leq r$ such that:

$$Mq + /F(x_0)/ = q \text{ and } M^k q \leq M^k r \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Hence, by Lemma 2.5 x_1 is well defined and we have $a_0 \leq q$. Then, we get the estimate

$$\begin{aligned} P_1(r - q) &= M(r - q) + Nr_0 \\ &\leq Mr - Mq + Nr = R_0(r) - q \\ &\leq R_0(r) - q = r - q. \end{aligned}$$

It follows with Lemma 2.4 that r_1 is well defined and

$$r_1 + a_0 \leq r - q + q = r = r_0.$$

Suppose that (I_j) is true for each $j = 1, 2, \dots, n$. We need to show the existence of x_{n+1} and to obtain a bound q for a_n . To achieve this notice that:

$$Mr_n + N(r_{n-1} - r_n) = Mr_n + Nr_{n-1} - Nr_n = P_n(r_n) - Nr_n \leq r_n.$$

Then, it follows from Lemma 2.4 that there exists $q \leq r_n$ such that

$$q = Mq + N(r_{n-1} - r_n) \text{ and } (M + N)^k q \rightarrow 0, \text{ as } k \rightarrow \infty. \quad (3.3)$$

By (I_j) it follows that

$$b_n = /x_n - x_0/ \leq \sum_{j=0}^{n-1} a_j \leq \sum_{j=0}^{n-1} (r_j - r_{j+1}) = r - r_n \leq r.$$

Hence, $x_n \in U(x_0, r) \subset D$ and by (H₁) M is a bound for $I - A(x_n)$.

We can write by (H₂) that

$$\begin{aligned} /F(x_n)/ &= /F(x_n) - F(x_{n-1}) - A(x_{n-1})(x_n - x_{n-1})/ \\ &\leq Na_{n-1} \leq N(r_{n-1} - r_n). \end{aligned} \quad (3.4)$$

It follows from (3.3) and (3.4) that

$$Mq + /F(x_n)/ \leq q.$$

By Lemma 2.5, x_{n+1} is well defined and $a_n \leq q \leq r_n$. In view of the definition of r_{n+1} we have that

$$P_{n+1}(r_n - q) = P_n(r_n) - q = r_n - q,$$

so that by Lemma 2.4, r_{n+1} is well defined and

$$r_{n+1} + a_n \leq r_n - q + q = r_n,$$

which proves (I_{n+1}) . The induction for (I_n) is complete. Let $m \geq n$, then we obtain in turn that

$$/x_{m+1} - x_n/ \leq \sum_{j=n}^m a_j \leq \sum_{j=n}^m (r_j - r_{j+1}) = r_n - r_{m+1} \leq r_n. \quad (3.5)$$

Moreover, we get inductively the estimate

$$r_{n+1} = P_{n+1}(r_{n+1}) \leq P_{n+1}(r_n) \leq (M + N)r_n \leq \dots \leq (M + N)^{n+1}r.$$

It follows from (H_5) that $\{r_n\}$ is a null-sequence. Hence, $\{x_n\}$ is a complete sequence in a Banach space X by (3.5) and as such it converges to some $x^* \in X$. By letting $m \rightarrow \infty$ in (3.5) we deduce that $x^* \in U(x_n, r_n)$. Furthermore, (3.4) shows that x^* is a zero of F . Hence, (C_1) and (C_2) are proved.

In view of the estimate

$$R_n(r_n) \leq P_n(r_n) \leq r_n$$

the apriori, bound of (C_3) is well defined by Lemma 2.4. That is s_n is smaller in general than r_n . The conditions of Theorem 3.1 are satisfied for x_n replacing x_0 . A solution of the inequality of (C_2) is given by s_n (see (3.4)). It follows from (3.5) that the conditions of Theorem 3.1 are easily verified. Then, it follows from (C_1) that $x^* \in U(x_n, s_n)$ which proves (C_3) . ■

In general the aposterior, estimate is of interest. Then, condition (H_5) can be avoided as follows:

Proposition 3.2 *Suppose: condition (H_1) of Theorem 3.1 is true.*

(H'_3) There exists $s \in K$, $\theta \in (0, 1)$ such that

$$R_0(s) = (M + N)s + /F(x_0)/ \leq \theta s.$$

$$(H'_4) \ U(x_0, s) \subset D.$$

Then, there exists $r \leq s$ satisfying the conditions of Theorem 3.1. Moreover, the zero x^ of F is unique in $U(x_0, s)$.*

Remark 3.3 (i) Notice that by Lemma 2.4 $R_n^\infty(0)$ is the smallest solution of $R_n(s) \leq s$. Hence any solution of this inequality yields on upper estimate for $R_n^\infty(0)$. Similar inequalities appear in (H_2) and (H'_2) .

(ii) The weak assumptions of Theorem 3.1 do not imply the existence of $A(x_n)^{-1}$. In practice the computation of $T_n^\infty(0)$ as a solution of a linear equation is no problem and the computation of the expensive or impossible to compute in general $A(x_n)^{-1}$ is not needed.

(iii) We can use the following result for the computation of the a posteriori estimates. The proof can be found in [12, Lemma 4.2] by simply exchanging the definitions of R .

Lemma 3.4 Suppose that the conditions of Theorem 3.1 are satisfied. If $s \in K$ is a solution of $R_n(s) \leq s$, then $q := s - a_n \in K$ and solves $R_{n+1}(q) \leq q$. This solution might be improved by $R_{n+1}^k(q) \leq q$ for each $k = 1, 2, \dots$.

4 Special cases and applications

Application 4.1 The results obtained in earlier studies such as [5, 6, 7, 12] require that operator F (i.e. G) is Fréchet-differentiable. This assumption limits the applicability of the earlier results. In the present study we only require that F is a continuous operator. Hence, we have extended the applicability of these methods to include classes of operators that are only continuous.

Example 4.2 The j -dimensional space \mathbb{R}^j is a classical example of a generalized Banach space. The generalized norm is defined by componentwise absolute values. Then, as ordered Banach space we set $E = \mathbb{R}^j$ with componentwise ordering with e.g. the maximum norm. A bound for a linear operator (a matrix) is given by the corresponding matrix with absolute values. Similarly, we can define the "N" operators. Let $E = \mathbb{R}$. That is we consider the case of a real normed space with norm denoted by $\|\cdot\|$. Let us see how the conditions of Theorem 3.1 look like.

Theorem 4.3 (H_1) $\|I - A(x)\| \leq M$ for some $M \geq 0$.

(H_2) $\|F(y) - F(x) - A(x)(y - x)\| \leq N\|y - x\|$ for some $N \geq 0$.

(H_3) $M + N < 1$,

$$r = \frac{\|F(x_0)\|}{1 - (M + N)}. \quad (4.1)$$

(H_4) $U(x_0, r) \subseteq D$.

(H_5) $(M + N)^k r \rightarrow 0$ as $k \rightarrow \infty$, where r is given by (4.1).

Then, the conclusions of Theorem 3.1 hold.

5 Applications to k -Fractional Calculus

Background

We apply Theorem 4.3 in this section.

Let $f \in L_\infty([a, b])$, the k -left Riemann-Liouville fractional integral ([15]) of order $\alpha > 0$ is defined as follows:

$${}_k J_{a+}^\alpha f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt, \quad (5.1)$$

all $x \in [a, b]$, where $k > 0$, and $\Gamma_k(\alpha)$ is the k -gamma function given by $\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt$.

It holds ([4]) $\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha)$, $\Gamma(\alpha) = \lim_{k \rightarrow 1} \Gamma_k(\alpha)$, and we set ${}_k J_{a+}^\alpha f(x) = f(x)$.

Similarly, we define the k -right Riemann-Liouville fractional integral as

$${}_k J_{b-}^\alpha f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t) dt, \quad (5.2)$$

for all $x \in [a, b]$, and we set ${}_k J_{b-}^\alpha f(x) = f(x)$.

Results

I) Here we work with ${}_k J_{a+}^\alpha f(x)$. We observe that

$$\begin{aligned} |{}_k J_{a+}^\alpha f(x)| &\leq \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} |f(t)| dt \\ &\leq \frac{\|f\|_\infty}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} dt = \frac{\|f\|_\infty}{k\Gamma_k(\alpha)} \frac{(x-a)^{\frac{\alpha}{k}}}{\frac{\alpha}{k}} \\ &= \frac{\|f\|_\infty}{\Gamma_k(\alpha+k)} (x-a)^{\frac{\alpha}{k}} \leq \frac{\|f\|_\infty}{\Gamma_k(\alpha+k)} (b-a)^{\frac{\alpha}{k}}. \end{aligned} \quad (5.3)$$

We have proved that

$${}_k J_{a+}^\alpha f(a) = 0, \quad (5.4)$$

and

$$\|{}_k J_{a+}^\alpha f\|_\infty \leq \frac{(b-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \|f\|_\infty, \quad (5.5)$$

proving that ${}_k J_{a+}^\alpha$ is a bounded linear operator.

By [3], p. 388, we get that $({}_k J_{a+}^\alpha f)$ is a continuous function over $[a, b]$ and in particular continuous over $[a^*, b]$, where $a < a^* < b$.

Thus, there exist $x_1, x_2 \in [a^*, b]$ such that

$$({}_k J_{a+}^\alpha f)(x_1) = \min({}_k J_{a+}^\alpha f)(x), \quad (5.6)$$

$$({}_k J_{a+}^\alpha f)(x_2) = \max({}_k J_{a+}^\alpha f)(x), \quad x \in [a^*, b]. \quad (5.7)$$

We assume that

$$\left({}_k J_{a+}^\alpha f\right)\left(x_1\right)>0. \quad (5.8)$$

Hence

$$\left\|{}_k J_{a+}^\alpha f\right\|_{\infty,\left[a^*, b\right]}=\left({}_k J_{a+}^\alpha f\right)\left(x_2\right)>0. \quad (5.9)$$

Here it is

$$J(x)=m x, \quad m \neq 0. \quad (5.10)$$

Therefore the equation

$$J f(x)=0, \quad x \in\left[a^*, b\right], \quad (5.11)$$

has the same solutions as the equation

$$F(x):=\frac{J f(x)}{2\left({}_k J_{a+}^\alpha f\right)\left(x_2\right)}=0, \quad x \in\left[a^*, b\right]. \quad (5.12)$$

Notice that

$${}_k J_{a+}^\alpha\left(\frac{f}{2\left({}_k J_{a+}^\alpha f\right)\left(x_2\right)}\right)(x)=\frac{\left({}_k J_{a+}^\alpha f\right)(x)}{2\left({}_k J_{a+}^\alpha f\right)\left(x_2\right)} \leq \frac{1}{2}<1, \quad x \in\left[a^*, b\right]. \quad (5.13)$$

Call

$$A(x):=\frac{\left({}_k J_{a+}^\alpha f\right)(x)}{2\left({}_k J_{a+}^\alpha f\right)\left(x_2\right)}, \quad \forall x \in\left[a^*, b\right]. \quad (5.14)$$

We notice that

$$0<\frac{\left({}_k J_{a+}^\alpha f\right)\left(x_1\right)}{2\left({}_k J_{a+}^\alpha f\right)\left(x_2\right)} \leq A(x) \leq \frac{1}{2}, \quad \forall x \in\left[a^*, b\right]. \quad (5.15)$$

Hence it holds

$$\left|1-A(x)\right|=1-A(x) \leq 1-\frac{\left({}_k J_{a+}^\alpha f\right)\left(x_1\right)}{2\left({}_k J_{a+}^\alpha f\right)\left(x_2\right)}=: \gamma_0, \quad \forall x \in\left[a^*, b\right]. \quad (5.16)$$

Clearly $\gamma_0 \in(0,1)$.

We have proved that

$$\left|1-A(x)\right| \leq \gamma_0, \quad \forall x \in\left[a^*, b\right]. \quad (5.17)$$

Next we assume that $F(x)$ is a contraction, i.e.

$$\left|F(x)-F(y)\right| \leq \lambda|x-y| ; \quad \forall x, y \in\left[a^*, b\right], \quad (5.18)$$

and $0<\lambda<\frac{1}{2}$.

Equivalently we have

$$\left|J f(x)-J f(y)\right| \leq 2 \lambda\left({}_k J_{a+}^\alpha f\right)\left(x_2\right)|x-y|, \quad \text { all } x, y \in\left[a^*, b\right]. \quad (5.19)$$

We observe that

$$\begin{aligned} |F(y) - F(x) - A(x)(y-x)| &\leq |F(y) - F(x)| + |A(x)||y-x| \leq \\ \lambda|y-x| + |A(x)||y-x| &= (\lambda + |A(x)|)|y-x| =: (\psi_1), \quad \forall x, y \in [a^*, b]. \end{aligned} \quad (5.20)$$

We have that

$$|({}_k J_{a+}^\alpha f)(x)| \leq \frac{(b-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \|f\|_\infty < \infty, \quad \forall x \in [a^*, b]. \quad (5.21)$$

Hence

$$|A(x)| = \frac{|({}_k J_{a+}^\alpha f)(x)|}{2({}_k J_{a+}^\alpha f)(x_2)} \leq \frac{(b-a)^{\frac{\alpha}{k}} \|f\|_\infty}{2\Gamma_k(\alpha+k)({}_k J_{a+}^\alpha f)(x_2)} < \infty, \quad \forall x \in [a^*, b]. \quad (5.22)$$

Therefore we get

$$(\psi_1) \leq \left(\lambda + \frac{(b-a)^{\frac{\alpha}{k}} \|f\|_\infty}{2\Gamma_k(\alpha+k)({}_k J_{a+}^\alpha f)(x_2)} \right) |y-x|, \quad \forall x, y \in [a^*, b]. \quad (5.23)$$

Call

$$0 < \gamma_1 := \lambda + \frac{(b-a)^{\frac{\alpha}{k}} \|f\|_\infty}{2\Gamma_k(\alpha+k)({}_k J_{a+}^\alpha f)(x_2)}, \quad (5.24)$$

choosing $(b-a)$ small enough we can make $\gamma_1 \in (0, 1)$.

We have proved that

$$|F(y) - F(x) - A(x)(y-x)| \leq \gamma_1 |y-x|, \quad \forall x, y \in [a^*, b], \quad \gamma_1 \in (0, 1). \quad (5.25)$$

Next we call and we need that

$$0 < \gamma := \gamma_0 + \gamma_1 = 1 - \frac{({}_k J_{a+}^\alpha f)(x_1)}{2({}_k J_{a+}^\alpha f)(x_2)} + \lambda + \frac{(b-a)^{\frac{\alpha}{k}} \|f\|_\infty}{2\Gamma_k(\alpha+k)({}_k J_{a+}^\alpha f)(x_2)} < 1, \quad (5.26)$$

equivalently,

$$\lambda + \frac{(b-a)^{\frac{\alpha}{k}} \|f\|_\infty}{2\Gamma_k(\alpha+k)({}_k J_{a+}^\alpha f)(x_2)} < \frac{({}_k J_{a+}^\alpha f)(x_1)}{2({}_k J_{a+}^\alpha f)(x_2)}, \quad (5.27)$$

equivalently,

$$2\lambda({}_k J_{a+}^\alpha f)(x_2) + \frac{(b-a)^{\frac{\alpha}{k}} \|f\|_\infty}{\Gamma_k(\alpha+k)} < ({}_k J_{a+}^\alpha f)(x_1), \quad (5.28)$$

which is possible for small $\lambda, (b-a)$. That is $\gamma \in (0, 1)$. So our numerical method converges and solves (5.11).

II) Here we act on ${}_k J_{b-}^\alpha f(x)$, see (5.2).

Let $f \in L_\infty([a, b])$. We have that

$$\begin{aligned} |{}_k J_{b-}^\alpha f(x)| &\leq \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} |f(t)| dt \\ &\leq \frac{\|f\|_\infty}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} dt = \frac{\|f\|_\infty}{k\Gamma_k(\alpha)} \frac{(b-x)^{\frac{\alpha}{k}}}{\frac{\alpha}{k}} \\ &= \frac{\|f\|_\infty}{\Gamma_k(\alpha+k)} (b-x)^{\frac{\alpha}{k}} \leq \frac{\|f\|_\infty}{\Gamma_k(\alpha+k)} (b-a)^{\frac{\alpha}{k}}. \end{aligned} \quad (5.29)$$

We observe that

$${}_k J_{b-}^\alpha f(b) = 0, \quad (5.30)$$

and

$$\|{}_k J_{b-}^\alpha f\|_\infty \leq \frac{(b-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \|f\|_\infty. \quad (5.31)$$

That is ${}_k J_{b-}^\alpha$ is a bounded linear operator.

Let here $a < b^* < b$.

By [4] we get that ${}_k J_{b-}^\alpha f$ is continuous over $[a, b]$, and in particular it is continuous over $[a, b^*]$.

Thus, there exist $x_1, x_2 \in [a, b^*]$ such that

$$({}_k J_{b-}^\alpha f)(x_1) = \min({}_k J_{b-}^\alpha f)(x), \quad (5.32)$$

$$({}_k J_{b-}^\alpha f)(x_2) = \max({}_k J_{b-}^\alpha f)(x), \quad x \in [a, b^*].$$

We assume that

$$({}_k J_{b-}^\alpha f)(x_1) > 0. \quad (5.33)$$

Hence

$$\|{}_k J_{b-}^\alpha f\|_{\infty, [a^*, b]} = ({}_k J_{b-}^\alpha f)(x_2) > 0. \quad (5.34)$$

Here it is

$$J(x) = mx, \quad m \neq 0. \quad (5.35)$$

Therefore the equation

$$Jf(x) = 0, \quad x \in [a, b^*], \quad (5.36)$$

has the same solutions as the equation

$$F(x) := \frac{Jf(x)}{2({}_k J_{b-}^\alpha f)(x_2)} = 0, \quad x \in [a, b^*]. \quad (5.37)$$

Notice that

$${}_k J_{b-}^\alpha \left(\frac{f}{2({}_k J_{b-}^\alpha f)(x_2)} \right)(x) = \frac{({}_k J_{b-}^\alpha f)(x)}{2({}_k J_{b-}^\alpha f)(x_2)} \leq \frac{1}{2} < 1, \quad x \in [a, b^*]. \quad (5.38)$$

Call

$$A(x) := \frac{({}_k J_{b-}^\alpha f)(x)}{2({}_k J_{b-}^\alpha f)(x_2)}, \quad \forall x \in [a, b^*]. \quad (5.39)$$

We notice that

$$0 < \frac{({}_k J_{b-}^\alpha f)(x_1)}{2({}_k J_{b-}^\alpha f)(x_2)} \leq A(x) \leq \frac{1}{2}, \quad \forall x \in [a, b^*]. \quad (5.40)$$

Hence we have

$$|1 - A(x)| = 1 - A(x) \leq 1 - \frac{({}_k J_{b-}^\alpha f)(x_1)}{2({}_k J_{b-}^\alpha f)(x_2)} =: \gamma_0, \quad \forall x \in [a, b^*]. \quad (5.41)$$

Clearly $\gamma_0 \in (0, 1)$.

We have proved that

$$|1 - A(x)| \leq \gamma_0, \quad \forall x \in [a, b^*], \quad \gamma_0 \in (0, 1). \quad (5.42)$$

Next we assume that $F(x)$ is a contraction, i.e.

$$|F(x) - F(y)| \leq \lambda |x - y|; \quad \forall x, y \in [a, b^*], \quad (5.43)$$

and $0 < \lambda < \frac{1}{2}$.

Equivalently we have

$$|Jf(x) - Jf(y)| \leq 2\lambda ({}_k J_{b-}^\alpha f)(x_2) |x - y|, \quad \text{all } x, y \in [a, b^*]. \quad (5.44)$$

We observe that

$$\begin{aligned} |F(y) - F(x) - A(x)(y - x)| &\leq |F(y) - F(x)| + |A(x)| |y - x| \leq \\ \lambda |y - x| + |A(x)| |y - x| &= (\lambda + |A(x)|) |y - x| =: (\psi_1), \quad \forall x, y \in [a, b^*]. \end{aligned} \quad (5.45)$$

We have that

$$|({}_k J_{b-}^\alpha f)(x)| \leq \frac{(b-a)^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \|f\|_\infty < \infty, \quad \forall x \in [a, b^*]. \quad (5.46)$$

Hence

$$|A(x)| = \frac{|({}_k J_{b-}^\alpha f)(x)|}{2({}_k J_{b-}^\alpha f)(x_2)} \leq \frac{(b-a)^{\frac{\alpha}{k}} \|f\|_\infty}{2\Gamma_k(\alpha+k)({}_k J_{b-}^\alpha f)(x_2)} < \infty, \quad \forall x \in [a, b^*]. \quad (5.47)$$

Therefore we get

$$(\psi_1) \leq \left(\lambda + \frac{(b-a)^{\frac{\alpha}{k}} \|f\|_\infty}{2\Gamma_k(\alpha+k)({}_k J_{b-}^\alpha f)(x_2)} \right) |y - x|, \quad \forall x, y \in [a, b^*]. \quad (5.48)$$

Call

$$0 < \gamma_1 := \lambda + \frac{(b-a)^{\frac{\alpha}{k}} \|f\|_{\infty}}{2\Gamma_k(\alpha+k)({}_k J_{b-}^{\alpha} f)(x_2)}, \quad (5.49)$$

choosing $(b-a)$ small enough we can make $\gamma_1 \in (0, 1)$.

We have proved that

$$|F(y) - F(x) - A(x)(y-x)| \leq \gamma_1 |y-x|, \quad \forall x, y \in [a, b^*], \quad \gamma_1 \in (0, 1). \quad (5.50)$$

Next we call and we need that

$$0 < \gamma := \gamma_0 + \gamma_1 = 1 - \frac{({}_k J_{b-}^{\alpha} f)(x_1)}{2({}_k J_{b-}^{\alpha} f)(x_2)} + \lambda + \frac{(b-a)^{\frac{\alpha}{k}} \|f\|_{\infty}}{2\Gamma_k(\alpha+k)({}_k J_{b-}^{\alpha} f)(x_2)} < 1, \quad (5.51)$$

equivalently,

$$\lambda + \frac{(b-a)^{\frac{\alpha}{k}} \|f\|_{\infty}}{2\Gamma_k(\alpha+k)({}_k J_{b-}^{\alpha} f)(x_2)} < \frac{({}_k J_{b-}^{\alpha} f)(x_1)}{2({}_k J_{b-}^{\alpha} f)(x_2)}, \quad (5.52)$$

equivalently,

$$2\lambda({}_k J_{b-}^{\alpha} f)(x_2) + \frac{(b-a)^{\frac{\alpha}{k}} \|f\|_{\infty}}{\Gamma_k(\alpha+k)} < ({}_k J_{b-}^{\alpha} f)(x_1), \quad (5.53)$$

which is possible for small $\lambda, (b-a)$. That is $\gamma \in (0, 1)$. So our numerical method converges and solves (5.36).

III) Here we deal with the fractional M. Caputo-Fabrizio derivative defined as follows (see [9]):

let $0 < \alpha < 1, f \in C^1([0, b])$,

$${}^{CF}D_*^{\alpha} f(t) = \frac{1}{1-\alpha} \int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) f'(s) ds, \quad (5.54)$$

for all $0 \leq t \leq b$.

Call

$$\gamma := \frac{\alpha}{1-\alpha} > 0. \quad (5.55)$$

I.e.

$${}^{CF}D_*^{\alpha} f(t) = \frac{1}{1-\alpha} \int_0^t e^{-\gamma(t-s)} f'(s) ds, \quad 0 \leq t \leq b. \quad (5.56)$$

We notice that

$$\begin{aligned} |{}^{CF}D_*^{\alpha} f(t)| &\leq \frac{1}{1-\alpha} \left(\int_0^t e^{-\gamma(t-s)} ds \right) \|f'\|_{\infty} \\ &= \frac{e^{-\gamma t}}{\alpha} (e^{\gamma t} - 1) \|f'\|_{\infty} = \frac{1}{\alpha} (1 - e^{-\gamma t}) \|f'\|_{\infty} \leq \left(\frac{1 - e^{-\gamma b}}{\alpha} \right) \|f'\|_{\infty}. \end{aligned} \quad (5.57)$$

That is

$$({}^{CF}D_*^\alpha f)(0) = 0, \quad (5.58)$$

and

$$|{}^{CF}D_*^\alpha f(t)| \leq \left(\frac{1 - e^{-\gamma b}}{\alpha} \right) \|f'\|_\infty, \quad \forall t \in [0, b]. \quad (5.59)$$

Notice here that $1 - e^{-\gamma t}$, $t \geq 0$ is an increasing function.

Thus the smaller the t , the smaller it is $1 - e^{-\gamma t}$. We rewrite

$${}^{CF}D_*^\alpha f(t) = \frac{e^{-\gamma t}}{1 - \alpha} \int_0^t e^{\gamma s} f'(s) ds, \quad (5.60)$$

proving that $({}^{CF}D_*^\alpha f)$ is a continuous function over $[0, b]$, in particular it is continuous over $[a, b]$, where $0 < a < b$.

Therefore there exist $x_1, x_2 \in [a, b]$ such that

$${}^{CF}D_*^\alpha f(x_1) = \min {}^{CF}D_*^\alpha f(x), \quad (5.61)$$

and

$${}^{CF}D_*^\alpha f(x_2) = \max {}^{CF}D_*^\alpha f(x), \quad \text{for } x \in [a, b].$$

We assume that

$${}^{CF}D_*^\alpha f(x_1) > 0. \quad (5.62)$$

(i.e. ${}^{CF}D_*^\alpha f(x) > 0$, $\forall x \in [a, b]$).

Furthermore

$$\|{}^{CF}D_*^\alpha f\|_{\infty, [a, b]} = {}^{CF}D_*^\alpha f(x_2). \quad (5.63)$$

Here it is

$$J(x) = mx, \quad m \neq 0. \quad (5.64)$$

The equation

$$Jf(x) = 0, \quad x \in [a, b], \quad (5.65)$$

has the same set of solutions as the equation

$$F(x) := \frac{Jf(x)}{{}^{CF}D_*^\alpha f(x_2)} = 0, \quad x \in [a, b]. \quad (5.66)$$

Notice that

$${}^{CF}D_*^\alpha \left(\frac{f(x)}{{}^{CF}D_*^\alpha f(x_2)} \right) = \frac{{}^{CF}D_*^\alpha f(x)}{{}^{CF}D_*^\alpha f(x_2)} \leq \frac{1}{2} < 1, \quad \forall x \in [a, b]. \quad (5.67)$$

We call

$$A(x) := \frac{{}^{CF}D_*^\alpha f(x)}{{}^{CF}D_*^\alpha f(x_2)}, \quad \forall x \in [a, b]. \quad (5.68)$$

We notice that

$$0 < \frac{{}^{CF}D_*^\alpha f(x_1)}{{}^{CF}D_*^\alpha f(x_2)} \leq A(x) \leq \frac{1}{2}. \quad (5.69)$$

Furthermore it holds

$$|1 - A(x)| = 1 - A(x) \leq 1 - \frac{{}^{CF}D_*^\alpha f(x_1)}{{}^{CF}D_*^\alpha f(x_2)} =: \gamma_0, \quad \forall x \in [a, b]. \quad (5.70)$$

Clearly $\gamma_0 \in (0, 1)$.

We have proved that

$$|1 - A(x)| \leq \gamma_0 \in (0, 1), \quad \forall x \in [a, b]. \quad (5.71)$$

Next we assume that $F(x)$ is a contraction over $[a, b]$, i.e.

$$|F(x) - F(y)| \leq \lambda |x - y|; \quad \forall x, y \in [a, b], \quad (5.72)$$

and $0 < \lambda < \frac{1}{2}$.

Equivalently we have

$$|Jf(x) - Jf(y)| \leq 2\lambda ({}^{CF}D_*^\alpha f(x_2)) |x - y|, \quad \forall x, y \in [a, b]. \quad (5.73)$$

We observe that

$$\begin{aligned} |F(y) - F(x) - A(x)(y - x)| &\leq |F(y) - F(x)| + |A(x)| |y - x| \leq \\ \lambda |y - x| + |A(x)| |y - x| &= (\lambda + |A(x)|) |y - x| =: (\xi_2), \quad \forall x, y \in [a, b]. \end{aligned} \quad (5.74)$$

Here we have

$$|({}^{CF}D_*^\alpha f)(x)| \leq \left(\frac{1 - e^{-\gamma b}}{\alpha} \right) \|f'\|_\infty, \quad \forall t \in [a, b]. \quad (5.75)$$

Hence, $\forall x \in [a, b]$ we get that

$$|A(x)| = \frac{|{}^{CF}D_*^\alpha f(x)|}{{}^{CF}D_*^\alpha f(x_2)} \leq \frac{(1 - e^{-\gamma b}) \|f'\|_\infty}{{}^{CF}D_*^\alpha f(x_2)} < \infty. \quad (5.76)$$

Consequently we observe

$$(\xi_2) \leq \left(\lambda + \frac{(1 - e^{-\gamma b}) \|f'\|_\infty}{{}^{CF}D_*^\alpha f(x_2)} \right) |y - x|, \quad \forall x, y \in [a, b]. \quad (5.77)$$

Call

$$0 < \gamma_1 := \lambda + \frac{(1 - e^{-\gamma b}) \|f'\|_\infty}{{}^{CF}D_*^\alpha f(x_2)}, \quad (5.78)$$

choosing b small enough, we can make $\gamma_1 \in (0, 1)$.

We have proved

$$|F(y) - F(x) - A(x)(y - x)| \leq \gamma_1 |y - x|, \quad \gamma_1 \in (0, 1), \quad \forall x, y \in [a, b]. \quad (5.79)$$

Next we call and need

$$0 < \gamma := \gamma_0 + \gamma_1 = 1 - \frac{{}^{CF}D_*^\alpha f(x_1)}{{}^{CF}D_*^\alpha f(x_2)} + \lambda + \frac{(1 - e^{-\gamma b}) \|f'\|_\infty}{2\alpha ({}^{CF}D_*^\alpha f)(x_2)} < 1, \quad (5.80)$$

equivalently,

$$\lambda + \frac{(1 - e^{-\gamma b}) \|f'\|_\infty}{2\alpha ({}^{CF}D_*^\alpha f)(x_2)} < \frac{{}^{CF}D_*^\alpha f(x_1)}{{}^{CF}D_*^\alpha f(x_2)}, \quad (5.81)$$

equivalently,

$$2\lambda {}^{CF}D_*^\alpha f(x_2) + \frac{(1 - e^{-\gamma b})}{\alpha} \|f'\|_\infty < {}^{CF}D_*^\alpha f(x_1), \quad (5.82)$$

which is possible for small λ, b .

We have proved that

$$\gamma = \gamma_0 + \gamma_1 \in (0, 1). \quad (5.83)$$

Hence equation (5.65) can be solved with our presented numerical methods.

Conclusion:

In all three applications we have proved that

$$|1 - A(x)| \leq \gamma_0 \in (0, 1), \quad (5.84)$$

and

$$|F(y) - F(x) - A(x)(y - x)| \leq \gamma_1 |y - x|, \quad (5.85)$$

where $\gamma_1 \in (0, 1)$, and

$$\gamma = \gamma_0 + \gamma_1 \in (0, 1), \quad (5.86)$$

for all $x, y \in [a^*, b], [a, b^*], [a, b]$, respectively.

Consequently, our presented Numerical methods here, Theorem 4.3, apply to solve

$$f(x) = 0. \quad (5.87)$$

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Some Sets of Sufficient Conditions for Carathéodory Functions

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Abstract

In this paper, we first investigate and present several sets of sufficient conditions for Carathéodory functions in the open unit disk \mathbb{U} . We then apply the main results proven here in order to derive some conditions for starlike functions in \mathbb{U} . Relevant connections with various known results are also considered.

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1. Introduction, Definitions and Preliminaries

Let \mathcal{P} denote the class of functions p of the form:

$$p(z) = \sum_{n=0}^{\infty} p_n z^n,$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

The function $p \in \mathcal{P}$ is called a *Carathéodory function* if it satisfies the following condition:

$$\Re \{p(z)\} > 0 \quad (z \in \mathbb{U}).$$

Let \mathcal{A} denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in \mathbb{U} . A function $f \in \mathcal{A}$ is in the class \mathcal{S}^* of starlike functions in \mathbb{U} , if it satisfies the following condition:

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in \mathbb{U}).$$

In recent years, many authors (see, for example, [1, 2, 3, 4, 6, 8, 9, 10, 12, 16, 18]) have investigated and derived sufficient conditions for Carathéodory functions and some of their results have been applied to find some sufficient conditions for starlikeness or convexity of analytic functions (see, for example, [5, 11, 13, 14, 15, 17]).

Following the principle of differential subordination, we say that a function f is subordinate to F in \mathbb{U} , written as $f \prec F$, if and only if

$$f(z) = F(w(z)) \quad (z \in \mathbb{U})$$

for some Schwarz function $w(z)$, with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U}).$$

If $F(z)$ is univalent in \mathbb{U} , then the subordination $f \prec F$ is equivalent to

$$f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}).$$

We denote by \mathcal{Q} the class of functions q that are analytic and injective on $\overline{\mathbb{U}} \setminus E(q)$, where

$$E(q) = \left\{ \zeta : \zeta \in \partial\mathbb{U} \quad \text{and} \quad \lim_{z \rightarrow \zeta} q(z) = \infty \right\},$$

and are such that

$$q'(\zeta) \neq 0 \quad (\zeta \in \partial\mathbb{U} \setminus E(q)).$$

Furthermore, let the subclass of \mathcal{Q} for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$.

The main object of this paper is to investigate and present several sets of sufficient conditions for Carathéodory functions in the open unit disk \mathbb{U} . The main results proven here are shown to lead to some conditions for starlike functions in \mathbb{U} . We also consider the relevant connections of our results with various known results.

2. A Set of Main Results

In order to prove our main results, we need the following lemma due to Miller and Mocanu [7, p. 24].

Lemma 1. *Let $q \in \mathcal{Q}(a)$ and let the function $p(z)$ given by*

$$p(z) = a + a_n z^n + \cdots \quad (n \geq 1)$$

be analytic in \mathbb{U} with $p(0) = a$. If p is not subordinate to q , then there exist points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U} \setminus E(q)$ for which

$$(i) \quad p(z_0) = q(\zeta_0) \quad \text{and}$$

$$(ii) \ z_0 p'(z_0) = m \zeta_0 q'(\zeta_0) \quad (m \geq n \geq 1).$$

Applying Lemma 1, we can obtain the following results.

Theorem 1. Let $P : \mathbb{U} \rightarrow \mathbb{C}$ with

$$\Re \{P(z)\} \geq \Im \{P(z)\} \tan \alpha \geq 0 \quad \left(0 \leq \alpha < \frac{\pi}{2}\right).$$

If the function p is an analytic in \mathbb{U} with $p(0) = 1$ and

$$\Re \{[p(z)]^2 + P(z)zp'(z)\} > \frac{B^2 \sin^2 \alpha}{4A \cos^2 \alpha} - \frac{B}{2 \cos \alpha}, \quad (1)$$

where

$$A = \cos 2\alpha + \frac{B}{2 \cos \alpha} \quad (2)$$

and

$$B = \Re \{P(z)\} \cos \alpha - \Im \{P(z)\} \sin \alpha, \quad (3)$$

then

$$|\arg \{p(z)\}| < \frac{\pi}{2} - \alpha \quad \left(0 \leq \alpha < \frac{\pi}{2}; \ z \in \mathbb{U}\right).$$

Proof. Let us define two functions $q(z)$ and $h_1(z)$ by

$$q(z) = e^{i\alpha} p(z) \quad \left(q(z) \neq e^{i\alpha}; \ 0 \leq \alpha < \frac{\pi}{2}; \ z \in \mathbb{U}\right) \quad (4)$$

and

$$h_1(z) = \frac{e^{i\alpha} + \overline{e^{i\alpha}}z}{1-z} \quad \left(0 \leq \alpha < \frac{\pi}{2}; \ z \in \mathbb{U}\right), \quad (5)$$

respectively. Then the functions $q(z)$ and $h_1(z)$ are analytic in \mathbb{U} with

$$q(0) = h_1(0) = e^{i\alpha} \in \mathbb{C} \quad \text{and} \quad h_1(\mathbb{U}) = \{w : w \in \mathbb{C} \text{ and } \Re \{w\} > 0\}.$$

We now suppose that the function q is not subordinate to h_1 . Then, by Lemma 1, there exist points $z_1 \in \mathbb{U}$ and $\zeta_1 \in \partial\mathbb{U} \setminus \{1\}$ such that

$$q(z_1) = h_1(\zeta_1) = i\rho \quad (\rho \in \mathbb{R}) \quad \text{and} \quad z_1 q'(z_1) = m \zeta_1 h'_1(\zeta_1) = m\sigma_1 \quad (m \geq 1), \quad (6)$$

where

$$\sigma_1 = -\frac{\rho^2 - 2\rho \sin \alpha + 1}{2 \cos \alpha}. \quad (7)$$

Using the equations (4), (5), (6) and (7), we obtain

$$\begin{aligned} & \Re \{[p(z_1)]^2 + P(z_1)z_1 p'(z_1)\} \\ &= \Re \left\{ [e^{-i\alpha} q(z_1)]^2 + P(z_1) e^{-i\alpha} z_1 q'(z_1) \right\} \\ &= \Re \left\{ e^{-2i\alpha} [h_1(\zeta_1)]^2 + P(z_1) e^{-i\alpha} m \zeta_1 h'_1(\zeta_1) \right\} \\ &= \Re \left\{ e^{-2i\alpha} (i\rho)^2 + P(z_1) e^{-i\alpha} m \sigma_1 \right\} \\ &= -\rho^2 \cos 2\alpha + m \sigma_1 B_1 \\ &\leq -\left(\cos 2\alpha + \frac{B_1}{2 \cos \alpha} \right) \rho^2 + \left(\frac{B_1 \sin \alpha}{\cos \alpha} \right) \rho - \frac{B_1}{2 \cos \alpha} \\ &= -A_1 \rho^2 + \left(\frac{B_1 \sin \alpha}{\cos \alpha} \right) \rho - \frac{B_1}{2 \cos \alpha} \\ &=: g(\rho), \end{aligned} \quad (8)$$

where B_1 and A_1 are given by

$$B_1 = \Re \{P(z_1)\} \cos \alpha - \Im \{P(z_1)\} \sin \alpha$$

and

$$A_1 = \cos 2\alpha + \frac{B_1}{2 \cos \alpha},$$

respectively. By a simple calculation, we see that the function $g_1(\rho)$ in (8) takes on the maximum value at ρ^* given by

$$\rho^* = \frac{B_1 \sin \alpha}{2A_1 \cos \alpha}.$$

Hence we have

$$\begin{aligned} & \Re \{[p(z_1)]^2 + P(z_1)z_1p'(z_1)\} \\ & \leq g_1(\rho^*) \\ & = \frac{B_1^2 \sin^2 \alpha}{4A_1 \cos^2 \alpha} - \frac{B_1}{2 \cos \alpha} \\ & \leq \frac{B^2 \sin^2 \alpha}{4A \cos^2 \alpha} - \frac{B}{2 \cos \alpha}, \end{aligned}$$

where A and B are given by (2) and (3), respectively. Moreover, this inequality is a contradiction to (1). Therefore, we obtain

$$\Re \{e^{i\alpha}p(z)\} > 0 \quad \left(0 \leq \alpha < \frac{\pi}{2}; z \in \mathbb{U}\right). \quad (9)$$

Next, let us define two analytic functions by

$$r(z) = e^{-i\alpha}p(z) \quad \left(0 \leq \alpha < \frac{\pi}{2}; z \in \mathbb{U}\right) \quad (10)$$

and

$$h_2(z) = \frac{e^{-i\alpha} + \overline{e^{-i\alpha}}z}{1-z} \quad \left(0 \leq \alpha < \frac{\pi}{2}; z \in \mathbb{U}\right). \quad (11)$$

Then the functions r and h_2 are analytic in \mathbb{U} with

$$r(0) = h_2(0) = e^{-i\alpha} \in \mathbb{C} \quad \text{and} \quad h_2(\mathbb{U}) = \{w : w \in \mathbb{C} \quad \text{and} \quad \Re \{w\} > 0\} = h_1(\mathbb{U}).$$

Suppose that r is not subordinate to h_2 . Then, by Lemma 1, there exist points $z_2 \in \mathbb{U}$ and $\zeta_2 \in \partial\mathbb{U} \setminus \{1\}$ such that

$$r(z_2) = h_2(\zeta_2) = i\rho \quad (\rho \in \mathbb{R}) \quad \text{and} \quad z_2r'(z_2) = m\zeta_2h_2'(\zeta_2) = m\sigma_2 \quad (m \geq 1), \quad (12)$$

where

$$\sigma_2 = -\frac{\rho^2 + 2\rho \sin \alpha + 1}{2 \cos \alpha}. \quad (13)$$

From the equations (10), (11), (12) and (13), we get

$$\begin{aligned}
 & \Re \{ [p(z_2)]^2 + P(z_2)z_2p'(z_2) \} \\
 &= \Re \{ e^{2i\alpha} [h_2(\zeta_2)]^2 + P(z_2)e^{i\alpha}m\zeta_2h_2'(\zeta_2) \} \\
 &= \Re \{ e^{2i\alpha} (i\rho)^2 + P(z_2)e^{i\alpha}m\sigma_2 \} \\
 &= -\rho^2 \cos 2\alpha + m\sigma_2 B \\
 &\leq -\rho^2 \cos 2\alpha + \sigma_2 B \\
 &= -A\rho^2 - \left(\frac{B \sin \alpha}{\cos \alpha} \right) \rho - \frac{B}{2 \cos \alpha} \\
 &= g_2(\rho) \\
 &\leq g_2 \left(-\frac{B \sin \alpha}{2A \cos \alpha} \right) \\
 &= \frac{B^2 \sin^2 \alpha}{4A \cos^2 \alpha} - \frac{B}{2 \cos \alpha},
 \end{aligned}$$

which is a contradiction to (1). Therefore, we have

$$\Re \{ e^{-i\alpha} p(z) \} > 0 \quad \left(0 \leq \alpha < \frac{\pi}{2}; z \in \mathbb{U} \right). \quad (14)$$

Hence, by applying the inequalities (9) and (14), we find that

$$|\arg \{p(z)\}| < \frac{\pi}{2} - \alpha \quad \left(0 \leq \alpha < \frac{\pi}{2}; z \in \mathbb{U} \right).$$

This evidently complete the proof of Theorem 1. □

If we take $P(z) \equiv \beta$ ($\beta > 0$) in Theorem 1, then we have the following corollary.

Corollary 1. *Let the function p be analytic in \mathbb{U} with $p(0) = 1$. If*

$$\begin{aligned}
 & \Re \{ [p(z)]^2 + \beta z p'(z) \} > \frac{1}{2\beta + 4 \cos 2\alpha} \{ (\beta^2 + 4\beta) \sin^2 \alpha - \beta^2 - 2\beta \} \\
 & \quad \left(\beta > 0; 0 \leq \alpha < \frac{\pi}{2}; z \in \mathbb{U} \right),
 \end{aligned}$$

then

$$|\arg \{p(z)\}| < \frac{\pi}{2} - \alpha \quad \left(0 \leq \alpha < \frac{\pi}{2}; z \in \mathbb{U} \right).$$

More specially, if we take $P(z) \equiv 1$ in Theorem 1 or set $\beta = 1$ in Corollary 1, we obtain the following corollary.

Corollary 2. *Let the function p be analytic in \mathbb{U} with $p(0) = 1$. If*

$$\Re \{ [p(z)]^2 + z p'(z) \} > \frac{5 \sin^2 \alpha - 3}{6 - 8 \sin^2 \alpha} \quad \left(0 \leq \alpha < \frac{\pi}{2}; z \in \mathbb{U} \right),$$

then

$$|\arg \{p(z)\}| < \frac{\pi}{2} - \alpha \quad \left(0 \leq \alpha < \frac{\pi}{2}; z \in \mathbb{U} \right).$$

Taking $\alpha = 0$ in Corollary 2, we have the following corollary.

Corollary 3. *Let the function p be analytic in \mathbb{U} with $p(0) = 1$. If*

$$\Re \{ [p(z)]^2 + zp'(z) \} > -\frac{1}{2} \quad (z \in \mathbb{U}),$$

then

$$\Re \{ p(z) \} > 0 \quad (z \in \mathbb{U}).$$

The following corollary presents a sufficient condition for starlikeness of analytic functions in \mathbb{U} . It follows easily by taking

$$p(z) = \frac{zf'(z)}{f(z)} \quad (f \in \mathcal{A})$$

in Corollary 3.

Corollary 4. *Let $f \in \mathcal{A}$. Then*

$$\Re \left\{ \frac{zf'(z)}{f(z)} \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} > -\frac{1}{2} \quad (z \in \mathbb{U})$$

implies that $f \in \mathcal{S}^$.*

3. Further Sufficient Conditions

We now find another another set of sufficient conditions for Carathéodory functions.

Theorem 2. *Let $p(z)$ be a nonzero analytic function in \mathbb{U} with $p(0) = 1$ and*

$$\left| \frac{zp'(z)}{[p(z)]^2} \right| < \frac{1}{2} \cos \alpha \quad \left(0 \leq \alpha < \frac{\pi}{2}; z \in \mathbb{U} \right). \quad (15)$$

Then

$$|\arg \{ p(z) \}| < \frac{\pi}{2} - \alpha \quad \left(0 \leq \alpha < \frac{\pi}{2}; z \in \mathbb{U} \right).$$

Proof. As before, we define the functions $q(z)$ and $h_1(z)$ by (4) and (5), respectively. We also suppose that q is not subordinate to h_1 . Then, by Lemma 1, there exist points $z_1 \in \mathbb{U}$ and $\zeta_1 \in \partial\mathbb{U} \setminus \{1\}$ satisfying (6). We note that $\rho \neq 0$ in (6), since the function $p(z)$ cannot vanish in \mathbb{U} . Thus, from the equations (4), (5), (6) and (7), we obtain

$$\left| \frac{z_1 p'(z_1)}{[p(z_1)]^2} \right| = \left| \frac{z_1 q'(z_1)}{[q(z_1)]^2} \right| = \left| \frac{m \zeta_1 h_1'(\zeta_1)}{[h_1(\zeta_1)]^2} \right| = \left| \frac{m \sigma_1}{(i\rho)^2} \right|.$$

We also have

$$\left| \frac{m \sigma_1}{(i\rho)^2} \right| = m \frac{|\sigma_1|}{\rho^2} \geq -\frac{\sigma_1}{\rho^2} = \frac{1}{2 \cos \alpha} g_1(\rho),$$

where

$$g_1(\rho) = \frac{\rho^2 - 2\rho \sin \alpha + 1}{\rho^2}.$$

For the case when $\alpha \neq 0$, since g_1 has its minimum at

$$\rho^* = \frac{1}{\sin \alpha},$$

we have

$$\left| \frac{z_1 p'(z_1)}{[p(z_1)]^2} \right| \geq \frac{1}{2 \cos \alpha} g_1(\rho^*) = \frac{1}{2} \cos \alpha,$$

which is a contradiction to (15). We thus have

$$q(z) \prec h_1(z) \quad (z \in \mathbb{U})$$

or, equivalently,

$$\Re \{e^{i\alpha} p(z)\} > 0 \quad (z \in \mathbb{U}). \quad (16)$$

We next define the functions r and h_2 by (10) and (11), respectively. By using a similar method as the above, we obtain

$$\Re \{e^{-i\alpha} p(z)\} > 0 \quad (z \in \mathbb{U}) \quad (17)$$

for the case when $\alpha \neq 0$. Thus, from (16) and (17), we have

$$|\arg \{p(z)\}| < \frac{\pi}{2} - \alpha \quad \left(0 < \alpha < \frac{\pi}{2}; z \in \mathbb{U}\right).$$

For the case when $\alpha = 0$, we have

$$g_1(\rho) = 1 + \rho^{-2} \geq 1 \quad (\rho \in \mathbb{R} \setminus \{0\}).$$

We thus have

$$\left| \frac{z_1 p'(z_1)}{[p(z_1)]^2} \right| \geq \frac{1}{2} g_1(\rho) \geq \frac{1}{2},$$

which is also a contradiction to (15). Finally, we have

$$q(z) \prec h_1(z) \quad (z \in \mathbb{U})$$

or, equivalently,

$$|\arg \{p(z)\}| < \frac{\pi}{2}.$$

We thus find that

$$|\arg \{p(z)\}| < \frac{\pi}{2} - \alpha \quad \left(0 \leq \alpha < \frac{\pi}{2}; z \in \mathbb{U}\right).$$

□

By setting

$$p(z) = \frac{zf'(z)}{f(z)} \quad (f \in \mathcal{A}) \quad \text{and} \quad \alpha = 0$$

in Theorem 2, we can deduce the following corollary.

Corollary 5. *Let $f \in \mathcal{A}$. Then*

$$\left| \frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} - 1 \right| < \frac{1}{2} \quad \left(0 \leq \alpha < \frac{\pi}{2}; z \in \mathbb{U}\right)$$

implies that $f \in \mathcal{S}^$.*

Theorem 3. Let $\beta \in \mathbb{C}$ with $u := \Re\{\beta\} > 0$. Let p be a nonzero analytic function with $p(0) = 1$ and

$$\delta_1(\alpha) < \Im \left\{ p(z) + \beta \frac{zp'(z)}{p(z)} \right\} < \delta_2(\alpha) \quad \left(0 \leq \alpha < \frac{\pi}{2}; z \in \mathbb{U} \right), \quad (18)$$

where

$$\delta_1(\alpha) = -\frac{\sqrt{(2\cos^2\alpha + u)u} + u\sin\alpha}{\cos\alpha}$$

and

$$\delta_2(\alpha) = \frac{\sqrt{(2\cos^2\alpha + u)u} - u\sin\alpha}{\cos\alpha}.$$

Then

$$|\arg\{p(z)\}| < \frac{\pi}{2} - \alpha \quad \left(0 \leq \alpha < \frac{\pi}{2}; z \in \mathbb{U} \right).$$

Proof. We define the functions q and h_1 by (4) and (5), respectively. We also suppose that q is not subordinate to h_1 . Then, by Lemma 1, there exist points $z_1 \in \mathbb{U}$ and $\zeta_1 \in \partial\mathbb{U} \setminus \{1\}$ satisfying (6). We also have $\rho \neq 0$ in (6). Thus, from the equations (4), (5), (6) and (7), we have

$$\begin{aligned} \Im \left\{ p(z_1) + \beta \frac{z_1 p'(z_1)}{p(z_1)} \right\} &= \Im \left\{ e^{-i\alpha} q(z_1) + \beta \frac{z_1 q'(z_1)}{q(z_1)} \right\} \\ &= \Im \left\{ e^{-i\alpha} h(\zeta_1) + \beta \frac{m\zeta_1 h'(\zeta_1)}{h(\zeta_1)} \right\} \\ &= \Im \left\{ e^{-i\alpha}(i\rho) + \beta \frac{m\sigma_1}{i\rho} \right\} \\ &= \rho \cos\alpha - \frac{m\sigma_1 u}{\rho}, \end{aligned}$$

where $u = \Re\{\beta\}$ and σ_1 is given by (7). For the case when $\rho > 0$, we have

$$\begin{aligned} &\rho \cos\alpha - \frac{m\sigma_1 u}{\rho} \\ &\geq \rho \cos\alpha - \frac{\sigma_1 u}{\rho} \\ &= \rho \cos\alpha + \frac{u(\rho^2 - 2\rho \sin\alpha + 1)}{2\rho \cos\alpha} \\ &= \frac{1}{2\cos\alpha} \left\{ (2\cos^2\alpha + u)\rho + \frac{u}{\rho} - 2u\sin\alpha \right\} \\ &\geq \frac{1}{2\cos\alpha} \left\{ 2\sqrt{(2\cos^2\alpha + u)u} - 2u\sin\alpha \right\} \\ &= \delta_2(\alpha). \end{aligned}$$

Therefore, we have

$$\Im \left\{ p(z_1) + \beta \frac{z_1 p'(z_1)}{p(z_1)} \right\} \geq \delta_2(\alpha),$$

which is a contradiction to (18). For the case when $\rho < 0$, we put

$$\tilde{\rho} = -\rho > 0.$$

Then, by using the same method as the above, we obtain

$$\begin{aligned}
 & \rho \cos \alpha - \frac{m\sigma_1 u}{\rho} \\
 & \leq -\tilde{\rho} \cos \alpha + \frac{\sigma_1 u}{\tilde{\rho}} \\
 & = -\frac{1}{2 \cos \alpha} \left\{ (2 \cos^2 \alpha + u) \tilde{\rho} + \frac{u}{\tilde{\rho}} + 2u \sin \alpha \right\} \\
 & \leq -\frac{1}{2 \cos \alpha} \left\{ 2\sqrt{(2 \cos^2 \alpha + u)u} + 2u \sin \alpha \right\} \\
 & = \delta_1(\alpha).
 \end{aligned}$$

Moreover, this last inequality yields

$$\Im \left\{ p(z_1) + \beta \frac{z_1 p'(z_1)}{p(z_1)} \right\} \leq \delta_1(\alpha),$$

which is a contradiction to (18). Hence we have

$$\Re \{ e^{i\alpha} p(z) \} > 0 \quad \left(0 \leq \alpha < \frac{\pi}{2}; z \in \mathbb{U} \right). \quad (19)$$

We next define the functions r and h_2 by (10) and (11), respectively. Then, by using a similar method as the above, we obtain

$$\Re \{ e^{-i\alpha} p(z) \} > 0 \quad \left(0 \leq \alpha < \frac{\pi}{2}; z \in \mathbb{U} \right). \quad (20)$$

Thus, from (19) and (20), we have

$$|\arg \{ p(z) \}| < \frac{\pi}{2} - \alpha \quad \left(0 \leq \alpha < \frac{\pi}{2}; z \in \mathbb{U} \right).$$

The proof of Theorem 3 is thus completed. \square

Remark 1. If we put $\beta = 1$ in Theorem 3, then we can obtain the result given earlier by Kim and Cho [4, Theorem 2].

By setting

$$p(z) = \frac{zf'(z)}{f(z)} \quad (f \in \mathcal{A}) \quad \text{and} \quad \alpha = 0$$

in Theorem 3, we can deduce the following corollary.

Corollary 6. Let $\beta \in \mathbb{C}$ with $u := \Re \{ \beta \} > 0$ and let $f \in \mathcal{A}$. Then

$$\left| \Im \left\{ (1 - \beta) \frac{zf'(z)}{f(z)} + \beta \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} \right| < \sqrt{u^2 + 2u} \quad (z \in \mathbb{U})$$

implies that $f \in \mathcal{S}^*$.

Theorem 4. Let $\gamma \in \mathbb{R}$ with $\gamma > 0$. Let p be a nonzero analytic function with $p(0) = 1$ and

$$\left| p(z) + \gamma \frac{zp'(z)}{p(z)} - 1 \right| < \left(\frac{\gamma}{2} + 1 \right) |p(z)| \cos \alpha \quad \left(0 \leq \alpha < \frac{\pi}{2}; z \in \mathbb{U} \right). \quad (21)$$

Then

$$|\arg \{p(z)\}| < \frac{\pi}{2} - \alpha \quad \left(0 \leq \alpha < \frac{\pi}{2}; z \in \mathbb{U} \right).$$

Proof. Let

$$q(z) = \frac{e^{i\alpha}}{p(z)} \quad \left(0 \leq \alpha < \frac{\pi}{2}; z \in \mathbb{U} \right).$$

Also let the function h_1 be defined by (5). If the function q is not subordinate to h_1 , then there exist points $z_1 \in \mathbb{U}$ and $\zeta_1 \in \partial\mathbb{U} \setminus \{1\}$ satisfying (4). By using the equations (4), (5), (6) and (7), we have

$$\begin{aligned} & \frac{|p(z_1) + \gamma \frac{z_1 p'(z_1)}{p(z_1)} - 1|}{|p(z_1)|} \\ &= |e^{-i\alpha} q(z_1) + e^{-i\alpha} \gamma z_1 q'(z_1) - 1| \\ &= |h(\zeta_1) + m\gamma \zeta_1 h'_1(\zeta_1) - e^{i\alpha}| \\ &= |i\rho + m\gamma \sigma_1 - e^{i\alpha}| \\ &= \sqrt{(m\gamma \sigma_1 - \cos \alpha)^2 + (\rho - \sin \alpha)^2} \\ &\geq \sqrt{(|\sigma_1|\gamma + \cos \alpha)^2 + (\rho - \sin \alpha)^2} \\ &= \sqrt{\left(\frac{\gamma}{2 \cos \alpha} (\rho - \sin \alpha)^2 + \frac{1}{2} \gamma \cos \alpha + \cos \alpha \right)^2 + (\rho - \sin \alpha)^2} \\ &\geq \left(\frac{\gamma}{2} + 1 \right) \cos \alpha. \end{aligned}$$

We thus find that

$$\left| p(z_1) + \gamma \frac{z_1 p'(z_1)}{p(z_1)} - 1 \right| \geq \left(\frac{\gamma}{2} + 1 \right) |p(z_1)| \cos \alpha,$$

which is a contradiction to (21). Therefore, we have

$$q(z) \prec h_1(z) \quad (z \in \mathbb{U}),$$

that is,

$$\Re \left\{ \frac{e^{i\alpha}}{p(z)} \right\} > 0 \quad \left(0 \leq \alpha < \frac{\pi}{2}; z \in \mathbb{U} \right). \quad (22)$$

We next consider the function $r(z)$ defined by

$$r(z) = \frac{e^{-i\alpha}}{p(z)} \quad \left(0 \leq \alpha < \frac{\pi}{2}; z \in \mathbb{U} \right)$$

and the function h_2 defined by (11). Using a similar method as the above, we obtain

$$\Re \left\{ \frac{e^{-i\alpha}}{p(z)} \right\} > 0 \quad \left(0 \leq \alpha < \frac{\pi}{2}; z \in \mathbb{U} \right). \quad (23)$$

Therefore, by (22) and (23), we have the assertion of Theorem 4. \square

Remark 2. If we put $\gamma = 1$ in Theorem 4, then we can obtain the result proven earlier by Kim and Cho [4, Theorem 3].

If we take

$$p(z) = \frac{zf'(z)}{f(z)} \quad (f \in \mathcal{A}) \quad \text{and} \quad \alpha = 0$$

in Theorem 4, we obtain the following result.

Corollary 7. Let $\gamma \in \mathbb{R}$ with $\gamma > 0$ and let $f \in \mathcal{A}$ with

$$\frac{f(z)}{z} \neq 0 \quad (z \in \mathbb{U}).$$

Then the following inequality:

$$\left| (1 - \gamma) \frac{zf'(z)}{f(z)} + \gamma \left(1 + \frac{zf''(z)}{f'(z)} \right) - 1 \right| < \left(\frac{\gamma}{2} + 1 \right) \left| \frac{zf'(z)}{f(z)} \right|$$

implies that $f \in \mathcal{S}^*$.

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